THE MASLOV AND MORSE INDICES FOR SYSTEM SCHRÖDINGER OPERATORS ON $\mathbb{R}$

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Abstract. Assuming a symmetric matrix-valued potential that approaches constant end-states with a sufficient asymptotic rate, we relate the Maslov and Morse indices for Schrödinger operators on $\mathbb{R}$. In particular, we show that with our choice of convention, the Morse index is precisely the negative of the Maslov index. Our analysis is motivated, in part, by applications to stability of nonlinear waves, for which the Morse index of an associated linear operator typically determines stability. In a series of three examples, we illustrate the role of our result in such applications.

1. Introduction

We consider eigenvalue problems

$$Hy := -y'' + V(x)y = \lambda y; \quad \mathrm{dom}(H) = H^2(\mathbb{R}), \tag{1.1}$$

and also (for any $s \in \mathbb{R}$)

$$H_s y := -y'' + sy' + V(x)y = \lambda y; \quad \mathrm{dom}(H_s) = H^2(\mathbb{R}), \tag{1.2}$$

where $\lambda \in \mathbb{R}$, $y(x) \in \mathbb{R}^n$ and $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ is a real-valued symmetric matrix potential satisfying the following asymptotic conditions:

(A1) The limits $\lim_{x\to\pm\infty} V(x) = V_\pm$ exist, and for each $M \in \mathbb{R}$,

$$\int_{-M}^{\infty} (1+|x|)|V(x) - V_+|dx < \infty; \quad \int_{-\infty}^{M} (1+|x|)|V(x) - V_-|dx < \infty.$$

(A2) The eigenvalues of $V_\pm$ are all positive. We denote the smallest among all these eigenvalues $\nu_{\min} > 0$.

The domain for H is often expressed as the set

$$\mathcal{D} := \{y \in L^2(\mathbb{R}) : y, y' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}),\ -y'' + V(x)y \in L^2(\mathbb{R})\},$$

and we note that in the current setting (i.e., under our assumptions on V) this is equivalent to $H^2(\mathbb{R})$ (see, e.g., [69]). With this domain, H is self-adjoint.

Our particular interest lies in counting the number of negative eigenvalues for H (i.e., the Morse index). We proceed by relating the Morse index to the Maslov index, which is described in Section 3. In essence, we find that the Morse index can be computed in terms of

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the Maslov index, and that while the Maslov index is less elementary than the Morse index, it can be computed (numerically) in a relatively straightforward way.

The Maslov index has its origins in the work of V. P. Maslov [56] and subsequent development by V. I. Arnol'd [3]. It has now been studied extensively, both as a fundamental geometric quantity [9, 20, 26, 28, 62, 64] and as a tool for counting the number of eigenvalues on specified intervals [10, 13, 15, 16, 18, 19, 24, 27, 43, 44, 46, 57]. In this latter context, there has been a strong resurgence of interest following the analysis by Deng and Jones (i.e., [24]) for multidimensional domains. Our aim in the current analysis is to rigorously develop a relationship between the Maslov index and the Morse index in the relatively simple setting of (1.1). Our approach is adapted from [19, 24, 42].

As a starting point, we define what we will mean by a *Lagrangian subspace* of $\mathbb{R}^{2n}$.

Definition 1.1. We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and

$$(Ju, v)_{\mathbb{R}^{2n}} = 0,$$

for all $u, v \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We sometimes adopt standard notation for symplectic forms, $\omega(u, v) = (Ju, v)_{\mathbb{R}^{2n}}$. In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and we will refer to this as the *Lagrangian Grassmannian*.

A simple example, important for intuition, is the case $n = 1$, for which $(Ju, v)_{\mathbb{R}^2} = 0$ if and only if u and v are linearly dependent. In this case, we see that any line through the origin is a Lagrangian subspace of $\mathbb{R}^2$. More generally, any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* (sometimes *Lagrangian frame*) for ℓ. Moreover, we will often write $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote paths of Lagrangian subspaces $\ell_i : I \to \Lambda(n)$, for some parameter interval I. The Maslov index associated with these paths, which we will denote $\operatorname{Mas}(\ell_1, \ell_2; I)$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (Precise definitions of what we mean in this context by *multiplicity* and *direction* will be given in Section 3.) In some cases, the Lagrangian subspaces will be defined along some contour in the (α, β)-plane

$$\Gamma = \{(\alpha(t), \beta(t)) : t \in I\},$$

and when it is convenient we will use the notation $\operatorname{Mas}(\ell_1, \ell_2; \Gamma)$.

We will verify in Section 2 that under our assumptions on $V(x)$, and for $\lambda < \nu_{\min}$, (1.1) has n linearly independent solutions that decay to zero as $x \to -\infty$ and n linearly independent solutions that decay to zero as $x \to +\infty$. We express these respectively as

$$\begin{aligned} \phi^-_{n+j}(x; \lambda) &= e^{\mu^-_{n+j}(\lambda) x}(r^-_j + \mathcal{E}^-_j(x; \lambda)), \\ \phi^+_j(x; \lambda) &= e^{\mu^+_j(\lambda) x}(r^+_{n+1-j} + \mathcal{E}^+_j(x; \lambda)), \end{aligned}$$

with also

$$\partial_x \phi^-_{n+j}(x;\lambda) = e^{\mu^-_{n+j}(\lambda)x}(\mu^-_{n+j} r^-_j + \tilde{\mathcal{E}}^-_j(x;\lambda)),$$
$$\partial_x \phi^+_j(x;\lambda) = e^{\mu^+_j(\lambda)x}(\mu^+_j r^+_{n+1-j} + \tilde{\mathcal{E}}^+_j(x;\lambda)),$$

for $j = 1, 2, \dots, n$, where the nature of the $\mu^\pm_j \in \mathbb{R}$, $r^\pm_j \in \mathbb{R}^n$, and the error terms $\mathcal{E}^\pm_j(x;\lambda), \tilde{\mathcal{E}}^\pm_j(x;\lambda)$ are developed in Section 2. The only detail we will need for this preliminary discussion is the observation that under assumptions (A1) and (A2)

$$\lim_{x \to \pm\infty} \mathcal{E}^\pm_j(x;\lambda) = 0; \quad \lim_{x \to \pm\infty} \tilde{\mathcal{E}}^\pm_j(x;\lambda) = 0. \tag{1.3}$$

We will verify in Section 2 that if we create a frame $\mathbf{X}^-(x;\lambda) = \binom{X^-(x;\lambda)}{Y^-(x;\lambda)}$ by taking $\{\phi^-_{n+j}(x;\lambda)\}_{j=1}^n$ as the columns of $X^-(x;\lambda)$ and $\{\partial_x \phi^-_{n+j}(x;\lambda)\}_{j=1}^n$ as the respective columns of $Y^-(x;\lambda)$ then $\mathbf{X}^-(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^-(x;\lambda)$. Likewise, we can create a frame $\mathbf{X}^+(x;\lambda) = \binom{X^+(x;\lambda)}{Y^+(x;\lambda)}$ by taking $\{\phi^+_j(x;\lambda)\}_{j=1}^n$ as the columns of $X^+(x;\lambda)$ and $\{\partial_x \phi^+_j(x;\lambda)\}_{j=1}^n$ as the respective columns of $Y^+(x;\lambda)$. Then $\mathbf{X}^+(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^+(x;\lambda)$.

In constucting our Lagrangian frames, we can view the exponential multipliers $e^{\mu^\pm_j x}$ as expansion coefficients, and if we drop these off we retain frames for the same spaces. That is, we can create an alternative frame for $\ell^-(x;\lambda)$ by taking the expressions $r^-_j + \mathcal{E}^-_j(x;\lambda)$ as the columns of (a modification of) $X^-(x;\lambda)$ and the expressions $\mu^-_{n+j} r^-_j + \tilde{\mathcal{E}}^-_j(x;\lambda)$ as the corresponding columns for (a modification of) $Y^-(x;\lambda)$. Using (1.3) we see that in the limit as x tends to $-\infty$ (of the resulting modified frames) we obtain the frame $\mathbf{R}^-(\lambda) = \binom{R^-}{S^-(\lambda)}$, where

$$R^- = \begin{pmatrix} r^-_1 & r^-_2 & \dots & r^-_n \end{pmatrix},$$
$$S^-(\lambda) = \begin{pmatrix} \mu^-_{n+1}(\lambda) r^-_1 & \mu^-_{n+2}(\lambda) r^-_2 & \dots & \mu^-_{2n}(\lambda) r^-_n \end{pmatrix}.$$

(The dependence on λ is specified here to emphasize the fact that $S^-(\lambda)$ depends on λ through the multipliers $\{\mu^-_{n+j}\}_{j=1}^n$.) We will verify in Section 2 that $\mathbf{R}^-(\lambda)$ is the frame for a Lagrangian subspace, and we denote this space $\ell^-_{\mathbf{R}}(\lambda)$.

Proceeding similarly with $\ell^+(x;\lambda)$, we obtain the asymptotic Lagrangian subspace $\ell^+_{\mathbf{R}}(\lambda)$ with frame $\mathbf{R}^+(\lambda) = \binom{R^+}{S^+(\lambda)}$, where

$$\begin{aligned} R^+ &= \begin{pmatrix} r^+_n & r^+_{n-1} & \dots & r^+_1 \end{pmatrix}, \\ S^+(\lambda) &= \begin{pmatrix} \mu^+_1(\lambda) r^+_n & \mu^+_2(\lambda) r^+_{n-1} & \dots & \mu^+_n(\lambda) r^+_1 \end{pmatrix}. \end{aligned} \tag{1.4}$$

(The ordering of the columns of $\mathbf{R}^+$ is simply a convention, which follows naturally from our convention for indexing $\{\phi^+_j\}_{j=1}^n$.)

Let $\bar{\Gamma}_0$ denote the contour in the (x,λ)-plane obtained by fixing $\lambda = 0$ and letting x run from $-\infty$ to $+\infty$. We stress that along $\bar{\Gamma}_0$ the path ℓ^- depends on x (i.e., we have $\ell^-(x;0)$, with x running from $-\infty$ to $+\infty$), while $\ell^+_{\mathbf{R}}$ does not depend on x (i.e., we have $\ell^+_{\mathbf{R}}(0)$, independent of x).

We next state the main result of our paper, which relates the Morse index $\mathrm{Mor}(H)$ to the Maslov index $\mathrm{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0)$. For purposes of exposition, we have elected to postpone a precise (somewhat technical) definition of the Maslov index until a later section (Section 3), but we note that intuitively $\operatorname{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0)$ can be viewed as the number of twists the path ℓ^- has relative to the fixed Lagrangian subspace $\ell^+_{\mathbf{R}}$, as x runs from $-\infty$ to $+\infty$. For example, in the case of a single equation, the frame for $\ell^-(x;\lambda)$ can be taken as $(\phi^-(x;\lambda), \phi^{-\prime}(x;\lambda))^t$, for any solution $\phi^-(x;\lambda)$ that decays as $x \to -\infty$, and such frames can be identified with points on S^1 in the obvious way. The number of twists (half cycles of S^1) can be counted as the number of values $x_* \in \mathbb{R}$ so that $\phi^-(x_*;\lambda) = 0$, corresponding with the usual Sturm oscillation count.

Theorem 1.2. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then*

$$\operatorname{Mor}(H) = -\operatorname{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0).$$

Remark 1.3. The advantage of this theorem resides in the fact that the Maslov index on the right-hand side is generally straightforward to compute numerically. See, for example, [13, 14, 15, 16, 18], and the examples we discuss in Section 6. The choice of $\lambda = 0$ for $\bar{\Gamma}_0$ is not necessary for the analysis, and indeed if we fix any $\lambda_0 < \nu_{\min}$ and denote by $\bar{\Gamma}_{\lambda_0}$ the contour in the (x,λ)-plane obtained by fixing $\lambda = \lambda_0$ and letting x run from $-\infty$ to $+\infty$ then $\operatorname{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_{\lambda_0})$ will be negative the count of eigenvalues of H strictly less than λ_0. Since $\bar{\Gamma}_0$ plays a distinguished role, we refer to $\operatorname{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0)$ as the *Principal Maslov Index* (following [42]). Here, we use $\bar{\Gamma}_0$ to emphasize that $x \in [-\infty, \infty]$; the notation Γ_0 will be reserved below for the contour obtained by fixing $\lambda = 0$ and letting x run from $-\infty$ to x_∞ for a fixed large x_∞.

Remark 1.4. Our restriction in Remark 1.3 to the case $\lambda_0 < \nu_{\min}$ keeps our analysis strictly below the essential spectrum, and we briefly note here the primary technical difficulty that arises if this restriction is not enforced. For simplicity, suppose $\nu_{\min} = 0$ and observe that in this case we will have at least one decay solution of the form

$$\mathbf{p}^+_n(x;\lambda) = e^{-\sqrt{-\lambda}x}\Big(\begin{pmatrix} r^+_1 \\ -\sqrt{-\lambda}r^+_1 \end{pmatrix} + \mathbf{E}^+_n(x;\lambda)\Big)$$

(typically referred to as a *slow* decay solution), and (as discussed in Lemma 2.2 below), a corresponding growth solution

$$\mathbf{p}^+_{n+1}(x;\lambda) = e^{+\sqrt{-\lambda}x}\Big(\begin{pmatrix} r^+_1 \\ +\sqrt{-\lambda}r^+_1 \end{pmatrix} + \mathbf{E}^+_{n+1}(x;\lambda)\Big),$$

(typically referred to as a *slow* growth solution). Fundamentally, the difficulty we run into in moving λ_0 to $\nu_{\min} = 0$ is that it's difficult to keep $\mathbf{p}^+_n(x;\lambda)$ and $\mathbf{p}^+_{n+1}(x;\lambda)$ separated for λ small and x large.

Remark 1.5. In Section 3 our definition of the Maslov index will be for compact intervals I. We will see that we are able to view $\ell^-(\cdot;0) : x \mapsto \Lambda(n)$ as a continuous path of Lagrangian subspaces on $[-1,1]$ by virtue of the change of variables

$$x = \ln(\frac{1+\tau}{1-\tau}), \quad \tau \in [-1,1]. \tag{1.5}$$

We will verify in Section 5 that for $s \in \mathbb{R}$, any eigenvalue of H_s with real part less than or equal to $\nu_{\min}$ must be real-valued. This observation will allow us to construct the Lagrangian subspaces $\ell^-(x;\lambda)$ and $\ell^+_{\mathbf{R}}(\lambda)$ in that case through a development that looks identical to the discussion above. We obtain the following theorem.

Theorem 1.6. *Let $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Let $s \in \mathbb{R}$, and let $\ell^-(x;0)$ and $\ell^+_{\mathbf{R}}(0)$ denote Lagrangian subspaces developed for (1.2). Then*

$$\operatorname{Mor}(H_s) = -\operatorname{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0).$$

Remark 1.7. As described in more detail in Sections 5 and 6, equations of forms (1.1) and (1.2) arise naturally when a gradient system

$$u_t + F'(u) = u_{xx}$$

is linearized about an asymptotically constant stationary solution $\bar{u}(x)$ or a traveling wave solution $\bar{u}(x - st)$ (respectively). The case of solitary waves, for which (without loss of generality)

$$\lim_{x\to\pm\infty} \bar{u}(x) = 0,$$

has been analyzed in [10, 15, 16, 17, 18] (with $s \neq 0$ in [10] and $s = 0$ in the others). In particular, theorems along the lines of our Theorem 1.2 (though restricted to the case of solitary waves) appear as Corollary 3.8 in [10] and Proposition 35 in Appendix C.2 of [18]. The same framework can also be applied in the context of periodic stationary solutions (see, e.g., [45, 60, 61]), but we do not pursue that here.

We conclude this section by stating a straightforward corollary of (the proofs of) Theorems 1.2 and 1.6 (see particularly Claim 4.11). We remark on this in part to observe the relationship between the current analysis and the elegant analyses in [12] and [66], in which the authors use exponential dichotomy techniques to establish that the spectrum of H on $\mathbb{R}$ (and indeed for a much larger class of operators) can be approximated by the spectrum of H on a bounded (sufficiently large) interval I, with appropriate boundary conditions. The primary differences between the results of [12] and the corollary stated here are: (1) our corollary is for a half-line problem instead of a bounded-interval problem, and (2) our boundary condition depends on the spectral parameter λ.

Before stating our corollary, we note that if λ_0 is an eigenvalue of H_s satisfying $\lambda_0 < \nu_{\min}$ then it must be isolated, because it is away from essential spectrum (see Section 2). In this way, we can find $\epsilon_0 > 0$ sufficiently small so that λ_0 is the only eigenvalue of H_s in the disk

$$D_{\epsilon_0} := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \epsilon_0\},$$

and we can additionally take ϵ_0 small enough so that for all $\lambda \in D_{\epsilon_0}$ we have $\lambda < \nu_{\min}$. Let E denote the eigenspace associated with λ_0 (as an eigenvalue of H_s). For L taken sufficiently large in the corollary, we will consider the half-line problem

$$\begin{aligned} \mathcal{H}_L y &:= -y'' + V(x)y = \lambda y \\ y &\in H^2((-\infty, L]); \quad y(L) \in \ell^+_{\mathbf{R}}(\lambda), \end{aligned}$$

and for any $0 < \epsilon \le \epsilon_0$, we denote by σ_L^ϵ the collection of eigenvalues of $\mathcal{H}_L$ in D_ϵ (i.e., $\sigma_L^\epsilon = \sigma(\mathcal{H}_L) \cap D_\epsilon$). Finally, we denote by E_L^ϵ the direct sum of eigenspaces associated with the eigenvalues of $\mathcal{H}_L$ in σ_L^ϵ.

Corollary 1.8. *Let the assumptions of Theorem* 1.6 *hold, as well as the notation in the preceding paragraph. Given any* $0 < \epsilon \le \epsilon_0$ *there exists* x_∞ *sufficiently large so that for all* $L \ge x_\infty$

$$\dim E = \dim E_L^\epsilon.$$

In particular, the number of eigenvalues of $\mathcal{H}_L$ *in* D_ϵ*, counted with multiplicity, is equal to the multiplicity of* λ_0 *as an eigenvalue of* H*.*

Plan of the paper. In Section 2 we develop several relatively standard results from ODE theory that will be necessary for our construction and analysis of the Maslov index. In Section 3, we define the Maslov index, and discuss some of its salient properties, and in Section 4 we prove Theorem 1.2. In Section 5, we verify that the analysis can be extended to the case of any $s \in \mathbb{R}$, and finally, in Section 6 we provide some illustrative applications.

2. ODE Preliminaries

In this section, we develop preliminary ODE results that will serve as the foundation of our analysis. This development is standard, and follows [70], pp. 779-781 (see, e.g., [7, 21] for similar analyses). We begin by clarifying our terminology.

Definition 2.1. We define the point spectrum of H, denoted $\sigma_{\mathrm{pt}}(H)$, as the set

$$\sigma_{\mathrm{pt}}(H) = \{\lambda \in \mathbb{R} : H\phi = \lambda\phi \text{ for some } \phi \in H^2(\mathbb{R})\backslash\{0\}\}.$$

We define the essential spectrum of H, denoted $\sigma_{\mathrm{ess}}(H)$, as the values in $\mathbb{R}$ that are not in the resolvent set of H and are not isolated eigenvalues of finite multiplicity.

We note that the total spectrum is $\sigma = \sigma_{\mathrm{pt}}(H) \cup \sigma_{\mathrm{ess}}(H)$, and the *discrete* spectrum is defined as $\sigma_{\mathrm{discrete}}(H) = \sigma \backslash \sigma_{\mathrm{ess}}(H)$. Since our analysis takes place entirely away from essential spectrum, the eigenvalues we are counting are elements of the discrete spectrum.

