Assignment 9 Selected Hints

2. Let $W_\epsilon$ be as in the hint and note that by Step 4 of the proof of Theorem A.C.6 we have
\[
\|\eta_\epsilon * D^\alpha u_\epsilon\|_{L^p(V)} \leq \|D^\alpha u_\epsilon\|_{L^p(W_\epsilon)}.
\]
Now $D^\alpha u \in L^p(U)$ so take $g_\alpha \in C(U)$ so that $\|D^\alpha u - g_\alpha\|_{L^p(U)} < \delta$, where $\delta > 0$ can be taken arbitrarily small by a choice of $g_\alpha$. We have (setting $g_\epsilon^\alpha(\vec{x}) = g_\alpha(\vec{x} + \epsilon \lambda \hat{e}_n)$)
\[
\|D^\alpha u_\epsilon - g_\epsilon^\alpha\|_{L^p(W_\epsilon)}^p = \int_{W_\epsilon} |D^\alpha u(\vec{x} + \epsilon \lambda \hat{e}_n) - g_\alpha(\vec{x} + \epsilon \lambda \hat{e}_n)|^p d\vec{x}
\leq \|D^\alpha u - g_\alpha\|_{L^p(U)}^p < \delta^p,
\]
where in the first inequality we simply changed the variable of integration. Now compute
\[
\|\eta_\epsilon * D^\alpha u_\epsilon - D^\alpha u_\epsilon\|_{L^p(V)} \leq \|\eta_\epsilon * D^\alpha u_\epsilon - \eta_\epsilon * g_\epsilon^\alpha\|_{L^p(V)} + \|\eta_\epsilon * g_\epsilon^\alpha - g_\alpha\|_{L^p(V)}
+ \|g_\alpha - D^\alpha u_\epsilon\|_{L^p(V)} = I_\epsilon + J_\epsilon + K_\epsilon.
\]
Now show that $I_\epsilon + J_\epsilon + K_\epsilon \to 0$ as $\epsilon \to 0$.

9. I said in class that I wasn’t going to ask about the material from Section 5.8.1, but that’s exactly what you need for this problem. In particular, this is a trivial application of Theorem 5.8.1. (Note that Evans assumes $\partial U$ is smooth at the beginning of the problem set.)