The Maslov index and spectral counts for linear Hamiltonian systems on $\mathbb{R}$

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Abstract

Working with a general class of linear Hamiltonian systems specified on $\mathbb{R}$, we develop a framework for relating the Maslov index to the number of eigenvalues the systems have on intervals of the form $[\lambda_1, \lambda_2)$ and $(-\infty, \lambda_2)$. We verify that our framework can be implemented for Sturm-Liouville systems, fourth-order potential systems, and a family of systems nonlinear in the spectral parameter. The analysis is primarily motivated by applications to the analysis of spectral stability for nonlinear waves, and aspects of such analyses are emphasized.

1 Introduction

For values λ confined to an interval $I \subset \mathbb{R}$, we consider linear Hamiltonian systems

$$Jy' = \mathbb{B}(x; \lambda)y; \quad x \in \mathbb{R}, \quad y(x; \lambda) \in \mathbb{C}^{2n}, \tag{1.1}$$

where J denotes the $2n \times n$ symplectic matrix

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

and throughout the analysis we will make the following assumptions on $\mathbb{B}(x; \lambda)$:

(A) For each $\lambda \in I$, $\mathbb{B}(\cdot; \lambda) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{2n \times 2n})$, with $\mathbb{B}(x; \lambda)$ self-adjoint for a.e. $x \in \mathbb{R}$, and additionally the partial derivative $\mathbb{B}_\lambda(x; \lambda)$ exists for a.e. $x \in \mathbb{R}$, with $\mathbb{B}_\lambda(\cdot; \lambda) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{2n \times 2n})$.

For this analysis, we will say that λ is an eigenvalue of (1.1) provided there exists a function

$$y(\cdot; \lambda) \in (\text{AC}_{\text{loc}}(\mathbb{R}, \mathbb{C}^{2n}) \cap L^2(\mathbb{R}, \mathbb{C}^{2n})) \backslash \{0\}$$

that satisfies (1.1) for a.e. $x \in \mathbb{R}$, and we will take the geometric multiplicity of λ to be the dimension of the space of such solutions. (Here, and throughout, $\text{AC}_{\text{loc}}(\cdot)$ refers to the space of functions absolutely continuous on compact subsets of $\mathbb{R}$.) Our primary goal for the analysis is to use the Maslov index to count the number of eigenvalues that (1.1) has on intervals of the form $[\lambda_1, \lambda_2)$ and $(-\infty, \lambda_2)$ (assumed, in each case, to be a subset of I).

We are primarily motivated by applications to the spectral stability of nonlinear waves arising in certain nonlinear evolutionary PDE such as Allen-Cahn systems

$$u_t + DF(u) = u_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n, \tag{1.2}$$

and higher-order analogues

$$u_t + DF(u) = -u_{xxxx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n.$$

(Here, D denotes the Jacobian operator.) In the former case, if $\bar{u}(x)$ denotes a stationary solution, then we can linearize about $\bar{u}(x)$ (setting $u = \bar{u} + v$ and dropping terms nonlinear in v) to obtain a linear equation

$$v_t + D^2F(\bar{u})v = v_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad v(x,t) \in \mathbb{C}^n.$$

In this setting, spectral stability is determined by the spectrum of the associated eigenvalue problem

$$-\phi_{xx} + D^2F(\bar{u})\phi = \lambda\phi, \quad x \in \mathbb{R}, \quad \phi(x;\lambda) \in \mathbb{C}^n, \tag{1.3}$$

which we can put in the form of (1.1) by setting $y = \binom{y_1}{y_2} = \binom{\phi}{\phi'}$. Precisely, we find

$$Jy' = \mathbb{B}(x;\lambda)y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I - D^2F(\bar{u}(x)) & 0 \\ 0 & I \end{pmatrix}.$$

We would like to determine whether (1.3) has any negative eigenvalues, and this information clearly follows from a count of the number of eigenvalues that (1.3) has on $(-\infty, 0)$.

We are particularly interested in stationary solutions that approach fixed endstates $u_\pm$ as $x \to \pm\infty$, and such cases provide us with additional structure that will be necessary for our general analysis. In order to keep the analysis as applicable as possible, we will make three general assumptions (in addition to Assumptions **(A)**), which we will subsequently verify in a selection of important cases. Prior to stating these assumptions, we need to develop some notation and terminology that will be used througout the discussion. We begin with the following definition.

Definition 1.1. *We say that a measurable function $y : \mathbb{R} \to \mathbb{C}^{2n}$ lies left in $\mathbb{R}$ if for any $c \in \mathbb{R}$, the restriction of $y(\cdot)$ to $(-\infty, c)$ is in $L^2((-\infty,c),\mathbb{C}^{2n})$. Likewise, we say that a measurable function $y : \mathbb{R} \to \mathbb{C}^{2n}$ lies right in $\mathbb{R}$ if for any $c \in \mathbb{R}$, the restriction of $y(\cdot)$ to $(c, +\infty)$ is in $L^2((c,+\infty),\mathbb{C}^{2n})$. If each column of a matrix-valued function lies left (resp. right) in $\mathbb{R}$, then we say the matrix-valued function lies left (resp. right) in $\mathbb{R}$.*

Our primary tool for this analysis will be the Maslov index, and as a starting point for a discussion of this object, we define what we will mean by a Lagrangian subspace of $\mathbb{C}^{2n}$.

Definition 1.2. *We say $\ell \subset \mathbb{C}^{2n}$ is a Lagrangian subspace of $\mathbb{C}^{2n}$ if ℓ has dimension n and*

$$(Ju, v) = 0, \tag{1.4}$$

for all $u, v \in \ell$. (Here, and throughout, $(\cdot,\cdot)$ denotes the usual inner product on $\mathbb{C}^{2n}$.) In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{C}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.

Any Lagrangian subspace ℓ of $\mathbb{C}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{C}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often coordinatize our frames as $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices. Following [25] (p. 274), we specify a metric on $\Lambda(n)$ in terms of appropriate orthogonal projections. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^* \mathbf{X}_i)^{-1} \mathbf{X}_i^*$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\| \cdot \|$ can denote any matrix norm. We will say that a path of Lagrangian subspaces $\ell : \mathcal{I} \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote continuous paths of Lagrangian subspaces $\ell_i : \mathcal{I} \to \Lambda(n)$, $i = 1, 2$, for some parameter interval $\mathcal{I}$. The Maslov index associated with these paths, which we will denote $\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I})$, is a count of the number of times t at which the Lagrangian subspaces $\ell_1(t)$ and $\ell_2(t)$ intersect as t ranges over $\mathcal{I}$, counted with both multiplicity and direction. (In this setting, if we let t_* denote the point of intersection (often referred to as a *crossing point*), then multiplicity corresponds with the dimension of the intersection $\ell_1(t_*) \cap \ell_2(t_*)$; a precise definition of what we mean in this context by *direction* will be given in Section 2.) For additional background on the Maslov index for evolving pairs of Lagrangian subspaces, we refer the reader to Section 3 in [46], Section 3.5 in [25], and Section 1 in [31].

We are now prepared to state the three general assumptions (in addition to Assumptions **(A)**) that will be required for our analysis. For convenient reference, some notational conventions will be embedded in the statements of these assumptions.

(B1) For each $\lambda \in I$, there exists an n-dimensional space of solutions to (1.1) that lie left in $\mathbb{R}$, and likewise an n-dimensional space of solutions to (1.1) that lie right in $\mathbb{R}$. We will denote by $\mathbf{X}(x;\lambda)$ a $2n \times n$ matrix solution of (1.1) comprising as its columns a choice of basis for the n-dimensional space of solutions to (1.1) that lie left in $\mathbb{R}$, and we will denote by $\tilde{\mathbf{X}}(x;\lambda)$ a $2n \times n$ matrix solution of (1.1) comprising as its columns a choice of basis for the n-dimensional space of solutions to (1.1) that lie right in $\mathbb{R}$. We will show that when constructed in this way, $\mathbf{X}(x;\lambda)$ and $\tilde{\mathbf{X}}(x;\lambda)$ constitute frames for Lagrangian subspaces of $\mathbb{C}^{2n}$, which we will respectively denote $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$. We assume that $\ell, \tilde{\ell} \in C(\mathbb{R} \times I, \Lambda(n))$, and additionally that for any $\lambda_0 \in I$ there exists a constant $r_0 > 0$ and a choice of frames $\mathbf{X}_0(x;\lambda)$ for $\ell(x;\lambda)$ (resp. $\tilde{\mathbf{X}}_0(x;\lambda)$ for $\tilde{\ell}(x;\lambda)$) such that for each $x \in \mathbb{R}$, $\mathbf{X}_0(x;\lambda)$ (resp. $\tilde{\mathbf{X}}_0(x;\lambda)$) is differentiable in λ in the interval $(\lambda_0 - r_0, \lambda_0 + r_0)$, with $\partial_\lambda \mathbf{X}_0(\cdot;\lambda)$ lying left in $\mathbb{R}$ and satisfying $(\partial_\lambda \mathbf{X}_0(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_0(x;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \mathbf{X}_0(x;\lambda)$ for a.e. $x \in \mathbb{R}$ (resp. $\partial_\lambda \tilde{\mathbf{X}}_0(x;\lambda)$ lying right in $\mathbb{R}$ and satisfying $(\partial_\lambda \tilde{\mathbf{X}}_0(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\tilde{\mathbf{X}}_0(x;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \tilde{\mathbf{X}}_0(x;\lambda)$ for a.e. $x \in \mathbb{R}$).

(B2) For each $\lambda \in I$, there exists a choice of frames $\mathbf{X}^\sharp(x;\lambda)$ and $\tilde{\mathbf{X}}^\sharp(x;\lambda)$ for $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$ (respectively) so that the asymptotic frames

$$\mathbf{X}_-(\lambda) := \lim_{x \to -\infty} \mathbf{X}^\sharp(x;\lambda); \quad \text{and} \quad \tilde{\mathbf{X}}_+(\lambda) := \lim_{x \to +\infty} \tilde{\mathbf{X}}^\sharp(x;\lambda)$$

are well defined, and are respectively frames for Lagrangian subspaces $\ell_-(\lambda)$ and $\tilde{\ell}_+(\lambda)$, satisfying

$$\ell_-(\lambda) \cap \tilde{\ell}_+(\lambda) = \{0\} \quad \forall\, \lambda \in I.$$

In addition, for fixed values $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, there exists a choice of frames $\mathbf{X}^\flat(x; \lambda_i)$, $i = 1, 2$, so that the asymptotic frames

$$\mathbf{X}_+(\lambda_i) := \lim_{x \to +\infty} \mathbf{X}^\flat(x; \lambda_i), \quad i = 1, 2, \tag{1.5}$$

are well defined, and are frames for Lagrangian subspaces $\ell_+(\lambda_i)$, $i = 1, 2$. In all of these statements, the subscripts $-$ and $+$ denote objects obtained in the asymptotic limit as $x \to -\infty$ or $x \to +\infty$ (respectively), and consequently expressions such as $\ell_-(\lambda)$ and $\tilde{\ell}_+(\lambda)$ refer to fixed Lagrangian subspaces rather than paths.

(B3) There exists a constant $c_0 > 0$ sufficiently large so that for any $c > c_0$, the matrix

$$\int_{-\infty}^{c} \mathbf{X}(x; \lambda)^* \mathbb{B}_\lambda(x; \lambda) \mathbf{X}(x; \lambda) dx$$

is positive definite for all $\lambda \in I$.

Assumptions **(B1)**, **(B2)**, and **(B3)**, along with Assumptions **(A)**, hold in many important cases. As specific examples, we will verify them for linear Hamiltonian systems associated with Sturm-Liouville Systems

$$-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi, \quad x \in \mathbb{R}, \quad \phi(x; \lambda) \in \mathbb{C}^n, \tag{1.6}$$

fourth-order potential equations,

$$\phi'''' + V(x)\phi = \lambda\phi, \quad x \in \mathbb{R}, \quad \phi(x; \lambda) \in \mathbb{C}^n, \tag{1.7}$$

and a family of systems nonlinear in the spectral parameter λ,

$$-(P_{11}(x)\phi')' + V_{11}(x)\phi + V_{12}(x)(\lambda I - V_{22}(x))^{-1} V_{12}(x)^*\phi = \lambda\phi, \quad x \in \mathbb{R}, \quad \phi(x; \lambda) \in \mathbb{C}^n, \tag{1.8}$$

with appropriate assumptions on the coefficient matrices in all cases. (Equation (1.8) arises in the analysis of differential-algebraic Sturm-Liouville systems; see Section 5 for details.)

We can state our main theorem as follows.

Theorem 1.1. *For (1.1), suppose that for some $I \subset \mathbb{R}$ and $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, Assumptions* **(A)**, **(B1)**, **(B2)**, *and* **(B3)** *hold. If $\mathcal{N}([\lambda_1, \lambda_2))$ denotes the number of eigenvalues that (1.1) has on the interval $[\lambda_1, \lambda_2)$, counted with geometric multiplicity, then*

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell(\cdot; \lambda_2), \tilde{\ell}_+(\lambda_2), (-\infty, +\infty]) + \operatorname{Mas}(\ell(\cdot; \lambda_1), \tilde{\ell}_+(\lambda_1), (-\infty, +\infty]).$$

Remark 1.1. *The inclusive bracket on $+\infty$ indicates that we use Assumption* **(B2)** *to compactify $\mathbb{R}$ for our Maslov index calculations. In particular, this means that, at least in principle, $\pm\infty$ can serve as crossing points. According to Assumption* **(B2)**, *we have $\ell_-(\lambda_1) \cap \tilde{\ell}_+(\lambda_1) = \{0\}$, so $-\infty$ will never serve as a crossing point for our analysis (hence the open parentheses on $-\infty$), but it may be the case that $\ell_+(\lambda_1) \cap \tilde{\ell}_+(\lambda_1) \neq \{0\}$, in which case $+\infty$ will serve as a crossing point. The same remark holds with λ_1 replaced by λ_2.*

For specific applications such as the ones we will discuss in detail, we can establish additional properties that may not hold in the generality Theorem 1.1. Among these, we will emphasize the following:

1. In some cases we can replace the target spaces $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ with target spaces for which the flow associated with the relevant Maslov index is monotonic (i.e., the sign associated with each crossing point is the same). As an important example, we will show in Section 4 that for Sturm-Liouville systems, this is the case for the Dirichlet Lagrangian subspace ℓ_D (with frame $\mathbf{X}_D = \binom{0}{I}$).

2. As discussed in our motivating applications to the stability of nonlinear waves, we are often interested in counting the number of eigenvalues that (1.1) has below some fixed value λ_2, and for this it's convenient to show that we can take λ_1 sufficiently negative so that
$$\operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1), (-\infty,+\infty]) = 0.$$
In this case,
$$\mathcal{N}((-\infty,\lambda_2)) = -\operatorname{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2), (-\infty,+\infty]).$$

3. In certain specialized cases, we can apply our results to operators that are not self-adjoint. The most important such case arises when (1.2) is linearized about a traveling wave solution $\bar{u}(x-st)$, leading to the eigenvalue problem
$$-\phi_{xx} - s\phi_x + D^2F(\bar{u})\phi = \lambda\phi, \quad x \in \mathbb{R}, \quad \phi(x) \in \mathbb{C}^n. \tag{1.9}$$
This case will be discussed in Section 4.3.

We now state specific results obtained for (1.6), (1.7), and (1.8). In all cases, we refer to later sections, where detailed assumptions are stated.

Theorem 1.2. *For (1.6), let Assumptions* **(SL1)** *and* **(SL2)** *from Section 4 hold, and express (1.6) in the form (1.1) (giving (4.8)). Then for κ specified as in (4.5),* **(A)**, **(B1)**, **(B2)**, *and* **(B3)** *all hold for (4.8) with $I = (-\infty,\kappa)$ and any $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, and so the result of Theorem 1.1 holds for all intervals $[\lambda_1,\lambda_2]$, $\lambda_1 < \lambda_2 < \kappa$. In addition, if $\mathcal{N}([\lambda_1,\lambda_2))$ denotes the number of eigenvalues, counted with multiplicity, that (1.6) has on the interval $[\lambda_1,\lambda_2)$, and we express the frame $\mathbf{X}(x;\lambda)$ from* **(B1)** *as $\mathbf{X}(x;\lambda) = \binom{X(x;\lambda)}{Y(x;\lambda)}$, then we have*
$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_2) - \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_1),$$
and
$$\mathcal{N}((-\infty,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_2).$$

Theorem 1.3. *For (1.8), let Assumptions* **(DA1)** *and* **(DA2)** *from Section 5 hold, and express (1.8) in the form (1.1) (giving (5.7)). Then for any interval $I \subset \mathbb{R}$ satisfying (5.5), Assumptions* **(A)**, **(B1)**, **(B2)**, *and* **(B3)** *all hold for (5.7) with any $\lambda_1,\lambda_2 \in I$, $\lambda_1 < \lambda_2$, and so the result of Theorem 1.1 holds for all intervals $[\lambda_1,\lambda_2] \subset I$, $\lambda_1 < \lambda_2$. In addition, if $\mathcal{N}([\lambda_1,\lambda_2))$ denotes the number of eigenvalues, counted with multiplicity, that (1.8) has on the interval $[\lambda_1,\lambda_2)$, and we express the frame $\mathbf{X}(x;\lambda)$ from* **(B1)** *as $\mathbf{X}(x;\lambda) = \binom{X(x;\lambda)}{Y(x;\lambda)}$, then we have*
$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_2) - \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_1).$$

Finally, if $\lambda_2 \in I$ lies entirely below $\sigma_{\text{ess}}(\mathcal{L}_a)$ (with $\mathcal{L}_a$ as specified in Section 5), then

$$\mathcal{N}((-\infty, \lambda_2)) = \sum_{x \in \mathbb{R}} \dim \ker X(x; \lambda_2).$$

Theorem 1.4. *For (1.7), let Assumptions* **(FP1)** *and* **(FP2)** *from Section 6 hold, and express (1.7) in the form (1.1) (giving (6.5)). Then for κ specified as in (6.3),* **(A)**, **(B1)**, **(B2)**, *and* **(B3)** *all hold for (6.5) with $I = (-\infty, \kappa)$ and any $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, and so the result of Theorem 1.1 holds for all intervals $[\lambda_1, \lambda_2]$, $\lambda_1 < \lambda_2 < \kappa$. In addition, if $\mathcal{N}([\lambda_1, \lambda_2))$ denotes the number of eigenvalues, counted with multiplicity, that (1.7) has on the interval $[\lambda_1, \lambda_2)$, then*

$$\mathcal{N}([\lambda_1, \lambda_2)) = \sum_{x \in \mathbb{R}} \dim \ker \Phi(x; \lambda_2) - \sum_{x \in \mathbb{R}} \dim \ker \Phi(x; \lambda_1),$$

where (for $i = 1, 2$)

$$\Phi(x; \lambda_i) = \begin{pmatrix} \phi_1(x; \lambda_i) & \phi_2(x; \lambda_i) & \dots & \phi_{2n}(x; \lambda_i) \\ \phi_1'(x; \lambda_i) & \phi_2'(x; \lambda_i) & \dots & \phi_{2n}'(x; \lambda_i) \end{pmatrix},$$

with $\{\phi_j(x; \lambda_i)\}_{j=1}^{2n}$, $i = 1, 2$, comprising a collection of $2n$ linearly independent solutions of (1.7) that lie left in $\mathbb{R}$. Finally,

$$\mathcal{N}((-\infty, \lambda_2)) = \sum_{x \in \mathbb{R}} \dim \ker \Phi(x; \lambda_2).$$

In the remainder of this introduction, we provide some background and context for our analysis and also set out a plan for the paper. For the former, our results serve as natural generalizations of Sturm's Oscillation Theorem and the Morse Index Theorem, and so go respectively back to [47] and [44]. The earliest result readily identifiable with our methods is due to R. Bott in [10], followed (chronologically) by the work of H.M. Edwards in [24], V. Maslov in [43] and V. I. Arnol'd in [3, 4]. In [10, 24], the authors work in the context of bounded domains, with [24] especially emphasizing a formulation of the problem via forms rather than differential operators. For the eponymous reference [43], the Maslov index is introduced as a tool in the development of asymptotic relations, and spectral counts such as those developed here aren't directly considered (see Part 2, Chapter 2, Section 2 of [43]). Likewise, in [3, 4], the author's interest lies primarily in developing properties of the Maslov index for fixed values of λ.