As discussed, for example, in [38, 52], the essential spectrum of H is determined by the asymptotic equations

$$-y'' + V_\pm y = \lambda y. \tag{2.1}$$

In particular, if we look for solutions of the form $y(x) = e^{ikx}r$, for some scalar constant $k \in \mathbb{R}$ and (non-zero) constant vector $r \in \mathbb{R}^n$ then the essential spectrum will be confined to the allowable values of λ. For (2.1), we find

$$(k^2 I + V_\pm)r = \lambda r,$$

so that

$$\lambda(k) \ge \frac{(V_\pm r, r)_{\mathbb{R}^n}}{\|r\|^2}.$$

Applying the min-max principle, we see that $\sigma_{\mathrm{ess}}(H) \subset [\nu_{\min}, \infty)$ (keeping in mind that $\nu_{\min} > 0$).

Away from essential spectrum, we begin our construction of asymptotically decaying solutions to (1.1) by looking for solutions of (2.1) of the form $\phi(x;\lambda) = e^{\mu x} r$, where in this case μ is a scalar function of λ, and r is again a constant vector in $\mathbb{R}^n$. In this case, we obtain the relation

$$(-\mu^2 I + V_\pm - \lambda I) r = 0,$$

from which we see that the values of $\mu^2 + \lambda$ will correspond with eigenvalues of $V_\pm$, and the vectors r will be eigenvectors of $V_\pm$. We denote the spectrum of $V_\pm$ by $\sigma(V_\pm) = \{\nu_j^\pm\}_{j=1}^n$, ordered so that $j < k$ implies $\nu_j^\pm \le \nu_k^\pm$, and we order the eigenvectors correspondingly so that $V_\pm r_j^\pm = \nu_j^\pm r_j^\pm$ for all $j \in \{1, 2, \dots, n\}$. Moreover, since $V_\pm$ are symmetric matrices, we can choose the set $\{r_j^-\}_{j=1}^n$ to be orthonormal, and similarly for $\{r_j^+\}_{j=1}^n$.

We have

$$\mu^2 + \lambda = \nu_j^\pm \implies \mu = \pm\sqrt{\nu_j^\pm - \lambda}.$$

We will denote the admissible values of μ by $\{\mu_j^\pm\}_{j=1}^{2n}$, and for consistency we choose our labeling scheme so that $j < k$ implies $\mu_j^\pm \le \mu_k^\pm$ (for $\lambda \le \nu_{\min}$). This leads us to the specifications

$$\begin{aligned} \mu_j^\pm(\lambda) &= -\sqrt{\nu_{n+1-j}^\pm - \lambda} \\ \mu_{n+j}^\pm(\lambda) &= \sqrt{\nu_j^\pm - \lambda}, \end{aligned}$$

for $j = 1, 2, \dots, n$.

We now express (1.1) as a first order system, with $\mathbf{p} = \binom{p}{q} = \binom{y}{y'}$. We find

$$\frac{d\mathbf{p}}{dx} = \mathbb{A}(x;\lambda)\mathbf{p}; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I \\ V(x) - \lambda I & 0 \end{pmatrix}, \tag{2.2}$$

and we additionally set

$$\mathbb{A}_\pm(\lambda) := \lim_{x \to \pm\infty} \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I \\ V_\pm - \lambda I & 0 \end{pmatrix}.$$

We note that the eigenvalues of $\mathbb{A}_\pm$ are precisely the values $\{\mu_j^\pm\}_{j=1}^{2n}$, and the associated eigenvectors are $\{\mathscr{r}_j^\pm\}_{j=1}^n = \{\binom{r_{n+1-j}^\pm}{\mu_j^\pm r_{n+1-j}^\pm}\}_{j=1}^n$ and $\{\mathscr{r}_{n+j}^\pm\}_{j=1}^n = \{\binom{r_j^\pm}{\mu_{n+j}^\pm r_j^\pm}\}_{j=1}^n$.

Lemma 2.2. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then for any $\lambda < \nu_{\min}$ there exist n linearly independent solutions of (2.2) that decay to zero as $x \to -\infty$ and n linearly independent solutions of (2.2) that decay to zero as $x \to +\infty$. Respectively, we can choose these so that they can be expressed as*

$$\begin{aligned} \mathbf{p}_{n+j}^-(x;\lambda) &= e^{\mu_{n+j}^-(\lambda)x}(\mathscr{r}_{n+j}^- + \mathbf{E}_{n+j}^-(x;\lambda)); \quad j = 1, 2, \dots, n, \\ \mathbf{p}_j^+(x;\lambda) &= e^{\mu_j^+(\lambda)x}(\mathscr{r}_j^+ + \mathbf{E}_j^+(x;\lambda)); \quad j = 1, 2, \dots, n, \end{aligned}$$

where for any fixed $\lambda_0 < \nu_{\min}$ and $\lambda_\infty > 0$ (with $-\lambda_\infty < \lambda_0$), $\mathbf{E}_{n+j}^-(x;\lambda) = \mathbf{O}((1 + |x|)^{-1})$, uniformly for $\lambda \in [-\lambda_\infty, \lambda_0]$, and similarly for $\mathbf{E}_j^+(x;\lambda)$.

Moreover, there exist n linearly independent solutions of (2.2) that grow to infinity as $x \to -\infty$ and n linearly independent solutions of (2.2) that grow to infinity as $x \to +\infty$. Respectively, we can choose these so that they can be expressed as

$$\mathbf{p}_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x}(\boldsymbol{r}_j^- + \mathbf{E}_j^-(x;\lambda)); \quad j = 1,2,\dots,n,$$
$$\mathbf{p}_{n+j}^+(x;\lambda) = e^{\mu_{n+j}^+(\lambda)x}(\boldsymbol{r}_{n+j}^+ + \mathbf{E}_{n+j}^+(x;\lambda)); \quad j = 1,2,\dots,n,$$

where for any fixed $\lambda_0 < \nu_{\min}$ and $\lambda_\infty > 0$ (with $-\lambda_\infty < \lambda_0$), $\mathbf{E}_j^-(x;\lambda) = \mathbf{O}((1+|x|)^{-1})$, uniformly for $\lambda \in [-\lambda_\infty, \lambda_0]$, and similarly for $\mathbf{E}_{n+j}^+(x;\lambda)$.

Proof. Focusing on solutions that decay as $x \to -\infty$, we express (2.2) as

$$\frac{d\mathbf{p}}{dx} = \mathbb{A}_-(\lambda)\mathbf{p} + \mathcal{R}_-(x)\mathbf{p}; \quad \mathcal{R}_-(x) = \mathbb{A}(x;\lambda) - \mathbb{A}_-(\lambda) = \begin{pmatrix} 0 & 0 \\ V(x) - V_- & 0 \end{pmatrix}. \tag{2.3}$$

We have seen that asymptotically decaying solutions to the asymptotic equation $\frac{d\mathbf{p}}{dx} = \mathbb{A}_-(\lambda)\mathbf{p}$ have the form $\mathbf{p}_{n+j}^-(x;\lambda) = e^{\mu_{n+j}^-(\lambda)}\boldsymbol{r}_{n+j}^-$, and so it's natural to look for solutions of the form

$$\mathbf{p}_{n+j}^-(x;\lambda) = e^{\mu_{n+j}^-(\lambda)x}\mathbf{z}_{n+j}^-(x;\lambda),$$

for which we have

$$\frac{d\mathbf{z}_{n+j}^-(x;\lambda)}{dx} = (\mathbb{A}_-(\lambda) - \mu_{n+j}^-(\lambda)I)\mathbf{z}_{n+j}^-(x;\lambda) + \mathcal{R}_-(x)\mathbf{z}_{n+j}^-(x;\lambda). \tag{2.4}$$

Let $P_{n+j}^-(\lambda)$ project onto the eigenspace of $\mathbb{A}_-(\lambda)$ associated with eigenvalues $\sigma(\mathbb{A}_-(\lambda)) \ni \mu \le \mu_{n+j}^-$, and let $Q_{n+j}^-(\lambda)$ likewise project onto the eigenspace of $\mathbb{A}_-(\lambda)$ associated with $\sigma(\mathbb{A}_-(\lambda)) \ni \mu > \mu_{n+j}^-$. Notice particularly that there exists some $\eta > 0$ so that $\mu - \mu_{n+j}^- \ge \eta$ for all μ associated with $Q_{n+j}^-(\lambda)$.

For some fixed $M > 0$, we will look for a solution to (2.4) in $L^\infty(-\infty, -M]$ of the form

$$\begin{aligned} \mathbf{z}_{n+j}^-(x;\lambda) = \boldsymbol{r}_{n+j}^- &+ \int_{-\infty}^x e^{(\mathbb{A}_-(\lambda)-\mu_{n+j}^-(\lambda)I)(x-\xi)} P_{n+j}^-(\lambda)\mathcal{R}_-(\xi)\mathbf{z}_{n+j}^-(\xi;\lambda)d\xi \\ &- \int_x^{-M} e^{(\mathbb{A}_-(\lambda)-\mu_{n+j}^-(\lambda)I)(x-\xi)} Q_{n+j}^-(\lambda)\mathcal{R}_-(\xi)\mathbf{z}_{n+j}^-(\xi;\lambda)d\xi. \end{aligned} \tag{2.5}$$

We proceed by contraction mapping, defining $\mathcal{T}\mathbf{z}_{n+j}^-(x;\lambda)$ to be the right-hand side of (2.5). Let $\mathbf{z}_{n+j}^-, \mathbf{w}_{n+j}^- \in L^\infty(-\infty, -M]$, so that

$$\begin{aligned} |\mathcal{T}\mathbf{z}_{n+j}^- - \mathcal{T}\mathbf{w}_{n+j}^-| &\le K\|\mathbf{z}_{n+j}^- - \mathbf{w}_{n+j}^-\|_{L^\infty(-\infty,-M]}\Big\{\int_{-\infty}^x |\mathcal{R}_-(\xi)|d\xi + \int_x^{-M} e^{\eta(x-\xi)}|\mathcal{R}_-(\xi)|d\xi\Big\} \\ &=: I_1 + I_2, \end{aligned} \tag{2.6}$$

for some constant $K > 0$.

By assumption (A1) we know

$$\int_{-\infty}^0 (1+|x|)|\mathcal{R}_-(x)|dx = C < \infty,$$

so that
$$\int_{-\infty}^{-M} (1+M)|\mathcal{R}_-(\xi)|d\xi \leq \int_{-\infty}^{-M} (1+|\xi|)|\mathcal{R}_-(\xi)|d\xi \leq C,$$
giving the inequality
$$\int_{-\infty}^{-M} |\mathcal{R}_-(\xi)|dx \leq \frac{C}{1+M}.$$
Likewise, we can check that for $x \in (-\infty, -M]$
$$\int_{x}^{-M} e^{\eta(x-\xi)}|\mathcal{R}_-(\xi)|dx \leq \frac{C}{1+M}.$$
We see that
$$|\mathcal{T}\mathbf{z}_{n+j}^- - \mathcal{T}\mathbf{w}_{n+j}^-| \leq \frac{2KC}{1+M}\|\mathbf{z}_{n+j}^- - \mathbf{w}_{n+j}^-\|_{L^\infty(-\infty,-M]},$$
for all $x \in (-\infty, -M]$ so that
$$\|\mathcal{T}\mathbf{z}_{n+j}^- - \mathcal{T}\mathbf{w}_{n+j}^-\|_{L^\infty(-\infty,-M]} \leq \frac{2KC}{1+M}\|\mathbf{z}_{n+j}^- - \mathbf{w}_{n+j}^-\|_{L^\infty(-\infty,-M]},$$
and for M large enough we have the desired contraction. Moreover, the exponential decay in I_2 allows us to see that
$$\lim_{x\to-\infty} \mathbf{z}_{n+j}^-(x;\lambda) = \boldsymbol{z}_{n+j}^-,$$
with the asymptotic rate indicated.

Finally, we note that the case $x \to +\infty$ is similar. □

Recall that we denote by $\mathbf{X}^-(x;\lambda)$ the $2n \times n$ matrix obtained by taking each $\mathbf{p}_{n+j}^-(x;\lambda)$ from Lemma 2.2 as a column. In order to check that $\mathbf{X}^-(x;\lambda)$ is the frame for a Lagrangian subspace, let $\phi, \psi \in \{\mathbf{p}_{n+j}^-(x;\lambda)\}_{j=1}^n$, and consider $\omega(\phi,\psi) = (J\phi, \psi)$. First,
$$\frac{d}{dx}\omega(\phi,\psi) = (J\frac{d\phi}{dx}, \psi) + (J\phi, \frac{d\psi}{dx}) = (J\mathbb{A}\phi, \psi) + (J\phi, \mathbb{A}\psi).$$
It's important to note at this point that we can express $\mathbb{A}$ as $\mathbb{A} = J\mathbb{B}$, for the symmetric matrix
$$\mathbb{B}(x;\lambda) = \begin{pmatrix} V(x) - \lambda I & 0 \\ 0 & -I \end{pmatrix}.$$
Consequently
$$\begin{aligned}\frac{d}{dx}\omega(\phi,\psi) &= (J^2\mathbb{B}\phi, \psi) + (J\phi, J\mathbb{B}\psi) = -(\mathbb{B}\phi,\psi) - (J^2\phi, \mathbb{B}\psi) \\ &= -(\mathbb{B}\phi,\psi) + (\phi, \mathbb{B}\psi) = 0,\end{aligned}$$
where the final equality follows from the symmetry of $\mathbb{B}$. We conclude that $\omega(\phi,\psi)$ is constant in x, but since
$$\lim_{x\to-\infty} \omega(\phi,\psi) = 0,$$
this constant must be 0.

Proceeding in the same way, we can verify that $\mathbf{X}^+(x;\lambda)$ is also a frame for a Lagrangian subspace.

We conclude this section by verifying that $\mathbf{R}^-(\lambda)$ (specified in the introduction) is the frame for a Lagrangian subspace. To see this, we change notation a bit from the previous calculation and take $\binom{\phi}{\mu\phi}, \binom{\psi}{\nu\psi} \in \{\mathbf{r}^-_{n+j}\}_{j=1}^n$. We compute

$$\omega(\begin{pmatrix} \phi \\ \mu\phi \end{pmatrix}, \begin{pmatrix} \psi \\ \nu\psi \end{pmatrix}) = (J\begin{pmatrix} \phi \\ \mu\phi \end{pmatrix}, \begin{pmatrix} \psi \\ \nu\psi \end{pmatrix}) = (\nu - \mu)(\phi, \psi) = 0,$$

where the final equality follows from orthogonality of the eigenvectors of V_-. Likewise, we find that $\mathbf{R}^+(\lambda)$ is a Lagrangian subspace.

3. The Maslov Index

Given any two Lagrangian subspaces ℓ_1 and ℓ_2, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we can define the complex $n \times n$ matrix

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{3.1}$$

As verified in [41], the matrices $(X_1 - iY_1)$ and $(X_2 + iY_2)$ are both invertible, and $\tilde{W}$ is unitary. We have the following theorem from [41].

Theorem 3.1. *Suppose $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ are Lagrangian subspaces, with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, and let $\tilde{W}$ be as defined in (3.1). Then*

$$\dim \ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2).$$

That is, the dimension of the eigenspace of $\tilde{W}$ associated with the eigenvalue -1 is precisely the dimension of the intersection of the Lagrangian subspaces ℓ_1 and ℓ_2.

Following [9, 28], we use Theorem 3.1, along with an approach to spectral flow introduced in [62], to define the Maslov index. Given a parameter interval $I = [a, b]$, which can be normalized to $[0, 1]$, we consider maps $\ell : I \to \Lambda(n)$, which will be expressed as $\ell(t)$. In order to specify a notion of continuity, we need to define a metric on $\Lambda(n)$, and following [28] (p. 274), we do this in terms of orthogonal projections onto elements $\ell \in \Lambda(n)$. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^t\mathbf{X}_i)^{-1}\mathbf{X}_i^t$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that $\ell : I \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Given two continuous maps $\ell_1(t), \ell_2(t)$ on a parameter interval I, we denote by $\mathcal{L}(t)$ the path

$$\mathcal{L}(t) = (\ell_1(t), \ell_2(t)).$$

In what follows, we will define the Maslov index for the path $\mathcal{L}(t)$, which will be a count, including both multiplicity and direction, of the number of times the Lagrangian paths ℓ_1 and ℓ_2 intersect. In order to be clear about what we mean by multiplicity and direction, we observe that associated with any path $\mathcal{L}(t)$ we will have a path of unitary complex matrices as described in (3.1). We have already noted that the Lagrangian subspaces ℓ_1 and

ℓ_2 intersect at a value $t_0 \in I$ if and only if $\tilde{W}(t_0)$ has -1 as an eigenvalue. In the event of such an intersection, we define the multiplicity of the intersection to be the multiplicity of -1 as an eigenvalue of $\tilde{W}$ (since $\tilde{W}$ is unitary the algebraic and geometric multiplicites are the same). When we talk about the direction of an intersection, we mean the direction the eigenvalues of $\tilde{W}$ are moving (as t varies) along the unit circle S^1 when they cross -1 (we take counterclockwise as the positive direction). We note that we will need to take care with what we mean by a crossing in the following sense: we must decide whether to increment the Maslov index upon arrival or upon departure. Indeed, there are several different approaches to defining the Maslov index (see, for example, [20, 64]), and they often disagree on this convention.

Following [9, 28, 62] (and in particular Definition 1.4 from [9]), we proceed by choosing a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $I = [a, b]$, along with numbers $\epsilon_j \in (0, \pi)$ so that $\ker\big(\tilde{W}(t) - e^{i(\pi+\epsilon_j)}I\big) = \{0\}$ for $t_{j-1} \le t \le t_j$; that is, $e^{i(\pi+\epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ and $j = 1, \dots, n$. Moreover, we notice that for each $j = 1, \dots, n$ and any $t \in [t_{j-1}, t_j]$ there are only finitely many values $\theta \in [0, \epsilon_j)$ for which $e^{i(\pi+\theta)} \in \sigma(\tilde{W}(t))$.

Fix some $j \in \{1, 2, \dots, n\}$ and consider the value

$$k(t, \epsilon_j) := \sum_{0 \le \theta < \epsilon_j} \dim \ker \big(\tilde{W}(t) - e^{i(\pi+\theta)}I\big). \tag{3.2}$$

for $t_{j-1} \le t \le t_j$. This is precisely the sum, along with multiplicity, of the number of eigenvalues of $\tilde{W}(t)$ that lie on the arc

$$A_j := \{e^{i(\pi+\theta)} : \theta \in [0, \epsilon_j)\}.$$

(See Figure 1.) The stipulation that $e^{i(\pi\pm\epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ asserts that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $t_{j-1} \le t \le t_j$. In this way, we see that $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that enter A_j in the counterclockwise direction (i.e., through -1) minus the number that leave in the clockwise direction (again, through -1) during the interval $[t_{j-1}, t_j]$.

In dealing with the catenation of paths, it's particularly important to understand the difference $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ when an eigenvalue resides at -1 at either $t = t_{j-1}$ or $t = t_j$ (i.e., if an eigenvalue begins or ends at a crossing). If an eigenvalue moving in the counterclockwise direction arrives at -1 at $t = t_j$, then we increment the difference forward, while if the eigenvalue arrives at -1 from the clockwise direction we do not (because it was already in A_j prior to arrival). On the other hand, suppose an eigenvalue resides at -1 at $t = t_{j-1}$ and moves in the counterclockwise direction. The eigenvalue remains in A_j, and so we do not increment the difference. However, if the eigenvalue leaves in the clockwise direction then we decrement the difference. In summary, the difference increments forward upon arrivals in the counterclockwise direction, but not upon arrivals in the clockwise direction, and it decrements upon departures in the clockwise direction, but not upon departures in the counterclockwise direction.

We are now ready to define the Maslov index.

