M641 Fall 2012, Assignment 1, due Wednesday Sept. 5

1. [10 pts] In this problem I want to collect several important differentiation formulas. First, recall that for a function $\vec{f}(\vec{x})$, with $\vec{x} \in \mathbb{R}^n$ and $\vec{f} \in \mathbb{R}^m$, we're using the notation

$$D_x \vec{f}(\vec{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Two important special cases are $n = 1$ and $m = 1$. First, for $m = 1$ we obtain the gradient vector

$$Df(\vec{x}) = (f_{x_1}, f_{x_2}, \dots, f_{x_n}).$$

(The gradient vector is often regarded as a column vector, and in certain contexts this is important, so I may not typically refer to this as the gradient vector.) For $n = 1$ we obtain

$$D\vec{f}(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \\ \vdots \\ \frac{\partial f_m}{\partial x} \end{pmatrix} = \frac{d\vec{f}}{dx}.$$

Theorem (Chain Rule). Suppose $\vec{y} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{x}_0 \in \mathbb{R}^n$ and $\vec{f} : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $\vec{y}(\vec{x}_0) \in \mathbb{R}^m$. Then $\vec{h} = \vec{f}(\vec{y}(\vec{x}))$ is differentiable at $\vec{x} = \vec{x}_0$ and

$$D_x \vec{h}(\vec{x}_0) = D_y \vec{f}(\vec{y}(\vec{x}_0)) D_x \vec{y}(\vec{x}_0).$$

In particular, notice that this is the multiplication of a $k \times m$ matrix by a $m \times n$ matrix, resulting in a $k \times m$ matrix.

For the following problems assume all functions are as differentiable as necessary. In all cases you can apply the chain rule without proof.

(a) Show that if A is any $n \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a column vector, then

$$D_x(A\vec{x}) = A,$$

and likewise

$$D_x(\vec{x}^{tr} A^{tr}) = A$$

(b) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{x} : \mathbb{R} \to \mathbb{R}^n$ then

$$\frac{d}{dt} f(\vec{x}(t)) = D_x f \cdot \frac{d\vec{x}}{dt}.$$

(c) Show that if $\vec{x} \in \mathbb{R}^n$ and $f : [0, \infty) \to \mathbb{R}$ then

$$D_x f(|\vec{x}|) = f'(|\vec{x}|) \frac{\vec{x}}{|\vec{x}|},$$

and in particular

$$D_x(|\vec{x}|^r) = r|\vec{x}|^{r-2}\vec{x}.$$

(d) Show that if $\vec{x} \in \mathbb{R}^n$ and $f : [0, \infty) \to \mathbb{R}$ then

$$\Delta f(|\vec{x}|) = f''(|\vec{x}|) + \frac{n-1}{|\vec{x}|} f'(|\vec{x}|).$$

Here, $\Delta := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$.

(e) Show that if $u, v : \mathbb{R}^n \to \mathbb{R}$ then

$$\Delta(uv) = v\Delta u + 2Du \cdot Dv + u\Delta v.$$

(f) Show that if $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g} : \mathbb{R}^n \to \mathbb{R}^m$ are regarded as row vectors then

$$D_x(\vec{f}(\vec{x}) \cdot \vec{g}(\vec{x})) = \vec{g}(\vec{x}) D_x \vec{f}(\vec{x}) + \vec{f}(\vec{x}) D_x \vec{g}(\vec{x}).$$

(g) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ then

$$D_x(f(\vec{x})\vec{g}(\vec{x})) = \vec{g}(\vec{x}) \otimes D_x f(\vec{x}) + f(\vec{x}) D_x \vec{g}(\vec{x}),$$

where $\otimes$ denotes the $m \times n$ tensor product matrix

$$\vec{g}(\vec{x}) \otimes D_x f(\vec{x}) = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix} (f_{x_1}, f_{x_2}, \dots, f_{x_n}).$$

2. [10 pts]

a. Show that for any positive integers $j \leq k$

$$\binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j}.$$

b. Prove the Binomial Theorem

$$(a+b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j},$$

for $k = 1, 2, \dots$.

3. [10 pts] Prove the Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} \vec{x}^\alpha.$$

4. [10 pts] Suppose $f \in C^{k+1}(B^o(0,r))$ for some $r > 0$ and prove that for any $\vec{x} \in B^o(0,r) \subset \mathbb{R}^n$

$$f(\vec{x}) = \sum_{|\alpha| \le k} \frac{D^\alpha f(0)}{\alpha!} \vec{x}^\alpha + \sum_{|\alpha| = k+1} \frac{D^\alpha f(\vec{\xi})}{\alpha!} \vec{x}^\alpha,$$

where $\vec{\xi}$ lies on a line between 0 and $\vec{x}$.

5. [10 pts] Solve the optimization problem

$$\begin{aligned} \max f(\vec{x}) &= x_1 x_2 x_3 \\ \text{s.t. } g(\vec{x}) &= 2(x_1 x_2 + x_1 x_3 + x_2 x_3) - A = 0. \end{aligned}$$