Specific applications to the stability of nonlinear waves were carried about by Chris Jones in [38, 39], by Jones and collaborators in [6, 8, 17, 40], and subsequently by numerous others, including [11, 14, 15, 21]. The Maslov index is amenable to numerical computations, and several analyses have emphasized this aspect of the theory, including [9, 12, 13, 16]. These results have all addressed applications to equations of the form (1.2) (though [14, 15, 21] address a skew-gradient reaction term) and to nonlinear waves associated with homoclinic orbits (i.e., with $u_- = u_+$). In addition, the target Lagrangian subspace in the relevant calculations has typically been taken to be the Dirichlet Lagrangian subspace rather than the "natural" targets $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ (an exception is [21]).

In [22], Jones and Jian Deng applied the Maslov index in the setting of multidimensional Schrödinger equations, instigating a resurgence of interest in the methods (see also [19, 41]). Motivated by this work, the author, along with Yuri Latushkin and Alim Sukhtayev revisited implementations of the Maslov index in a single space dimension, adapting the spectral-flow formulation of [45] to obtain a specification of the Maslov index especially suitable to general linear Hamiltonian systems associated with either homoclinic or heteroclinic orbits [30, 34]. This approach was employed in [31] to establish a result for heteroclinic traveling-wave solutions arising in equations of the form (1.2), and was employed in [29] in an analysis of general linear Hamiltonian systems of the form (1.1) on finite domains.

The primary goal of the current analysis is to adapt the approach taken in [31] (addressing equations of the form (1.2)) to the more general setting of (1.1). Auxiliary to this, we hope to clarify the mechanism by which the target Lagrangian subspaces $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ can be replaced by target Lagrangian subspaces for which all crossing points for the Maslov index calculations have the same direction (i.e., target Lagrangian subspaces for which the flow is monotonic). To the author's knowledge this is the most general setting in which these types of "Maslov equals Morse" results have been developed. In particular, the author is not aware of any prior such results on unbounded domains for higher order equations such as (1.7), or for equations such as (1.8) for which the dependence on λ in the Hamiltonian formulation is nonlinear. In addition, the only previous work in the case of heteroclinic orbits appears to be [31].

Having stated the novel aspects of the current analysis, we emphasize that these comments refer to a specific approach for which a target space is fixed and the Maslov index is computed as a count of intersections between a path of Lagrangian subspaces and this fixed target. Alternative approaches have been based on computing Maslov indices for appropriate pairs of evolving Lagrangian subspaces (i.e., both paths of Lagrangian subspaces evolve as the independent variable varies, and there is no fixed target). In [33], the authors evolve one path of Lagrangian subspaces forward from $-\infty$ and another backward from $+\infty$, and the associated spectral flow is captured where the two meet at $x = 0$ (see Theorem 1 in [33], which is formulated for a general class of linear Hamiltonian systems and stated in terms of the spectral flow of an appropriately defined operator pencil). In [26, 27, 37], the authors use *renormalized oscillation theory*, in which the Maslov index is computed for a pair of Lagrangian paths, with one specified at some value λ_1 and the other specified at $\lambda_2 > \lambda_1$, leading to a count of the number of eigenvalues the operator has on (λ_1, λ_2). This latter method has the advantage of providing a naturally monotonic flow as x increases from $-\infty$ to $+\infty$ and being applicable in a wider range of cases than either the current approach or the approach of [33] (perhaps most notably, in the renormalized oscillation setting, there's no requirement on the existence of asymptotic endstates as assumed here in Assumption **(B1)**).

The paper is organized as follows. In Section 2, we review elements of the Maslov index that will be used in our development, and in Section 3 we prove Theorem 1.1. In the subsequent three sections, we apply Theorem 1.1 to prove (respectively) Theorems 1.2, 1.4, and 1.3.

2 The Maslov Index

Our framework for computing the Maslov index is adapted from Section 2 of [35], which is based on the spectral flow formulation of [45]. Rather than repeating that development here, we will only highlight the points most salient to the current analysis. For a full discussion of the Maslov index in the current setting, we refer the reader to [35], and for a broader view of the Maslov index we refer the reader to [7, 20, 46].

Given any pair of Lagrangian subspaces ℓ_1 and ℓ_2 with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we consider the matrix

$$\tilde{W} := -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{2.1}$$

In [35], the authors establish: (1) the inverses appearing in (2.1) exist; (2) $\tilde{W}$ is independent of the specific frames $\mathbf{X}_1$ and $\mathbf{X}_2$ (as long as these are indeed frames for ℓ_1 and ℓ_2); (3) $\tilde{W}$ is unitary; and (4) the identity

$$\dim(\ell_1 \cap \ell_2) = \dim(\ker(\tilde{W} + I)). \tag{2.2}$$

Given two continuous paths of Lagrangian subspaces $\ell_i : [0,1] \to \Lambda(n)$, $i = 1, 2$, with respective frames $\mathbf{X}_i : [0,1] \to \mathbb{C}^{2n \times n}$, relation (2.2) allows us to compute the Maslov index $\text{Mas}(\ell_1, \ell_2; [0,1])$ as a spectral flow through -1 for the path of matrices

$$\tilde{W}(t) := -(X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_2(t) - iY_2(t))(X_2(t) + iY_2(t))^{-1}. \tag{2.3}$$

If $-1 \in \sigma(\tilde{W}(t_*))$ for some $t_* \in [0,1]$, then we refer to t_* as a crossing point, and we see from (2.2) that the multiplicity of -1 as an eigenvalue of $\tilde{W}(t_*)$ corresponds with $\dim(\ell_1(t_*) \cap \ell_2(t_*))$. We compute the Maslov index $\text{Mas}(\ell_1, \ell_2; [0,1])$ by allowing t to increase from 0 to 1 and incrementing the index whenever an eigenvalue crosses -1 in the counterclockwise direction, while decrementing the index whenever an eigenvalue crosses -1 in the clockwise direction. These increments/decrements are counted with multiplicity, so for example, if a pair of eigenvalues crosses -1 together in the counterclockwise direction, then a net amount of $+2$ is added to the index. Regarding behavior at the endpoints, if an eigenvalue of $\tilde{W}$ rotates away from -1 in the clockwise direction as t increases from 0, then the Maslov index decrements (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates away from -1 in the counterclockwise direction as t increases from 0, then the Maslov index does not change. Likewise, if an eigenvalue of $\tilde{W}$ rotates into -1 in the counterclockwise direction as t increases to 1, then the Maslov index increments (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates into -1 in the clockwise direction as t increases to 1, then the Maslov index does not change. Finally, it's possible that an eigenvalue of $\tilde{W}$ will arrive at -1 for $t = t_*$ and stay. In these cases, the Maslov index only increments/decrements upon arrival or departure, and the increments/decrements are determined as for the endpoints (departures determined as with $t = 0$, arrivals determined as with $t = 1$).

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we let $\mathcal{I}$ be a closed interval in $\mathbb{R}$, and we denote by $\mathcal{P}(\mathcal{I})$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t,s) \in \mathcal{I} \times [0,1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties.

(P1) (Path Additivity) If $\mathcal{L} \in \mathcal{P}(\mathcal{I})$ and $a, b, c \in \mathcal{I}$, with $a < b < c$, then

$$\operatorname{Mas}(\mathcal{L}; [a,c]) = \operatorname{Mas}(\mathcal{L}; [a,b]) + \operatorname{Mas}(\mathcal{L}; [b,c]).$$

(P2) (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ for some $a, b \in \mathcal{I}$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\operatorname{Mas}(\mathcal{L}; [a,b]) = \operatorname{Mas}(\mathcal{M}; [a,b]).$$

Straightforward proofs of these properties appear in [30] for Lagrangian subspaces of $\mathbb{R}^{2n}$, and proofs in the current setting of Lagrangian subspaces of $\mathbb{C}^{2n}$ are essentially identical.

As noted previously, the direction we associate with a crossing point is determined by the direction in which eigenvalues of $\tilde{W}$ rotate through -1 (counterclockwise is positive, while clockwise is negative). In order to understand the nature of this rotation in specific cases, we will use the following lemma from [30]. (In [30], the statement takes the frames to be C^1, but the proof only requires differentiability, as asserted here.)

Lemma 2.1. *Suppose $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ denote paths of Lagrangian subspaces of $\mathbb{C}^{2n}$ with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$ that are differentiable at $t_0 \in \mathcal{I}$. If the matrices*

$$-\mathbf{X}_1(t_0)^* J \mathbf{X}_1'(t_0) = X_1(t_0)^* Y_1'(t_0) - Y_1(t_0)^* X_1'(t_0)$$

and (noting the sign change)

$$\mathbf{X}_2(t_0)^* J \mathbf{X}_2'(t_0) = -(X_2(t_0)^* Y_2'(t_0) - Y_2(t_0)^* X_2'(t_0))$$

are both non-negative, and at least one is positive definite, then the eigenvalues of $\tilde{W}(t)$ rotate in the counterclockwise direction as t increases through t_0. Likewise, if both of these matrices are non-positive, and at least one is negative definite, then the eigenvalues of $\tilde{W}(t)$ rotate in the clockwise direction as t increases through t_0.

Remark 2.1. *In Theorem 1.1, the Maslov indices are computed on the unbounded interval $(-\infty, +\infty)$, and the notation $(-\infty, +\infty]$ is used to signify that the limit $+\infty$ can serve as a crossing point. Precisely, under our limit assumptions in* **(B2)**, *we can compactify $(-\infty, \infty)$ with a map such as*

$$x = \ln\left(\frac{1+\tau}{1-\tau}\right); \quad \tau \in [-1, 1],$$

and subsequently compute the relevant Maslov indices on the bounded interval $\mathcal{I} = [-1, 1]$, employing the considerations discussed in this section. We recall from Remark 1.1 that due to our Assumption **(B2)**, *$-\infty$ cannot serve as a crossing point, and so is omitted from the square-bracket notation.*

3 Proof of Theorem 1.1

Before proving Theorem 1.1, we verify the assertion made in the statement of Assumption **(B1)** that $\mathbf{X}(x;\lambda)$ and $\tilde{\mathbf{X}}(x;\lambda)$ are necessarily frames for Lagrangian subspaces of $\mathbb{C}^{2n}$ for all $(x,\lambda) \in \mathbb{R} \times I$. We will carry out the demonstration for $\mathbf{X}(x;\lambda)$; the case of $\tilde{\mathbf{X}}(x;\lambda)$ is essentially identical. First, we note that under our Assumption **(A)** we have $\mathbf{X}(\cdot;\lambda), \tilde{\mathbf{X}}(\cdot;\lambda) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{2n \times n})$ for all $\lambda \in I$ (see, e.g., Theorem 2.1 in [48]). Next, according to Proposition 2.1 of [35], it's sufficient to show that

$$\mathbf{X}(x;\lambda)^* J \mathbf{X}(x;\lambda) = 0, \quad \forall\, (x,\lambda) \in \mathbb{R} \times I. \tag{3.1}$$

In order to verify this, we fix any $\lambda \in I$ and compute (with prime denoting differentiation with respect to x)

$$\begin{aligned}
\frac{\partial}{\partial x}(\mathbf{X}(x;\lambda)^* J \mathbf{X}(x;\lambda)) &= \mathbf{X}'(x;\lambda)^* J \mathbf{X}(x;\lambda) + \mathbf{X}(x;\lambda)^* J \mathbf{X}'(x;\lambda) \\
&= -(J\mathbf{X}'(x;\lambda))^* \mathbf{X}(x;\lambda) + \mathbf{X}(x;\lambda)^* J \mathbf{X}'(x;\lambda) \\
&= -(\mathbb{B}(x;\lambda)\mathbf{X}(x;\lambda))^* \mathbf{X}(x;\lambda) + \mathbf{X}(x;\lambda)^* \mathbb{B}(x;\lambda) \mathbf{X}(x;\lambda) \\
&= 0, \quad \text{a.e. } x \in \mathbb{R},
\end{aligned}$$

where in obtaining the final equality we've observed from Assumption **(A)** that $\mathbb{B}(x;\lambda)$ is self-adjoint for a.e. $x \in \mathbb{R}$. Recalling that $\mathbf{X}(x;\lambda)^* J \mathbf{X}(x;\lambda)$ is locally absolutely continuous in $\mathbb{R}$, we see that it is constant on $\mathbb{R}$. But the columns of $\mathbf{X}(x;\lambda)$ lie left in $\mathbb{R}$, so

$$\lim_{x \to -\infty} \mathbf{X}(x;\lambda)^* J \mathbf{X}(x;\lambda) = 0.$$

This calculation holds for all $\lambda \in I$, allowing us to conclude (3.1).

Turning now to the proof of Theorem 1.1, we begin by fixing any pair $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, and for all $(x,\lambda) \in \mathbb{R} \times [\lambda_1, \lambda_2]$, we let $\ell(x,\lambda)$ and $\tilde{\ell}(x;\lambda)$ denote the Lagrangian subspaces described in **(B1)** and **(B2)**. We will fix some $c > 0$ to be chosen sufficiently large during the analysis, and we will establish Theorem 1.1 by considering the Maslov index for $\ell(x;\lambda)$ and $\tilde{\ell}(c;\lambda)$ along a path designated as the *Maslov box* in the next paragraph. As described in Section 2, this Maslov index is computed as a spectral flow for the matrix

$$\begin{aligned}
\tilde{W}_c(x;\lambda) := &-(X(x;\lambda) + iY(x;\lambda))(X(x;\lambda) - iY(x;\lambda))^{-1} \\
&\times (\tilde{X}(c;\lambda) - i\tilde{Y}(c;\lambda))(\tilde{X}(c;\lambda) + i\tilde{Y}(c;\lambda))^{-1}.
\end{aligned} \tag{3.2}$$

By Maslov Box, in this case we mean the following sequence of contours: (1) fix $x = -c$ and let λ increase from λ_1 to λ_2 (the *bottom shelf*); (2) fix $\lambda = \lambda_2$ and let x increase from $-c$ to c (the *right shelf*); (3) fix $x = c$ and let λ decrease from λ_2 to λ_1 (the *top shelf*); and (4) fix $\lambda = \lambda_1$ and let x decrease from c to $-c$ (the *left shelf*).

The Bottom Shelf. For the bottom shelf, the Maslov index detects intersections between $\ell(-c;\lambda)$ and $\tilde{\ell}(c,\lambda)$ as λ increases from λ_1 to λ_2. Since $[\lambda_1, \lambda_2]$ is compact, it follows from our Assumption **(B2)** that we can take c sufficiently large so that

$$\ell(-c;\lambda) \cap \tilde{\ell}(c;\lambda) = \{0\}, \quad \forall\, \lambda \in [\lambda_1, \lambda_2].$$

In this way, we see that

$$\operatorname{Mas}(\ell(-c;\cdot),\tilde{\ell}(c;\cdot);[\lambda_1,\lambda_2])=0.$$

The Top Shelf. For the top shelf, the Maslov index detects intersections between $\ell(c;\lambda)$ and $\tilde{\ell}(c;\lambda)$ as λ decreases from λ_2 to λ_1. These Lagrangian subspaces will intersect if and only if λ is an eigenvalue of (1.1), and the multiplicity of this intersection will correspond with the geometric multiplicity of λ as an eigenvalue of (1.1). We would like to conclude that the Maslov index for the top shelf is precisely a count, including geometric multiplicity, of the number of eigenvalues that (1.1) has on the interval $[\lambda_1,\lambda_2)$, and in order to draw this conclusion we need to know that crossing points in this case all have the same direction. For this, we observe from Lemma 2.1 that the direction of rotation associated with the Maslov index along the top shelf will be determined by the signs of the matrices

$$-\mathbf{X}(c;\lambda)^*J\partial_\lambda\mathbf{X}(c;\lambda) \tag{3.3}$$

and

$$\tilde{\mathbf{X}}(c;\lambda)^*J\partial_\lambda\tilde{\mathbf{X}}(c;\lambda) \tag{3.4}$$

in the following sense: if both of these matrices are non-positive at some $\lambda\in[\lambda_1,\lambda_2]$, and at least one of them is negative definite at λ, then the rotation at that value λ for all eigenvalues of $\tilde{W}(c;\lambda)$ will be in the clockwise direction (with λ increasing).

For the first of these matrices, we compute

$$\begin{aligned}\frac{\partial}{\partial x}\mathbf{X}(x;\lambda)^*J\partial_\lambda\mathbf{X}(x;\lambda)&=\mathbf{X}'(x;\lambda)^*J\partial_\lambda\mathbf{X}(x;\lambda)+\mathbf{X}(x;\lambda)^*J\partial_\lambda\mathbf{X}'(x;\lambda)\\&=-(J\mathbf{X}'(x;\lambda))^*\partial_\lambda\mathbf{X}(x;\lambda)+\mathbf{X}(x;\lambda)^*\partial_\lambda(J\mathbf{X}'(x;\lambda))\\&=-(\mathbb{B}(x;\lambda)\mathbf{X}(x;\lambda))^*\partial_\lambda\mathbf{X}(x;\lambda)+\mathbf{X}(x;\lambda)^*\partial_\lambda(\mathbb{B}(x;\lambda)\mathbf{X}(x;\lambda))\\&=-\mathbf{X}(x;\lambda)^*\mathbb{B}(x;\lambda)\partial_\lambda\mathbf{X}(x;\lambda)+\mathbf{X}(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}(x;\lambda)\\&\qquad+\mathbf{X}(x;\lambda)^*\mathbb{B}(x;\lambda)\partial_\lambda\mathbf{X}(x;\lambda)\\&=\mathbf{X}(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}(x;\lambda).\end{aligned}$$

Upon integrating this relation on $(-\infty,c)$ and observing from **(B1)** that

$$\lim_{x\to-\infty}\mathbf{X}(x;\lambda)^*J\partial_\lambda\mathbf{X}(x;\lambda)=0,$$

we obtain the relation

$$\mathbf{X}(c;\lambda)^*J\partial_\lambda\mathbf{X}(c;\lambda)=\int_{-\infty}^c\mathbf{X}(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}(x;\lambda)d\xi.$$

According to Assumption **(B3)**, this matrix is positive definite for c sufficiently large, and we can conclude that (3.3) is negative definite. By a similar calculation, we can check that (3.4) is negative definite as well. We can conclude from Lemma 2.1 that the eigenvalues of $\tilde{W}(c;\lambda)$ rotate monotonically in the clockwise direction as λ increases from λ_1 to λ_2, and consequently that

$$\mathcal{N}([\lambda_1,\lambda_2))=-\operatorname{Mas}(\ell(c;\cdot),\tilde{\ell}(c;\cdot);[\lambda_1,\lambda_2]).$$

The inclusion of λ_1 on the left-hand side is due to the clockwise rotation as λ increases, leading to a decrement of the Maslov index if (c, λ_1) corresponds with a crossing point, and the exclusion of λ_2 follows similarly.