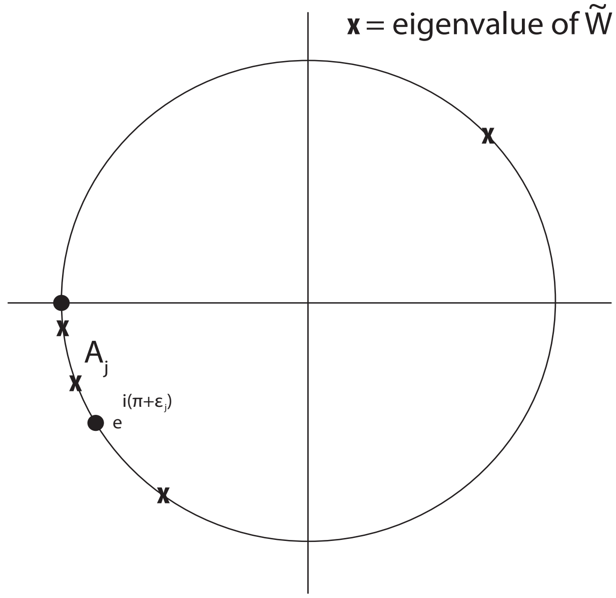

FIGURE 1. The arc A_j.

Definition 3.2. Let $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. The Maslov index $\operatorname{Mas}(\mathcal{L}; I)$ is defined by

$$\operatorname{Mas}(\mathcal{L}; I) = \sum_{j=1}^{n} (k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)). \tag{3.3}$$

Remark 3.3. As discussed in [9], the Maslov index does not depend on the choices of $\{t_j\}_{j=0}^n$ and $\{\epsilon_j\}_{j=1}^n$, so long as they follow the specifications above.

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(I)$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t, s) \in I \times [0, 1]$ into $\Lambda(n)$.

The Maslov index has the following properties (see, for example, [41] in the current setting, or Theorem 3.6 in [28] for a more general result).

(P1) (Path Additivity) If $a < b < c$ then

$$\operatorname{Mas}(\mathcal{L}; [a, c]) = \operatorname{Mas}(\mathcal{L}; [a, b]) + \operatorname{Mas}(\mathcal{L}; [b, c]).$$

(P2) (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\operatorname{Mas}(\mathcal{L}; [a, b]) = \operatorname{Mas}(\mathcal{M}; [a, b]).$$

4. Application to Schrödinger Operators

For H in (1.1), a value $\lambda \in \mathbb{R}$ is an eigenvalue (see Definition 2.1) if and only if there exist coefficient vectors $\alpha(\lambda), \beta(\lambda) \in \mathbb{R}^n$ and an eigenfunction $\phi(x;\lambda)$ so that $\mathbf{p} = \binom{\phi}{\phi'}$ satisfies

$$\mathbf{X}^-(x;\lambda)\alpha(\lambda) = \mathbf{p}(x;\lambda) = \mathbf{X}^+(x;\lambda)\beta(\lambda).$$

This clearly holds if and only if the Lagrangian subspaces $\ell^-(x;\lambda)$ and $\ell^+(x;\lambda)$ have non-trivial intersection. Moreover, the dimension of intersection will correspond with the geometric multiplicity of λ as an eigenvalue. In this way, we can fix any $x \in \mathbb{R}$ and compute the number of negative eigenvalues of H, including multiplicities, by counting the intersections of $\ell^-(x;\lambda)$ and $\ell^+(x;\lambda)$, including multiplicities. Our approach will be to choose $x = x_\infty$ for a sufficiently large value $x_\infty > 0$. Our tool for counting the number and multiplicity of intersections will be the Maslov index, and our two Lagrangian subspaces (in the roles of ℓ_1 and ℓ_2 above) will be $\ell^-(x;\lambda)$ and $\ell^+_\infty(\lambda) := \ell^+(x_\infty;\lambda)$. We will denote the Lagrangian frame associated with ℓ^+_∞ by

$$\mathbf{X}^+_\infty(\lambda) = \begin{pmatrix} X^+_\infty(\lambda) \\ Y^+_\infty(\lambda) \end{pmatrix},$$

where $X^+_\infty(\lambda) = X^+(x_\infty;\lambda)$ and $Y^+_\infty(\lambda) = Y^+(x_\infty;\lambda)$.

Remark 4.1. We will verify in the appendix that while the limit

$$\ell^-_{+\infty}(\lambda) := \lim_{x \to +\infty} \ell^-(x;\lambda)$$

is well defined for each $\lambda < \nu_{\min}$, the resulting limit is not necessarily continuous as a function of λ. This is our primary motivation for working with x_∞ rather than with the asymptotic limit.

Our analysis will be based on computing the Maslov index along a closed path in the x-λ plane, determined by sufficiently large values $x_\infty, \lambda_\infty > 0$. First, if we fix $\lambda = 0$ and let x run from $-\infty$ to x_∞, we denote the resulting path Γ_0 (the *right shelf*). Next, we fix $x = x_\infty$ and let Γ_+ denote a path in which λ decreases from 0 to $-\lambda_\infty$. Continuing counterclockwise along our path, we denote by Γ_∞ the path obtained by fixing $\lambda = -\lambda_\infty$ and letting x run from x_∞ to $-\infty$ (the *left shelf*). Finally, we close the path in an asymptotic sense by taking a final path, Γ_-, with λ running from $-\lambda_\infty$ to 0 (viewed as the asymptotic limit as $x \to -\infty$; we refer to this as the *bottom shelf*). See Figure 2.

We recall that we can take the vectors in our frame $\mathbf{X}^-(x;\lambda)$ to be

$$\mathbf{r}^-_{n+j} + \mathbf{E}^-_{n+j}(x;\lambda),$$

from which we see that $\ell^-(x;\lambda)$ approaches the asymptotic frame $\mathbf{R}^-(\lambda)$ as $x \to -\infty$. Introducing the change of variables

$$x = \ln\left(\frac{1+\tau}{1-\tau}\right) \iff \tau = \frac{e^x - 1}{e^x + 1},$$

we see that ℓ^- can be viewed as a continuous map on the compact domain

$$[-1, \frac{e^{x_\infty} - 1}{e^{x_\infty} + 1}] \times [-\lambda_\infty, 0].$$

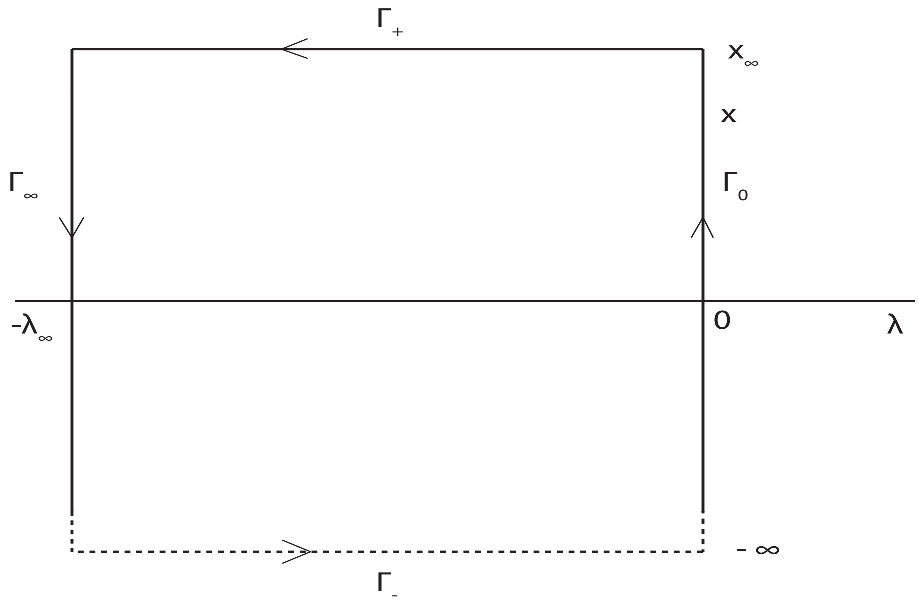

FIGURE 2. The Maslov Box.

In the setting of (1.1), our evolving Lagrangian subspaces have frames $\mathbf{X}^-(x;\lambda)$ and $\mathbf{X}^+_\infty(\lambda)$, so that $\tilde{W}$ from (3.1) becomes

$$\begin{aligned}\tilde{W}(x;\lambda) &= -(X^-(x;\lambda)+iY^-(x;\lambda))(X^-(x;\lambda)-iY^-(x;\lambda))^{-1} \\ &\quad \times (X^+_\infty(\lambda)-iY^+_\infty(\lambda))(X^+_\infty(\lambda)+iY^+_\infty(\lambda))^{-1}.\end{aligned} \tag{4.1}$$

Since $\tilde{W}(x;\lambda)$ is unitary, its eigenvalues are confined to the until circle in $\mathbb{C}$, S^1. In the limit as $x \to -\infty$ we obtain

$$\begin{aligned}\tilde{W}^-_{x_\infty}(\lambda) := \lim_{x\to-\infty} \tilde{W}(x;\lambda) &= -(R^- + iS^-(\lambda))(R^- - iS^-(\lambda))^{-1} \\ &\quad \times (X^+_\infty(\lambda)-iY^+_\infty(\lambda))(X^+_\infty(\lambda)+iY^+_\infty(\lambda))^{-1}.\end{aligned} \tag{4.2}$$

4.1. **Monotonicity.** Our first result for this section asserts that the eigenvalues of $\tilde{W}(x;\lambda)$ and $\tilde{W}^-_{x_\infty}(\lambda)$ rotate monotonically as λ varies along $\mathbb{R}$. In order to prove this, we will use a lemma from [42], which we state as follows (see also Theorem V.6.1 in [5]).

Lemma 4.2 ([42], Lemma 3.11.)**.** *Let $\tilde{W}(\tau)$ be a C^1 family of unitary $n \times n$ matrices on some interval I, satisfying a differential equation $\frac{d}{d\tau}\tilde{W}(\tau) = i\tilde{W}(\tau)\tilde{\Omega}(\tau)$, where $\tilde{\Omega}(\tau)$ is a continuous, self-adjoint and negative-definite $n \times n$ matrix. Then the eigenvalues of $\tilde{W}(\tau)$ move (strictly) monotonically clockwise on the unit circle as τ increases.*

We are now prepared to state and prove our monotonicity lemma.

Lemma 4.3. *Let $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then for each fixed $x \in \mathbb{R}$ the eigenvalues of $\tilde{W}(x;\lambda)$ rotate (strictly)* *monotonically clockwise as $\lambda \in (-\infty, \nu_{\min})$ increases. Moreover, the eigenvalues of $\tilde{W}^-_{x_\infty}(\lambda)$ rotate (strictly) monotonically clockwise as $\lambda \in (-\infty, \nu_{\min})$ increases.*

Remark 4.4. The monotonicity described in Lemma 4.3 seems to be generic for self-adjoint operators in a broad range of settings (see, for example, [42]); monotonicity in x is not generic.

Proof. Following [42], we begin by computing $\frac{\partial \tilde{W}}{\partial \lambda}$, and for this calculation it's convenient to write $\tilde{W}(x;\lambda) = -\tilde{W}_1(x;\lambda)\tilde{W}_2(\lambda)$, where

$$\begin{aligned} \tilde{W}_1(x;\lambda) &= (X^-(x;\lambda) + iY^-(x;\lambda))(X^-(x;\lambda) - iY^-(x;\lambda))^{-1} \\ \tilde{W}_2(\lambda) &= (X^+_\infty(\lambda) - iY^+_\infty(\lambda))(X^+_\infty(\lambda) + iY^+_\infty(\lambda))^{-1}. \end{aligned}$$

For $\tilde{W}_1$, we have (suppressing independent variables for notational brevity)

$$\begin{aligned} \frac{\partial \tilde{W}_1}{\partial \lambda} &= (X^-_\lambda + iY^-_\lambda)(X^- - iY^-)^{-1} - (X^- + iY^-)(X^- - iY^-)^{-1}(X^-_\lambda - iY^-_\lambda)(X^- - iY^-)^{-1} \\ &= (X^-_\lambda + iY^-_\lambda)(X^- - iY^-)^{-1} - \tilde{W}_1(X^-_\lambda - iY^-_\lambda)(X^- - iY^-)^{-1}. \end{aligned}$$

If we multiply by $\tilde{W}_1^*$ we find

$$\begin{aligned} \tilde{W}_1^* \frac{\partial \tilde{W}_1}{\partial \lambda} &= (X^{-t} + iY^{-t})^{-1}(X^{-t} - iY^{-t})(X^-_\lambda + iY^-_\lambda)(X^- - iY^-)^{-1} \\ &\quad - (X^-_\lambda - iY^-_\lambda)(X^- - iY^-)^{-1} \\ &= (X^{-t} + iY^{-t})^{-1}\Big\{(X^{-t} - iY^{-t})(X^-_\lambda + iY^-_\lambda) \\ &\qquad - (X^{-t} + iY^{-t})(X^-_\lambda - iY^-_\lambda)\Big\}(X^- - iY^-)^{-1} \\ &= \Big((X^- - iY^-)^{-1}\Big)^*\Big\{2iX^{-t}Y^-_\lambda - 2iY^{-t}X^-_\lambda\Big\}(X^- - iY^-)^{-1}. \end{aligned}$$

Multiplying back through by $\tilde{W}_1$, we conclude

$$\frac{\partial \tilde{W}_1}{\partial \lambda} = i\tilde{W}_1\tilde{\Omega}_1,$$

where

$$\tilde{\Omega}_1 = \Big((X^- - iY^-)^{-1}\Big)^*\Big\{2X^{-t}Y^-_\lambda - 2Y^{-t}X^-_\lambda\Big\}\Big((X^- - iY^-)^{-1}\Big).$$

Likewise, we find that

$$\frac{\partial \tilde{W}_2}{\partial \lambda} = i\tilde{W}_2\tilde{\Omega}_2,$$

where

$$\tilde{\Omega}_2 = \Big((X^+_\infty + iY^+_\infty)^{-1}\Big)^*\Big\{2Y^{+\,t}_\infty\partial_\lambda X^+_\infty - 2X^{+\,t}_\infty\partial_\lambda Y^+_\infty\Big\}\Big((X^+_\infty + iY^+_\infty)^{-1}\Big). \tag{4.3}$$

Combining these observations, we find

$$\begin{aligned}\frac{\partial \tilde{W}}{\partial \lambda} &= -\frac{\partial \tilde{W}_1}{\partial \lambda}\tilde{W}_2 - \tilde{W}_1\frac{\partial \tilde{W}_2}{\partial \lambda} = -i\tilde{W}_1\tilde{\Omega}_1\tilde{W}_2 - i\tilde{W}_1\tilde{W}_2\tilde{\Omega}_2 \\ &= -i\tilde{W}_1\tilde{W}_2(\tilde{W}_2^*\tilde{\Omega}_1\tilde{W}_2) - i\tilde{W}_1\tilde{W}_2\tilde{\Omega}_2 = i\tilde{W}\tilde{\Omega},\end{aligned} \tag{4.4}$$

where (recalling that $\tilde{W} = -\tilde{W}_1\tilde{W}_2$)

$$\tilde{\Omega} = \tilde{W}_2^*\tilde{\Omega}_1\tilde{W}_2 + \tilde{\Omega}_2.$$

We see that the behavior of $\frac{\partial \tilde{W}}{\partial \lambda}$ will be determined by the quantities ${X^-}^t Y^-_\lambda - {Y^-}^t X^-_\lambda$ and ${Y^+_\infty}^t \partial_\lambda X^+_\infty - {X^+_\infty}^t \partial_\lambda Y^+_\infty$. For the former, we differentiate with respect to x to find

$$\begin{aligned}\frac{\partial}{\partial x}\Big\{{X^-}^t Y^-_\lambda - {Y^-}^t X^-_\lambda\Big\} &= {X^-_x}^t Y^-_\lambda + {X^-}^t Y^-_{\lambda x} - {Y^-_x}^t X^-_\lambda - {Y^-}^t X^-_{\lambda x} \\ &= {Y^-}^t Y^-_\lambda + {X^-}^t (VX^- - \lambda X^-)_\lambda - (VX^- - \lambda X^-)^t X^-_\lambda - {Y^-}^t Y^-_\lambda \\ &= -{X^-}^t X^-,\end{aligned}$$

where we've used $X^-_x = Y^-$ and $Y^-_x = V(x)X^- - \lambda X^-$. Integrating from $-\infty$ to x, we find

$${X^-}^t Y^-_\lambda - {Y^-}^t X^-_\lambda = -\int_{-\infty}^x {X^-}^t(y;\lambda)X^-(y;\lambda)dy,$$

from which it is clear that ${X^-}^t Y^-_\lambda - {Y^-}^t X^-_\lambda$ is negative definite, which implies that $\tilde{\Omega}_1$ is negative definite.

Likewise, even though x_∞ is fixed, we can differentiate

$$Y^+(x;\lambda)^t X^+_\lambda(x;\lambda) - X^+(x;\lambda)^t Y^+_\lambda(x;\lambda)$$

with respect to x and evaluate at $x = x_\infty$ to find

$${Y^+_\infty}^t \partial_\lambda X^+_\infty - {X^+_\infty}^t \partial_\lambda Y^+_\infty = -\int_{x_\infty}^{+\infty} {X^-}^t(y;\lambda)X^-(y;\lambda)dy,$$

from which it is clear that ${Y^+_\infty}^t \partial_\lambda X^+_\infty - {X^+_\infty}^t \partial_\lambda Y^+_\infty$ is negative definite, which implies that $\tilde{\Omega}_2$ is negative definite.

We conclude that $\tilde{\Omega}$ is negative definite, at which point we can employ Lemma 3.11 from [42] to obtain the claim.

For the case of $\tilde{W}^-_{x_\infty}(\lambda)$, we have $\tilde{W}^-_{x_\infty}(\lambda) = -\tilde{W}_1(\lambda)\tilde{W}_2(\lambda)$, where

$$\tilde{W}_1(\lambda) = (R^- + iS^-)(R^- - iS^-)^{-1},$$

and $\tilde{W}_2(\lambda)$ is as above. Computing as before, we find

$$\frac{\partial \tilde{W}_1}{\partial \lambda} = i\tilde{W}_1\tilde{\Omega}_1,$$

where in this case

$$\tilde{\Omega}_1 = \Big((R^- - iS^-)^{-1}\Big)^* \Big\{2{R^-}^t S^-_\lambda - 2{S^-}^t R^-_\lambda\Big\}\Big((R^- - iS^-)^{-1}\Big).$$

Recalling that $R_\lambda^- = 0$, we see that the nature of $\tilde{\Omega}_1$ is determined by $R^{-t} S_\lambda^-$. Recalling that

$$S^-(\lambda) = \begin{pmatrix} \mu_{n+1}^-(\lambda) r_1^- & \mu_{n+2}^-(\lambda) r_2^- & \dots & \mu_{2n}^-(\lambda) r_n^- \end{pmatrix},$$

we have (recalling $\mu_{n+j}^+(\lambda) = \sqrt{\nu_j^- - \lambda}\,$)

$$S_\lambda^-(\lambda) = -\frac{1}{2} \begin{pmatrix} \frac{1}{\mu_{n+1}^-(\lambda)} r_1^- & \frac{1}{\mu_{n+2}^-(\lambda)} r_2^+ & \dots & \frac{1}{\mu_{2n}^-(\lambda)} r_n^- \end{pmatrix}.$$

In this way, orthogonality of the $\{r_j^-\}_{j=1}^n$ leads to the relation

$$R^{-t} S_\lambda^- = -\frac{1}{2} \begin{pmatrix} \frac{1}{\mu_{n+1}^-(\lambda)} & 0 & \dots & 0 \\ 0 & \frac{1}{\mu_{n+2}^-(\lambda)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{\mu_{2n}^-(\lambda)} \end{pmatrix}. \tag{4.5}$$

Since the $\{\mu_{n+j}^-\}_{j=1}^n$ are all positive (for $\lambda < \nu_{\min}$), we see that $\tilde{\Omega}_1$ is self-adjoint and negative definite.

