M612 Spring 2015 Assignment 2, due Friday Feb. 6

1. [10 pts] This problem fills in the final step of our derivation from class of the Hamilton–Jacobi equation. Suppose $L \in C^2(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$, and for each pair $(\bar{\beta}, T)$, $\bar{q}(t; \bar{\beta}, T)$ is a unique critical point of the action

$$A[\bar{q}] = \int_0^T L(\bar{q}(t), \dot{\bar{q}}(t), t) dt.$$ 

As we did in class, set

$$u(\beta, T) = \int_0^T L(\bar{q}(t; \bar{\beta}, T), \dot{\bar{q}}(t; \bar{\beta}, T), t) dt.$$ 

Show that

$$\frac{\partial u}{\partial T} = -H(\rho(T; \bar{\beta}, T), \bar{q}(T; \bar{\beta}, T), T).$$

2. [10 pts] Consider an object of mass $m$ fired directly up from the earth’s surface (with relatively small velocity), so that the Lagrangian is

$$L = \frac{1}{2} my'(t)^2 - mgy,$$

where $y(t)$ denotes height above the earth’s surface at time $t$ and $g$ denotes gravitational acceleration at the earth’s surface.

a. Write down the Euler-Lagrange equations for $L$.

b. Fix $y(0) = 0$, and find a parametrized family of solutions to the Euler-Lagrange equations $y(t; \beta, T)$ satisfying $y(0; \beta, T) = 0$ and $y(T; \beta, T) = \beta$. Is $y(t; \beta, T)$ uniquely determined? For $T > 0$ is $y(t; \beta, T)$ continuously differentiable in $\beta$ and $T$?

c. Explicitly compute the action

$$u(\beta, T) = \int_0^T \frac{1}{2} my'(t; \beta, T) - mgy(t; \beta, T) dt,$$

and verify by direct calculation that it satisfies the Hamilton-Jacobi equation

$$u_T + \frac{1}{2m}u_\beta^2 + mg\beta = 0.$$ 

3. [10 pts] Use the method of characteristics to solve the PDE

$$u_t + |Du|^2 = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+$$

$$u(x, 0) = g(x), \quad x \in \mathbb{R}^n,$$

for $g \in C^1(\mathbb{R}^n)$. For the case $n = 1$, explain how the solution could fail to exist.

Note. There will be an implicit nature to your final solution.

4. [10 pts] (Evans 3.5.10.) If $H : \mathbb{R}^n \to \mathbb{R}$ is convex, we write $L = H^*$,
a. Let \( H(\vec{p}) = \frac{1}{r} |\vec{p}|^r \), for \( 1 < r < \infty \). Show

\[
L(\vec{v}) = \frac{1}{s} |\vec{v}|^s, \quad \text{where} \quad \frac{1}{r} + \frac{1}{s} = 1.
\]

b. Let \( H(\vec{p}) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j + \sum_{i=1}^{n} b_i p_i \), where \( A = ((a_{ij})) \) is a symmetric, positive definite matrix, \( \vec{b} \in \mathbb{R}^n \). Compute \( L(\vec{v}) \).

**Note.** In Part (b) you may find it convenient to observe that if we regard \( \vec{p} \) as a row vector

\[
H(\vec{p}) = \frac{1}{2} \vec{p} A \vec{p}^T + \vec{b} \cdot \vec{p}.
\]