The left and right shelves. The left and right shelves are both left as computations in Theorem 1.1, but in order to eliminate the arbitrary value c, we need to show that by taking c sufficiently large we can ensure that

$$\operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}(c;\lambda_1); [-c, c]) = \operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]), \tag{3.5}$$

and similarly with λ_1 replaced by λ_2. In order to understand why (3.5) holds, it's convenient to observe that the left-hand side is computed via the matrix $\tilde{W}_c(x;\lambda_1)$ (i.e., (3.2) with $\lambda = \lambda_1$), while the right-hand side is computed via the matrix

$$\begin{aligned}\tilde{\mathcal{W}}(x;\lambda_1) = &-(X(x;\lambda_1) + iY(x;\lambda_1))(X(x;\lambda_1) - iY(x;\lambda_1))^{-1} \\ &\times (\tilde{X}_+(\lambda_1) - i\tilde{Y}_+(\lambda_1))(\tilde{X}_+(\lambda_1) + i\tilde{Y}_+(\lambda_1))^{-1}.\end{aligned}$$

Comparing expressions for $\tilde{W}_c(x;\lambda_1)$ and $\tilde{\mathcal{W}}(x;\lambda_1)$, we see that we can write $\tilde{W}_c(x;\lambda_1) = \tilde{\mathcal{W}}(x;\lambda_1)\tilde{V}(c;\lambda_1)$, where

$$\begin{aligned}\tilde{V}(c;\lambda_1) = &(\tilde{X}_+(\lambda_1) + i\tilde{Y}_+(\lambda_1))(\tilde{X}_+(\lambda_1) - i\tilde{Y}_+(\lambda_1))^{-1} \\ &\times (\tilde{X}(c;\lambda_1) - i\tilde{Y}(c;\lambda_1))(\tilde{X}(c;\lambda_1) + i\tilde{Y}(c;\lambda_1))^{-1}.\end{aligned}$$

Here, $\tilde{V}(c;\lambda_2)$ is a continuous matrix-valued function of c, satisfying

$$\lim_{c \to +\infty} \tilde{V}(c;\lambda_2) = I.$$

Let $\{w_j^c(x;\lambda_1)\}_{j=1}^n$ denote the eigenvalues of $\tilde{W}_c(x;\lambda_1)$, and let $\{\omega_j(x;\lambda_1)\}_{j=1}^n$ denote the eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$. Using Assumption **(B2)**, we see that the limits

$$\begin{aligned}\tilde{W}_c^-(\lambda_1) &:= \lim_{x \to -\infty} \tilde{W}_c(x;\lambda_1); \quad \tilde{\mathcal{W}}^-(\lambda_1) := \lim_{x \to -\infty} \tilde{\mathcal{W}}(x;\lambda_1) \\ \tilde{W}_c^+(\lambda_1) &:= \lim_{x \to +\infty} \tilde{W}_c(x;\lambda_1); \quad \tilde{\mathcal{W}}^+(\lambda_1) := \lim_{x \to +\infty} \tilde{\mathcal{W}}(x;\lambda_1)\end{aligned}$$

are all well defined. It follows that we can effectively view $\tilde{W}_c(x;\lambda_1)$ and $\tilde{\mathcal{W}}(x;\lambda_1)$ as continuous matrix-valued functions on a compact interval (as discussed in Remark 2.1). Precisely, given any $\epsilon > 0$, we can find $L, c_0 > 0$ sufficiently large so that for each $j \in \{1, 2, \dots, n\}$ (with an appropriate choice of labeling)

$$|w_j^c(x;\lambda_1) - \omega_j(x;\lambda_1)| < \epsilon, \quad \forall\, |x| > L,\ c > c_0. \tag{3.6}$$

To be clear about this important relation, we introduce the notation $\{w_j^{c,+}(\lambda_1)\}_{j=1}^n$ for the eigenvalues of $\tilde{W}_c^+(\lambda_1)$, and observe by continuity that given any $\epsilon_1 > 0$ we can find L_1 sufficiently large so that

$$|w_j^c(x;\lambda_1) - w_j^{c,+}(\lambda_1)| < \epsilon_1, \quad \forall\, x > L_1,$$

and likewise we introduce the notation $\{\omega_j^+(\lambda_1)\}_{j=1}^n$ for the eigenvalues of $\tilde{\mathcal{W}}^+(\lambda_1)$, and observe by continuity that given any $\epsilon_2 > 0$ we can find L_2 sufficiently large so that

$$|\omega_j^+(\lambda_1) - \omega_j(x;\lambda_1)| < \epsilon_2, \quad \forall\, x > L_2.$$

Finally, we notice that given any $\epsilon_3 > 0$ there exists c_3 sufficiently large so that

$$|w_j^{c,+}(\lambda_1) - \omega_j^+(\lambda_1)| < \epsilon_3, \quad \forall\, c > c_3.$$

Combining these observations, we compute

$$\begin{aligned}
|w_j^c(x;\lambda_1) - \omega_j(x;\lambda_1)| &= |w_j^c(x;\lambda_1) - w_j^{c,+}(\lambda_1) + w_j^{c,+}(\lambda_1) - \omega_j^+(\lambda_1) + \omega_j^+(\lambda_1) - \omega_j(x;\lambda_1)| \\
&\le |w_j^c(x;\lambda_1) - w_j^{c,+}(\lambda_1)| + |w_j^{c,+}(\lambda_1) - \omega_j^+(\lambda_1)| + |\omega_j^+(\lambda_1) - \omega_j(x;\lambda_1)| \\
&< \epsilon_1 + \epsilon_3 + \epsilon_2.
\end{aligned}$$

Given any $\epsilon > 0$, we can now choose L_1, L_2, and c_3 sufficiently large so that $\epsilon_i < \epsilon/3$, $i = 1, 2, 3$. If we then take $L = \max\{L_1, L_2\}$ and $c_0 = c_3$ we obtain the part of (3.6) with $x > L$. The case $x < -L$ can be handled similarly, possibly by choosing larger values of L and c_0. In all of these calculations, it has been critical that λ_1 remains fixed, as it can be shown that the path of asymptotic Lagrangian subspaces $\ell_+(\lambda)$ is not necessarily continuous in λ (see, e.g., the appendix of [31] or Lemma 3.7 of [1]).

Similarly as with the previous calculation, given any $\epsilon > 0$ we can use compactness of $[-L, L]$ to take $c_1 > c_0$ sufficiently large so that

$$|w_j^c(x;\lambda_1) - \omega_j(x;\lambda_1)| < \epsilon, \quad \forall\, x \in [-L, L],\ c > c_1.$$

Combining these observations, we see that given any $\epsilon > 0$ we can take c_1 sufficiently large so that

$$|w_j^c(x;\lambda_1) - \omega_j(x;\lambda_1)| < \epsilon, \quad \forall\, x \in \mathbb{R},\ c > c_1. \tag{3.7}$$

We also note that according to the second part of Assumption **(B2)**, we can take c large enough so that we have both $-1 \notin \sigma(\tilde{W}_c(-c;\lambda_1))$ and $-1 \notin \sigma(\tilde{\mathcal{W}}(-c;\lambda_1))$.

At this point, we divide the analysis into two cases: (1) λ_1 is not an eigenvalue of (1.1); and (2) λ_1 is an eigenvalue of (1.1). For Case (1), suppose λ_1 is not an eigenvalue of (1.1). Then we immediately have $-1 \notin \sigma(\tilde{W}_c(c;\lambda_1))$ (for any $c \in \mathbb{R}$), and since $\ell_+(\lambda_1) \cap \tilde{\ell}_+(\lambda_1) = \{0\}$, we can take c large enough so that $-1 \notin \sigma(\tilde{\mathcal{W}}(c;\lambda_1))$. In summary, the situation is as follows: for $\tilde{W}_c(x;\lambda_1)$ we have both $-1 \notin \sigma(\tilde{W}_c(-c;\lambda_1))$ and $-1 \notin \sigma(\tilde{W}_c(c;\lambda_1))$, and likewise for $\tilde{\mathcal{W}}(x;\lambda_1)$ we have both $-1 \notin \sigma(\tilde{\mathcal{W}}(-c;\lambda_1))$ and $-1 \notin \sigma(\tilde{\mathcal{W}}(c;\lambda_1))$. It follows that there exists some $\delta > 0$ so that

$$|w_j^c(-c;\lambda_1) + 1| > \delta, \quad |w_j^c(c;\lambda_1) + 1| > \delta, \quad \forall\, j \in \{1, 2, \dots, n\},$$

and

$$|\omega_j(-c;\lambda_1) + 1| > \delta, \quad |\omega_j(c;\lambda_1) + 1| > \delta, \quad \forall\, j \in \{1, 2, \dots, n\}.$$

See Figure 3.1, sketched for the case $n = 2$.

Using (3.7), we can take c large enough so that $\epsilon < \delta$. In this way, as x increases from $-c$ to c, an eigenvalue of $\tilde{W}_c(x;\lambda)$ can complete a full loop around S^1 if and only

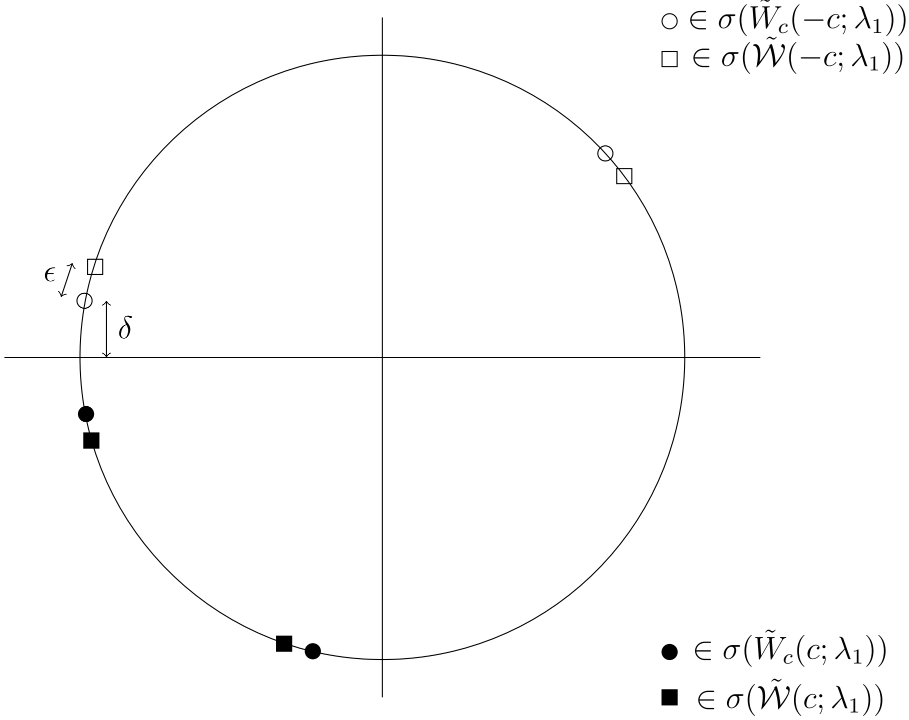

Figure 3.1: Eigenvalues of $\tilde{W}_c(\pm c;\lambda_1)$ and $\tilde{\mathcal{W}}(\pm c;\lambda_1)$.

if a corresponding eigenvalue of $\tilde{\mathcal{W}}(x;\lambda_1)$ also completes a full loop. In addition, since the distance between eigenvalues of $\tilde{W}_c(x;\lambda)$ and eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$ is less than the initial and final distances of the eigenvalues of these matrices from -1, the total count of crossing points associated with the Maslov index computed via $\tilde{W}_c(x;\lambda_1)$ must be precisely the corresponding count computed via $\tilde{\mathcal{W}}(x;\lambda_1)$. We can conclude that

$$\operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}(c;\lambda_1);[-c,c]) = \operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);[-c,c]), \tag{3.8}$$

for all c sufficiently large. According to **(B2)**, we can take c sufficiently large so that $\ell(x;\lambda_1)\cap\tilde{\ell}_+(\lambda_1)=\{0\}$ for all $x<-c$, and since λ_1 is not an eigenvalue of (1.1), we can take c sufficiently large so that $\ell(x;\lambda_1)\cap\tilde{\ell}_+(\lambda_1)=\{0\}$ for all $x>c$. We conclude that

$$\operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);(-\infty,-c]) = 0,$$

and

$$\operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);[c,+\infty]) = 0.$$

This allows us to conclude (3.5) in the case that λ_1 is not an eigenvalue of (1.1).

For Case (2), we assume λ_1 is an eigenvalue of (1.1), and in order to be definite we will specify its geometric multiplicity as m. We continue to have $-1\notin\sigma(\tilde{W}_c(-c;\lambda_1))$ and $-1\notin\sigma(\tilde{\mathcal{W}}(-c;\lambda_1))$ (for c sufficiently large), but now $-1\in\sigma(\tilde{W}_c(c;\lambda_1))$ with multiplicity m, and it's not definite whether -1 is in the spectrum of $\tilde{\mathcal{W}}(c;\lambda_1)$. The matrix $\tilde{W}_c(c;\lambda_1)$ will have $n-m$ eigenvalues located away from -1, and there will correspond $n-m$ eigenvalues of $\tilde{\mathcal{W}}(c;\lambda_1)$ located away from -1 (with the remaining eigenvalues of $\tilde{\mathcal{W}}(c;\lambda_1)$ necessarily near -1). The flow associated with these $n-m$ eigenvalues can be analyzed precisely as in Case (1).

We now consider the m eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$ near -1 at $x=c$. For this, we first recall that $-1 \in \sigma(\tilde{W}_c(c;\lambda_1))$ with multiplicity m, and that c has been chosen large enough so that the eigenvalues of $\tilde{\mathcal{W}}(c;\lambda_1)$ will be near the eigenvalues of $\tilde{W}_c(c;\lambda_1)$, with "near" determined by the value of ϵ in (3.7). It follows that exactly m eigenvalues of $\tilde{\mathcal{W}}(c;\lambda_1)$ will satisfy

$$|\omega_j(c;\lambda_1)+1| < \epsilon. \tag{3.9}$$

Moreover, since the limit

$$\lim_{x\to\infty} \tilde{\mathcal{W}}(x;\lambda_1)$$

is well defined, c can be taken large enough so that

$$|\omega_j(x;\lambda_1)+1| < \epsilon, \quad \forall\, x > c. \tag{3.10}$$

This group of m eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$ near -1 in the sense of (3.9) will track the group of eigenvalues of $\tilde{W}_c(x;\lambda_1)$ that approach -1 as $x \to c^-$. The evolution will proceed as in the previous case, except that in this case the right and left sides of (3.8) won't necessarily agree. For $\tilde{W}_c(x;\lambda_1)$, the evolution stops at $x=c$, but for $\tilde{\mathcal{W}}(x;\lambda_1)$, it continues as x tends to $+\infty$. Moreover, as x tends to $+\infty$ the m eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$ that are not bounded away from -1 will necessarily approach -1 in the asymptotic limit. At this point, it's critical to observe that this set of m eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1)$ cannot complete a loop of S^1 as x increases from c (because they must remain near -1), and so the signs associated with their approaches to -1 have already been determined by the time x arrives at c. In particular, these signs must agree with those of the eigenvalues of $\tilde{W}_c(x;\lambda_1)$ that approach -1 as $x \to c^-$. In this way, we conclude

$$\operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}(c;\lambda_1);[-c,c]) = \operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);[-c,+\infty]),$$

and extension of the right-hand side to $(-\infty,+\infty]$ is precisely as before. This gives (3.5) in Case (2). The same considerations hold for λ_2.

Combining these observations, we can use path additivity along with homotopy invariance to write (respectively)

$$\begin{aligned}
0 &= \text{bottom shelf} + \text{right shelf} + \text{top shelf} + \text{left shelf} \\
&= 0 + \operatorname{Mas}(\ell(\cdot;\lambda_2),\tilde{\ell}(c;\lambda_2);[-c,c]) + \mathcal{N}([\lambda_1,\lambda_2)) - \operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}(c;\lambda_1);[-c,c]) \\
&= \operatorname{Mas}(\ell(\cdot;\lambda_2),\tilde{\ell}_+(\lambda_2);(-\infty,+\infty]) + \mathcal{N}([\lambda_1,\lambda_2)) - \operatorname{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);(-\infty,+\infty]).
\end{aligned}$$

Rearranging terms, we obtain precisely the claim of Theorem 1.1. □

4 Sturm-Liouville Systems

In this section, we apply Theorem 1.1 to Sturm-Liouville systems

$$-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi; \quad x \in \mathbb{R},\ \phi(x;\lambda) \in \mathbb{C}^n, \tag{4.1}$$

and we also establish the additional properties stated in Theorem 1.2. In order to ensure that our general assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** hold, we make the following assumptions on the coefficient matrices P, V, and Q.

(SL1) We take $P \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{n\times n})$ and $V, Q \in C(\mathbb{R}, \mathbb{C}^{n\times n})$, with $P(x)$, $V(x)$, and $Q(x)$ self-adjoint for each $x \in \mathbb{R}$. Moreover, we assume that there exist constants $\theta_P, \theta_Q > 0$ so that for any $v \in \mathbb{C}^n$

$$(P(x)v, v) \geq \theta_P |v|^2; \quad (Q(x)v, v) \geq \theta_Q |v|^2,$$

for all $x \in \mathbb{R}$.

(SL2) Each of the matrices P, V, and Q approaches well-defined asymptotic endstates at exponential rate as $x \to \pm\infty$. Precisely, there exist self-adjoint matrices $P_\pm, V_\pm, Q_\pm \in \mathbb{C}^{n\times n}$, with $P_\pm, Q_\pm$ positive definite, along with constants $C, M \geq 0$, $\eta > 0$, so that

$$|P(x) - P_\pm| \leq Ce^{-\eta|x|}; \quad |V(x) - V_\pm| \leq Ce^{-\eta|x|}; \quad |Q(x) - Q_\pm| \leq Ce^{-\eta|x|} \quad x \gtrless \pm M,$$
$$|P'(x)| \leq Ce^{-\eta|x|} \quad \text{a.e.}\, x \gtrless \pm M.$$

Remark 4.1. *Our assumption of continuity on $V(x)$ and $Q(x)$ is only used in establishing the final two claims in Theorem 1.2, and in particular Theorem 1.1 can be applied under the weaker assumptions $V(\cdot), Q(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{n\times n})$.*

We can associate with (4.1) the operator

$$\mathcal{L}\phi := Q(x)^{-1}\Big\{ - (P(x)\phi')' + V(x)\phi \Big\}, \tag{4.2}$$

for which we assign the domain

$$\mathcal{D} := \Big\{ \phi \in L^2(\mathbb{R}, \mathbb{C}^n) : \phi, \phi' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n),\, \mathcal{L}\phi \in L^2(\mathbb{R}, \mathbb{C}^n) \Big\}, \tag{4.3}$$

and we also introduce the inner product

$$\langle \phi, \psi \rangle_Q := \int_{\mathbb{R}} (Q(x)\phi(x), \psi(x)) dx.$$

With this choice of domain and inner product, $\mathcal{L}$ is densely defined, closed, and self-adjoint, so $\sigma(\mathcal{L}) \subset \mathbb{R}$ (see, e.g., [48]).