The matrix $\tilde{W}_2$ is unchanged, so we can draw the same conclusion about monotonicity. $\square$

4.2. **Lower Bound on the Spectrum of H.** We have already seen that the essential spectrum of H is confined to the interval $[\nu_{\min}, +\infty)$. For the point spectrum, if λ is an eigenvalue of H then there exists a corresponding eigenfunction $\phi(\cdot;\lambda) \in H^2(\mathbb{R})$. If we take an $L^2(\mathbb{R})$ inner product of (1.1) with ϕ we find

$$\lambda \|\phi\|_2^2 = \|\phi'\|_2^2 + \langle V\phi, \phi\rangle \geq -C\|\phi\|_2^2,$$

for some contant $C > 0$ taken so that $|\langle V\phi, \phi\rangle| \leq C\|\phi\|_2^2$ for all $\phi \in H^2(\mathbb{R})$. We conclude that $\sigma_{pt}(H) \subset [-C, \infty)$. For example, $C = \|V\|_\infty$ clearly works. In what follows, we will take a value λ_∞ sufficiently large, and in particular we will take $\lambda_\infty > C$ (additional requirements will be added as well, but they can all be accommodated by taking λ_∞ larger, so that this initial restriction continues to hold).

4.3. **The Top Shelf.** Along the top shelf Γ_+, the Maslov index counts intersections of the Lagrangian subspaces $\ell^-(x_\infty;\lambda)$ and $\ell_\infty^+(\lambda) = \ell^+(x_\infty;\lambda)$. Such intersections will correspond with solutions of (1.1) that decay at both $\pm\infty$, and hence will correspond with eigenvalues of H. Moreover, the dimension of these intersections will correspond with the dimension of the space of solutions that decay at both $\pm\infty$, and so will correspond with the geometric multiplicity of the eigenvalues. Finally, we have seen that the eigenvalues of $\tilde{W}(x;\lambda)$ rotate monotonically counterclockwise as λ decreases from 0 to $-\lambda_\infty$ (i.e., as Γ_+ is traversed), and so the Maslov index on Γ_+ is a direct count of the crossings, including multiplicity (with no cancellations arising from crossings in opposite directions). We conclude that the Maslov index associated with this path will be a count, including multiplicity, of the negative eigenvalues of H; i.e., of the Morse index. We can express these considerations as

$$\mathrm{Mor}(H) = \mathrm{Mas}(\ell^-, \ell_\infty^+; \Gamma_+).$$

4.4. **The Bottom Shelf.** For the bottom shelf, we have

$$\begin{aligned}\tilde{W}^-_{x_\infty}(\lambda) = &-(R^- + iS^-(\lambda))(R^- - iS^-(\lambda))^{-1} \\ &\times (X^+_\infty(\lambda) - iY^+_\infty(\lambda))(X^+_\infty(\lambda) + iY^+_\infty(\lambda))^{-1}.\end{aligned} \tag{4.6}$$

By choosing x_∞ suitably large, we can ensure that the frame $\mathbf{X}^+_\infty(\lambda)$ is as close as we like to the frame $\mathbf{R}^+(\lambda)$, where we recall $\mathbf{R}^- = \binom{R^-}{S^-}$ and $\mathbf{R}^+ = \binom{R^+}{S^+}$. (As noted in Remark 4.1 $\ell^-_{+\infty}(\lambda)$ is not necessarily continuous in λ, but $\ell^+_{\mathbf{R}}(\lambda)$ certainly is continuous in λ.) We will proceed by analyzing the matrix

$$\tilde{\mathcal{W}}^-_{\mathbf{R}}(\lambda) := -(R^- + iS^-(\lambda))(R^- - iS^-(\lambda))^{-1}(R^+ - iS^+(\lambda))(R^+ + iS^+(\lambda))^{-1}, \tag{4.7}$$

for which we will be able to conclude that for $\lambda < \nu_{\min}$, -1 is never an eigenvalue. By continuity, we will be able to draw conclusions about $\tilde{W}^-_{x_\infty}(\lambda)$ as well.

Lemma 4.5. *For any $\lambda < \nu_{\min}$ the spectrum of $\tilde{\mathcal{W}}^-_{\mathbf{R}}(\lambda)$ does not include -1. I.e., for all such λ $\dim(\ell^-_{\mathbf{R}}(\lambda) \cap \ell^+_{\mathbf{R}}(\lambda)) = 0$.*

Proof. We need only show that for any $\lambda < \nu_{\min}$ the $2n$ vectors comprising the columns of $\mathbf{R}^-$ and $\mathbf{R}^+$ are linearly independent. We proceed by induction, first establishing that any single column of $\mathbf{R}^-$ is linearly independent of the columns of $\mathbf{R}^+$. Suppose not. Then there is some $j \in \{1, 2, \dots, n\}$, along with some collection of constants $\{c_k\}_{k=1}^n$ so that

$$\boldsymbol{\mathscr{r}}^-_{n+j} = \sum_{k=1}^n c_k \, \boldsymbol{\mathscr{r}}^+_k. \tag{4.8}$$

Recalling the definitions of $\boldsymbol{\mathscr{r}}^-_{n+j}$ and $\boldsymbol{\mathscr{r}}^+_k$, we have the two equations

$$\begin{aligned} r^-_j &= \sum_{k=1}^n c_k r^+_{n+1-k} \\ \mu^-_{n+j} r^-_j &= \sum_{k=1}^n c_k \mu^+_k r^+_{n+1-k}. \end{aligned}$$

Multiplying the first of these equations by μ^-_{n+j}, and subtracting the second equation from the result, we find

$$0 = \sum_{k=1}^n (\mu^-_{n+j} - \mu^+_k) c_k r^+_{n+1-k}.$$

Since the collection $\{r^+_{n+1-k}\}_{k=1}^n$ is linearly independent, and since $\mu^-_{n+j} - \mu^+_k > 0$ for all $k \in \{1, 2, \dots, n\}$ (for $\lambda < \nu_{\min}$), we conclude that the constants $\{c_k\}_{k=1}^n$ must all be zero, but this contradicts (4.8).

For the induction step, suppose that for some $1 \le m < n$, any m elements of the collection $\{\boldsymbol{\mathscr{r}}^-_{n+j}\}_{j=1}^n$ are linearly independent of the set $\{\boldsymbol{\mathscr{r}}^+_k\}_{k=1}^n$. We want to show that any $m+1$ elements of the collection $\{\boldsymbol{\mathscr{r}}^-_{n+j}\}_{j=1}^n$ are linearly independent of the set $\{\boldsymbol{\mathscr{r}}^+_k\}_{k=1}^n$. If not, then by a change of labeling if necessary there exist constants $\{c_l^-\}_{l=2}^{m+1}$ and $\{c_k^+\}_{k=1}^{n}$ so that

$$\boldsymbol{v}_{n+1}^- = \sum_{l=2}^{m+1} c_l^- \boldsymbol{v}_{n+l}^- + \sum_{k=1}^{n} c_k^+ \boldsymbol{v}_k^+. \tag{4.9}$$

Again, we have two equations

$$\begin{aligned} r_1^- &= \sum_{l=2}^{m+1} c_l^- r_l^- + \sum_{k=1}^{n} c_k^+ r_{n+1-k}^+ \\ \mu_{n+1}^- r_1^- &= \sum_{l=2}^{m+1} c_l^- \mu_{n+l}^- r_l^- + \sum_{k=1}^{n} c_k^+ \mu_k^+ r_{n+1-k}^+. \end{aligned}$$

Multiplying the first of these equations by μ_{n+1}^-, and subtracting the second equation from the result, we obtain the relation

$$0 = \sum_{l=2}^{m+1} c_l^- (\mu_{n+1}^- - \mu_{n+l}^-) r_l^- + \sum_{k=1}^{n} c_k^+ (\mu_{n+1}^- - \mu_k^+) r_{n+1-k}^+.$$

By our induction hypothesis, the vectors on the right-hand side are all linearly independent, and since $\mu_{n+1}^- - \mu_k^+ > 0$ for all $k \in 1, 2, \dots, n$, we can conclude that $c_k^+ = 0$ for all $k \in 1, 2, \dots, n$. (Notice that we make no claim about the c_l^-.) Returning to (4.9), we obtain a contradiction to the linear independence of the collection $\{\boldsymbol{v}_{n+j}^-\}_{j=1}^{n}$.

Continuing the induction up to $m = n - 1$ gives the claim. □

Remark 4.6. It is important to note that we do not include the case $\lambda = \nu_{\min}$ in our lemma, and indeed the lemma does not generally hold in this case. For example, consider the case in which $V(x)$ vanishes identically at both $\pm\infty$ (i.e., $V_- = V_+ = 0$, so in particular $\nu_{\min} = 0$). In this case, we can take $R^- = I$, $S^- = \sqrt{-\lambda} I$, $R^+ = \check{I}$, and $S^+ = -\sqrt{-\lambda}\check{I}$, where

$$\check{I} = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

We easily find

$$\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda) = -\frac{(1 + i\sqrt{-\lambda})^2}{(1 - i\sqrt{-\lambda})^2} I,$$

and we see explicitly that $\tilde{W}_{x_\infty}^-(0) = -I$, so that all n eigenvalues reside at -1. Moreover, as λ proceeds from 0 toward $-\infty$ the eigenvalues of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$ remain coalesced, and move monotonically counterclockwise around S^1, returning to -1 in the limit as $\lambda \to -\infty$. In this case, we can conclude that for the path from 0 to $-\lambda_\infty$, the Maslov index does not increment.

We immediately obtain the following lemma.

Lemma 4.7. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then for any $\lambda_0 < \nu_{\min}$ we can choose x_∞ sufficiently large so that we will have*

$$\dim(\ell_{\mathbf{R}}^-(\lambda) \cap \ell_\infty^+(\lambda)) = 0$$

for all $\lambda \in [-\lambda_\infty, \lambda_0]$. It follows by taking $\lambda_0 = 0$ that

$$\operatorname{Mas}(\ell^-, \ell_\infty^+; \Gamma_-) = 0.$$

Proof. First, according to Lemma 4.5, none of the eigenvalues of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$ is -1 for any $\lambda \in [-\lambda_\infty, \lambda_0]$. In particular, since the interval $[-\lambda_\infty, \lambda_0]$ is compact there exists some $\epsilon > 0$ so that each eigenvalue $\tilde{\omega}(\lambda)$ of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$ satisfies

$$|\tilde{\omega}(\lambda) + 1| > \epsilon$$

for all $\lambda \in [-\lambda_\infty, \lambda_0]$.

Similarly as above, we can make the change of variables

$$x_\infty = \ln(\frac{1+\tau_\infty}{1-\tau_\infty}), \iff \tau_\infty = \frac{e^{x_\infty} - 1}{e^{x_\infty} + 1}.$$

This allows us to view $\tilde{W}_{x_\infty}^-$ as a continuous function on the compact domain $(x_\infty, \lambda) \in [1-\delta, 1] \times [-\lambda_\infty, \lambda_0]$, where $\delta > 0$ is small, indicating that x_∞ is taken to be large. We see that $\tilde{W}_{x_\infty}^-$ is uniformly continuous and so by choosing τ_∞ sufficiently close to 1, we can force the eigenvalues of $\tilde{W}_{x_\infty}^-$ to be as close to the eigenvalues of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$ as we like. We take τ_∞ sufficiently close to 1 so that for each $\lambda \in [-\lambda_\infty, \lambda_0]$ and each eigenvalue $\tilde{\omega}$ of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$ there is a corresponding eigenvalue of $\tilde{W}_{x_\infty}^-$, which we denote $\omega(\lambda)$, so that $|\tilde{\omega}(\lambda) - \omega(\lambda)| < \epsilon/2$. But then

$$\begin{aligned} \epsilon < |\tilde{\omega}(\lambda) + 1| &= |\tilde{\omega}(\lambda) - \omega(\lambda) + \omega(\lambda) + 1| \\ &\le |\tilde{\omega}(\lambda) - \omega(\lambda)| + |\omega(\lambda) + 1| < \frac{\epsilon}{2} + |\omega(\lambda) + 1|, \end{aligned}$$

from which we conclude that

$$|\omega(\lambda) + 1| \ge \frac{\epsilon}{2},$$

for all $\lambda \in [-\lambda_\infty, \lambda_0]$. □

4.5. **The Left Shelf.** For the left shelf Γ_∞, we need to understand the Maslov index associated with $\tilde{W}(x; -\lambda_\infty)$ (with λ_∞ sufficiently large) as x goes from $-\infty$ to x_∞ (keeping in mind that the path Γ_∞ reverses this flow). In order to accomplish this, we follow the approach of [36, 70] in developing large-$|\lambda|$ estimates on solutions of (1.1), uniformly in x. For $\lambda < 0$, we set

$$\xi = \sqrt{-\lambda}x; \quad \phi(\xi) = y(x),$$

so that (1.1) becomes

$$\phi''(\xi) + \frac{1}{\lambda} V(\frac{\xi}{\sqrt{-\lambda}})\phi = \phi.$$

Setting $\Phi_1 = \phi$, $\Phi_2 = \phi'$, and $\Phi = \binom{\Phi_1}{\Phi_2} \in \mathbb{R}^{2n}$, we can express this equation as

$$\Phi' = \mathbb{A}(\xi;\lambda)\Phi; \quad \mathbb{A}(\xi;\lambda) = \begin{pmatrix} 0 & I \\ I - \frac{1}{\lambda}V(\frac{\xi}{\sqrt{-\lambda}}) & 0 \end{pmatrix}.$$

We begin by looking for solutions that decay as $x \to -\infty$ (and so as $\xi \to -\infty$); i.e., we begin by constructing the frame $\mathbf{X}^-(x; -\lambda_\infty)$. It's convenient to write

$$\mathbb{A}(\xi;\lambda) = \mathbb{A}_-(\lambda) + \mathbb{E}_-(\xi;\lambda),$$

where

$$\mathbb{A}_-(\lambda) = \begin{pmatrix} 0 & I \\ I - \frac{1}{\lambda}V_- & 0 \end{pmatrix}; \quad \mathbb{E}_-(\xi;\lambda) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\lambda}(V_- - V(\frac{\xi}{\sqrt{-\lambda}})) & 0 \end{pmatrix}.$$

Fix any $M \gg 0$ and note that according to (A1), we have

$$\begin{aligned} \int_{-\infty}^{M} |\mathbb{E}_-(\xi;\lambda)| d\xi &\leq \frac{1}{|\lambda|} \int_{-\infty}^{M} |V(\frac{\xi}{\sqrt{-\lambda}}) - V_-| d\xi \\ &= \frac{1}{|\lambda|} \int_{-\infty}^{\frac{M}{\sqrt{-\lambda}}} |V(x) - V_-| \sqrt{-\lambda} dx \leq \frac{K}{\sqrt{-\lambda}}, \end{aligned}$$

for some constant $K = K(M)$. Recalling that we are denoting the eigenvalues of V_- by $\{\nu_j^-\}_{j=1}^n$, we readily check that the eigenvalues of $\mathbb{A}_-(\lambda)$ can be expressed as

$$\hat{\mu}_j^-(\lambda) = -\sqrt{1 - \frac{\nu_{n+1-j}^-}{\lambda}} = \frac{1}{\sqrt{-\lambda}} \mu_j^-$$

$$\hat{\mu}_{n+j}^-(\lambda) = \sqrt{1 - \frac{\nu_j^-}{\lambda}} = \frac{1}{\sqrt{-\lambda}} \mu_{n+j}^-,$$

for $j = 1, 2, \dots, n$ (ordered, as usual, so that $j < k$ implies $\hat{\mu}_j^- \leq \hat{\mu}_k^-$). In order to select a solution decaying with rate $\hat{\mu}_{n+j}^-$ (as $\xi \to -\infty$), we look for solutions of the form $\Phi_{n+j}^-(\xi;\lambda) = e^{\hat{\mu}_{n+j}^-(\lambda)\xi} Z_{n+j}^-(\xi;\lambda)$, for which Z_{n+j}^- satisfies

$$Z_{n+j}^{-\,\prime} = (\mathbb{A}_-(\lambda) - \hat{\mu}_{n+j}(\lambda) I) Z_{n+j}^- + \mathbb{E}_-(\xi;\lambda) Z_{n+j}^-.$$

Proceeding similarly as in the proof of Lemma 2.2, we obtain a collection of solutions

$$Z_{n+j}^-(\xi;\lambda) = \hat{\mathfrak{r}}_{n+j}^- + \mathbf{O}(|\lambda|^{-1/2}),$$

where the $\mathbf{O}(\cdot)$ terms are uniform for $x \in (-\infty, M]$. These lead to

$$\Phi_{n+j}^-(\xi;\lambda) = e^{\hat{\mu}_{n+j}^-(\lambda)\xi} (\hat{\mathfrak{r}}_{n+j}^- + \mathbf{O}(|\lambda|^{-1/2})),$$

where $\hat{\mathfrak{r}}$ corresponds with $\mathfrak{r}$, with μ replaced by $\hat{\mu}$. Returning to original coordinates, we construct the frame $\mathbf{X}^-(x;\lambda)$ out of basis elements

$$\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} = e^{\sqrt{-\lambda}\hat{\mu}_{n+j}^-(\lambda)x} \Big(\begin{pmatrix} r_j^- \\ \sqrt{-\lambda}\hat{\mu}_{n+j}^- r_j^- \end{pmatrix} + \begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) \\ \mathbf{O}(1) \end{pmatrix} \Big).$$

Recalling that when specifying a frame for ℓ^- we can view the exponential multipliers as expansion coefficients, we see that we can take as our frame for ℓ^- the matrices

$$X^-(x;\lambda) = R^- + \mathbf{O}(|\lambda|^{-1/2})$$
$$Y^-(x;\lambda) = S^- + \mathbf{O}(1),$$

where the $\mathbf{O}(\cdot)$ terms are uniform for $x \in (-\infty, M]$, and we have observed that $\mu_j^- = \sqrt{-\lambda}\hat{\mu}_j^-$, for $j = 1, 2, \dots, 2n$. Likewise, we find that for $-\lambda > 0$ sufficiently large

$$X_\infty^+(\lambda) = R^+ + \mathbf{O}(|\lambda|^{-1/2})$$
$$Y_\infty^+(\lambda) = S^+ + \mathbf{O}(1).$$

Turning to $\tilde{W}(x;\lambda)$, we first observe that $S^-(\lambda)^{-1}$ can easily be identified, using the orthogonality of R^-; in particular, the i-th row of $S^-(\lambda)^{-1}$ is $\frac{1}{\mu_{n+i}^-}(r_i^-)^t$, which is $\mathbf{O}(|\lambda|^{-1/2})$. In this way, we see that

$$\begin{aligned} X^-(x;\lambda) - iY^-(x;\lambda) &= R^- + \mathbf{O}(|\lambda|^{-1/2}) - iS^-(\lambda) + \mathbf{O}(1) \\ &= -iS^-(\lambda)\Big\{ iS^-(\lambda)^{-1}(R^- + \mathbf{O}(1)) + I \Big\} \\ &= -iS^-(\lambda)(I + \mathbf{O}(|\lambda|^{-1/2})), \end{aligned}$$

and so

$$\begin{aligned} (X^-(x;\lambda) - iY^-(x;\lambda))^{-1} &= i(I + \mathbf{O}(|\lambda|^{-1/2}))^{-1} S^-(\lambda)^{-1} \\ &= i(I + \mathbf{O}(|\lambda|^{-1/2}))S^-(\lambda)^{-1}, \end{aligned}$$

by Neumann approximation. Likewise,

$$\begin{aligned} X^-(x;\lambda) + iY^-(x;\lambda) &= R^- + \mathbf{O}(|\lambda|^{-1/2}) + iS^-(\lambda) + \mathbf{O}(1) \\ &= (iI + (R^- + \mathbf{O}(1))S^-(\lambda)^{-1})S^-(\lambda) \\ &= (iI + \mathbf{O}(|\lambda|^{-1/2}))S^-(\lambda). \end{aligned}$$

In this way, we see that

$$\begin{aligned} &(X^-(x;\lambda) + iY^-(x;\lambda))(X^-(x;\lambda) - iY^-(x;\lambda))^{-1} \\ &\quad = (iI + \mathbf{O}(|\lambda|^{-1/2}))S^-(\lambda) i(I + \mathbf{O}(|\lambda|^{-1/2}))S^-(\lambda)^{-1} \\ &\quad = (iI + \mathbf{O}(|\lambda|^{-1/2}))(iI + \mathbf{O}(|\lambda|^{-1/2})) \\ &\quad = -I + \mathbf{O}(|\lambda|^{-1/2}). \end{aligned}$$

Proceeding similarly for $\mathbf{X}_\infty(\lambda)$, we have

$$(X_\infty^+(\lambda) + iY_\infty^+(\lambda)(X_\infty^+(\lambda) - iY_\infty^+(\lambda))^{-1} = -I + \mathbf{O}(|\lambda|^{-1/2}),$$

and so

$$\tilde{W}(x;\lambda) = -I + \mathbf{O}(|\lambda|^{-1/2})$$

uniformly for $x \in (-\infty, M]$. We see that for λ_∞ sufficiently large the eigenvalues of $\tilde{W}(x; -\lambda_\infty)$ are near -1 uniformly for $x \in (-\infty, M]$.

