M641 Assignment 6, due Wed. Oct. 10

1 [10 pts]. Show that for any matrix $A \in \mathbb{C}^{m\times n}$ the least squares solution obtained by solving the normal equation

$$A^*A\vec{x} = A^*\vec{b}$$

subject to the condition $\langle \vec{x}, \vec{v} \rangle = 0$ for all $\vec{v} \in \mathcal{N}(A)$ is unique.

2 [10 pts]. Solve the following:

a. Suppose $x_n \to x$ as $n \to \infty$ in some inner product space X, and suppose $\|y\| \le C$, where $\|\cdot\|$ is the induced norm. Show that

$$\lim_{n\to\infty} \langle x_n, y \rangle = \langle x, y \rangle.$$

b. Let X denote a Banach space and suppose $\{x_n\}_{n=1}^\infty \subset X$. Show that if $\sum_{n=1}^\infty x_n$ is absolutely convergent (i.e., $\sum_{n=1}^\infty \|x_n\|$ converges) then $\sum_{n=1}^\infty x_n$ is convergent.

3 [10 pts]. Solve the following:

a. (**Keener Problem 2.1.1.**) Verify that ℓ^2 is a normed vector space. Show that for all sequences $x, y \in \ell^2$ the inner product

$$\langle x, y \rangle = \sum_{i=1}^\infty x_i \bar{y}_i$$

is defined and satisfies the requisite properties.

b. (**Keener Problem 2.1.3.**) Show that the sequence $\{x_n\}_{n=1}^\infty$, $x_n = \sum_{k=1}^n \frac{1}{k!}$ is a Cauchy sequence using the measure of distance $d(x, y) = |x - y|$.

4 [10 pts]. In this problem, we'll consider *metric spaces.* Intuitively, we can think of a metric as a measure of distance between two objects in a set.

Definition. Let X be a set. A metric on X is a map $\rho : X \times X \to [0, \infty)$ with the following properties:

(i) $\rho(x, y) = 0$ if and only if $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

(iii) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

A *metric space* is a set X with a metric ρ, typically denoted (X, ρ).

a. Show that $\rho(x, y) = |e^x - e^y|$ is a metric on $\mathbb{R}$.

b. Show that any normed linear space is a metric space.

c. Show that the converse of (b) is false.

5 [10 pts]. Define a map on square matrices by setting

$$\rho(A, B) := \text{rank } (A - B),$$

for any square matrices $A, B \in \mathbb{C}^{n\times n}$. Show that ρ defines a metric.