As shown in [28, 42] (though see Remark 4.2, immediately following this paragraph), the essential spectrum of $\mathcal{L}$ is entirely determined by the asymptotic systems

$$-P_\pm \phi'' + V_\pm \phi = \lambda Q_\pm \phi \tag{4.4}$$

in the following way: the essential spectrum is precisely the collection of values $\lambda \in \mathbb{R}$ for which there exists a solution to (4.4) of the form $\phi(x) = e^{ikx} r$ for some constant scalar $k \in \mathbb{R}$ and some constant non-zero vector $r \in \mathbb{C}^n$. Upon substitution of $\phi(x) = e^{ikx} r$ into (4.4), we obtain the relation

$$(k^2 P_\pm + V_\pm) r = \lambda Q_\pm r.$$

If we compute an inner product of this equation with r, we find

$$k^2(P_\pm r, r) + (V_\pm r, r) = \lambda(Q_\pm r, r).$$

Since $P_\pm, Q_\pm$ are positive definite, we see that

$$\lambda \geq \frac{(V_\pm r, r)}{(Q_\pm r, r)}, \quad \forall\, k \in \mathbb{R}.$$

We'll set

$$\kappa := \min\Big\{ \inf_{r \in \mathbb{C}^n \setminus \{0\}} \frac{(V_- r, r)}{(Q_- r, r)}, \inf_{r \in \mathbb{C}^n \setminus \{0\}} \frac{(V_+ r, r)}{(Q_+ r, r)} \Big\}. \tag{4.5}$$

Then

$$\sigma_{\text{ess}}(\mathcal{L}) = [\kappa, +\infty),$$

and we can conclude that for (4.1) we can take the interval I described in Assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** to be $I = (-\infty, \kappa)$.

Remark 4.2. *Strictly speaking, our references [28, 42] assume slightly more on the coefficients $P(x)$, $Q(x)$, and $V(x)$ than we assume here. (See Theorem A.2 in the appendix to Chapter 5 of [28] and Theorem 3.1.11 in [42]). In the current setting, however, we can verify directly that the results of [28, 42] extend to $\mathcal{L}$. Briefly, we do this by using Lemma 4.1 (see below) to directly compute the resolvent kernel (i.e., the Green's function) for the operator $\mathcal{L} - \lambda I$, and then verifying directly that for $\lambda < \kappa$, and λ not an eigenvalue of $\mathcal{L}$, the resolvent expressed via this Green's function is indeed a bounded linear operator on our Hilbert space $L^2(\mathbb{R}, \mathbb{C}^n)$. Finally, the fact that $\sigma_{\text{ess}}(\mathcal{L})$ is precisely $[\kappa, \infty)$ (not just a subset) follows from Theorem 11.5(c) in [48]. This program is carried out in detail in the closely-related setting of Sturm-Liouville operators on the half-line at the end of Section 2 in [36].*

Next, in order to describe the Lagrangian subspaces $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$ specified in our general assumptions **(B1)**, **(B2)**, and **(B3)** we'll need a characterization of solutions to (4.1) that lie left in $\mathbb{R}$, along with a characterization of solutions to (4.1) that lie right in $\mathbb{R}$. For this, we begin by fixing some $\lambda < \kappa$ and looking for solutions of (4.4) of the form $\phi(x;\lambda) = e^{\mu(\lambda)x} r(\lambda)$, where in this case μ is a scalar function of λ and r is a vector-valued function of λ with $r(\lambda) \in \mathbb{C}^n$. We find that

$$(-\mu^2 P_\pm + V_\pm - \lambda Q_\pm) r = 0,$$

which we can rearrange as

$$P_\pm^{-1}(V_\pm - \lambda Q_\pm) r = \mu^2 r.$$

Since the matrices $P_\pm$ are positive definite, it's natural to work with the inner products

$$(r, s)_\pm := (P_\pm r, s), \tag{4.6}$$

and it's clear that for $\lambda < \kappa$, the matrices $P_\pm^{-1}(V_\pm - \lambda Q_\pm)$ are self-adjoint and positive definite with these inner products (respectively). We conclude that the values μ^2 will be positive real values, and that the associated eigenvectors can be chosen to be orthonormal with respect to (4.6). For each of the n values of μ^2, we can associate two values $\pm\sqrt{\mu^2}$. By a choice of labeling, we can split these into n negative values $\{\mu_k^\pm\}_{k=1}^n$ and n positive values $\{\mu_k^\pm\}_{k=n+1}^{2n}$, with the correspondence (again, by labeling convention)

$$\mu_{n+k}^\pm(\lambda) = -\mu_k^\pm(\lambda), \quad k = 1, 2, \dots, n.$$

For each $k \in \{1, 2, \dots, n\}$, we denote by $r_k^\pm(\lambda)$ the eigenvector of $P_\pm^{-1}(V_\pm - \lambda Q_\pm)$ with associated eigenvalue $\mu_k^\pm(\lambda)^2 = \mu_{n+k}^\pm(\lambda)^2$. I.e., (with dependence on λ temporarily suppressed)

$$P_\pm^{-1}(V_\pm - \lambda Q_\pm) r_k^\pm = \mu_k^{\pm\,2} r_k^\pm, \quad \forall\, k \in \{1, 2, \dots, n\}.$$

In order to place (4.1) in our general framework, we set $y = \binom{y_1}{y_2} = \binom{\phi}{P(x)\phi'}$, so that we have

$$y' = \mathbb{A}(x;\lambda) y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & P(x)^{-1} \\ V(x) - \lambda Q(x) & 0 \end{pmatrix}, \tag{4.7}$$

or equivalently

$$Jy' = \mathbb{B}(x;\lambda) y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V(x) & 0 \\ 0 & P(x)^{-1} \end{pmatrix}. \tag{4.8}$$

We see immediately that under Assumption **(SL1)**, our general Assumptions **(A)** hold.

If we set

$$\mathbb{A}_\pm(\lambda) := \begin{pmatrix} 0 & P_\pm^{-1} \\ V_\pm - \lambda Q_\pm & 0 \end{pmatrix},$$

then under our Assumptions **(SL2)** we have the relations

$$|\mathbb{A}(x;\lambda) - \mathbb{A}_\pm(\lambda)| \le \tilde{C} e^{-\tilde{\eta}|x|} \quad x \gtrless \pm\tilde{M}.$$

for some constants $\tilde{C}, \tilde{M} \ge 0$, $\tilde{\eta} > 0$.

The values $\{\mu_k^\pm(\lambda)\}_{k=1}^{2n}$ described above comprise a labeling of the eigenvalues of $\mathbb{A}_\pm(\lambda)$. If we let $\{\mathbf{r}_k^\pm\}_{k=1}^{2n}$ denote the eigenvectors of $\mathbb{A}_\pm(\lambda)$ respectively associated with these eigenvectors, then we find

$$\mathbf{r}_k^\pm(\lambda) = \begin{pmatrix} r_k^\pm(\lambda) \\ \mu_k^\pm(\lambda) P_\pm r_k^\pm(\lambda) \end{pmatrix}; \quad \mathbf{r}_{n+k}^\pm(\lambda) = \begin{pmatrix} r_k^\pm(\lambda) \\ -\mu_k^\pm(\lambda) P_\pm r_k^\pm(\lambda) \end{pmatrix}; \quad k = 1, 2, \dots, n. \tag{4.9}$$

We'll set

$$R_\pm(\lambda) = (r_1^\pm(\lambda) \;\; r_2^\pm(\lambda) \;\; \dots \;\; r_n^\pm(\lambda)),$$

noting that by orthonormality of the vectors $\{r_j^\pm(\lambda)\}_{j=1}^n$ with respect to the inner products (4.6), we have the relations

$$R_\pm(\lambda)^* P_\pm R_\pm(\lambda) = I. \tag{4.10}$$

If we also set

$$D_\pm(\lambda) = \operatorname{diag}(\mu_1^\pm(\lambda) \;\; \mu_2^\pm(\lambda) \;\; \dots \;\; \mu_n^\pm(\lambda)),$$

then we can express a frame for the eigenspace of $\mathbb{A}_-(\lambda)$ associated with its positive eigenvalues as

$$\mathbf{X}_-(\lambda) = \begin{pmatrix} R_-(\lambda) \\ -P_- R_-(\lambda) D_-(\lambda) \end{pmatrix}.$$

Likewise, we can express a frame for the eigenspace of $\mathbb{A}_-(\lambda)$ associated with its negative eigenvalues as

$$\mathbf{X}_-^g(\lambda) = \begin{pmatrix} R_-(\lambda) \\ P_-R_-(\lambda)D_-(\lambda) \end{pmatrix},$$

where the superscript g indicates that solutions to (4.4) associated with negative eigenvalues of $\mathbb{A}_-$ will grow as x tends to $-\infty$. In the same way, we can express a frame for the eigenspace of $\mathbb{A}_+(\lambda)$ associated with its negative eigenvalues as

$$\tilde{\mathbf{X}}_+(\lambda) = \begin{pmatrix} R_+(\lambda) \\ P_+R_+(\lambda)D_+(\lambda) \end{pmatrix},$$

and a frame for the eigenspace of $\mathbb{A}_+(\lambda)$ associated with its positive eigenvalues as

$$\tilde{\mathbf{X}}_+^g(\lambda) = \begin{pmatrix} R_+(\lambda) \\ -P_+R_+(\lambda)D_+(\lambda) \end{pmatrix}.$$

It's straightforward to check that the matrices $\mathbf{X}_-(\lambda)$, $\mathbf{X}_-^g(\lambda)$, $\tilde{\mathbf{X}}_+(\lambda)$, and $\tilde{\mathbf{X}}_+^g(\lambda)$ are all frames for Lagrangian subspaces of $\mathbb{C}^{2n}$. To see this for $\mathbf{X}_-(\lambda)$, we recall from Lemma 2.2 of [35] that we only need to show that $\operatorname{rank} \mathbf{X}_-(\lambda) = n$ and $\mathbf{X}_-(\lambda)^* J \mathbf{X}_-(\lambda) = 0$. The statement about rank is clear from the invertibility of $R_-(\lambda)$, and for the latter requirement we can compute directly to see that

$$\begin{aligned} \mathbf{X}_-(\lambda)^* J \mathbf{X}_-(\lambda) &= \begin{pmatrix} R_-(\lambda)^* & -D_-(\lambda)R_-(\lambda)^*P_- \end{pmatrix} \begin{pmatrix} P_-R_-(\lambda)D_-(\lambda) \\ R_-(\lambda) \end{pmatrix} \\ &= R_-(\lambda)^*P_-R_-(\lambda)D_-(\lambda) - D_-(\lambda)R_-(\lambda)^*P_-R_-(\lambda) = D_-(\lambda) - D_-(\lambda) = 0, \end{aligned}$$

where we have used (4.10).

The following lemma can be adapted directly from Lemma 2.1 in [36], and we refer the reader to that reference for the proof.

Lemma 4.1. *Assume* **(SL1)** *and* **(SL2)** *hold, and let* $\{\mu_k^\pm(\lambda)\}_{k=1}^{2n}$ *and* $\{\mathbf{r}_k^\pm(\lambda)\}_{k=1}^{2n}$ *be as described just above. Then there exists a family of bases* $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=n+1}^{2n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (4.7) that lie left in* $\mathbb{R}$, *and a family of bases* $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=1}^{n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (4.7) that lie right in* $\mathbb{R}$. *Respectively, we can choose these so that*

$$\begin{aligned} \mathbf{y}_{n+k}^-(x;\lambda) &= e^{-\mu_k^-(\lambda)x}(\mathbf{r}_{n+k}^-(\lambda) + \mathbf{E}_{n+k}^-(x;\lambda)), \quad k = 1, 2, \dots, n, \\ \mathbf{y}_k^+(x;\lambda) &= e^{\mu_k^+(\lambda)x}(\mathbf{r}_k^+(\lambda) + \mathbf{E}_k^+(x;\lambda)), \quad k = 1, 2, \dots, n, \end{aligned}$$

where for any fixed interval $[\lambda_1, \lambda_2]$, *with* $\lambda_1 < \lambda_2 < \kappa$, *there exist a constant* $\delta > 0$ *so that for each* $k \in \{1, 2, \dots, n\}$

$$\mathbf{E}_{n+k}^-(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to -\infty; \qquad \mathbf{E}_k^+(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1, \lambda_2]$.

Moreover, there exists a λ*-dependent family of bases* $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=1}^{n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (4.7) that do not lie left in* $\mathbb{R}$, *and a* λ*-dependent family of bases*

$\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=n+1}^{2n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (4.7) that do not lie right in* $\mathbb{R}$. *Respectively, we can choose these so that*

$$\mathbf{y}_k^-(x;\lambda) = e^{\mu_k^-(\lambda)x}(\mathbf{r}_k^-(\lambda) + \mathbf{E}_k^-(x;\lambda)), \quad k = 1, 2, \dots, n,$$
$$\mathbf{y}_{n+k}^+(x;\lambda) = e^{-\mu_k^+(\lambda)x}(\mathbf{r}_{n+k}^+(\lambda) + \mathbf{E}_{n+k}^+(x;\lambda)), \quad k = 1, 2, \dots, n,$$

where for any fixed interval $[\lambda_1, \lambda_2]$, *with* $\lambda_1 < \lambda_2 < \kappa$, *there exist a constant* $\delta > 0$ *so that for each* $k \in \{1, 2, \dots, n\}$

$$\mathbf{E}_k^-(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to -\infty; \qquad \mathbf{E}_{n+k}^+(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1, \lambda_2]$.

Lemmas along the lines of of Lemma 4.1 are quite standard (e.g., Theorem 10.1.2 of [42]), but the statement has been given in full to emphasize the amount of information we have in this case about solutions to (4.7). We note that the proof of Lemma 4.1 does not require that we know a priori that $\sigma_{\text{ess}}(\mathcal{L}) = [\kappa, \infty)$.

In addition to the structural assertions of Lemma 4.1, we need to establish the continuity and differentiability in λ specified in Assumption **(B1)**. For this, we take advantage of the observation that we can work with any valid frames for $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$.

Lemma 4.2. *Assume* **(SL1)** *and* **(SL2)** *hold, and for each* $\lambda \in (-\infty, \kappa)$ *let* $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=n+1}^{2n}$ *and* $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=1}^{n}$ *be as described in Lemma 4.1. If* $\ell(x;\lambda)$ *and* $\tilde{\ell}(x;\lambda)$ *respectively denote the Lagrangian subspaces with frames*

$$\mathbf{X}(x;\lambda) = (\mathbf{y}_{n+1}^-(x;\lambda) \ \ \mathbf{y}_{n+2}^-(x;\lambda) \ \ \cdots \ \ \mathbf{y}_{2n}^-(x;\lambda)), \tag{4.11}$$

and

$$\tilde{\mathbf{X}}(x;\lambda) = (\mathbf{y}_1^+(x;\lambda) \ \ \mathbf{y}_2^+(x;\lambda) \ \ \cdots \ \ \mathbf{y}_n^+(x;\lambda)), \tag{4.12}$$

then $\ell, \tilde{\ell} \in C(\mathbb{R} \times (-\infty, \kappa), \Lambda(n))$.

Lemma 4.3. *Assume* **(SL1)** *and* **(SL2)** *hold, and for some fixed* $\lambda_0 \in (-\infty, \kappa))$ *let the elements* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=n+1}^{2n}$ *and* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{n}$ *be as described in Lemma 4.1. Then there exists a constant* $r_0 > 0$ *so that the elements* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=n+1}^{2n}$ *(resp.* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{n}$*) can be analytically extended on* $B(\lambda_0, r_0)$ *(the disk in* $\mathbb{C}$ *with center* λ_0 *and radius* r_0*) to a basis for the space of solutions of (4.8) that lie left in* $\mathbb{R}$ *(resp. lie right in* $\mathbb{R}$*). Moreover, The* λ*-derivatives of these extensions lie left in* $\mathbb{R}$ *(resp. right in* $\mathbb{R}$*) and respectively satisfy* $(\partial_\lambda \mathbf{y}_k^\pm(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\mathbf{y}_k^\pm(x;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \mathbf{y}_k^\pm(x;\lambda)$ *for all* $\lambda \in B(\lambda_0, r_0)$ *and a.e.* $x \in \mathbb{R}$.

Remark 4.3. *The verification that the frames* $\mathbf{X}(x;\lambda)$ *and* $\tilde{\mathbf{X}}(x;\lambda)$ *specified in (4.11) and (4.12) are indeed frames for Lagrangian subspaces is precisely as in the calculation immediately following (3.1) in Section 3.*

The significance of Lemma 4.2 lies in the assertion that in addition to being continuous in x, ℓ *and* $\tilde{\ell}$ *are continuous in* λ *as well. The significance of Lemma 4.3 lies in the assertion that we can find frames for* ℓ *and* $\tilde{\ell}$, *possibly alternative to (4.11) and (4.12), that are locally* *analytic in λ, with λ-derivatives that lie respectively left and right in $\mathbb{R}$. The final assertion of Lemma 4.3 is straightforward to see by integrating (4.7) to see that*

$$y(x;\lambda) = y(-M;\lambda) + \int_{-M}^{x} \mathbb{A}(\xi;\lambda)y(\xi;\lambda)d\xi,$$

and using the properties of $\mathbb{A}(x;\lambda)$ and the analytic extensions of $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=n+1}^{2n}$ to justify differentiating under the integral sign in λ, followed by differentiation in x (and similarly for the analytic extensions of $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{n}$).

These lemmas are both proven under more general assumptions in Section 2.3 of [37].

Since $e^{D_-(\lambda)x}$ is an invertible matrix, we can replace the frame $\mathbf{X}(x;\lambda)$ specified in Lemma 4.2 with $\mathbf{X}(x;\lambda)e^{D_-(\lambda)x}$ (i.e., each of these matrices $\mathbf{X}(x;\lambda)$ and $\mathbf{X}(x;\lambda)e^{D_-(\lambda)x}$ is a valid frame for $\ell(x;\lambda)$). From this observation and the estimates of Lemma 4.1 we see that

$$\lim_{x\to-\infty} \mathbf{X}(x;\lambda)e^{D_-(\lambda)x} = \mathbf{X}_-(\lambda).$$

We conclude that the asymptotic Lagrangian subspace $\ell_-(\lambda)$ described in **(B2)** exists, with the choice of frame $\mathbf{X}_-(\lambda)$. Likewise, since $e^{-D_+(\lambda)x}$ is an invertible matrix, we can replace the frame $\tilde{\mathbf{X}}(x;\lambda)$ specified in Lemma 4.2 with $\tilde{\mathbf{X}}(x;\lambda)e^{-D_+(\lambda)x}$. From this observation and the estimates of Lemma 4.1 we see that

$$\lim_{x\to+\infty} \tilde{\mathbf{X}}(x;\lambda)e^{-D_+(\lambda)x} = \tilde{\mathbf{X}}_+(\lambda).$$

We conclude that the asymptotic Lagrangian subspace $\tilde{\ell}_+(\lambda)$ described in **(B2)** exists, with the choice of frame $\tilde{\mathbf{X}}_+(\lambda)$.