Turning to the behavior of $\tilde{W}(x;\lambda)$ as x tends to $+\infty$ (i.e., for $x \geq M$), we recall from Section 4.2 that if λ_∞ is large enough then $-\lambda_\infty$ will not be an eigenvalue of H. This means the evolving Lagrangian subspace ℓ^- cannot intersect the space of solutions asymptotically decaying as $x \to +\infty$, and so the frame $X^-(x;\lambda)$ must be comprised of solutions that grow as x tends to $+\infty$. The construction of these growing solutions is almost identical to our construction of the decaying solutions Φ_j^-, and we'll be brief.

In this case, it's convenient to write

$$\mathbb{A}(\xi;\lambda) = \mathbb{A}_+(\lambda) + \mathbb{E}_+(\xi;\lambda),$$

where

$$\mathbb{A}_+(\lambda) = \begin{pmatrix} 0 & I \\ I - \frac{1}{\lambda}V_+ & 0 \end{pmatrix}; \quad \mathbb{E}_+(\xi;\lambda) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\lambda}(V_+ - V(\frac{\xi}{\sqrt{-\lambda}})) & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbb{A}_+(\lambda)$ can be expressed as

$$\hat{\mu}_j^+(\lambda) = -\sqrt{1 - \frac{\nu_{m+1-j}^+}{\lambda}} = \frac{1}{\sqrt{-\lambda}}\mu_j^+$$

$$\hat{\mu}_{n+j}^+(\lambda) = \sqrt{1 - \frac{\nu_j^+}{\lambda}} = \frac{1}{\sqrt{-\lambda}}\mu_{n+j}^+,$$

for $j = 1, 2, \dots, n$ (ordered, as usual, so that $j < k$ implies $\hat{\mu}_j^+ \leq \hat{\mu}_k^+$). In order to select a solution growing with rate $\hat{\mu}_{n+j}^+$ (as $\xi \to +\infty$), we look for solutions of the form $\Phi_{n+j}^+(\xi;\lambda) = e^{\hat{\mu}_{n+j}^+(\lambda)\xi} Z_{n+j}^+(\xi;\lambda)$, for which Z_{n+j}^+ satisfies

$$Z_{n+j}^{+'} = (\mathbb{A}_+(\lambda) - \hat{\mu}_{n+j}^+(\lambda)I)Z_{n+j}^+ + \mathbb{E}_+(\xi;\lambda)Z_{n+j}^+.$$

Proceeding as with the frame of solutions that decay as $x \to -\infty$, we find that for M sufficiently large (so that asymptotically decaying solutions become negligible), we can take as our frame for ℓ^-

$$X^+(x;\lambda) = R^+ + \mathbf{O}(|\lambda|^{-1/2})$$
$$Y^+(x;\lambda) = \tilde{S}^+ + \mathbf{O}(1),$$

where

$$\tilde{S}^+ = \begin{pmatrix} \mu_{n+1}^+ r_n^+ & \mu_{n+2}^+ r_{n-1}^+ & \dots & \mu_{2n}^+ r_1^+ \end{pmatrix},$$

and the $\mathbf{O}(\cdot)$ terms are uniform for $x \in [M,\infty)$. Proceeding now almost exactly as we did for the interval $(-\infty, M]$ we find that for λ_∞ sufficiently large the eigenvalues of $\tilde{W}(x;-\lambda_\infty)$ are near -1 uniformly for $x \in [M,\infty)$.

We summarize these considerations in a lemma.

Lemma 4.8. *Let $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then given any $\epsilon > 0$ there exists $\lambda_\infty > 0$ sufficiently large so that for all $x \in \mathbb{R}$ and for any eigenvalue $\omega(x;-\lambda_\infty)$ of $\tilde{W}(x;-\lambda_\infty)$ we have*

$$|\omega(x;-\lambda_\infty) + 1| < \epsilon.$$

Remark 4.9. We note that it would be insufficient to simply take $M = x_\infty$ in our argument (thus avoiding the second part of the argument, based on asymptotically growing solutions). This is because our overall argument is structured in such a way that we choose λ_∞ first, and then choose x_∞ sufficiently large, based on this value. (This if for the bottom shelf argument.) But λ_∞ must be chosen based on M, so M should not depend on the value of x_∞.

We now make the following claim.

Lemma 4.10. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose (A1) and (A2) hold. Then given any $M > 0$ there exists $\lambda_\infty > 0$ sufficiently large so that*

$$\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_\infty) = 0,$$

for any $x_\infty > M$.

Proof. We begin by observing that by taking λ_∞ sufficiently large, we can ensure that for all $x_\infty > M$ the eigenvalues of $\tilde{W}(x_\infty; -\lambda_\infty)$ are all near -1. To make this precise, given any $\epsilon > 0$ we can take λ_∞ sufficiently large so that the eigenvalues of $\tilde{W}(x_\infty; -\lambda_\infty)$ are confined to the arc $\mathcal{A}_\epsilon = \{e^{i\theta} : |\theta - \pi| < \epsilon\}$. Moreover, we know from Lemma 4.3 that as λ decreases toward $-\lambda_\infty$ the eigenvalues of $\tilde{W}(x_\infty; \lambda)$ will monotonically rotate in the counterclockwise direction, and so the eigenvalues of $\tilde{W}(x_\infty; -\lambda_\infty)$ will in fact be confined to the arc $\mathcal{A}^+_\epsilon = \{e^{i\theta} : -\epsilon < \theta - \pi < 0\}$. (See Figure 3; we emphasize that none of the eigenvalues can cross -1, because such a crossing would correspond with an eigenvalue of H, and we have assumed λ_∞ is large enough so that there are no eigenvalues for $\lambda \le -\lambda_\infty$.) Likewise, by the same monotonicity argument, we see that the eigenvalues of $\tilde{W}^-_{x_\infty}(-\lambda_\infty)$ (characterizing behavior on the bottom shelf) are also confined to $\mathcal{A}^+_\epsilon$.

Turning now to the flow of eigenvalues as x proceeds from x_∞ to $-\infty$ (i.e., along Γ_∞), we note by uniformity of our large-$|\lambda|$ estimates that we can take λ_∞ large enough so that the eigenvalues of $\tilde{W}(x; -\lambda_\infty)$ are confined to $\mathcal{A}_\epsilon$ (not necessarily $\mathcal{A}^+_\epsilon$) for all $x \in \mathbb{R}$. Combining these observations, we conclude that the eigenvalues of $\tilde{W}(x; -\lambda_\infty)$ must begin and end in $\mathcal{A}^+_\epsilon$, without completing a loop of S^1, and consequently the Maslov index along the entirety of Γ_∞ must be 0. ∎

4.6. **Proof of Theorem 1.2.** Let Γ denote the contour obtained by proceeding counterclockwise along the paths Γ_0, Γ_+, Γ_∞, Γ_-. By the catenation property of the Maslov index, we have

$$\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma) = \operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_0) + \operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_+) + \operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_\infty) + \operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_-).$$

Moreover, by the homotopy property, and by noting that Γ is homotopic to an arbitrarily small cycle attached to any point of Γ, we can conclude that $\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma) = 0$. By Lemma 4.7 we have $\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_0) = 0$, and by Lemma 4.10 we have $\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_\infty) = 0$. Since $\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_+) = \operatorname{Mor}(H)$, it follows immediately that

$$\operatorname{Mas}(\ell^-, \ell^+_\infty; \Gamma_0) + \operatorname{Mor}(H) = 0. \tag{4.10}$$

We will complete the proof with the following claim.

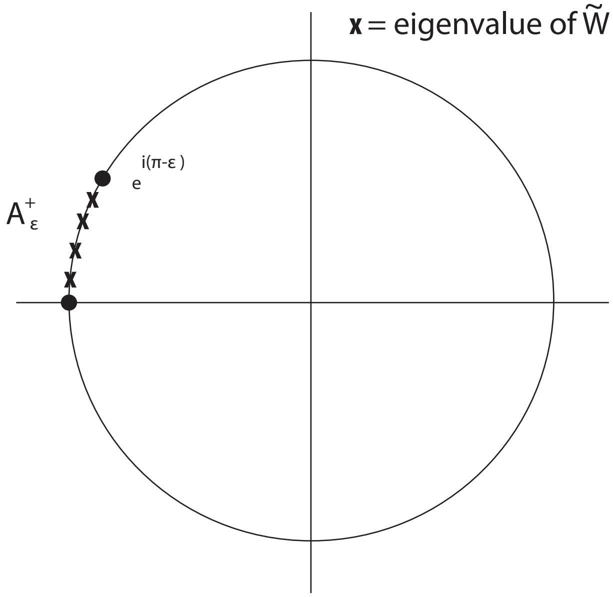

FIGURE 3. Eigenvalues confined to A_ϵ^+.

Claim 4.11. Under the assumptions of Theorem 1.2, and for x_∞ sufficiently large,

$$\operatorname{Mas}(\ell^-, \ell_\infty^+; \Gamma_0) = \operatorname{Mas}(\ell^-, \ell_{\mathbf{R}}^+; \bar{\Gamma}_0).$$

Proof. As usual, let $\tilde{W}(x;\lambda)$ denote the unitary matrix (4.1) (which we recall depends on x_∞), and let $\tilde{\mathcal{W}}(x;\lambda)$ denote the unitary matrix

$$\begin{aligned} \tilde{\mathcal{W}}(x;\lambda) &= -(X^-(x;\lambda) + iY^-(x;\lambda))(X^-(x;\lambda) - iY^-(x;\lambda))^{-1} \\ &\times (R^+ - iS^+(\lambda))(R^+ + iS^+(\lambda))^{-1}. \end{aligned} \tag{4.11}$$

I.e., $\tilde{W}(x;\lambda)$ is the unitary matrix used in the calculation of $\operatorname{Mas}(\ell^-, \ell_\infty^+; \Gamma_0)$ and $\tilde{\mathcal{W}}(x;\lambda)$ is the unitary matrix used in the calculation of $\operatorname{Mas}(\ell^-, \ell_{\mathbf{R}}^+; \bar{\Gamma}_0)$. Likewise, set

$$\begin{aligned} \tilde{W}_{x_\infty}^-(\lambda) &= \lim_{x \to -\infty} \tilde{W}(x;\lambda) \\ \tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda) &= \lim_{x \to -\infty} \tilde{\mathcal{W}}(x;\lambda), \end{aligned}$$

both of which are well defined. (Notice that while the matrix $\tilde{\mathcal{W}}(x;\lambda)$ has not previously appeared, the other matrices here, including $\tilde{\mathcal{W}}_{\mathbf{R}}^-(\lambda)$, are the same as before.)

By taking x_∞ sufficiently large we can ensure that the spectrum of $\tilde{W}_{x_\infty}^-(0)$ is arbitrarily close to the spectrum of $\tilde{\mathcal{W}}_{\mathbf{R}}^-(0)$ in the following sense: given any $\epsilon > 0$ we can take x_∞ sufficiently large so that for any $\omega \in \sigma(\tilde{W}_{x_\infty}^-(0))$ there exists $\tilde{\omega} \in \sigma(\tilde{\mathcal{W}}_{\mathbf{R}}^-(0))$ so that $|\omega - \tilde{\omega}| < \epsilon$.

Turning to the other end of our contours, we recall that $\tilde{W}(x_\infty; 0)$ will have -1 as an eigenvalue if and only if $\lambda = 0$ is an eigenvalue of H, and the multiplicity of -1 as an eigenvalue of $\tilde{W}(x_\infty; 0)$ will correspond with the geometric multiplicity of $\lambda = 0$ as an eigenvalue of H. For $\tilde{\mathcal{W}}(x; 0)$ set

$$\tilde{\mathcal{W}}^+(0) = \lim_{x \to \infty} \tilde{\mathcal{W}}(x; 0),$$

which is well defined by our construction in the appendix.

Claim 4.12. As with $\tilde{W}(x_\infty; 0)$, $\tilde{\mathcal{W}}^+(0)$ will have -1 as an eigenvalue if and only if $\lambda = 0$ is an eigenvalue of H, and the multiplicity of -1 as an eigenvalue of $\tilde{\mathcal{W}}^+(0)$ will correspond with the geometric multiplicity of $\lambda = 0$ as an eigenvalue of H.

Proof. We first consider the case in which $\lambda = 0$ is not an eigenvalue of H. In this case, the construction in our appendix shows that we can construct a frame for $\ell^-(x; \lambda)$ entirely from the solutions of (2.2) that grow as $x \to +\infty$. In this way, we see that $\ell^-_{+\infty}(0) = \ell^+_{\tilde{\mathbf{R}}}(0)$, where $\ell^+_{\tilde{\mathbf{R}}}(0)$ denotes the Lagrangian subspace with frame $\tilde{\mathbf{R}} = \binom{R^+}{\tilde{S}^+}$ (i.e., the asymptotic space associated with growing solutions; see Section 4.5). It follows that $\tilde{\mathcal{W}}^+(0)$ detects intersections of $\ell^+_{\mathbf{R}}(0)$ and $\ell^+_{\tilde{\mathbf{R}}}(0)$. But $\dim(\ell^+_{\mathbf{R}}(0) \cap \ell^+_{\tilde{\mathbf{R}}}(0)) = 0$, so in this case $\tilde{\mathcal{W}}^+(0)$ will not have -1 as an eigenvalue.

On the other hand, suppose $\lambda = 0$ is an eigenvalue of H. Then using our construction from the appendix, we find that the frame for $\ell^-(x; \lambda)$ must contain at least one element that decays as $x \to +\infty$. But then $\dim(\ell^-_{+\infty}(0) \cap \ell^+_{\mathbf{R}}(0)) \neq 0$, and -1 will be an eigenvalue of $\tilde{\mathcal{W}}^+(0)$. Moreover, suppose $\lambda = 0$ has geometric multiplicity m so that there are m solutions of (2.2) that decay at both $-\infty$ and $+\infty$. In order for our construction in the appendix to include all of these solutions it must include at least m solutions that decay as $x \to +\infty$ (and it cannot include more, or the multiplicity would be greater than m). We conclude that $\dim(\ell^-_{+\infty}(0) \cap \ell^+_{\mathbf{R}}(0)) = m$.

We see from the preceding discussion that -1 is an eigenvalue of $\tilde{\mathcal{W}}^+(0)$ if and only if $\lambda = 0$ is an eigenvalue of H, and that if $\lambda = 0$ is an eigenvalue of H with multiplicity m then -1 will be an eigenvalue of $\tilde{\mathcal{W}}^+(0)$ with multiplicity m. It remains to show that if -1 is an eigenvalue of $\tilde{\mathcal{W}}^+(0)$ with multiplicity m then $\lambda = 0$ will be an eigenvalue of H with multiplicity m. First, if -1 is an eigenvalue of $\tilde{\mathcal{W}}^+(0)$ then there exists some

$$\zeta \in \ell^-_{+\infty}(0) \cap \ell^+_{\mathbf{R}}(0).$$

Since $\zeta \in \ell^+_{\mathbf{R}}(0)$ there exist constants $\{d^+_j\}_{j=1}^n$ so that

$$\zeta = \sum_{j=1}^n d^+_j \boldsymbol{r}^+_j, \tag{4.12}$$

and likewise since $\zeta \in \ell^-_{+\infty}(0)$ there exist constants $\{d^-_j\}_{j=1}^n$ so that

$$\zeta = \sum_{j=1}^n d^-_j s^+_{k(j)}. \tag{4.13}$$

In this way, we see that we must have the relation

$$\sum_{j=1}^{n} d_j^+ \mathscr{r}_j^+ - \sum_{j=1}^{n} d_j^- s_{k(j)}^+ = 0.$$

The full set $\{\mathscr{r}_j^+\}_{j=1}^{2n}$ is linearly independent, so the set $\{s_{k(j)}^+\}_{j=1}^{n}$ must contain at least one of the $\{\mathscr{r}_j^+\}_{j=1}^{n}$, say with $j = j^*$. (A bit more precisely, there must be some index j^* so that $s_{k(j^*)}^+$ can be expressed as a linear combination of $\{\mathscr{r}_j^+\}_{j=1}^{n}$.) But then

$$e^{\mu_{k(j^*)}^+ x}(s_{k(j^*)}^+ + \tilde{E}_{k(j^*)}^+(x;\lambda)) \tag{4.14}$$

will decay at both $-\infty$ and $+\infty$. We conclude that this serves as an eigenfunction for H with $\lambda = 0$.

Likewise, if

$$\eta \in \ell_{+\infty}^-(0) \cap \ell_{\mathbf{R}}^+(0),$$

is linearly independent of ζ then we can find constants $\{e_j^+\}_{j=1}^{n}$ and $\{e_j^-\}_{j=1}^{n}$ so that

$$\eta = \sum_{j=1}^{n} e_j^+ \mathscr{r}_j^+; \quad \eta = \sum_{j=1}^{n} d_j^- s_{k(j)}^+.$$

Just as for ζ, there must exist some j^{**} so that $s_{k(j^{**})}^+$ can be expressed as a linear combination of $\{\mathscr{r}_j^+\}_{j=1}^{n}$. If $j^{**} \neq j^*$, then we obtain a second eigenfunction for $\lambda = 0$,

$$e^{\mu_{k(j^{**})}^+ x}(s_{k(j^{**})}^+ + \tilde{E}_{k(j^{**})}^+(x;\lambda)), \tag{4.15}$$

which is necessarily linearly independent of (4.14), and so the geometric multiplicity of $\lambda = 0$ as an eigenvalue of H is at least 2. On the other hand, if $j^* = j^{**}$ then since ζ and η are linearly independent, we can find a linear combination $\chi = a\zeta + b\eta$ from which $s_{k(j^*)}^+$ has been eliminated. But we will still have $\chi \in \ell_{+\infty}^-(0) \cap \ell_{\mathbf{R}}^+(0)$, and so there must exist some $j^{***} \neq j^*$ so that $s_{k(j^{***})}^+$ can be expressed as a linear combination of the $\{\mathscr{r}_j^+\}_{j=1}^{n}$. Again, we obtain a second eigenfunction for H (for $\lambda = 0$), and conclude that $\lambda = 0$ has multiplicity at least 2. We have already seen that if $\lambda = 0$ is an eigenvalue of multiplicity greater than 2 then the dimension of $\ell_{+\infty}^-(0) \cap \ell_{\mathbf{R}}^+(0)$ will be greater than 2, so if the dimension of this intersection is 2 then the multiplicity of $\lambda = 0$ as an eigenvalue of H will be exactly 2.

This completes the argument for multiplicities 1 and 2, and the argument for higher order multiplicities follows similarly.

□

By choosing x_∞ sufficiently large, we can ensure that the eigenvalues of $\tilde{W}(x_\infty; 0)$ are arbitrarily close to the eigenvalues of $\tilde{\mathcal{W}}^+(0)$. I.e., -1 repeats as an eigenvalue the same number of times for these two matrices, and the eigenvalues aside from -1 can be made arbitrarily close.

