M641 Fall 2012 Assignment 7, due Wed. Oct. 17

1 [10 pts]. (**Keener Problem 2.1.10.**) Show that the Lebesgue dominated convergence theorem fails to apply for the sequence of functions $f_n(x) = n^2 x e^{-nx}$ for $x \in [0,1]$.

2 [10 pts]. We define the convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$f * g(\vec{x}) := \int_{\mathbb{R}^n} f(\vec{x} - \vec{y}) g(\vec{y}) d\vec{y}.$$

Establish the following properties of convolution:

(i)

$$f * g = g * f$$

(ii)

$$f, g \in L^2(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^\infty} \le \|f\|_{L^2} \|g\|_{L^2}$$

(iii)

$$f, g \in L^1(\mathbb{R}^n) \Rightarrow \|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}$$

(iv)

$$f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow f * g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

3 [10 pts]. Prove the following theorem: Suppose $U \subset \mathbb{R}^n$ is open and bounded. Then for $1 \le q \le p$ we have

$$\|u\|_{L^q(U)} \le C_{p,q} \|u\|_{L^p(U)},$$

for some constant $C_{p,q}$ that you should identify in your proof.

4 [10 pts]. Show that if $f \in L^1(\mathbb{R}^n)$ then given any $\epsilon > 0$ there exists $\delta > 0$ so that

$$\Big| E \Big| < \delta \Rightarrow \int_E |f| d\vec{x} < \epsilon.$$

5 [10 pts]. Suppose $k \in \{0, 1, 2, \ldots\}$ and $0 < \gamma \le 1$. Prove that $C^{k,\gamma}(\bar{U})$ is a Banach space.

Note. You can assume that for each $k \in \{0, 1, 2, \ldots\}$ $C^k(\bar{U})$ is a Banach space. This can be proved similarly as for the case $C([a, b])$, which Keener carries out on p. 61.