We also need to verify that appropriate limiting frames $\mathbf{X}_+(\lambda_1)$ and $\mathbf{X}_+(\lambda_2)$ exist. Focusing on λ_1, we first note that if λ_1 is not an eigenvalue of $\mathcal{L}$ then the columns of $\mathbf{X}(x;\lambda_1)$ must necessarily be a basis for the space of solutions to (4.7) that do not lie right in $\mathbb{R}$. Using Lemma 4.1, we see that in this case we can take as our frame for $\ell(x;\lambda_1)$ the matrix

$$\mathbf{X}^g(x;\lambda) := (\mathbf{y}_{n+1}^+(x;\lambda) \;\; \mathbf{y}_{n+2}^+(x;\lambda) \;\; \cdots \;\; \mathbf{y}_{2n}^+(x;\lambda)),$$

comprising solutions to (4.7) that do not lie right in $\mathbb{R}$. Proceeding similarly as above, we can replace $\mathbf{X}^g(x;\lambda)$ with the alternative frame $\mathbf{X}^g(x;\lambda)e^{D_+(\lambda)x}$, from which we see that

$$\lim_{x\to+\infty} \mathbf{X}^g(x;\lambda)e^{D_+(\lambda)x} = \tilde{\mathbf{X}}_+^g(\lambda_1).$$

The case in which λ_1 is an eigenvalue of $\mathcal{L}$ requires more care, and we refer the reader to the appendix of [31] for a full discussion. (The analysis of [31] is for the case in which $P(x)$ and $Q(x)$ are identity matrices, but it carries immediately to the current setting.)

We've now established that Assumptions **(A)** and **(B1)** hold, along with the first part of **(B2)**. For the second part of **(B2)**, we need to show that

$$\ell_-(\lambda) \cap \tilde{\ell}_+(\lambda) = \{0\}, \quad \forall\, \lambda \in (-\infty, \kappa).$$

According to Lemma 2.2 of [35], it suffices to show that the matrix $\mathbf{X}_-(\lambda)^* J\tilde{\mathbf{X}}_+(\lambda)$ has a trivial kernel for all $\lambda \in (-\infty, \kappa)$. To verify this, we compute

$$\begin{aligned}\mathbf{X}_-(\lambda)^* J\tilde{\mathbf{X}}_+(\lambda) &= \begin{pmatrix} R_-(\lambda)^* & -D_-(\lambda)R_-(\lambda)^*P_-^* \end{pmatrix} \begin{pmatrix} -P_+R_+(\lambda)D_+(\lambda) \\ R_+(\lambda) \end{pmatrix} \\ &= -R_-(\lambda)^*P_+R_+(\lambda)D_+(\lambda) - D_-(\lambda)^*R_-(\lambda)^*P_-^*R_+(\lambda).\end{aligned} \tag{4.13}$$

Using (4.10) we can compute

$$(R_-(\lambda)^*P_+R_+(\lambda))^{-1} = (P_+R_+(\lambda))^{-1}(R_-(\lambda)^*)^{-1} = R_+(\lambda)^*P_-R_-(\lambda).$$

If we multiply the right-hand side of (4.13) on the left by $(R_-(\lambda)^*P_+R_+(\lambda))^{-1}$ we obtain

$$D_+(\lambda) + R_+(\lambda)^*P_-R_-(\lambda)D_-(\lambda)R_-(\lambda)^*P_-R_+(\lambda),$$

which is self-adjoint and negative definite (since the eigenvalues of the diagonal matrices $D_\pm(\lambda)$ are all strictly negative). In particular, this matrix is non-singular, and we can conclude that $\mathbf{X}_-(\lambda)^* J\tilde{\mathbf{X}}_+(\lambda)$ is non-singular as well, which is what we hoped to show.

This leaves us with **(B3)**, for which we first observe that

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} Q(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

We see that

$$\mathbf{X}(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}(x;\lambda) = X(x;\lambda)^*Q(x)X(x;\lambda),$$

so that

$$\int_{-\infty}^{c} \mathbf{X}(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}(x;\lambda)dx = \int_{-\infty}^{c} X(x;\lambda)^*Q(x)X(x;\lambda)dx.$$

Since $Q(x)$ is positive definite for a.e. $x \in \mathbb{R}$, the right-hand side of this last relation is positive definite for any $c \in \mathbb{R}$, which is more than we need for **(B3)**. We've now established that under our Assumptions **(SL1)** and **(SL2)** on (4.1), our general Assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** all hold. This establishes the first part of Theorem 1.2.

4.1 Exchanging the Target Space

Next, we will use Hörmander's index to show that in the calculation of the Maslov indices $\text{Mas}(\ell(\cdot;\lambda_i), \tilde{\ell}_+(\lambda_i); (-\infty, +\infty])$, $i = 1, 2$, the target spaces $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ can be replaced by the Dirichlet plane ℓ_D with frame $\mathbf{X}_D = \binom{0}{I}$, and that the resulting flow in this case is monotonic (i.e., the direction is the same for each crossing point). We begin with a brief discussion of Hörmander's index, and in particular the approach of [32] for its evaluation. For this, we fix any four Lagrangian subspaces ν, σ, $\tilde{\nu}$, and $\tilde{\sigma}$, using Greek letters to distinguish this background discussion from the notation currently in use for analyzing (4.7). In addition, we denote by $\mathcal{P}(\nu, \sigma)$ the collection of all continuous paths of Lagrangian subspaces $\rho : [0,1] \to \Lambda(n)$ such that $\rho(0) = \nu$ and $\rho(1) = \sigma$. As verified in Section 3 of [32], the difference

$$\text{Mas}(\rho(\cdot), \tilde{\sigma}; [0,1]) - \text{Mas}(\rho(\cdot), \tilde{\nu}; [0,1])$$

is independent of $\rho \in \mathcal{P}(\nu,\sigma)$, and so this difference is an integer depending only on the fixed Lagrangian subspaces ν, σ, $\tilde{\nu}$, and $\tilde{\sigma}$. Following standard terminology and notation (e.g., equation (2.9) in [23] and Definition 3.9 in [49]) we refer to this value as *Hörmander's index* and express it as

$$s(\tilde{\nu},\tilde{\sigma};\nu,\sigma) := \operatorname{Mas}(\rho(\cdot),\tilde{\sigma};[0,1]) - \operatorname{Mas}(\rho(\cdot),\tilde{\nu};[0,1]). \tag{4.14}$$

We see immediately from (4.14) that if we are given any $\rho \in \mathcal{P}(\nu,\sigma)$, and would like to change its target from $\tilde{\sigma}$ to $\tilde{\nu}$, then we can simply rearrange (4.14) to write

$$\operatorname{Mas}(\rho(\cdot),\tilde{\sigma};[0,1]) = \operatorname{Mas}(\rho(\cdot),\tilde{\nu};[0,1]) + s(\tilde{\nu},\tilde{\sigma};\nu,\sigma). \tag{4.15}$$

In order to make practical use of (4.15), we need to compute $s(\tilde{\nu},\tilde{\sigma};\nu,\sigma)$, and one straightforward approach for this, developed in [32], is as follows. First, a Maslov-box argument similar to the one in Section 3 of the current analysis can be used to show that for any $\tilde{\rho} \in \mathcal{P}(\tilde{\nu},\tilde{\sigma})$, we have

$$s(\tilde{\nu},\tilde{\sigma};\nu,\sigma) = \operatorname{Mas}(\tilde{\rho}(\cdot),\nu;[0,1]) - \operatorname{Mas}(\tilde{\rho}(\cdot),\sigma;[0,1]). \tag{4.16}$$

In this calculation, we are free to choose any $\tilde{\rho} \in \mathcal{P}(\tilde{\nu},\tilde{\sigma})$ we like, and one convenient choice is to let $\tilde{\rho}(t)$ denote (for each $t \in [0,1]$) the Lagrangian subspace with frame

$$\tilde{\mathbf{X}}(t) := t\mathbf{X}_{\tilde{\sigma}} + (1-t)\mathbf{X}_{\tilde{\nu}}. \tag{4.17}$$

To be sure, $\tilde{\mathbf{X}}(t)$ specified in this way is not always the frame for a Lagrangian subspace for all $t \in [0,1]$, but for a broad class of frames $\mathbf{X}_{\tilde{\sigma}}$ and $\mathbf{X}_{\tilde{\nu}}$, including all those we will need to work with in the current analysis, it is. (A convenient criterion for verifying that this holds for a pair of frames $\mathbf{X}_{\tilde{\sigma}}$ and $\mathbf{X}_{\tilde{\nu}}$ is given in Section 3.2 of [32].)

In order to organize calculations along these lines, we follow the convention of [32] and define

$$\mathcal{I}(\nu;\mathbf{X}_{\tilde{\nu}},\mathbf{X}_{\tilde{\sigma}}) := -\operatorname{Mas}(\tilde{\rho}(\cdot),\nu;[0,1]),$$

where in this specification $\tilde{\rho} : [0,1] \to \Lambda(n)$ must be precisely the path of Lagrangian subspaces associated with the frames $\tilde{\mathbf{X}}(t)$ specified in (4.17). This development allows us to compute Hörmander's index as

$$s(\tilde{\nu},\tilde{\sigma};\nu,\sigma) = \mathcal{I}(\sigma;\mathbf{X}_{\tilde{\nu}},\mathbf{X}_{\tilde{\sigma}}) - \mathcal{I}(\nu;\mathbf{X}_{\tilde{\nu}},\mathbf{X}_{\tilde{\sigma}}). \tag{4.18}$$

In Section 3 of [32], the author computes $\mathcal{I}(\nu;\mathbf{X}_{\tilde{\nu}},\mathbf{X}_{\tilde{\sigma}})$ for several typical cases, including all those encountered in the current analysis. One important such case, which we will use in this section, supposes ν has a frame $\binom{I}{M_\nu}$, for some self-adjoint matrix M_ν, $\mathbf{X}_{\tilde{\nu}}$ is the Dirichlet frame $\binom{0}{I}$, and $\mathbf{X}_{\tilde{\sigma}}$ has the frame $\binom{I}{M_{\tilde{\sigma}}}$, for some self-adjoint matrix $M_{\tilde{\sigma}}$. Then, from Section 3.3.1 of [32], we have the relation

$$\mathcal{I}(\nu;\mathbf{X}_{\tilde{\nu}},\mathbf{X}_{\tilde{\sigma}}) = n_-(M_{\tilde{\nu}} - M_\nu) + n_0(M_{\tilde{\nu}} - M_\nu), \tag{4.19}$$

where for any self-adjoint $n \times n$ matrix M, $n_-(M)$ denotes the number of negative eigenvalues of M and $n_0(M)$ denotes the dimension of the kernel of M.

Returning now to the current analysis, since the calculations will be the same for $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$, we will focus on the latter. Our goal, we recall, is to exchange the target $\tilde{\ell}_+(\lambda_2)$ for the Dirichlet target ℓ_D in the calculation of $\operatorname{Mas}(\ell(\cdot;\lambda_2);\tilde{\ell}_+(\lambda_2);(-\infty,+\infty])$, and according to our general development above we can accomplish this by writing

$$\begin{aligned}\operatorname{Mas}(\ell(\cdot;\lambda_2);\tilde{\ell}_+(\lambda_2);(-\infty,+\infty]) &= \operatorname{Mas}(\ell(\cdot;\lambda_2);\ell_D;(-\infty,+\infty]) \\ &+ s(\ell_D,\tilde{\ell}_+(\lambda_2);\ell_-(\lambda_2),\ell_+(\lambda_2)).\end{aligned}$$

In order to evaluate Hörmander's index in this case, we'll use (4.18) to write

$$\begin{aligned}s(\ell_D,\tilde{\ell}_+(\lambda_2);\ell_-(\lambda_2),\ell_+(\lambda_2)) &= \mathcal{I}(\ell_+(\lambda_2);\mathbf{X}_D,\begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1}\end{pmatrix}) \\ &- \mathcal{I}(\ell_-(\lambda_2);\mathbf{X}_D,\begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1}\end{pmatrix}),\end{aligned} \tag{4.20}$$

where in order to place ourselves in the context of [32], we have replaced $\tilde{\mathbf{X}}_+(\lambda_2)$ with its normalized frame. According to (4.19), we can write

$$\begin{aligned}\mathcal{I}(\ell_+(\lambda_2);\mathbf{X}_D,\begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1}\end{pmatrix}) &= n_-(\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_+(\lambda_2)X_+(\lambda_2)^{-1}) \\ &+ n_0(\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_+(\lambda_2)X_+(\lambda_2)^{-1}),\end{aligned} \tag{4.21}$$

for which we need to understand the matrix $\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_+(\lambda_2)X_+(\lambda_2)^{-1}$. First, we see immediately that

$$\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} = P_+R_+(\lambda_2)D_+(\lambda_2)R_+(\lambda_2)^*P_+.$$

At this point, we divide the analysis into two cases: (1) λ_2 is not an eigenvalue of (4.1); and (2) λ_2 is an eigenvalue of (4.1). In the event that λ_2 is not an eigenvalue of (4.1), we can take $\mathbf{X}_+(\lambda_2) = \tilde{\mathbf{X}}_+^g(\lambda_2)$ so that

$$Y_+(\lambda_2)X_+(\lambda_2)^{-1} = -P_+R_+(\lambda_2)D_+(\lambda_2)R_+(\lambda_2)^*P_+.$$

In this way, we see that

$$\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_+(\lambda_2)X_+(\lambda_2)^{-1} = 2P_+R_+(\lambda_2)D_+(\lambda_2)R_+(\lambda_2)^*P_+,$$

and this final matrix is self-adjoint and negative definite. We conclude that

$$\mathcal{I}(\ell_+(\lambda_2);\mathbf{X}_D,\begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1}\end{pmatrix}) = n. \tag{4.22}$$

Likewise,

$$\begin{aligned}\mathcal{I}(\ell_-(\lambda_2);\mathbf{X}_D,\begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1}\end{pmatrix}) &= n_-(\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_-(\lambda_2)X_-(\lambda_2)^{-1}) \\ &+ n_0(\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_-(\lambda_2)X_-(\lambda_2)^{-1}).\end{aligned}$$

In this case,

$$\tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} - Y_-(\lambda_2)X_-(\lambda_2)^{-1} = P_+R_+(\lambda_2)D_+(\lambda_2)R_+(\lambda_2)^*P_+ \\ + P_-R_-(\lambda_2)D_-(\lambda_2)R_-(\lambda_2)^*P_-.$$

This is a sum of two self-adjoint negative definite operators, and so it is negative definite. We conclude that

$$\mathcal{I}(\ell_-(\lambda_2); \mathbf{X}_D, \begin{pmatrix} I \\ \tilde{Y}_+(\lambda_2)\tilde{X}_+(\lambda_2)^{-1} \end{pmatrix}) = n,$$

and combining this with (4.22), we see that

$$s(\ell_D, \tilde{\ell}_+(\lambda_2); \ell_-(\lambda_2), \ell_+(\lambda_2)) = 0,$$

so that

$$\text{Mas}(\ell(\cdot; \lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) = \text{Mas}(\ell(\cdot; \lambda_2), \ell_D; (-\infty, +\infty]). \tag{4.23}$$

Since we are currently assuming that λ_2 is not an eigenvalue of (4.1), $\ell_+(\lambda_2) \cap \ell_D = \{0\}$ (because $\ell_+(\lambda_2) = \tilde{\ell}_+^g(\lambda_2)$, and $\tilde{\ell}_+^g(\lambda_2) \cap \ell_D = \{0\}$). (These claims are easily checked by using the frames for $\ell_-(\lambda_2)$ and $\tilde{\ell}_+^g(\lambda_2)$.) We conclude that $+\infty$ cannot serve as a crossing point for the calculation on either side of (4.23), allowing us to write

$$\text{Mas}(\ell(\cdot; \lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) = \text{Mas}(\ell(\cdot; \lambda_2), \ell_D; (-\infty, +\infty)).$$

As a transition to the case in which λ_2 is an eigenvalue of (4.1), we claim that in either case (i.e., whether or not λ_2 is an eigenvalue of (4.1)), the crossing points arising in the calculation of $\text{Mas}(\ell(\cdot; \lambda_2), \ell_D; (-\infty, +\infty])$ all have the same direction (negative). To see this, we employ Lemma 1.1 of [32], which in the current setting can be stated as follows:

Lemma 4.4 (Lemma 1.1 from [32])**.** *Fix $a, b \in \mathbb{R}$, $a < b$, and for values of λ confined to an interval $I \subset \mathbb{R}$ consider the linear Hamiltonian system*

$$Jy' = \mathbb{B}(x; \lambda)y, \quad x \in (a, b), \quad y(x; \lambda) \in \mathbb{C}^{2n}, \tag{4.24}$$

for which we assume the following:

($\mathcal{A}$) *For each $\lambda \in I$, $\mathbb{B}(\cdot; \lambda) \in C([a, b], \mathbb{C}^{2n \times 2n})$, with $\mathbb{B}(x; \lambda)$ self-adjoint for each $x \in [a, b]$, and additionally the partial derivative $\mathbb{B}_\lambda(x; \lambda)$ exists for all $x \in \mathbb{R}$, with $\mathbb{B}_\lambda(\cdot; \lambda) \in L^1((a, b), \mathbb{C}^{2n \times 2n})$;*

($\mathcal{B}$1) *If $\mathbf{Z}$ is a frame for a Lagrangian subsapace of $\mathbb{C}^{2n}$ and $\mathbf{X}_a(x; \lambda)$ is a matrix solution of (4.24) such that $\mathbf{X}_a(a; \lambda) = \mathbf{Z}$, then for each $x \in (a, b]$ the matrix*

$$\int_a^x \mathbf{X}_a(\xi; \lambda)^* \mathbb{B}_\lambda(\xi; \lambda) \mathbf{X}_a(\xi; \lambda) d\xi$$

is positive definite for all $\lambda \in I$;

($\mathcal{B}$2) *For a fixed target Lagrangian subspace ℓ_T, the restriction $\mathbb{B}(x; \lambda)|_{\ell_T}$ is non-negative for all $(x, \lambda) \in [a, b] \times I$, and moreover if $y(x; \lambda)$ is any non-trivial solution of (4.24) with $y(x; \lambda) \in \ell_T$ for all x in some interval $[c, d] \subset [a, b]$, $c < d$, then*

$$\int_c^d (\mathbb{B}(x; \lambda)y(x; \lambda), y(x; \lambda))dx > 0.$$

(We note that this final condition can be satisfied in the vacuous case that there are no such non-trivial solutions.)