We see that the path of matrices $\tilde{W}(x;0)$, as x runs from $-\infty$ to x_∞ can be viewed as a small perturbation from the path of matrices $\tilde{\mathcal{W}}(x;0)$, as x runs from $-\infty$ to $+\infty$. In order to clarify this, we recall that by using the change of variables (1.5) we can specify our path of Lagrangian subspaces on the compact interval $[-1,1]$. Likewise, the interval $(-\infty, x_\infty]$ compactifies to $[-1,(e^{x_\infty}-1)/(e^{x_\infty}+1)]$. For this latter interval, we can make the further change of variables

$$\xi = \frac{2}{1+r_\infty}\tau + \frac{1-r_\infty}{1+r_\infty},$$

where $r_\infty = (e^{x_\infty}-1)/(e^{x_\infty}+1)$, so that $\tilde{W}(x;0)$ and $\tilde{\mathcal{W}}(x;0)$ can both be specfied on the interval $[-1,1]$. Finally, we see that

$$|\xi - \tau| = (1+\tau)\frac{1-r_\infty}{1+r_\infty},$$

so by choosing x_∞ sufficiently large (and hence r_∞ sufficiently close to 1), we can take the values of ξ and τ as close as we like. By uniform continuity the eigenvalues of the adjusted path will be arbitrarily close to those of the original path. Since the endstates associated with these paths are arbitrarily close, and since the eigenvalues of one path end at -1 if and only if the eigenvalues of the other path do, we can conclude by a continuity argument that the spectral flow must be the same along each of these paths, and this establishes the claim. □

Theorem 1.2 is now an immediate consequence of Claim 4.11. □

5. Equations with Constant Convection

For a traveling wave solution $\bar{u}(x+st)$ to the Allen-Cahn equation

$$u_t + F'(u) = u_{xx}, \tag{5.1}$$

it's convenient to switch to a shifted coordinate frame in which the wave is a stationary solution $\bar{u}(x)$ for the equation

$$u_t + su_x + F'(u) = u_{xx}. \tag{5.2}$$

In this case, linearization about the wave leads to an eigenvalue problem

$$H_s y := -y'' + sy' + V(x)y = \lambda y, \tag{5.3}$$

where $V(x) = F''(\bar{u}(x))$. Our goal in this section is to show that our development for (1.1) can be extended to the case (5.3) in a straightforward manner. For this discussion, we take any real number $s \neq 0$, and we continue to let assumptions (A1) and (A2) hold.

The main issues we need to address are as follows: (1) we need to show that the point spectrum for H_s is real-valued; (2) we need to show that the n-dimensional subspaces associated with H_s are Lagrangian; and (3) we need to show that the eigenvalues of the associated unitary matrix $\tilde{W}(x;\lambda)$ rotate monotonically as λ increases (or decreases). Once these items have been verified, the remainder of our analysis carries over directly from the case $s = 0$.

5.1. **Essential Spectrum.** As for the case $s = 0$ the essential spectrum for $s \neq 0$ can be identified from the asymptotic equations

$$-y'' + sy' + V_{\pm}y = \lambda y. \tag{5.4}$$

Precisely, the essential spectrum will correspond with values of λ for which (5.4) admits a solution of the form $y(x) = e^{ikx}r$ for some constant non-zero vector $r \in \mathbb{C}^n$. Upon substitution of this ansatz into (5.3) we obtain the relations

$$(k^2 I + iskI + V_{\pm})r = \lambda r.$$

We see that for $k \in \mathbb{R}$ the admissible values for

$$\lambda(k) - k^2 - isk$$

will be precisely the eigenvalues of $V_{\pm}$, which we continue to denote $\{\nu_j^{\pm}\}_{j=1}^n$. We can conclude that the essential spectrum will lie on or to the right of the family of parabolas

$$\lambda(k) = \nu_j^{\pm} + k^2 + isk,$$

which can be characterized as

$$\operatorname{Re}\lambda = \nu_j^{\pm} + \frac{1}{s^2}(\operatorname{Im}\lambda)^2.$$

In particular, we see that if $\operatorname{Re}\lambda < \nu_{\min}$ then λ will not be in the essential spectrum. For notational convenience, we denote by Ω the region in $\mathbb{C}$ on or two the right of these parabolas.

5.2. **In $\mathbb{C}\backslash\Omega$ the Point Spectrum of H_s is Real-Valued.** For any $\lambda \in \mathbb{C}\backslash\Omega$, we can look for ODE solutions with asymptotic behavior $y(x) = e^{\mu x}r$. Upon substitution into (5.4) we obtain the eigenvalue problem

$$(-\mu^2 + s\mu + V_{\pm} - \lambda)r = 0.$$

As in Section 2 we denote the eigenvalues of $V_{\pm}$ by $\{\nu_j^{\pm}\}_{j=1}^n$, with associated eigenvectors $\{r_j^{\pm}\}_{j=1}^n$. We see that the possible growth/decay rates μ will satisfy

$$\mu^2 - s\mu + \lambda = \nu_j^{\pm} \implies \mu = \frac{s \pm \sqrt{s^2 - 4(\lambda - \nu_j^{\pm})}}{2}.$$

We label the $2n$ growth/decay rates similarly as in Section 2

$$\mu_j^{\pm}(\lambda) = \frac{s - \sqrt{s^2 - 4(\lambda - \nu_{n+1-j}^{\pm})}}{2}$$
$$\mu_{n+j}^{\pm}(\lambda) = \frac{s + \sqrt{s^2 - 4(\lambda - \nu_j^{\pm})}}{2},$$

for $j = 1, 2, \dots, n$.

Now, suppose $\lambda \in \mathbb{C}\backslash\Omega$ is an eigenvalue for H_s. For this fixed value, we can obtain asymptotic ODE estimates on solutions of (5.3) with precisely the same form as those described in Lemma 2.2 (keeping in mind that the specifications of $\{\mu_j^{\pm}\}_{j=1}^{2n}$ are different). Letting $\psi(x;\lambda)$ denote the eigenfunction associated with λ, we conclude that $\psi(x;\lambda)$ can be expressed both as a linear combination of the solutions that decay as $x \to -\infty$ (i.e., those associated with rates $\{\mu_{n+j}^-\}_{j=1}^n$) and as a linear combination of the solutions that decay as $x \to +\infty$ (i.e., those associated with rates $\{\mu_j^+\}_{j=1}^n$).

Keeping in mind that we are in the case $s \neq 0$, we make the change of variable $\phi(x) = e^{-\frac{s}{2}x} y(x)$ (following a similar analysis in [10]), for which a direct calculation yields

$$\mathcal{H}_s \phi := e^{-\frac{s}{2}x} H_s e^{\frac{s}{2}x} \phi = -\phi'' + (\frac{s^2}{4} + V(x))\phi = \lambda \phi.$$

Moreover, if $y(x)$ is a solution of $H_s y = \lambda y$ that decays with rate $\mu_{n+j}^-(\lambda)$ as $x \to -\infty$ then the corresponding $\phi(x)$ will decay as $x \to -\infty$ with rate

$$-\frac{s}{2} + \frac{s + \sqrt{s^2 - 4(\lambda - \nu_j^\pm)}}{2} = \frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_j^\pm)} > 0, \tag{5.5}$$

and likewise if $y(x)$ is a solution of $H_s y = \lambda y$ that decays with rate $\mu_j^+(\lambda)$ as $x \to +\infty$ then the corresponding $\phi(x)$ will decay as $x \to +\infty$ with rate

$$-\frac{s}{2} + \frac{s - \sqrt{s^2 - 4(\lambda - \nu_j^\pm)}}{2} = -\frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_j^\pm)} < 0. \tag{5.6}$$

In this way we see that $\varphi(x;\lambda) = e^{-\frac{s}{2}x}\psi(x;\lambda)$ is an eigenfunction for $\mathcal{H}_s$, associated with the eigenvalue λ. But $\mathcal{H}_s$ is self-adjoint, and so its spectrum is confined to $\mathbb{R}$. We conclude that $\lambda \in \mathbb{R}$.

Finally, we observe that although the real value $\lambda = \nu_{\min}$ is embedded in the essential spectrum, it is already in $\mathbb{R}$. In this way, we conclude that any eigenvalue λ of H_s with $\operatorname{Re} \lambda \leq \nu_{\min}$ must be real-valued.

5.3. **Bound on the Point Spectrum of H_s.** Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of H_s with associated eigenvector $\psi(x;\lambda)$. Taking an $L^2(\mathbb{R})$ inner product of $H_s\psi = \lambda\psi$ with ψ we obtain the relation

$$\|\psi'\|^2 + s\langle \psi', \psi \rangle + \langle V\psi, \psi \rangle = \lambda \|\psi\|^2.$$

We see that

$$\begin{aligned}
\lambda\|\psi\|^2 &\geq \|\psi'\|^2 - |s|\|\psi'\|\|\psi\| + \langle V\psi, \psi\rangle \\
&\geq \|\psi'\|^2 - |s|(\frac{\epsilon}{2}\|\psi'\|^2 + \frac{1}{2\epsilon}\|\psi\|^2) - \|V\|_\infty \|\psi\|^2 \\
&\geq -(\frac{1}{2\epsilon} + \|V\|_\infty)\|\psi\|^2,
\end{aligned}$$

from which we conclude that λ is bounded below. (In this calculation, $\epsilon > 0$ has been taken sufficiently small.)

5.4. **The Spaces $\ell^-(x;\lambda)$ and $\ell^-_{\mathbf{R}}(\lambda)$ are Lagrangian.** Since $\sigma_p(H) \subset \mathbb{R}$, we can focus on $\lambda \in \mathbb{R}$, in which case the growth/decay rates $\{\mu_j^\pm\}_{j=1}^{2n}$ remain ordered as λ varies. In light of this, the estimates of Lemma 2.2 remain valid precisely as stated, with our revised definitions of these rates. The Lagrangian property for $\mathbf{R}^- = \binom{R^-}{S^-}$ can be verified precisely as before, but for $\mathbf{X}^-(x;\lambda) = \binom{X^-(x;\lambda)}{Y^-(x;\lambda)}$ the calculation changes slightly. For this, take $\lambda < \nu_{\min}$ and temporarily set

$$A(x;\lambda) := X^-(x;\lambda)^t Y^-(x;\lambda) - Y^-(x;\lambda)^t X^-(x;\lambda),$$

and compute (letting prime denote differentiation with respect to x)

$$\begin{aligned} A'(x;\lambda) &= X^{-\prime}(x;\lambda)^t Y^-(x;\lambda) + X^-(x;\lambda)^t Y^{-\prime}(x;\lambda) \\ &\quad - Y^{-\prime}(x;\lambda)^t X^-(x;\lambda) - Y^-(x;\lambda)^t X^{-\prime}(x;\lambda). \end{aligned}$$

Using the relations

$$X^{-\prime}(x;\lambda) = Y^-(x;\lambda); \quad Y^{-\prime}(x;\lambda) = (V(x) - \lambda I)X^-(x;\lambda) + sY^-(x;\lambda), \tag{5.7}$$

we find that

$$A'(x;\lambda) = sA(x;\lambda).$$

It follows immediately that $e^{-sx}A(x;\lambda) = c$ for some constant c. But the rates of decay associated with $A(x;\lambda)$ have the form

$$\mu_{n+j}^-(\lambda) + \mu_{n+k}^-(\lambda) = s + \frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_j^-)} + \frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_k^-)},$$

from which we see that the exponents associated with $e^{-sx}A(x;\lambda)$ take the form

$$\frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_j^-)} + \frac{1}{2}\sqrt{s^2 - 4(\lambda - \nu_k^-)} > 0.$$

It is now clear that by taking $x \to -\infty$ we can conclude that $c = 0$. We conclude that $A(x;\lambda) = 0$ for all $x \in \mathbb{R}$, and it follows that $\mathbf{X}^-(x;\lambda)$ is the frame for a Lagrangian subspace (see Proposition 2.1 of [41]).

5.5. **Monotoncity.** In this case, according to Lemma 4.2 in [41] monotonicity of $\tilde{W}(x;\lambda)$ (in λ) will be determined by the matrices

$$X^-(x;\lambda)\partial_\lambda Y^-(x;\lambda) - Y^-(x;\lambda)\partial_\lambda X^-(x;\lambda) \tag{5.8}$$

and

$$X^+_\infty(\lambda)\partial_\lambda Y^+_\infty(\lambda) - Y^+_\infty(\lambda)\partial_\lambda X^+_\infty(\lambda). \tag{5.9}$$

(On the bottom shelf, (5.8) will be replaced by $(R^-)^t\partial_\lambda S^-(\lambda) - S^-(\lambda)^t\partial_\lambda R^-$.)

Let's temporarily set

$$B(x;\lambda) := X^-(x;\lambda)\partial_\lambda Y^-(x;\lambda) - Y^-(x;\lambda)\partial_\lambda X^-(x;\lambda),$$

and compute (letting prime denote differentiation with respect to x)

$$\begin{aligned} B'(x;\lambda) &:= X^{-\prime}(x;\lambda)\partial_\lambda Y^-(x;\lambda) + X^-(x;\lambda)\partial_\lambda Y^{-\prime}(x;\lambda) \\ &\quad - Y^{-\prime}(x;\lambda)\partial_\lambda X^-(x;\lambda) - Y^-(x;\lambda)\partial_\lambda X^{-\prime}(x;\lambda) \\ &= -X^-(x;\lambda)^t X^-(x;\lambda) + sB(x;\lambda), \end{aligned}$$

where we have used (5.7) to get this final relation. Integrating this last expression, we find that

$$B(x;\lambda) = -\int_{-\infty}^{x} e^{s(x-y)} X^-(y;\lambda)^t X^-(y;\lambda) dy,$$

from which we conclude that $B(x;\lambda)$ is negative definite. We can proceed similarly to verify that (5.9) is positive definite, and the matrix associated with the bottom shelf can be analyzed as in the case $s = 0$.

6. Applications

In this section, we discuss three illustrative examples that we hope will clarify the analysis. For the first two, which are adapted from [15], we will be able to carry out explicit calculations for a range of values of λ. The third example, adapted from [39], will employ Theorem 1.2 more directly, in that we will determine that a certain operator has no negative eigenvalues by computing only the principal Maslov index.

6.1. **Example 1.**

We consider the Allen-Cahn equation

$$u_t = u_{xx} - u + u^2,$$

which is known to have a pulse-type stationary solution

$$\bar{u}(x) = \frac{3}{2}\operatorname{sech}^2(\frac{x}{2}).$$

(See [15].) Linearizing about $\bar{u}(x)$ we obtain the eigenvalue problem

$$-y'' + (1 - 2\bar{u}(x))y = \lambda y,$$

which has the form (1.1) with $n = 1$ and $V(x) = 1 - 2\bar{u}(x)$ (for which (A1)-(A2) are clearly satisfied). Setting $\Phi = \binom{y}{y'}$, we can express this equation as a first order system $\Phi' = \mathbb{A}(x;\lambda)\Phi$, with

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 1 \\ 1 - 2\bar{u}(x) - \lambda & 0 \end{pmatrix}; \quad \mathbb{A}_\pm = \begin{pmatrix} 0 & 1 \\ 1 - \lambda & 0 \end{pmatrix}. \tag{6.1}$$

As observed in [15], this equation can be solved exactly for all $x \in \mathbb{R}$ and $\lambda < 1$ (in this case $\sigma_{\text{ess}}(H) \subset [1,\infty)$). In particular, if we set $s = \frac{x}{2}$, $\gamma = 2\sqrt{1-\lambda}$, and

$$H^\pm(s,\lambda) = \mp a_0 + a_1 \tanh s \mp \tanh^2 s + \tanh^3 s,$$

with

$$a_0 = \frac{\gamma}{15}(4 - \gamma^2); \quad a_1 = \frac{1}{5}(2\gamma^2 - 3); \quad a_2 = -\gamma,$$

then (6.1) has (up to multiplication by a constant) exactly one solution that decays as $x \to -\infty$,

$$\Phi^-(x;\lambda) = e^{\gamma s}\begin{pmatrix} H^-(s,\lambda) \\ \frac{1}{2}(H_s^-(s,\lambda) + \gamma H^-(s,\lambda)) \end{pmatrix},$$

and exactly one solution that decays as $x \to +\infty$,

$$\Phi^+(x;\lambda) = e^{-\gamma s} \begin{pmatrix} H^+(s,\lambda) \\ \frac{1}{2}(H_s^+(s,\lambda) - \gamma H^+(s,\lambda)) \end{pmatrix}.$$

The target space can be obtained either from $\Phi^+(x;\lambda)$ (by taking $x \to \infty$) or by working with $\mathbb{A}_+(\lambda)$ directly (as discussed during our analysis), and in either case we find that a frame for the target space is $\mathbf{R}^+ = \binom{R^+}{S^+} = \binom{1}{-\sqrt{1-\lambda}}$. Computing directly, we see that

$$(R^+ - iS^+(\lambda))(R^+ + iS^+(\lambda))^{-1} = \frac{1 + i\sqrt{1-\lambda}}{1 - i\sqrt{1-\lambda}}.$$

Likewise, the evolving frame in this case can be taken to be

$$\mathbf{X}^-(x;\lambda) = \begin{pmatrix} X^-(x;\lambda) \\ Y^-(x,\lambda) \end{pmatrix} = \begin{pmatrix} H^-(s,\lambda) \\ \frac{1}{2}(H_s^-(s,\lambda) + \gamma H^-(s,\lambda)) \end{pmatrix}.$$

We set

$$\tilde{\mathcal{W}}(x;\lambda) = -(X^-(x;\lambda) + iY^-(x;\lambda))(X^-(x;\lambda) - iY^-(x;\lambda))^{-1}(R^+ - iS^+(\lambda))(R^+ + iS^+(\lambda))^{-1},$$

which in this case we can compute directly. The results of such a calculation, carried out in MATLAB, are depicted in Figure 4.

Remark 6.1. For the Maslov Box, we should properly use $\tilde{W}(x;\lambda)$ as defined in (4.1) for some sufficiently large x_∞, but for the purpose of graphical illustration (see Figure 4) there is essentially no difference between working with $\tilde{W}(x;\lambda)$ and working with $\tilde{\mathcal{W}}(x;\lambda)$ (i.e., taking the target Lagrangian subspace to be $\ell^+_{\mathbf{R}}(\lambda)$). The lines at ± 3 in Figure 4 are only drawn for illustration.

Referring to Figure 4, the curves comprise x-λ pairs for which $\tilde{\mathcal{W}}(x;\lambda)$ has -1 as an eigenvalue. The eigenvalues in this case are known to be $-\frac{5}{4}$, 0, and $\frac{3}{4}$, and we see that these are the locations of crossings along the top shelf. We note particularly that the Principal Maslov Index is -1, because the path $\bar{\Gamma}_0$ is only crossed once (the middle curve approaches $\bar{\Gamma}_0$ asymptotically, but this does not increment the Maslov index).