If $\ell_a(x;\lambda)$ *denotes the Lagrangian subspace of* $\mathbb{C}^{2n}$ *with frame* $\mathbf{X}_a(x;\lambda)$*, then the crossing points for the calculation* $\operatorname{Mas}(\ell_a(\cdot;\lambda),\ell_T;[a,b])$ *all have the same sign (negative), and in particular we can write*

$$\begin{aligned}\operatorname{Mas}(\ell_a(\cdot;\lambda),\ell_T;[a,b]) &= -\sum_{x\in[a,b)} \dim(\ell_a(x;\lambda)\cap\ell_T)\\ &= -\sum_{x\in[a,b)} \dim\ker(\mathbf{X}_a(x;\lambda)^*J\mathbf{X}_T).\end{aligned}$$

For the current setting, it's clear from **(SL1)** and **(SL2)** that **(**$\mathcal{A}$**)** and **(**$\mathcal{B}$**1)** hold for any finite interval $[a,b]\subset\mathbb{R}$. To verify **(**$\mathcal{B}$**2)**, we need to check two items: (1) If P_D denotes projection onto the Dirichlet subspace, then the matrix $P_D\mathbb{B}(x;\lambda_2)P_D$ is non-negative for all $x\in\mathbb{R}$; and (2) if $y(x;\lambda_2)$ is any non-trivial solution of (4.8) with $y(x;\lambda_2)\in\ell_D$ for all x in some interval $[c,d]$, $c<d$, then

$$\int_c^d (\mathbb{B}(x;\lambda_2)y(x;\lambda_2),y(x;\lambda_2))dx > 0.$$

For (1), we observe that for any $v=\binom{v_1}{v_2}\in\mathbb{C}^{2n}$, we have $P_Dv=\binom{0}{v_2}$, so that

$$v^*P_D\mathbb{B}(x;\lambda_2)P_Dv = (0\ \ v_2^*)\begin{pmatrix}\lambda_2Q(x)-V(x) & 0\\ 0 & P(x)^{-1}\end{pmatrix}\begin{pmatrix}0\\ v_2\end{pmatrix} = v_2^*P(x)^{-1}v_2\geq 0,$$

where the final inequality follows from Assumption **(SL2)**. For (2), suppose $y(x;\lambda_2)$ is any non-trivial solution of (4.8) so that $y(x;\lambda_2)\in\ell_D$ for all x in some interval $[c,d]$, $c<d$. Then, in particular, $\phi(x;\lambda_2)=0$ for all such x, and since $\phi(x;\lambda_2)$ is absolutely continuous on $\mathbb{R}$ we can conclude that $\phi'(x;\lambda_2)=0$ for a.e. $x\in(a,b)$. But then $y(x;\lambda_2)=0$ for a.e. $x\in(a,b)$, contradicting our assumption that $y(x;\lambda_2)$ is non-trivial. We conclude that Items (1) and (2) both hold, and from Lemma 4.4 we can conclude that crossing points arising in the calculation of $\operatorname{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty])$ all have the same sign (negative). In addition, the well-defined limits of $\mathbb{B}(x;\lambda_2)$ and $\mathbf{X}(x;\lambda_2)$ as $x\to+\infty$ allow us to extend Lemma 4.4 to possible crossing points at $+\infty$.

Turning now to the case in which λ_2 is an eigenvalue of (4.1), we observe that we can no longer rule out the possibility that $+\infty$ serves as a crossing point for the calculation of $\operatorname{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty])$. Nonetheless, by our monotonicity considerations if $+\infty$ serves as a crossing point, it must correspond with one or more eigenvalues of

$$\tilde{W}_D(x;\lambda_2) := (X(x;\lambda_2)+iY(x;\lambda_2))(X(x;\lambda_2)-iY(x;\lambda_2))$$

arriving at -1 in the clockwise direction as $x\to+\infty$. (Here, $\tilde{W}_D(x;\lambda_2)$ is just our usual matrix of the form (2.1) with Dirichlet target.) Such arrivals do not increment the Maslov index, and so in this case we have again

$$\operatorname{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty]) = \operatorname{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty)).$$

Since λ_2 lies below the essential spectrum there must exist $\epsilon > 0$ sufficiently small so that $\lambda_2 - \epsilon$ is not an eigenvalue of (4.1), and moreover (4.1) has no eigenvalues between $\lambda_2 - \epsilon$ and λ_2. We claim that we must have

$$\operatorname{Mas}(\ell(\cdot;\lambda_2 - \epsilon), \ell_D; (-\infty, +\infty)) = \operatorname{Mas}(\ell(\cdot;\lambda_2), \ell_D; (-\infty, +\infty)). \tag{4.25}$$

To see this, we first observe by monotonicity that the calculation of the Maslov index $\operatorname{Mas}(\ell(\cdot;\lambda_2), \ell_D; (-\infty, +\infty))$ can involve at most a finite number of crossing points. This is because any eigenvalue of $\tilde{W}_D(x;\lambda_2)$ must complete a full loop of S^1 between crossings, and by continuity and compactification no eigenvalue of $\tilde{W}_D(x;\lambda_2)$ can compute more than a finite number of such loops.

Next, if we combine our monotonicity as λ varies with our monotonicity as x varies, we find that the crossing points for $\ell(x;\lambda)$ and ℓ_D form monotonic *spectral curves* as depicted in Figure 4.1 (see Section 2.3 of [32] for a detailed discussion of this point). We see that every finite crossing along the vertical shelf at λ_2 must correspond with exactly one crossing point along the vertical shelf at $\lambda_2 - \epsilon$. This still leaves open the possibility that $(+\infty, \lambda_2)$ serves as a crossing point for the vertical shelf at λ_2 while $(+\infty, \lambda_2 - \epsilon)$ does not serve as a crossing point for the vertical shelf at $\lambda_2 - \epsilon$. As noted above, however, such an asymptotic crossing doesn't contribute to the Maslov index in either case, establishing (4.25).

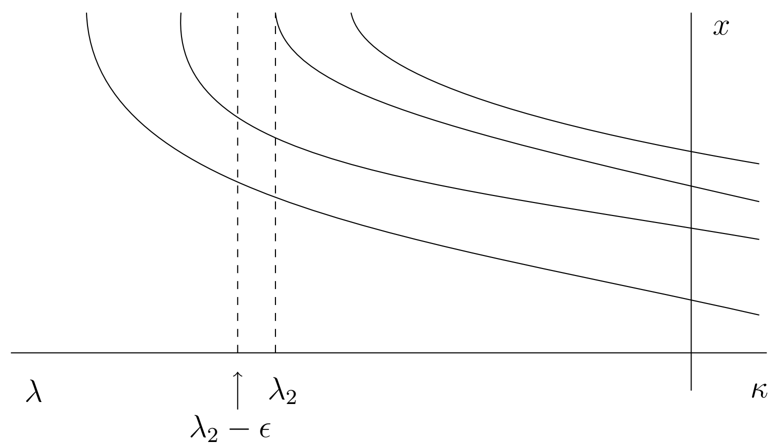

Figure 4.1: Monotonic spectral curves when the target space is ℓ_D.

Now, using the observation that $\lambda_2 - \epsilon$ is not an eigenvalue of (4.1), we can compute (with ϵ small enough so that $\lambda_1 < \lambda_2 - \epsilon$)

$$\begin{aligned}
\mathcal{N}([\lambda_1, \lambda_2)) &= \mathcal{N}([\lambda_1, \lambda_2 - \epsilon)) \\
&= -\operatorname{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2 - \epsilon); (-\infty, +\infty]) + \operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]) \\
&= -\operatorname{Mas}(\ell(\cdot;\lambda_2 - \epsilon), \ell_D; (-\infty, +\infty)) + \operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]) \\
&= -\operatorname{Mas}(\ell(\cdot;\lambda_2), \ell_D; (-\infty, +\infty)) + \operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]).
\end{aligned}$$

Since we also have (directly from Theorem 1.1)

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) + \operatorname{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]),$$

we can conclude that

$$\mathrm{Mas}(\ell(\cdot;\lambda_2),\tilde{\ell}_+(\lambda_2);(-\infty,+\infty]) = \mathrm{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty)).$$

Using again monotonicity of this last count, along with our observation above that the calculation of $\mathrm{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty))$ can involve at most a finite number of crossing points, we can write

$$\mathrm{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty)) = -\sum_{x\in\mathbb{R}} \dim(\ell(x;\lambda_2)\cap\ell_D) = -\sum_{x\in\mathbb{R}} \dim\ker(\mathbf{X}(x;\lambda_2)^* J\mathbf{X}_D), \tag{4.26}$$

where in obtaining this second equality, we have observed from Lemma 2.2 of [35] that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are frames for any two Lagrangian subspaces ℓ_1 and ℓ_2 then

$$\dim(\ell_1\cap\ell_2) = \dim\ker(\mathbf{X}_1^* J\mathbf{X}_2).$$

The same considerations hold for λ_1, allowing us to conclude

$$\begin{aligned}
\mathcal{N}([\lambda_1,\lambda_2)) &= -\mathrm{Mas}(\ell(\cdot;\lambda_2),\tilde{\ell}_+(\lambda_2);(-\infty,+\infty]) + \mathrm{Mas}(\ell(\cdot;\lambda_1),\tilde{\ell}_+(\lambda_1);(-\infty,+\infty])\\
&= -\mathrm{Mas}(\ell(\cdot;\lambda_2),\ell_D;(-\infty,+\infty)) + \mathrm{Mas}(\ell(\cdot;\lambda_1),\ell_D;(-\infty,+\infty))\\
&= \sum_{x\in\mathbb{R}} \dim(\ell(x;\lambda_2)\cap\ell_D) - \sum_{x\in\mathbb{R}} \dim(\ell(x;\lambda_1)\cap\ell_D)\\
&= \sum_{x\in\mathbb{R}} \dim\ker(\mathbf{X}(x;\lambda_2)^* J\mathbf{X}_D) - \sum_{x\in\mathbb{R}} \dim\ker(\mathbf{X}(x;\lambda_1)^* J\mathbf{X}_D),
\end{aligned}$$

For these latter calculations, we have (recalling $\mathbf{X}(x;\lambda_2) = \binom{X(x;\lambda_2)}{Y(x;\lambda_2)}$)

$$\mathbf{X}(x;\lambda_2)^* J\mathbf{X}_D = -X(x;\lambda_2)^*,$$

and since $\dim\ker(-X(x;\lambda_2)^*) = \dim\ker(X(x;\lambda_2))$, we can write

$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_2) - \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_1), \tag{4.27}$$

which is precisely the second assertion in Theorem 1.2.

4.2 Eliminating the Left Shelf

In this section, we will check that for Sturm-Liouville systems under assumptions **(SL1)** and **(SL2)** we can take λ_1 sufficiently negative so that there are no crossing points along the left shelf. We begin by noting that a point $(s,\lambda_1)\in\mathbb{R}\times(-\infty,\kappa)$ will be a crossing point for the Lagrangian subspaces $\ell(x;\lambda)$ and $\tilde{\ell}_+(\lambda_1)$ if and only if λ_1 is an eigenvalue for the half-line problem

$$\begin{aligned}
-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi; \quad x\in(-\infty,s)\\
\tilde{\mathbf{X}}_+(\lambda_1)^* J\begin{pmatrix}\phi(s)\\ P(s)\phi(s)\end{pmatrix} = 0.
\end{aligned}$$

We will use an energy argument to show that the set of eigenvalues for this problem is bounded below, independently of s. To this end, suppose λ is an eigenvalue, and let $\phi(x;\lambda)$ denote a corresponding eigenfunction. If we take an $L^2((-\infty,s),\mathbb{C}^n)$ inner product of the system with ϕ, we obtain

$$-\int_{-\infty}^{s}((P(x)\phi')',\phi)dx+\int_{-\infty}^{s}(V(x)\phi,\phi)dx=\lambda\int_{-\infty}^{s}(Q(x)\phi,\phi)dx.$$

For the first integral, we can integrate by parts to write

$$-\int_{-\infty}^{s}((P(x)\phi')',\phi)dx=-(P(s)\phi'(s),\phi(s))+\int_{-\infty}^{s}(P(x)\phi',\phi')dx.$$

The key point here is the boundary term, and for this, we observe that our boundary condition can be expressed as

$$\begin{aligned}0&=(R_+(\lambda_1)^*\ \ D_+(\lambda_1)R_+(\lambda_1)^*P_+)\begin{pmatrix}-P(s)\phi'(s)\\ \phi(s)\end{pmatrix}\\&=-R_+(\lambda_1)^*P(s)\phi'(s)+D_+(\lambda_1)R_+(\lambda_1)^*P_+\phi(s).\end{aligned}$$

Recalling the relation $(R_+(\lambda_1)^*)^{-1}=P_+R_+(\lambda_1)$, we can solve for $P(s)\phi'(s)$ in terms of $\phi(s)$ to get

$$P(s)\phi'(s)=P_+R_+(\lambda_1)D_+(\lambda_1)R_+(\lambda_1)^*P_+\phi(s).$$

We see that the boundary term can be expressed as

$$-(P(s)\phi'(s),\phi(s))=-(P_+R_+(\lambda_1)D_+(\lambda_1)R_+(\lambda_1)^*P_+\phi(s),\phi(s)).$$

The matrix $P_+R_+(\lambda_1)D_+(\lambda_1)R_+(\lambda_1)^*P_+$ is negative definite, so we can conclude that

$$-(P(s)\phi'(s),\phi(s))\geq 0$$

for all $\phi(s)\in\mathbb{C}^n$. It follows that

$$-\int_{-\infty}^{s}((P(x)\phi')',\phi)dx\geq\int_{-\infty}^{s}(P(x)\phi',\phi')dx\geq\theta_P\|\phi'\|^2_{L^2((-\infty,s),\mathbb{C}^n)}.$$

We also have

$$\int_{-\infty}^{s}(Q(x)\phi,\phi)dx\geq\theta_Q\|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)},$$

and by combining **(SL1)** and **(SL2)** we see that there exists a constant $C_V\geq 0$ sufficiently large so that

$$\Big|\int_{-\infty}^{s}(V(x)\phi,\phi)dx\Big|\leq C_V\|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)}.$$

Combining these observations, we see that for $\lambda<0$ we can write

$$\begin{aligned}\lambda\theta_Q\|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)}&\geq\lambda\int_{-\infty}^{s}(Q(x)\phi,\phi)dx=-\int_{-\infty}^{s}((P(x)\phi')',\phi)dx+\int_{-\infty}^{s}(V(x)\phi,\phi)dx\\&\geq\theta_P\|\phi'\|^2_{L^2((-\infty,s),\mathbb{C}^n)}-C_V\|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)},\end{aligned}$$

from which we see that

$$\lambda \geq -\frac{C_V}{\theta_Q}.$$

We conclude that for $\lambda < -(C_V/\theta_Q)$ there are no crossing points. In particular, if $\lambda_1 < -(C_V/\theta_Q)$, then

$$\sum_{x\in\mathbb{R}} \dim\ker(X(x;\lambda_1)) = 0,$$

so by taking $\lambda_1 < -(C_V/\theta_Q)$ in (4.27) we see that

$$\mathcal{N}((-\infty,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker X(x;\lambda_2),$$

which is precisely the final claim of Theorem 1.2. □

4.3 Traveling Waves

As noted in the Introduction, if we want to analyze the stability of a traveling-wave solution $\bar{u}(x-st)$ to the Allen-Cahn equation (1.2), we need to understand the eigenvalues of

$$H_s\phi := -\phi'' - s\phi' + V(x)\phi = \lambda\phi, \quad V(x) = D^2F(\bar{u}(x)), \tag{4.28}$$

which is not self-adjoint for $s \neq 0$ (even if $V(x)$ is self-adjoint). If we set $y = \binom{y_1}{y_2} = \binom{\phi}{\phi'}$, then we obtain

$$Jy' = \mathcal{B}(x;\lambda)y; \quad \mathcal{B}(x;\lambda) = \begin{pmatrix} \lambda I - V(x) & sI \\ 0 & I \end{pmatrix},$$

where we're using $\mathcal{B}(x;\lambda)$ in order to reserve the notation $\mathbb{B}(x;\lambda)$ for self-adjoint matrices.

In this case, we can readily place the analysis in the setting of (1.1) by making the change of variables $\zeta = e^{\frac{s}{2}x}y$, for which we find

$$J\zeta' = \mathbb{B}(x;\lambda)\zeta; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I - V(x) & \frac{s}{2}I \\ \frac{s}{2}I & I \end{pmatrix}. \tag{4.29}$$

If V satisfies the same assumptions as stated in **(SL1)** and **(SL2)**, then our analysis of (4.8) can be carried out with only minor adjustments, and we can conclude precisely the claims stated in Theorem 1.2. In fact, as shown in [31], the limit conditions can be relaxed from exponential rate to the following.

(SL2)$'$ There exist self-adjoint matrices $V_\pm$ so that the limits $\lim_{x\to\pm\infty} V(x) = V_\pm$ exist, and for each $M \in \mathbb{R}$,

$$\int_M^{+\infty} (1+|x|)|V(x)-V_+|dx < \infty, \quad \int_{-\infty}^M (1+|x|)|V(x)-V_-|dx < \infty.$$

For convenient reference, we state this assertion as a theorem. For a full proof, though by different calculations in some places, the reader is referred to [31].

Theorem 4.1. *For (4.28), let Assumptions* **(SL1)** *(on V) and* **(SL2)$'$** *hold. Then $\sigma_p(H_s) \subset \mathbb{R}$, and for κ specified as in (4.5), except with $Q_\pm = I$,* **(A)**, **(B1)**, **(B2)**, *and* **(B3)** *all hold for (4.29) with $I = (-\infty, \kappa)$. We conclude that the result of Theorem 1.1 holds for all intervals $[\lambda_1, \lambda_2]$, $\lambda_1 < \lambda_2 < \kappa$. In addition, if $\mathcal{N}([\lambda_1, \lambda_2))$ denotes the number of eigenvalues, counted with multiplicity, that (4.28) has on the interval $[\lambda_1, \lambda_2)$, and we express the frame $\mathbf{X}(x;\lambda)$ from* **(B1)** *as $\mathbf{X}(x;\lambda) = \binom{X(x;\lambda)}{Y(x;\lambda)}$, then*

$$\mathcal{N}([\lambda_1, \lambda_2)) = \sum_{x \in \mathbb{R}} \dim \ker X(x; \lambda_2) - \sum_{x \in \mathbb{R}} \dim \ker X(x; \lambda_1),$$

and

$$\mathcal{N}((-\infty, \lambda_2)) = \sum_{x \in \mathbb{R}} \dim \ker X(x; \lambda_2).$$

Unfortunately, our approach to handling traveling waves $\bar{u}(x - st)$ does not readily extend to more general Allen-Cahn type systems such as

$$u_t + DF(u) = Bu_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n,$$

for which the diffusion matrix B is not the identity matrix. For an interesting step in this direction, we refer the reader to the recent result [5].

We conclude this section by mentioning a second, more complicated, case in which the current method can be applied in the analysis of traveling waves. In particular, we consider equations

$$u_t + MDF(u) = u_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n, \tag{4.30}$$

for which M denotes a constant, invertible, self-adjoint $n \times n$ matrix. In order to analyze the stability of a traveling-wave solution $\bar{u}(x - st)$ to (4.30), we use moving coordinates as before and linearize, leading to the eigenvalue problem

$$-\phi'' - s\phi' + MD^2F(\bar{u}(x))\phi = \lambda\phi, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n. \tag{4.31}$$

The additional complication here is that $MD^2F(\bar{u}(x))$ may not be a self-adjoint potential. In order to place (4.31) in the current framework, we set $y = \binom{y_1}{y_2} = \binom{\phi}{M^{-1}\phi'}$, so that

$$Jy' = \mathcal{B}(x;\lambda)y; \quad \mathcal{B}(x;\lambda) = \begin{pmatrix} \lambda M^{-1} - D^2F(\bar{u}(x)) & sI \\ 0 & M \end{pmatrix}.$$

If we now set $\zeta = e^{\frac{s}{2}x}y$ as before, we obtain the system

$$J\zeta' = \mathbb{B}(x;\lambda)\zeta; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda M^{-1} - D^2F(\bar{u}(x)) & \frac{s}{2}I \\ \frac{s}{2}I & M \end{pmatrix},$$

which has the form of (1.1).

The verification of our general assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** requires additional assumptions on M, and we won't pursue a full analysis here. We note, however, that the following important case was analyzed in [18, 21]: $M = QS$, where Q is a diagonal matrix with either $+1$ or -1 in each diagonal entry and S is a positive diagonal matrix.

5 Differential-Algebraic Sturm-Liouville Systems

Following Section 5.4 in [35], we consider differential-algebraic Sturm-Liouville systems

$$\mathcal{L}_a \phi = -(P(x)\phi')' + V(x)\phi = \lambda \phi, \quad x \in \mathbb{R}, \quad \phi(x;\lambda) \in \mathbb{C}^n, \tag{5.1}$$

with degenerate matrix

$$P(x) = \begin{pmatrix} P_{11}(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

We make the following assumptions on P and V:

(DA1) For some $0 < m < n$, $P_{11} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{m\times m})$, with $P_{11}(x)$ self-adjoint for all $x \in \mathbb{R}$; also, $V \in C(\mathbb{R}, \mathbb{C}^{n\times n})$, with $V(x)$ self-adjoint for all $x \in \mathbb{R}$. In addition, there exists a constant $\theta_{P_{11}} > 0$ so that for any $v \in \mathbb{C}^n$,

$$(P_{11}(x)v, v) \geq \theta_{P_{11}} |v|^2$$

for all $x \in \mathbb{R}$.

(DA2) There exist self-adjoint matrices $P_{11}^{\pm}, V^{\pm}$, along with constants C, M, and $\eta > 0$, so that

$$\begin{aligned} |P_{11}(x) - P_{11}^{\pm}| &\leq Ce^{-\eta|x|}, \quad x \gtreqless \pm M; \qquad |P_{11}'(x)| \leq Ce^{-\eta|x|}, \quad x \gtreqless \pm M; \\ |V(x) - V^{\pm}| &\leq Ce^{-\eta|x|}, \quad x \gtreqless \pm M. \end{aligned}$$

For notational convenience, we'll write

$$V(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{12}(x)^* & V_{22}(x) \end{pmatrix},$$

where $V_{11}(x)$ is an $m \times m$ matrix, $V_{12}(x)$ is an $m \times (n-m)$ matrix, and $V_{22}(x)$ is an $(n-m) \times (n-m)$ matrix. We'll write $\phi = \binom{\phi_1}{\phi_2}$, where $\phi_1(x;\lambda) \in \mathbb{C}^m$ and $\phi_2(x;\lambda) \in \mathbb{C}^{n-m}$, allowing us to express (5.1) as

$$\begin{aligned} -(P_{11}(x)\phi_1')' + V_{11}(x)\phi_1 + V_{12}(x)\phi_2 &= \lambda\phi_1 \\ V_{12}(x)^*\phi_1 + V_{22}(x)\phi_2 &= \lambda\phi_2. \end{aligned} \tag{5.2}$$

We will take as our domain for $\mathcal{L}_a$, the set

$$\begin{aligned} \mathcal{D}_a := \{\phi = (\phi_1, \phi_2) \in L^2(\mathbb{R}, \mathbb{C}^m) \times L^2(\mathbb{R}, \mathbb{C}^{n-m}) : \\ \phi_1, \phi_1' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m),\ \mathcal{L}_a\phi \in L^2(\mathbb{R}, \mathbb{C}^m) \times L^2(\mathbb{R}, \mathbb{C}^{n-m})\}. \end{aligned}$$

According to [2], the essential spectrum of $\mathcal{L}_a$ will contain the ranges of the eigenvalues of $V_{22}(x)$ as x ranges over $\mathbb{R}$. Precisely, we'll let $\{\nu_k(x)\}_{k=1}^{n-m}$ denote the eigenvalues of $V_{22}(x)$, and we'll denote by $\mathcal{R}_k$ the closure of the range of $\nu_k : \mathbb{R} \to \mathbb{R}$. Then

$$\bigcup_{k=1}^{n-m} \mathcal{R}_k \subset \sigma_{\mathrm{ess}}(\mathcal{L}_a).$$

For all $\lambda \notin \cup_{k=1}^{n-m} \mathcal{R}_k$, we can solve the second equation in (5.2) for ϕ_2, giving

$$\phi_2(x;\lambda) = (\lambda I - V_{22}(x))^{-1} V_{12}(x)^* \phi_1(x;\lambda).$$

Upon substitution of this expression for ϕ_2 into the first equation in (5.2), we obtain an equation for ϕ_1,

$$-(P_{11}(x)\phi_1')' + \mathbf{V}(x;\lambda)\phi_1 = \lambda\phi_1, \tag{5.3}$$

where we've set

$$\mathbf{V}(x;\lambda) := V_{11}(x) + V_{12}(x)(\lambda I - V_{22}(x))^{-1} V_{12}(x)^*.$$

We can now analyze (5.3) similarly as we analyzed (4.1). First, for $\lambda \notin \cup_{k=1}^{n-m} \mathcal{R}_k$, the matrices $(\lambda I - V_{22}^{\pm})$ are non-singular, and we can consider the limiting system

$$-P_{11}^{\pm}\phi_1'' + \mathbf{V}^{\pm}(\lambda)\phi_1 = \lambda\phi_1, \tag{5.4}$$

where

$$\mathbf{V}^{\pm}(\lambda) := V_{11}^{\pm} + V_{12}^{\pm}(\lambda I - V_{22}^{\pm})^{-1} V_{12}^{\pm\,*}.$$

Similarly as with (4.1), we can check that in addition to the set $\cup_{k=1}^{n-m} \mathcal{R}_k$, the essential spectrum of $\mathcal{L}_a$ includes all values λ for which $\phi_1(x) = e^{ikx} r_1$ solves this equation for some constant scalar $k \in \mathbb{R}$ and constant vector $r_1 \in \mathbb{C}^m$. In this case, we have

$$(k^2 P_{11}^{\pm} + \mathbf{V}^{\pm}(\lambda)) r_1 = \lambda r_1.$$

Computing an inner product of this system with r_1, we see that

$$k^2 (P_{11}^{\pm} r_1, r_1) + (\mathbf{V}^{\pm}(\lambda) r_1, r_1) = \lambda |r_1|^2.$$

Since the matrices $P_{11}^{\pm}$ are positive definite, we see that in order for λ to satisfy this relationship, we must have

$$\lambda \geq \frac{(\mathbf{V}^{\pm}(\lambda) r_1, r_1)}{|r_1|^2}.$$

If we set

$$\kappa^{\pm}(\lambda) := \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(\mathbf{V}^{\pm}(\lambda) r_1, r_1)}{|r_1|^2}$$

(i.e., the lowest eigenvalues of the matrices $\mathbf{V}^{\pm}(\lambda)$), then we can characterize this part of the essential spectrum with the set

$$\mathcal{R}_0 := \Big\{ \lambda \in \mathbb{R} : \lambda \geq \min\{\kappa^-(\lambda), \kappa^+(\lambda)\} \Big\}.$$

With this notation in place, we see that we can consider any interval $I \subset \mathbb{R}$ so that

$$I \cap \bigcup_{k=0}^{n-m} \mathcal{R}_k = \emptyset. \tag{5.5}$$

As an important example case, we observe that we can take any interval I that lies entirely below the essential spectrum. In order to characterize the bottom of the essential spectrum more precisely, we begin by setting

$$\kappa_1^{\pm} := \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(V_{11}^{\pm} r_1, r_1)}{|r_1|^2}$$

and

$$\kappa_2^{\pm} := \inf_{r_2 \in \mathbb{C}^{n-m} \setminus \{0\}} \frac{(V_{22}^{\pm} r_2, r_2)}{|r_2|^2}.$$

By spectral mapping, the eigenvalues of $(\lambda I - V_{22}^{\pm})^{-1}$ will be $(\lambda - \nu_k^{\pm})^{-1}$, where $\{\nu_k^{\pm}\}_{k=1}^{n-m}$ denote the eigenvalues of $V_{22}^{\pm}$. In this case, we're taking λ below the set $\cup_{k=1}^{n-m} \mathcal{R}_k$, so in particular below the eigenvalues of $V_{22}^{\pm}$. It follows that

$$\inf_{r_2 \in \mathbb{C}^{n-m} \setminus \{0\}} \frac{((\lambda I - V_{22}^{\pm})^{-1} r_2, r_2)}{|r_2|^2} = (\lambda - \kappa_2^{\pm})^{-1} < 0,$$

and so

$$((\lambda I - V_{22}^{\pm})^{-1} r_2, r_2) \geq (\lambda - \kappa_2^{\pm})^{-1} |r_2|^2.$$

This allows us to compute

$$\begin{aligned}
\inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(V_{12}^{\pm} (\lambda I - V_{22}^{\pm})^{-1} V_{12}^{\pm *} r_1, r_1)}{|r_1|^2} &= \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{((\lambda I - V_{22}^{\pm})^{-1} V_{12}^{\pm *} r_1, V_{12}^{\pm *} r_1)}{|r_1|^2} \\
&\geq \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(\lambda - \kappa_2^{\pm})^{-1} |V_{12}^{\pm *} r_1|^2}{|r_1|^2} = (\lambda - \kappa_2^{\pm})^{-1} \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(V_{12}^{\pm} V_{12}^{\pm *} r_1, r_1)}{|r_1|^2} \\
&= (\lambda - \kappa_2^{\pm})^{-1} \rho^{\pm},
\end{aligned}$$

where $\rho^{\pm}$ denote the lowest eigenvalues of $V_{12}^{\pm} V_{12}^{\pm *}$. In summary, we can ensure that $\lambda \notin \mathcal{R}_0$ by taking λ to satisfy the pair of inequalities

$$\lambda < \kappa_1^{-} + \frac{\rho^{-}}{(\lambda - \kappa_2^{-})}, \quad \text{and} \quad \lambda < \kappa_1^{+} + \frac{\rho^{+}}{(\lambda - \kappa_2^{+})}.$$

Since $\lambda - \kappa_2^{\pm} < 0$, we can express these relations as the quadratic inequalities

$$\lambda^2 - (\kappa_1^{\pm} + \kappa_2^{\pm})\lambda + \kappa_1^{\pm} \kappa_2^{\pm} - \rho^{\pm,2} > 0.$$

(We emphasize that we are taking λ below $\cup_{k=1}^{n-m} \mathcal{R}_k$, so this does not assert that large positive values of λ are admissible.) Upon solving this quadratic inequality, we find that admissible values of λ include those values below $\cup_{k=1}^{n-m} \mathcal{R}_k$ that also satisfy the inequality

$$\lambda < \min_{\pm} \Big\{ \frac{1}{2} \Big((\kappa_1^{\pm} + \kappa_2^{\pm}) - \sqrt{(\kappa_1^{\pm} - \kappa_2^{\pm})^2 + 4\rho^{\pm 2}} \Big) \Big\}.$$

Returning to the general case, we let I denote any interval satisfying (5.5), not necessarily below $\sigma_{\text{ess}}(\mathcal{L}_a)$. For $\lambda \in I$, we are now in a position to develop frames $\mathbf{X}(x;\lambda)$ and $\tilde{\mathbf{X}}(x;\lambda)$

as described in **(B1)**. For this, we begin by looking for solutions to (5.4) of the form $\phi(x;\lambda) = e^{\mu(\lambda)x} r(\lambda)$, where $\mu : I \to \mathbb{R}$ and $r_1 : I \to \mathbb{C}^n$. We find,

$$\{-\mu^2 P_{11}^{\pm} + \mathbf{V}^{\pm}(\lambda) - \lambda I\} r_1 = 0.$$

The allowable values of μ^2 are precisely the eigenvalues of $(P_{11}^{\pm})^{-1}(\mathbf{V}^{\pm}(\lambda) - \lambda I)$, which is self-adjoint with respect to the inner product $(r,s)_{P_{11}^{\pm}} := (P_{11}^{\pm} r, s)$. We conclude that these eigenvalues will be real-valued, and that we can choose the associated eigenvectors to be orthonormal with respect to this inner product. In addition, for $\lambda \in I$, we have

$$\lambda < \inf_{r_1 \in \mathbb{C}^m \setminus \{0\}} \frac{(\mathbf{V}^{\pm}(\lambda) r_1, r_1)}{|r_1|^2},$$

so that $\mathbf{V}^{\pm}(\lambda) - \lambda I$ is a positive matrix. We conclude that μ^2 takes only positive real values, leading to n negative values for μ and n positive values. We will denote these values $\{\mu_k^{\pm}(\lambda)\}_{k=1}^{2n}$, with the first n values negative, the second n values positive, and (by a choice of labeling) the relation $\mu_{n+k}^{\pm}(\lambda) = -\mu_k^{\pm}(\lambda)$ for all $k \in \{1, 2, \dots, n\}$. We denote the corresponding eigenvectors $\{r_k^{\pm}(\lambda)\}_{k=1}^{n}$ so that

$$(P_{11}^{\pm})^{-1}(\mathbf{V}^{\pm}(\lambda) - \lambda I) r_k^{\pm} = (\mu_k^{\pm})^2 r_k^{\pm}, \quad \forall\, k \in \{1, 2, \dots, n\}.$$

In order to place (5.3) in our general framework, we will set $y = \binom{y_1}{y_2} = \binom{\phi_1}{P_{11}(x)\phi_1'}$ so that we have

$$y' = \mathbb{A}(x;\lambda) y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & P_{11}(x)^{-1} \\ \mathbf{V}(x;\lambda) - \lambda I & 0 \end{pmatrix}, \tag{5.6}$$

or equivalently

$$J y' = \mathbb{B}(x;\lambda) y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I - \mathbf{V}(x;\lambda) & 0 \\ 0 & P_{11}(x)^{-1} \end{pmatrix}. \tag{5.7}$$

For $\lambda \in I$, we see that $\mathbb{B}(\cdot;\lambda) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{2m \times 2m})$, and it's clear that $\mathbb{B}(x;\lambda)$ is self-adjoint for all $x \in \mathbb{R}$. We also need to compute $\mathbb{B}_\lambda(x;\lambda)$, and for this, we first observe that

$$\begin{aligned} \mathbf{V}_\lambda(x;\lambda) &= -V_{12}(x)(\lambda I - V_{22}(x))^{-2} V_{12}(x)^* \\ &= -\Big((\lambda I - V_{22}(x))^{-1} V_{12}(x)^*\Big)^* (\lambda I - V_{22}(x))^{-1} V_{12}(x)^*. \end{aligned} \tag{5.8}$$

Recalling that $V \in C(\mathbb{R}, \mathbb{C}^{n \times n})$, and that for $\lambda \in I$, we have $\lambda \notin \sigma(V_{22}(x)) \cup \sigma(V_{22}^{\pm})$ for all $x \in \mathbb{R}$, we see that $\mathbf{V}_\lambda(\cdot;\lambda) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{m \times m})$, and consequently $\mathbb{B}_\lambda(\cdot;\lambda) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^{2m \times 2m})$. This establishes Assumptions **(A)**.

For **(B1)**, we will proceed precisely as we did with Sturm-Liouville Systems. Similarly as in (4.9), the values $\{\mu_k^{\pm}(\lambda)\}_{k=1}^{2n}$ described above comprise a labeling of the eigenvalues of $\mathbb{A}_{\pm}(\lambda) := \lim_{x \to \pm\infty} \mathbb{A}(x;\lambda)$. If we let $\{\mathbf{r}_k^{\pm}(\lambda)\}_{k=1}^{2n}$ denote the eigenvectors of $\mathbb{A}_{\pm}(\lambda)$ respectively associated with these eigenvectors, then we find

$$\mathbf{r}_k^{\pm}(\lambda) = \begin{pmatrix} r_k^{\pm}(\lambda) \\ \mu_k^{\pm}(\lambda) P_{11}^{\pm} r_k^{\pm}(\lambda) \end{pmatrix}; \quad \mathbf{r}_{n+k}^{\pm}(\lambda) = \begin{pmatrix} r_k^{\pm}(\lambda) \\ -\mu_k^{\pm}(\lambda) P_{11}^{\pm} r_k^{\pm}(\lambda) \end{pmatrix}; \quad k = 1, 2, \dots, n. \tag{5.9}$$

The following lemma can be established by a proof almost identical to the proof of Lemma 4.1.

Lemma 5.1. *Assume* **(DA1)** *and* **(DA2)** *hold, and let I be as in (5.5). Also, let $\{\mu_k^\pm(\lambda)\}_{k=1}^{2m}$ and $\{\mathbf{r}_k^\pm(\lambda)\}_{k=1}^{2m}$ be as described just above. Then there exists a family of bases $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=m+1}^{2m}$, $\lambda \in I$, for the spaces of solutions to (5.6) that lie left in $\mathbb{R}$, and a family of bases $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=1}^{m}$, $\lambda \in I$, for the spaces of solutions to (5.6) that lie right in $\mathbb{R}$. Respectively, we can choose these so that*

$$\mathbf{y}_{m+k}^-(x;\lambda) = e^{-\mu_k^-(\lambda)x}(\mathbf{r}_{m+k}^-(\lambda) + \mathbf{E}_{m+k}^-(x;\lambda)), \quad k = 1,2,\dots,m,$$
$$\mathbf{y}_k^+(x;\lambda) = e^{\mu_k^+(\lambda)x}(\mathbf{r}_k^+(\lambda) + \mathbf{E}_k^+(x;\lambda)), \quad k = 1,2,\dots,m,$$

where for any fixed interval $[\lambda_1,\lambda_2] \subset I$, there exists a constant $\delta > 0$ so that for each $k \in \{1,2,\dots,m\}$

$$\mathbf{E}_{m+k}^-(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to -\infty; \qquad \mathbf{E}_k^+(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1,\lambda_2]$.

Moreover, there exists a λ-dependent family of bases $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=1}^{m}$, $\lambda \in I$, for the spaces of solutions to (5.6) that do not lie left in $\mathbb{R}$, and a λ-dependent family of bases $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=m+1}^{2m}$, $\lambda \in I$, for the spaces of solutions to (5.6) that do not lie right in $\mathbb{R}$. Respectively, we can choose these so that

$$\mathbf{y}_k^-(x;\lambda) = e^{\mu_k^-(\lambda)x}(\mathbf{r}_k^-(\lambda) + \mathbf{E}_k^-(x;\lambda)), \quad k = 1,2,\dots,m,$$
$$\mathbf{y}_{m+k}^+(x;\lambda) = e^{-\mu_k^+(\lambda)x}(\mathbf{r}_{m+k}^+(\lambda) + \mathbf{E}_{m+k}^+(x;\lambda)), \quad k = 1,2,\dots,m,$$

where for any fixed interval $[\lambda_1,\lambda_2] \subset I$, there exist a constant $\delta > 0$ so that for each $k \in \{1,2,\dots,m\}$

$$\mathbf{E}_k^-(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to -\infty; \qquad \mathbf{E}_{m+k}^+(x;\lambda) = \mathbf{O}(e^{-\delta|x|}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1,\lambda_2]$.

Precisely as in the case of Sturm-Liouville Systems, we require the following two auxiliary lemmas, which are again adapted from [37] (with a straightforward modification in this case, extending the result from cases in which $\mathbb{B}(x;\lambda)$ is linear in λ to cases in which it is analytic in λ).