6.2. **Example 2.** We consider the Allen-Cahn system

$$\begin{aligned} u_t &= u_{xx} - 4u + 6u^2 - c(u - v) \\ v_t &= v_{xx} - 4v + 6v^2 + c(u - v), \end{aligned} \tag{6.2}$$

where $c > -2$, with also $c \neq 0$. System (6.2) is known to have a stationary solution

$$\begin{aligned} \bar{u}(x) &= \operatorname{sech}^2 x \\ \bar{v}(x) &= \operatorname{sech}^2 x \end{aligned}$$

(see [15]). Linearizing about this vector solution, we obtain the eigenvalue system

$$\begin{aligned} -\phi'' + (4 - 12\bar{u}(x) + c)\phi - c\psi &= \lambda\phi \\ -\psi'' - c\phi + (4 - 12\bar{v}(x) + c)\psi &= \lambda\psi, \end{aligned} \tag{6.3}$$

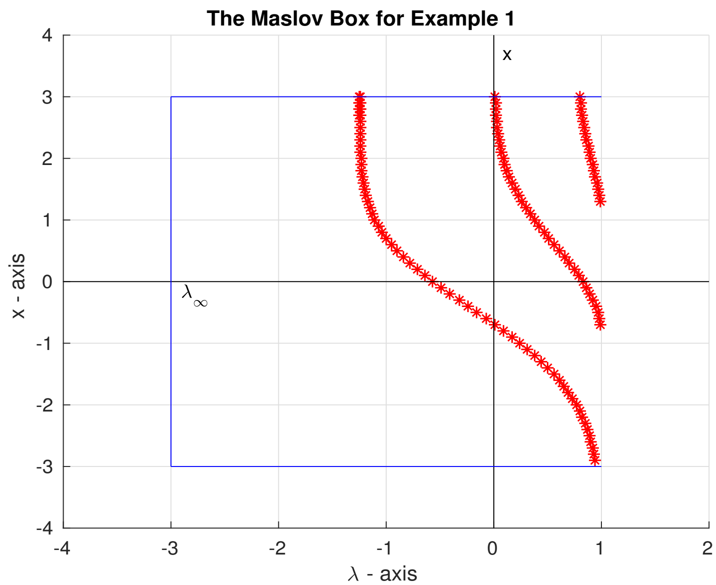

FIGURE 4. Figure for Example 1.

which can be expressed in form (1.1) with $y = \binom{\phi}{\psi}$ and

$$V(x) = \begin{pmatrix} (4 - 12\bar{u}(x) + c) & -c \\ -c & (4 - 12\bar{v} + c) \end{pmatrix}.$$

Following [15] we can solve this system explicity in terms of functions

$$\begin{aligned} w^-(x;\kappa) &= e^{\sqrt{\kappa}x}\Big(a_0 + a_1 \tanh x + a_2 \tanh^2 x + \tanh^3 x\Big) \\ w^+(x;\kappa) &= e^{-\sqrt{\kappa}x}\Big(-a_0 + a_1 \tanh x - a_2 \tanh^2 x + \tanh^3 x\Big), \end{aligned}$$

where

$$a_0 = \frac{\kappa}{15}(4 - \kappa); \quad a_1 = \frac{1}{5}(2\kappa - 3); \quad a_2 = -\sqrt{\kappa},$$

and the values of κ will be specified below.

We can now construct a basis for solutions decaying as $x \to -\infty$ as

$$\mathbf{p}_3^-(x;\lambda) = \begin{pmatrix} w^-(x; -\lambda + 4) \\ w^-(x; -\lambda + 4) \end{pmatrix}; \quad \mathbf{p}_4^-(x;\lambda) = \begin{pmatrix} -w^-(x; -\lambda + 4 + 2c) \\ w^-(x; -\lambda + 4 + 2c) \end{pmatrix},$$

and a basis for solutions decaying as $x \to +\infty$ as

$$\mathbf{p}_1^+(x;\lambda) = \begin{pmatrix} w^+(x; -\lambda + 4) \\ w^+(x; -\lambda + 4) \end{pmatrix}; \quad \mathbf{p}_2^+(x;\lambda) = \begin{pmatrix} -w^+(x; -\lambda + 4 + 2c) \\ w^+(x; -\lambda + 4 + 2c) \end{pmatrix}.$$

These considerations allow us to construct

$$X^-(x;\lambda) = \begin{pmatrix} w^-(x;-\lambda+4) & -w^-(x;-\lambda+4+2c) \\ w^-(x;-\lambda+4) & +w^-(x;-\lambda+4+2c) \end{pmatrix};$$

$$X^+(x;\lambda) = \begin{pmatrix} w^+(x;-\lambda+4) & -w^+(x;-\lambda+4+2c) \\ w^+(x;-\lambda+4) & +w^+(x;-\lambda+4+2c) \end{pmatrix},$$

with then $Y^-(x;\lambda) = X^-_x(x;\lambda)$ and $Y^+(x;\lambda) = X^+_x(x;\lambda)$.

In order to construct the target space, we write (6.3) as a first-order system by setting $\Phi_1 = \phi$, $\Phi_2 = \psi$, $\Phi_3 = \phi'$, and $\Phi_4 = \psi'$. This allows us to write

$$\mathbf{\Phi}' = \mathbb{A}(x;\lambda)\mathbf{\Phi}; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f(x;\lambda) & -c & 0 & 0 \\ -c & -f(x;\lambda) & 0 & 0 \end{pmatrix},$$

where $f(x;\lambda) = \lambda - 4 - c + 12\bar{u}$. We set

$$\mathbb{A}_+(\lambda) := \lim_{x\to+\infty} \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda+4+c & -c & 0 & 0 \\ -c & -\lambda+4+c & 0 & 0 \end{pmatrix}.$$

If we follow our usual ordering scheme for indices then for $-2 < c < 0$ we have $\nu_1^+ = 4+2c$ and $\nu_2^+ = 4$, with corresponding eigenvectors $r_1^+ = \binom{1}{-1}$ and $r_2^+ = \binom{1}{1}$. Accordingly, we have $\mu_1^+(\lambda) = -\sqrt{-\lambda+4}$, $\mu_2^+(\lambda) = -\sqrt{-\lambda+4+2c}$, $\mu_3^+(\lambda) = \sqrt{-\lambda+4+2c}$, and $\mu_4^+(\lambda) = \sqrt{-\lambda+4}$. We conclude that a frame for $\ell^+_{\mathbf{R}}(\lambda)$ is $\mathbf{R}^+ = \binom{R^+}{S^+(\lambda)}$, where

$$R^+ = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad S^+(\lambda) = \begin{pmatrix} \mu_1^+(\lambda) & \mu_2^+(\lambda) \\ \mu_1^+(\lambda) & -\mu_2^+(\lambda) \end{pmatrix}.$$

The resulting spectral curves are plotted in Figure 5 for $c = -1$. In this case, it is known that H has exactly six eigenvalues: -7, -5, -2, 0, 1 and 3 (the eigenvalues 1 and 3 are omitted from our window). We see that the three crossings along the line $\lambda = 0$ correspond with the count of three negative eigenvalues.

6.3. **Example 3.** Consider the Allen-Cahn system

$$u_t = u_{xx} - D_u F(u), \tag{6.4}$$

where

$$F(u_1,u_2) = u_1^2u_2^2 + u_1^2(1-u_1-u_2)^2 + u_2^2(1-u_1-u_2)^2,$$

which is adapted from p. 39 of [39]. In this setting, stationary solutions $\bar{u}(x)$ satisfying endstate conditions

$$\lim_{x\to\pm\infty} \bar{u}(x) = u_\pm,$$

for $u_- \neq u_+$ are called *transition waves.* A transition wave solution for (6.4) (numerically generated) is depicted in Figure 6.3. In this case, we have $u_1^- = 1$, $u_2^- = 0$, $u_1^+ = 0$, and $u_2^+ = 1$.

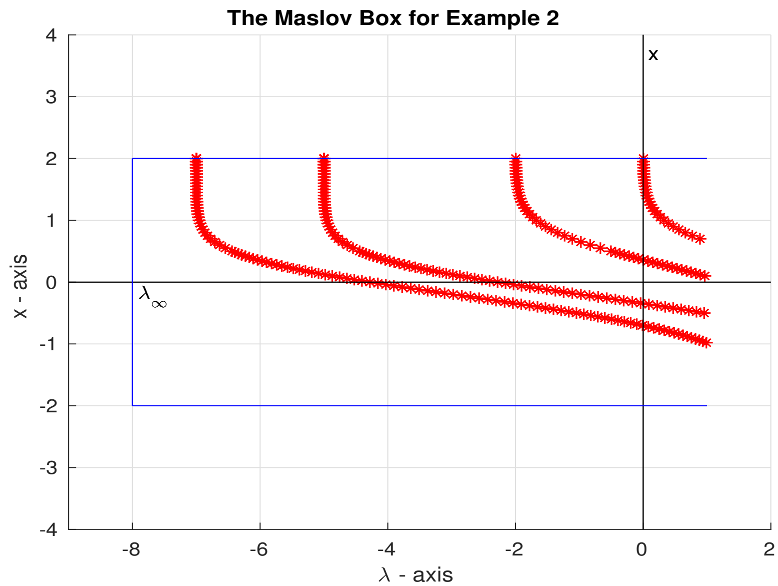

FIGURE 5. Figure for Example 2.

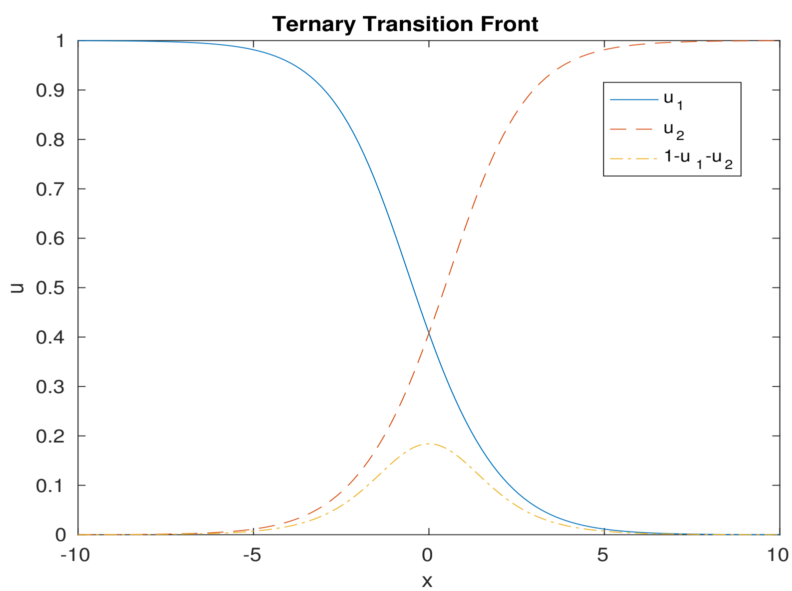

FIGURE 6. Transition wave solution for a ternary Cahn-Hilliard system.

Upon linearization of (6.4) about $\bar{u}(x)$, we obtain the eigenvalue problem

$$H\phi = -\phi'' + V(x)\phi = \lambda\phi, \tag{6.5}$$

where $V(x) := D_u^2 F(\bar{u})$ denotes the usual Hessian matrix. In this case,

$$V_- = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}; \quad V_+ = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}.$$

Using our usual labeling scheme, we have $\nu_1^+ = 3 - \sqrt{5}$ and $\nu_2^+ = 3 + \sqrt{5}$, with respective eigenvectors

$$r_1^+ = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}; \quad r_2^+ = \begin{pmatrix} 1 \\ -\frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

The corresponding values $\{\mu_j^+\}_{j=1}^4$ are $\mu_1^+ = -\sqrt{\nu_2^+ - \lambda}$, $\mu_2^+ = -\sqrt{\nu_1^+ - \lambda}$, $\mu_3^+ = \sqrt{\nu_1^+ - \lambda}$, $\mu_4^+ = \sqrt{\nu_2^+ - \lambda}$.

For the target space $\ell_{\mathbf{R}}^+$ we use the frame

$$\mathbf{R}^+(\lambda) = \begin{pmatrix} R^+ \\ S^+(\lambda) \end{pmatrix} = \begin{pmatrix} r_2^+ & r_1^+ \\ \mu_1^+ r_2^+ & \mu_2^+ r_1^+ \end{pmatrix}.$$

For the evolving Lagrangian subspace $\ell^-(x;\lambda)$ we need a basis for the two-dimensional space of solutions that decay as $x \to -\infty$. Generally, we construct this basis from the solutions

$$\mathbf{p}_{2+j}^-(x;\lambda) = e^{\mu_{2+j}^-(\lambda)x}(r_{2+j}^- + \mathbf{E}_{2+j}^-(x;\lambda)); \quad j = 1, 2,$$

from Lemma 2.2, but computationally it is easier to note that for $\lambda = 0$, $\bar{u}_x$ is a solution of (1.1) that decays as $x \to -\infty$. In [39] the authors check that $\bar{u}_x(x)$ decays at the slower rate (i.e., the rate of $\mathbf{p}_3^-$), so we can take as our frame

$$\mathbf{X}^-(x;0) = \begin{pmatrix} p_4^-(x;0) & \bar{u}_x(x) \\ p_4^{-\prime}(x;0) & \bar{u}_{xx}(x) \end{pmatrix},$$

which we scale to

$$\mathbf{X}^-(x;0) = \begin{pmatrix} e^{-\mu_4^-(\lambda)x} p_4^-(x;0) & e^{-\mu_3^-(\lambda)x} \bar{u}_x(x) \\ e^{-\mu_4^-(\lambda)x} p_4^{-\prime}(x;0) & e^{-\mu_3^-(\lambda)x} \bar{u}_{xx}(x) \end{pmatrix}.$$

The advantage of this is that $\bar{u}(x)$ is already known, and the faster-decaying solution $\mathbf{p}_4^-(x;0)$ can be generated numerically in a straightforward way (see [39]).

In practice, we compute $\tilde{W}(x;0)$ for x running from -10 to 10, and compute $\operatorname{Mas}(\ell^-, \ell_{\mathbf{R}}^+; \bar{\Gamma}_0)$ by following the eigenvalues of $\tilde{W}(x;0)$ as they move along S^1. For $x = -10$ the two eigenvalues of $\tilde{W}(x;0)$ both have positive real part, as depicted on the left side of Figure 7. As x increases from -10, the eigenvalue in the first quadrant remains confined to the first quadrant, and so does not contribute to the Maslov index. The eigenvalue in the fourth quadrant rotates monotonically clockwise, nearing -1 as x approaches 10 (and without ever making a complete loop). The final configuration of the eigenvalues of $\tilde{W}(x;0)$ is depicted on the right side of Figure 7. In calculations of this sort, we must take care with eigenvalues that start or stop near -1, because a crossing could be indicated by roundoff error. In this case, we know that $\lambda = 0$ is an eigenvalue for H (with eigenfunction $\bar{u}_x(x)$), and we can conclude from this that (at least) one of the eigenvalues of $\tilde{W}(x;0)$ will approach -1 as $x \to +\infty$. Since there is only one such eigenvalue, we can conclude that it indeed approaches -1 as

$x \to +\infty$. Since it is approaching in the clockwise direction, it does not contribute to the Maslov index, and we conclude that

$$\mathrm{Mas}(\ell^-, \ell^+_{\mathbf{R}}; \bar{\Gamma}_0) = 0,$$

from which Theorem 1.2 asserts that H has no eigenvalues below $\lambda = 0$. (Strictly speaking, this is only strong numerical evidence, not numerical proof.) Moreover, since only one eigenvalue of $\tilde{W}(x;0)$ approaches -1 as $x \to +\infty$ we can conclude that $\lambda = 0$ is a simple eigenvalue of H. It follows from Theorem 1.2 of [40] that $\bar{u}(x)$ is asymptically stable as a solution to (6.4).

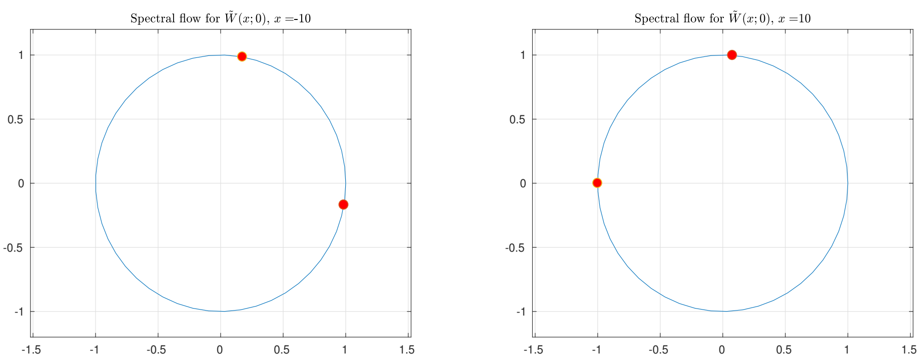

FIGURE 7. Spectral flow for $\tilde{W}(x;0)$ in Example 3.

We note that our ability to efficiently compute the Maslov index for the single shelf $\lambda = 0$ hinged on the observation that the known value $\bar{u}_x(x)$ provided the slowly decaying solution needed for the calculation. Computing the Maslov index associated with any other fixed λ, as x runs from $-\infty$ to $+\infty$ is more computationally intensive, and so we have not computed the full Maslov box for this example. Indeed, one strength of Theorem 1.2 is precisely that it obviates the need for such a calculation.

APPENDIX

In this short appendix, we construct the asymptotic Lagrangian path

$$\ell^-_{+\infty}(\lambda) = \lim_{x \to +\infty} \ell^-(x;\lambda),$$

for $\lambda < \nu_{\min}$ and show that it is not generally continuous in λ. For a related discussion from a different point of view, we refer to Lemma 3.7 of [2].

As a start, we recall that one choice of frame for $\ell^-(x;\lambda)$ is

$$\mathbf{X}^-(x;\lambda) = \begin{pmatrix} \mathbf{p}^-_{n+1}(x;\lambda) & \mathbf{p}^-_{n+2}(x;\lambda) & \dots & \mathbf{p}^-_{2n}(x;\lambda) \end{pmatrix},$$

where we have from Lemma 2.2

$$\mathbf{p}^-_{n+j}(x;\lambda) = e^{\mu^-_{n+j}(\lambda)x}(\mathbf{r}^-_{n+j} + \mathbf{E}^-_{n+j}(x;\lambda)); \quad j = 1, 2, \dots, n.$$

Each of the $\mathbf{p}^-_{n+j}$ can be expressed as a linear combination of the basis of solutions $\{\mathbf{p}^+_k\}_{k=1}^{2n}$, where we recall from Lemma 2.2 that the solutions $\{\mathbf{p}^+_k\}_{k=1}^{n}$ decay as $x \to +\infty$, while the solutions $\{\mathbf{p}^+_k\}_{k=n+1}^{2n}$ grow as $x \to +\infty$. I.e., for each $j = 1, 2, \dots, n$, there exist coefficients $\{c_{jk}(\lambda)\}_{k=1}^{2n}$ so that

$$\mathbf{p}^-_{n+j}(x;\lambda) = \sum_{k=1}^{2n} c_{jk}(\lambda)\mathbf{p}^+_k(x;\lambda),$$

and so the collection of vector functions on the right-hand side provides an alternative way to express the same frame $\mathbf{X}^-(x;\lambda)$.

Fix $\lambda \in [-\lambda_\infty, \nu_{\min})$, and suppose the fastest growth mode $\mathbf{p}^+_{2n}(x;\lambda)$ appears in the expansion of at least one of the $\mathbf{p}^-_{n+j}$ (i.e., the coefficient associated with this mode is non-zero). (There may be additional modes that grow at the same rate μ^+_{2n}, but they will have different, and linearly independent, associated eigenvectors $\mathscr{r}^-_{n+j}$, allowing us to distinguish them from $\mathbf{p}^+_{2n}(x;\lambda)$.) By taking appropriate linear combinations, we can identify a new frame for $\ell^-(x;\lambda)$ for which $\mathbf{p}^+_{2n}$ only appears in one column. If $\mathbf{p}^+_{2n}(x;\lambda)$ does not appear in the sum for any $\mathbf{p}^-_{n+j}$ we can start with $\mathbf{p}^+_{2n-1}$ and proceed similarly, continuing until we get to the first mode that appears. Since the $\{\mathbf{p}^-_{n+j}\}_{j=1}^n$ form a basis for an n-dimensional space, we will be able to distinguish n modes in this way. At the end of this process, we will have created a new frame for $\mathbf{X}^-(x;\lambda)$ with columns $\{\tilde{\mathbf{p}}^-_j\}_{j=1}^n$, where

$$\tilde{\mathbf{p}}^-_j(x;\lambda) = e^{\mu^+_{k(j)}x}\Big(s^+_{k(j)} + \tilde{\mathbf{E}}^+_{k(j)}(x;\lambda)\Big),$$

for some appropriate map $j \mapsto k(j)$. If the rate $\mu^+_{k(j)}$ is distinct as an eigenvalue of $\mathbb{A}_+(\lambda)$ then we will have $s^+_{k(j)} = \mathscr{r}^+_{k(j)}$, but if $\mu^+_{k(j)}$ is not distinct then $s^+_{k(j)}$ will generally be a linear combination of eigenvectors of $\mathbb{A}_+(\lambda)$ (and so, of course, still an eigenvector of $\mathbb{A}_+(\lambda)$). This process may also introduce an expansion coefficient in front of $s^+_{k(j)}$, but this can be factored out in the specification of the frame.

As usual, we can view the exponential scalings $e^{\mu^+_{k(j)}x}$ as expansion coefficients, and take as our frame for $\ell^-(x;\lambda)$ the $2n \times n$ matrix with columns $s^+_{k(j)} + \tilde{\mathbf{E}}^+_{k(j)}(x;\lambda)$. Taking now the limit $x \to \infty$ we see that we obtain the asymptotic frame

$$\mathbf{X}^-_{+\infty}(\lambda) = \begin{pmatrix} s^+_{k(1)} & s^+_{k(2)} & \cdots & s^+_{k(n)} \end{pmatrix}. \tag{6.6}$$

In order to see that $\mathbf{X}^-_{+\infty}(\lambda)$ is indeed the frame for a Lagrangian subspace, we first note that by construction the vectors $\{s_{k(j)}\}_{j=1}^n$ will be linearly independent. For the Lagrangian property, we have already verified that $\ell^-(x;\lambda)$ is a Lagrangian subspace, and that the matrix with columns $s^+_{k(j)} + \tilde{\mathbf{E}}^+_{k(j)}(x;\lambda)$ is a frame for $\ell^-(x;\lambda)$. This means that for any $i, j \in \{1, 2, \dots, n\}$

$$\omega(s^+_{k(i)} + \tilde{\mathbf{E}}^+_{k(i)}(x;\lambda), s^+_{k(j)} + \tilde{\mathbf{E}}^+_{k(j)}(x;\lambda)) = 0$$

for all $x \in \mathbb{R}$. Taking $x \to +\infty$ we see that

$$\omega(s^+_{k(i)}, s^+_{k(j)}) = 0$$

for all $i, j \in \{1, 2, \dots, n\}$, and this is precisely the Lagrangian property. We can now associate $\ell^-_{+\infty}(\lambda)$ as the Lagrangian subspace with this frame, verifying that this Lagrangian subspace is well-defined.

Last, we verify our comment that $\ell^-_{+\infty}(\lambda)$ is not generally continuous as a function of λ. To see this, we begin by noting that if $\lambda_0 \in [-\lambda_\infty, \nu_{\min})$ is not an eigenvalue of H then the leading modes selected in our process must all be growth modes, and we obtain $\mathbf{X}^-_{+\infty}(\lambda_0) = \mathbf{R}^+(\lambda_0)$, in agreement with Lemma 3.7 in [2]. Suppose, however, that $\lambda_0 \in [-\lambda_\infty, \nu_{\min})$ is an eigenvalue of H, and for simplicity assume λ_0 has geometric multiplicity 1. Away from essential spectrum, λ_0 will be isolated, and so we know that any λ sufficiently close to λ_0 will not be in the spectrum of H. We conclude that the frame for λ_0 will comprise $n-1$ of the eigenvectors $\{\mathscr{r}^+_{n+j}\}_{j=1}^n$, along with one of the $\{\mathscr{r}^+_j\}_{j=1}^n$. Since the exchanged vectors will lead to bases of different spaces, we can conclude that $\ell^-_{+\infty}(\lambda)$ is not continuous at λ_0.

In order to clarify the discussion, we briefly consider the simple case $n = 1$. In this case, we have (for $\lambda < \nu_{\min}$) a single solution $\mathbf{p}_2^-(x;\lambda)$ that decays as $x \to -\infty$, and we can write

$$\mathbf{p}_2^-(x;\lambda) = c_{11}(\lambda)\mathbf{p}_1^+(x;\lambda) + c_{12}(\lambda)\mathbf{p}_2^+(x;\lambda),$$

where $\mathbf{p}_1^+(x;\lambda)$ decays as $x \to +\infty$ and $\mathbf{p}_2^+(x;\lambda)$ grows as $x \to +\infty$. If λ_0 is not an eigenvalue of H we must have $c_{12}(\lambda_0) \neq 0$, and so

$$\begin{aligned}
\mathbf{p}_2^-(x;\lambda_0) &= c_{11}(\lambda_0)e^{\mu_1^+(\lambda_0)x}(\mathscr{r}_1^+ + \mathbf{E}_1^+(x;\lambda_0)) + c_{12}(\lambda_0)e^{\mu_2^+(\lambda_0)x}(\mathscr{r}_2^+ + \mathbf{E}_2^+(x;\lambda_0)) \\
&= c_{12}(\lambda_0)e^{\mu_2^+(\lambda_0)x}\Big(\mathscr{r}_2^+ + \mathbf{E}_2^+(x;\lambda_0) + \frac{c_{11}(\lambda_0)}{c_{12}(\lambda_0)}e^{(\mu_1^+(\lambda_0)-\mu_2^+(\lambda_0))x}(\mathscr{r}_1^+ + \mathbf{E}_1^+(x;\lambda_0))\Big) \\
&= c_{12}(\lambda_0)e^{\mu_2^+(\lambda_0)x}(\mathscr{r}_2^+ + \tilde{\mathbf{E}}_2^+(x;\lambda_0)),
\end{aligned}$$

where $\tilde{\mathbf{E}}_2^+(x;\lambda_0)) = \mathbf{O}((1+|x|)^{-1})$.

We can view $\mathscr{r}_2^+ + \tilde{\mathbf{E}}_2^+(x;\lambda_0)$ as a frame for $\ell^-(x;\lambda_0)$, and it immediately follows that as $x \to \infty$ the path of Lagrangian subspaces $\ell^-(x;\lambda_0)$ approaches the Lagrangian subspace with frame $\mathscr{r}_2^+$ (denoted $\ell^-_{+\infty}(\lambda_0)$ above). Moreover, since $\mathscr{r}_1^+$ serves as a frame for $\ell^+_{\mathbf{R}}(\lambda_0)$ we can construct $\tilde{\mathcal{W}}(x;\lambda_0)$ from this pair. Taking the limit as $x \to \infty$ we see that

$$\tilde{\mathcal{W}}^+(\lambda_0) := \lim_{x\to\infty} \tilde{\mathcal{W}}(x;\lambda_0) = -\frac{r_1^+ + i\mu_2^+(\lambda_0)r_1^+}{r_1^+ - i\mu_2^+(\lambda_0)r_1^+} \cdot \frac{r_2^+ - i\mu_1^+(\lambda_0)r_2^+}{r_2^+ + i\mu_1^+(\lambda_0)r_2^+}.$$

By normalization, we can take both r_1^+ and r_2^+ to be 1, and we must have $\mu_1^+ = -\mu_2^+$, so

$$\tilde{\mathcal{W}}^+(\lambda_0) = -\frac{1 + i\mu_2^+(\lambda_0)}{1 - i\mu_2^+(\lambda_0)} \cdot \frac{1 + i\mu_2^+(\lambda_0)}{1 - i\mu_2^+(\lambda_0)} = -\frac{(1 + i\mu_2^+(\lambda_0))^2}{(1 - i\mu_2^+(\lambda_0))^2},$$

which can only be -1 if $\mu_2^+(\lambda_0) = 0$ (a case ruled out in this calculation).

On the other hand, if $\lambda_0 \in \sigma_{pt}(H)$ we will have $c_{12}(\lambda_0) = 0$ (and $c_{11}(\lambda_0) \neq 0$). In this case, the frame for $\ell^-(x;\lambda_0)$ will be $\mathscr{r}_1^+ + \tilde{\mathbf{E}}_1^+(x;\lambda_0)$, and taking $x \to +\infty$ we see that $\ell^-(x;\lambda_0)$ will approach the Lagrangian subspace with frame $\mathscr{r}_1^+$. Recalling again that $\mathscr{r}_1^+$ serves as a frame for $\ell^+_{\mathbf{R}}(\lambda_0)$ we see that

$$\begin{aligned}\tilde{\mathcal{W}}^+(\lambda_0) &:= \lim_{x\to\infty} \tilde{\mathcal{W}}(x;\lambda_0) = -\frac{r_2^+ + i\mu_1^+(\lambda_0) r_2^+}{r_2^+ - i\mu_1^+(\lambda_0) r_2^+} \cdot \frac{r_2^+ - i\mu_1^+(\lambda_0) r_2^+}{r_2^+ + i\mu_1^+(\lambda_0) r_2^+} \\ &= -1.\end{aligned}$$

For Example 1 in Section 6, we have $\mu_1^+(\lambda) = -\sqrt{1-\lambda}$ and $\mu_2^+(\lambda) = +\sqrt{1-\lambda}$. We know that in that example $\lambda_0 = 0$ is an eigenvalue, so we have $\tilde{\mathcal{W}}^+(0) = -1$, but for $\lambda \neq 0$, $|\lambda| < 1$, we have

$$\tilde{\mathcal{W}}^+(\lambda) = -\frac{(1 + i\sqrt{1-\lambda})^2}{(1 - i\sqrt{1-\lambda})^2}.$$

If we substitute $\lambda = 0$ into this relation, we obtain $+1$, and so we see that $\tilde{\mathcal{W}}^+(\lambda)$ is not continuous in λ (at $\lambda = 0$ in this case).

References

[1] A. Abbondandolo, *Morse Theory for Hamiltonian Systems.* Chapman & Hall/CRC Res. Notes Math. **425**, Chapman & Hall/CRC, Boca Raton, FL, 2001.

[2] J. Alexander, R. Gardner, and C. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. reine angew. Math. **410** (1990) 167–212.

[3] V. I. Arnold, *Characteristic class entering in quantization conditions,* Func. Anal. Appl. **1** (1967) 1 – 14.

[4] V. I. Arnold, *The Sturm theorems and symplectic geometry,* Func. Anal. Appl. **19** (1985) 1–10.

[5] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, in the series *Mathematics in Science and Engineering* (vol. 8), Academic Press 1964.

[6] R. Bott, *On the iteration of closed geodesics and the Sturm intersection theory,* Comm. Pure Appl. Math. **9** (1956) 171 – 206.

[7] R. Beals, P. Deift, and C. Tomei, *Direct and inverse scattering on the line*, Mathematical surveys and monographs no. 28, AMS 1988.

[8] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill, Sydney, 1978.

[9] B. Booss-Bavnbek and K. Furutani, *The Maslov index: a functional analytical definition and the spectral flow formula,* Tokyo J. Math. **21** (1998), 1–34.

[10] A. Bose and C. K. R. T. Jones, *Stability of the in-phase traveling wave solution in a pair of coupled nerve fibers,* Indiana U. Math. J. **44** (1995) 189 – 220.

[11] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs **186**, AMS 2013.

[12] W.-J. Beyn and J. Lorenz, *Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals*, Numerical functional analysis and optimization **20** (1999) 201-244.

[13] M. Beck and S. Malham, *Computing the Maslov index for large systems*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 21592173.

[14] F. Chardard, F. Dias and T. J. Bridges, *Fast computation of the Maslov index for hyperbolic linear systems with periodic coefficients.* J. Phys. A **39** (2006) 14545 – 14557.

[15] F. Chardard, F. Dias and T. J. Bridges, *Computing the Maslov index of solitary waves. I. Hamiltonian systems on a four-dimensional phase space*, Phys. D **238** (2009) 1841 – 1867.

[16] F. Chardard, F. Dias and T. J. Bridges, *Computing the Maslov index of solitary waves, Part 2: Phase space with dimension greater than four.* Phys. D **240** (2011) 1334 – 1344.

[17] C-N Chen and X. Hu, *Maslov index for homoclinic orbits in Hamiltonian systems*, Ann. I. H. Poincaré – AN **24** (2007) 589 – 603.

[18] F. Chardard, *Stability of Solitary Waves*, Doctoral thesis, Centre de Mathematiques et de Leurs Applications, 2009. Advisor: T. J. Bridges.

[19] G. Cox, C. K. R. T. Jones, Y. Latushkiun, and A. Sukhtayev, *The Morse and Maslov indices for multidimensional Schrödinger operators with matrix-valued potentials*, Trans. Amer. Math. Soc. **368** (2016) 8145-8207.

[20] S. Cappell, R. Lee and E. Miller, *On the Maslov index*, Comm. Pure Appl. Math. **47** (1994), 121–186.

[21] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, MA 1965.

[22] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations.* Comm. Pure Appl. Math. **37** (1984) 207 – 253.

[23] J. J. Duistermaat, *On the Morse index in variational calculus.* Advances in Math. **21** (1976) 173 – 195.

[24] J. Deng and C. Jones, *Multi-dimensional Morse Index Theorems and a symplectic view of elliptic boundary value problems,* Trans. Amer. Math. Soc. **363** (2011) 1487 – 1508.

[25] N. Dunford and J. T. Schwartz, *Linear Operators Part II: Spectral Theory*, John Wiley & Sons, Inc., 1988 reprint of 1963 edition.

[26] D. McDuff and D. Salamon, *Introduction to symplectic topology*, Oxford University Press, 2nd Ed., 1999.

[27] R. Fabbri, R. Johnson and C. Núñez, *Rotation number for non-autonomous linear Hamiltonian systems I: Basic properties,* Z. angew. Math. Phys. **54** (2003) 484 – 502.

[28] K. Furutani, *Fredholm-Lagrangian-Grassmannian and the Maslov index,* Journal of Geometry and Physics **51** (2004) 269 – 331.

[29] R. A. Gardner, *On the structure of the spectra of periodic travelling waves,* J. Math. Pures Appl. **72** (1993) 415 – 439.

[30] F. Gesztesy, *Inverse spectral theory as influenced by Barry Simon*, In: *Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon's 60th Birthday*, pp. 741 – 820, Proc. Sympos. Pure Math. **76**, Part 2, AMS, Providence, RI, 2007.

[31] F. Gesztesy, Y. Latushkin and K. Zumbrun, *Derivatives of (modified) Fredholm determinants and stability of standing and traveling waves*, J. Math. Pures Appl. **90** (2008), 160–200.

[32] F. Gesztesy and M. Mitrea, *Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschuitz domains*, in *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications*, D. Mitrea and M. Mitrea (eds.), Proceedings of Symposia in Pure Mathematics, American Mathematical Society, RI 2008.

[33] F. Gesztesy, B. Simon and G. Teschl, *Zeros of the Wronskian and renormalized oscillation theory*, Amer. J. Math. **118** (1996) 571 – 594.

[34] F. Gesztesy and V. Tkachenko, *A criterion for Hill operators to be spectral operators of scalar type.* J. Anal. Math. **107** (2009) 287 – 353.

[35] F. Gesztesy and R. Weikard, *Picard potentials and Hill's equation on a torus.* Acta Math. **176** (1996) 73 – 107.

[36] R. Gardner and K. Zumbrun, *The gap lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math. **51** (1998) 797-855.

[37] P. Howard, *Asymptotic behavior near transition fronts for equations of generalized Cahn-Hilliard form*, Comm. Math. Phys. **269** (2007) 765-808.

[38] D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. **840**, Springer-Verlag, Berlin-New York, 1981.

[39] P. Howard and B. Kwon, *Spectral analysis for transition front solutions to Cahn-Hilliard systems*, Discrete and Continuous Dynamical Systems A **32** (2012) 126-166.

[40] P. Howard and B. Kwon, *Asymptotic stability analysis for transition front solutions in Cahn-Hilliard systems*, Physica D **241** (2012), no. 14, 1193-1222.

[41] P. Howard, Y. Latushkin, and A. Sukhtayev, *The Maslov index for Lagrangian pairs on* $\mathbb{R}^{2n}$, to appear in J. Math. Anal. Appl., arXiv 1608.00632.

[42] P. Howard and A. Sukhtayev, *The Maslov and Morse indices for Schrödinger operators on* $[0,1]$, J. Differential Equations **260** (2016), no. 5, 4499-4549.

[43] C. K. R. T. Jones, *Instability of standing waves for nonlinear Schrödinger-type equations*, Ergodic Theory Dynam. Systems **8** (1988) 119 – 138.

[44] C. K. R. T. Jones, *An instability mechanism for radially symmetric standing waves of a nonlinear Schrödinger equation*, J. Differential Equations **71** (1988) 34 – 62.

[45] C. K. R. T. Jones, Y. Latushkin, and S. Sukhtaiev, *Counting spectrum via the Maslov index for one dimensional θ-periodic Schrödinger operators*, Proceedings of the AMS **145** (2017) 363-377.

[46] C. K. R. T. Jones and R. Marangell, *The spectrum of travelling wave solutions to the Sine-Gordon equation*, Discrete and Cont. Dyn. Sys. **5** (2012) 925 – 937.

[47] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction to Dynamical Systems.* Oxford App. and Engin. Math., Oxford, 1999.

[48] Y. Karpeshina, *Perturbation Theory for the Schrödinger Operator with a Periodic Potential.* Lect. Notes Math. **1663**, Springer-Verlag, Berlin, 1997.

[49] A. Krall, *Hilbert Space, Boundary Value Problems and Orthogonal Polynomials. Operator Theory: Advances and Applications*, **133**, Birkhauser Verlag, Basel, 2002.

[50] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1980.

[51] J. P. Keener, *Principles of Applied Mathematics: Transformation and Approximation*, 2nd Ed., Westview 2000.

[52] T. Kapitula and K. Promislow, *Spectral and dynamical stability of nonlinear waves*, Springer 2013.

[53] P. Kuchment, *Quantum graphs: I. Some basic structures*, Waves in random media **14**.

[54] Y. Latushkin and A. Sukhtayev, *The Evans function and the Weyl-Titchmarsh function*, in *Special issue on stability of travelling waves*, Disc. Cont. Dynam. Syst. Ser. S 5 (2012), no. 5, 939 - 970.

[55] J. Milnor, *Morse Theory*, Annals of Math. Stud. **51**, Princeton Univ. Press, Princeton, N.J., 1963.

[56] V. P. Maslov, *Theory of perturbations and asymptotic methods*, Izdat. Moskov. Gos. Univ. Moscow, 1965. French tranlation Dunod, Paris, 1972.

[57] K. Meyer, G. Hall, and D. Offin, *Introduction to Hamiltonian systems and the N-body problem*, 2nd Ed., Applied Mathematical Sciences **90**, Springer, New York 2009.

[58] W. Magnus and S. Winkler, *Hill's Equation*, Dover, New York, 1979.

[59] V. Yu. Ovsienko, *Selfadjoint differential operators and curves on a Lagrangian Grassmannian that are subordinate to a loop,* Math. Notes **47** (1990) 270 – 275.

[60] D. Offin, *A spectral theorem for reversible second order equations with periodic coefficients*, Diff. Int. Eqns. **5** (1992) 615-629.

[61] D. Offin, *Variational structure of the zones of instability*, Diff. Int. Eqns. **14** (2001) 1111-1127.

[62] J. Phillips, Selfadjoint Fredholm operators and spectral flow, *Canad. Math. Bull.* **39** (1996), 460–467.

[63] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press, New York, 1978.

[64] J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993) 827 – 844.

[65] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995) 1–33.

[66] B. Sandstede and A. Scheel, *Absolute and convective instabilities of waves on unbounded and large bounded domains*, Physica D **145** (2000) 233-277.

[67] B. Sandstede and A. Scheel, *Relative Morse indices, Fredholm indices, and group velocities*, Discrete Contin. Dyn. Syst. **20** (2008) 139 – 158.

[68] J. Weidman, *Spectral theory of Sturm-Liouville operators. Approximation by regular problems.* In: *Sturm-Liouville Theory: Past and Present,* pp. 75–98, W. O. Amrein, A. M. Hinz and D. B. Pearson, edts, Birkhäuser, 2005.

[69] *Linear operators on Hilbert spaces*, Graduate texts in mathematics **68**, Springer 1980.

[70] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana U. Math. J. **47** (1998) 741-871. See also the errata for this paper: Indiana U. Math. J. **51** (2002) 1017–1021.

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