Lemma 5.2. *Assume* **(DA1)** *and* **(DA2)** *hold, and for each $\lambda \in I$ (with I as in (5.5)) let $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=m+1}^{2m}$ and $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=1}^{m}$ be as described in Lemma 5.1. If $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$ respectively denote the Lagrangian subspaces with frames*

$$\mathbf{X}(x;\lambda) = (\mathbf{y}_{m+1}^-(x;\lambda) \;\; \mathbf{y}_{m+2}^-(x;\lambda) \;\; \cdots \;\; \mathbf{y}_{2m}^-(x;\lambda)), \tag{5.10}$$

and

$$\tilde{\mathbf{X}}(x;\lambda) = (\mathbf{y}_1^+(x;\lambda) \;\; \mathbf{y}_2^+(x;\lambda) \;\; \cdots \;\; \mathbf{y}_m^+(x;\lambda)), \tag{5.11}$$

then $\ell, \tilde{\ell} \in C(\mathbb{R} \times I, \Lambda(n))$.

Lemma 5.3. *Assume* **(DA1)** *and* **(DA2)** *hold, and for some fixed* $\lambda_0 \in I$ *(with* I *as in (5.5)) let* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=m+1}^{2m}$ *and* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{m}$ *be as described in Lemma 5.1. Then there exists a constant* $r_0 > 0$ *so that the elements* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=m+1}^{2m}$ *(resp.* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{m}$*) can be analytically extended on* $B(\lambda_0, r_0)$ *to a basis for the space of solutions of (5.7) that lie left in* $\mathbb{R}$ *(resp. lie right in* $\mathbb{R}$*). Moreover, The* λ*-derivatives of these extensions lie left in* $\mathbb{R}$ *(resp. right in* $\mathbb{R}$*) and respectively satisfy* $(\partial_\lambda \mathbf{y}_k^\pm(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\mathbf{y}_k^\pm(x;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \mathbf{y}_k^\pm(x;\lambda)$ *for all* $\lambda \in B(\lambda_0, r_0)$ *and a.e.* $x \in \mathbb{R}$.

Proceeding similarly as with Sturm-Liouville Systems, we can respectively replace the frames $\mathbf{X}(x;\lambda)$ and $\tilde{\mathbf{X}}(x;\lambda)$ specified in (5.10) and (5.11) with $\mathbf{X}(x;\lambda)e^{D_-(\lambda)x}$ and $\tilde{\mathbf{X}}(x;\lambda)e^{-D_+(\lambda)x}$, where

$$D_\pm(\lambda) = \mathrm{diag}(\mu_1^\pm(\lambda) \;\; \mu_2^\pm(\lambda) \;\; \dots \;\; \mu_m^\pm(\lambda)).$$

It follows that the frames for $\ell_-(\lambda)$ and $\tilde{\ell}_+(\lambda)$ can be taken respectively to be

$$\mathbf{X}_-(\lambda) = \begin{pmatrix} R_-(\lambda) \\ -P_{11}^- R_-(\lambda) D_-(\lambda) \end{pmatrix}; \quad \tilde{\mathbf{X}}_+(\lambda) = \begin{pmatrix} R_+(\lambda) \\ P_{11}^+ R_+(\lambda) D_+(\lambda) \end{pmatrix},$$

where

$$R_\pm(\lambda) = (r_1^\pm(\lambda) \;\; r_2^\pm(\lambda) \;\; \dots \;\; r_m^\pm(\lambda)).$$

This establishes **(B1)** and the first part of **(B2)**, and the second part of **(B2)** can be established precisely as for Sturm-Liouville systems.

For **(B3)**, we have

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} I - \mathbf{V}_\lambda(x;\lambda) & 0 \\ 0 & 0 \end{pmatrix},$$

from which we see that

$$\mathbf{X}(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}(x;\lambda) = X(x;\lambda)^*(I - \mathbf{V}_\lambda(x;\lambda)) X(x;\lambda).$$

It's clear from (5.8) that $-\mathbf{V}_\lambda(x;\lambda)$ is non-negative, and so $I - \mathbf{V}_\lambda(x;\lambda)$ is positive definite. From this observation, **(B3)** follows immediately as in Section 4. We conclude that the assumptions of Theorem 1.1 hold in this case, and this gives the first part of Theorem 1.3.

For the remainder of Theorem 1.3, it follows from the structure of $\mathbf{X}(x;\lambda)$, $\tilde{\mathbf{X}}(x;\lambda)$, and $\mathbb{B}(x;\lambda)$ that the relevant calculations from Section 4 can be used to show that the target spaces $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ can be replaced with the Dirichlet space ℓ_D, and also that the Maslov index with ℓ_D as the target space has monotonic crossing points.

Last, suppose $[\lambda_1, \lambda_2] \subset I$ lies entirely below the essential spectrum of $\mathcal{L}_a$. Then λ_1 can be chosen sufficiently negative so that there are no crossing points along the vertical shelf at λ_1. To see this, we again proceed as in Section 4, observing that a point $(s, \lambda_1) \in \mathbb{R} \times I$ will correspond with a crossing point if and only if λ_1 is an eigenvalue of the half-line problem

$$\begin{aligned} -(P_{11}(x)\phi_1')' + \mathbf{V}(x;\lambda)\phi_1 &= \lambda\phi_1 \\ \tilde{\mathbf{X}}_+(\lambda_1)^* J \begin{pmatrix} \phi_1(s) \\ P_{11}(s)\phi_1'(s) \end{pmatrix} &= 0. \end{aligned}$$

Proceeding as in Section 4, the only new aspect is the term

$$\int_{-\infty}^{s} (\mathbf{V}(x;\lambda)\phi_1(x;\lambda), \phi_1(x;\lambda)) dx,$$

which we bound (in absolute value) by $\|\mathbf{V}(\cdot;\lambda)\|_{L^\infty(-\infty,s)}\|\phi_1(\cdot;\lambda)\|^2_{L^2(-\infty,s)}$. In the current setting,

$$|\mathbf{V}(x;\lambda)| \le |V_{11}(x)| + |V_{12}(x)||(\lambda I - V_{22}(x))^{-1}||V_{12}(x)^*|,$$

where $|\cdot|$ denotes any matrix norm. Using the facts that $\lambda \in I$, $V \in C(\mathbb{R}, \mathbb{C}^{n\times n})$, and the limit conditions **(DA2)**, we conclude that $|\mathbf{V}(x;\lambda)|$ is bounded independently of x and λ (for $\lambda < \lambda_2$). We conclude that if we take λ_1 sufficiently negative, there will be no crossing points along the vertical shelf at λ_1. This completes the proof of Theorem 1.3. □

6 Fourth Order Potential Systems

In this section, we apply Theorem 1.1 to fourth-order potential systems

$$\mathcal{L}\phi := \phi'''' + V(x)\phi = \lambda\phi; \quad x \in \mathbb{R}, \quad \phi(x;\lambda) \in \mathbb{C}^n. \tag{6.1}$$

In order to ensure that our general assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** hold, we make the following assumptions on the coefficient matrix V.

(FP1) We take $V \in C(\mathbb{R}, \mathbb{C}^{n\times n})$, with $V(x)$ self-adjoint for all $x \in \mathbb{R}$.

(FP2) We assume the limits $\lim_{x\to\pm\infty} V(x) = V_a$ exist and agree, and

$$\int_{-\infty}^{+\infty} (1+|x|)(V(x) - V_a)dx < \infty.$$

Remark 6.1. *We emphasize that in this case we take the endstates* $\lim_{x\to\pm\infty} V(x) = V_\pm$ *to agree. This corresponds with cases in which the PDE*

$$u_t + DF(u) = -u_{xxxx}; \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+, \quad u(x,t) \in \mathbb{C}^n,$$

is linearized about a stationary solution $\bar{u}(x)$ *for which the endstates* $u_\pm$ *agree. If* $V_- \neq V_+$*, the analysis becomes substantially more technical, and we leave such cases to future studies.*

We take as our domain for $\mathcal{L}$ the set $H^4(\mathbb{R}, \mathbb{C}^n)$, noting from [48] that with this choice $\mathcal{L}$ is self-adjoint. As with Sturm-Liouville systems, the essential spectrum of $\mathcal{L}$ is determined by the asymptotic problem

$$\phi'''' + V_a\phi = \lambda\phi. \tag{6.2}$$

Precisely, if we look for solutions of the form $\phi(x) = e^{ikx}r$, then the essential spectrum of (6.2) is precisely the collection of $\lambda \in \mathbb{R}$ for which $\phi(x) = e^{ikx}r$ solves (6.2) for some $k \in \mathbb{R}$ and $r \in \mathbb{C}^n$. Upon substitution of $\phi(x) = e^{ikx}r$ in to (6.2), we obtain

$$(k^4 I + V_a)r = \lambda r \quad \Longrightarrow \quad k^4|r|^2 + (V_a r, r) = \lambda|r|^2.$$

We see from this that if we set

$$\kappa := \inf_{r\neq 0} \frac{(V_a r, r)}{|r|^2}, \tag{6.3}$$

then $\sigma_{\text{ess}}(\mathcal{L}) = [\kappa, \infty)$. This will allow us to take the interval I in Assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** to be $I = (-\infty, \kappa)$.

In order to characterize the Lagrangian subspaces $\ell(x;\lambda)$ and $\tilde{\ell}(x;\lambda)$ described in Assumption **(B1)**, we will need a lemma analogous to Lemma 4.1. In order to develop such a lemma, we begin by looking for solutions of (6.2) of the form $\phi(x;\lambda) = e^{\mu(\lambda)x} r$, where in this case μ is a real-valued function of λ, and r is a constant vector $r \in \mathbb{C}^n$. We see that

$$(\mu^4 I + V_a - \lambda I) r = 0,$$

so in particular, the allowable values of $\lambda - \mu^4$ are eigenvalues of the matrix V_a. I.e., if we denote the eigenvalues of V_a by $\{\nu_k\}_{k=1}^n$, then each allowable value of μ^4 must satisfy

$$\lambda - \mu^4 = \nu_k$$

for some $\nu_k \in \sigma(V_a)$. Since $\lambda < \kappa$ (with κ the lowest eigenvalue of V_a), we see that $\lambda - \nu_k < 0$ for all $k \in \{1, \dots, n\}$, so that $\mu^4 > 0$. Each such ν_k will correspond with four values of μ, and we will denote the full collection of such values $\{\mu_k\}_{k=1}^{4n}$, indexed so that for each $k \in \{1, 2 \dots, n\}$,

$$\mu_k(\lambda) = (-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) \sqrt[4]{\nu_k - \lambda}; \quad \mu_{n+k}(\lambda) = (-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \sqrt[4]{\nu_k - \lambda};$$
$$\mu_{2n+k}(\lambda) = (\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \sqrt[4]{\nu_k - \lambda}; \quad \mu_{3n+k}(\lambda) = (\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) \sqrt[4]{\nu_k - \lambda}.$$

We note that with this choice of indexing, we have the relations

$$\mu_{2n+k}(\lambda) = -\mu_k(\lambda); \quad \mu_{3n+k}(\lambda) = -\mu_{n+k}(\lambda); \quad \forall\, k \in \{1, 2, \dots, n\}.$$

For each $k \in \{1, 2 \dots, n\}$, the values $\mu_k(\lambda)$, $\mu_{n+k}(\lambda)$, $\mu_{2n+k}(\lambda)$, and $\mu_{3n+k}(\lambda)$ all correspond with the same eigenvector of V_a, which we denote r_k (independent of λ, since V_a is independent of λ). For the set $\{\mu_k(\lambda)\}_{k=1}^n$, we can express this as

$$(\mu_k(\lambda)^4 I + V_a - \lambda I) r_k = 0.$$

Since the matrix V_a is self-adjoint, we can choose the collection $\{r_k\}_{k=1}^n$ to be orthonormal. We will set

$$R = (r_1 \ \ r_2 \ \ \cdots \ \ r_n),$$

for which orthonormality can be expressed as $R^* R = I$.

In order to place (6.1) in our general framework, we will express it as a first-order system. For this, it will be convenient to make the choices $y_1 = \phi$, $y_2 = \phi''$, $y_3 = -\phi'''$, and $y_4 = -\phi'$, for which we find

$$y' = \mathbb{A}(x;\lambda) y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \\ V(x) - \lambda I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, \tag{6.4}$$

or equivalently

$$J y' = \mathbb{B}(x;\lambda) y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I - V(x) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{pmatrix}. \tag{6.5}$$

(We refer the reader to [29] for a full discussion of the motivation behind these choices for the vector y.) The values $\{\mu_k\}_{k=1}^{4n}$ are precisely the eigenvalues of the matrix

$$\mathbb{A}_a(\lambda) := \lim_{x \to \pm\infty} \mathbb{A}(x; \lambda),$$

and it's straightforward to check that the associated eigenvectors are respectively

$$\mathbf{r}_{pn+k}(\lambda) = \begin{pmatrix} r_k \\ (\mu_{pn+k})^2 r_k \\ -(\mu_{pn+k})^3 r_j \\ -\mu_{pn+k} r_k \end{pmatrix}; \quad p = 0, 1, 2, 3.$$

The following lemma can be proven in almost precisely the same way as Lemma 2.2 in [31].

Lemma 6.1. *Assume* **(FP1)** *and* **(FP2)** *hold, and let* $\{\mu_k(\lambda)\}_{k=1}^{4n}$ *and* $\{\mathbf{r}_k(\lambda)\}_{k=1}^{4n}$ *be as described just above. Then there exists a family of bases* $\{\mathbf{y}_k^-(\cdot; \lambda)\}_{k=2n+1}^{2n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (6.4) that lie left in* $\mathbb{R}$, *and a family of bases* $\{\mathbf{y}_k^+(\cdot; \lambda)\}_{k=1}^{2n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (6.4) that lie right in* $\mathbb{R}$. *Respectively, we can choose these so that*

$$\mathbf{y}_{2n+k}^-(x; \lambda) = e^{-\mu_k(\lambda)x}(\mathbf{r}_{2n+k}(\lambda) + \mathbf{E}_{2n+k}^-(x; \lambda)), \quad k = 1, 2, \dots, 2n,$$
$$\mathbf{y}_k^+(x; \lambda) = e^{\mu_k(\lambda)x}(\mathbf{r}_k(\lambda) + \mathbf{E}_k^+(x; \lambda)), \quad k = 1, 2, \dots, 2n,$$

where for any fixed interval $[\lambda_1, \lambda_2] \subset (-\infty, \kappa)$, $\lambda_1 < \lambda_2$, *and for any* $k \in \{1, 2, \dots, 2n\}$

$$\mathbf{E}_{2n+k}^-(x; \lambda) = \mathbf{O}((1 + |x|)^{-1}), \quad x \to -\infty; \qquad \mathbf{E}_k^+(x; \lambda) = \mathbf{O}((1 + |x|)^{-1}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1, \lambda_2]$.

Moreover, there exists a λ*-dependent family of bases* $\{\mathbf{y}_k^-(\cdot; \lambda)\}_{k=1}^{2n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of solutions to (6.4) that do not lie left in* $\mathbb{R}$, *and a* λ*-dependent family of bases* $\{\mathbf{y}_k^+(\cdot; \lambda)\}_{k=2n+1}^{4n}$, $\lambda \in (-\infty, k)$, *for the spaces of solutions to (6.4) that do not lie right in* $\mathbb{R}$. *Respectively, we can choose these so that*

$$\mathbf{y}_k^-(x; \lambda) = e^{\mu_k(\lambda)x}(\mathbf{r}_k(\lambda) + \mathbf{E}_k^-(x; \lambda)), \quad k = 1, 2, \dots, 2n,$$
$$\mathbf{y}_{2n+k}^+(x; \lambda) = e^{-\mu_k(\lambda)x}(\mathbf{r}_{2n+k}(\lambda) + \mathbf{E}_{2n+k}^+(x; \lambda)), \quad k = 1, 2, \dots, 2n,$$

where for any fixed interval $[\lambda_1, \lambda_2] \subset (-\infty, \kappa)$, $\lambda_1 < \lambda_2$, *and for any* $k \in \{1, 2, \dots, 2n\}$

$$\mathbf{E}_k^-(x; \lambda) = \mathbf{O}((1 + |x|)^{-1}), \quad x \to -\infty; \qquad \mathbf{E}_{2n+k}^+(x; \lambda) = \mathbf{O}((1 + |x|)^{-1}), \quad x \to +\infty,$$

uniformly for $\lambda \in [\lambda_1, \lambda_2]$.

Precisely as in the previous cases, we require the following two auxiliary lemmas, which are again adapted from [37].

Lemma 6.2. *Assume* **(FP1)** *and* **(FP2)** *hold, and for each* $\lambda \in (-\infty, \kappa)$ *let* $\{\mathbf{y}_k^-(\cdot;\lambda)\}_{k=2n+1}^{4n}$ *and* $\{\mathbf{y}_k^+(\cdot;\lambda)\}_{k=1}^{2n}$ *be as described in Lemma 6.1. If* $\ell(x;\lambda)$ *and* $\tilde{\ell}(x;\lambda)$ *respectively denote the Lagrangian subspaces with frames*

$$\mathbf{X}(x;\lambda) = (\mathbf{y}_{2n+1}^-(x;\lambda) \;\; \mathbf{y}_{2n+2}^-(x;\lambda) \;\; \cdots \;\; \mathbf{y}_{4n}^-(x;\lambda)), \tag{6.6}$$

and

$$\tilde{\mathbf{X}}(x;\lambda) = (\mathbf{y}_1^+(x;\lambda) \;\; \mathbf{y}_2^+(x;\lambda) \;\; \cdots \;\; \mathbf{y}_{2n}^+(x;\lambda)), \tag{6.7}$$

then $\ell, \tilde{\ell} \in C(\mathbb{R} \times (-\infty, \kappa), \Lambda(n))$.

Lemma 6.3. *Assume* **(FP1)** *and* **(FP2)** *hold, and for some fixed* $\lambda_0 \in (-\infty, \kappa)$ *let the elements* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=2n+1}^{4n}$ *and* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{2n}$ *be as described in Lemma 6.1. Then there exists a constant* $r_0 > 0$ *so that the elements* $\{\mathbf{y}_k^-(\cdot;\lambda_0)\}_{k=2n+1}^{4n}$ *(resp.* $\{\mathbf{y}_k^+(\cdot;\lambda_0)\}_{k=1}^{2n}$*) can be analytically extended in* $B(\lambda_0, r_0)$ *to a basis for the space of solutions of (6.5) that lie left in* $\mathbb{R}$ *(resp. lie right in* $\mathbb{R}$*). Moreover, The* λ*-derivatives of these extensions lie left in* $\mathbb{R}$ *(resp. right in* $\mathbb{R}$*) and respectively satisfy* $(\partial_\lambda \mathbf{y}_k^\pm(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\mathbf{y}_k^\pm(x;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \mathbf{y}_k^\pm(x;\lambda)$ *for all* $\lambda \in B(\lambda_0, r_0)$ *and a.e.* $x \in \mathbb{R}$.

We will set

$$\mathcal{D}(\lambda) = \text{diag}(\mu_1(\lambda) \;\; \mu_2(\lambda) \;\; \ldots \;\; \mu_{2n}(\lambda)),$$

and we note that our labeling conventions have been chosen so that

$$-\mathcal{D}(\lambda) = \text{diag}(\mu_{2n+1}(\lambda) \;\; \mu_{2n+2}(\lambda) \;\; \ldots \;\; \mu_{4n}(\lambda)).$$

If we replace $\mathbf{X}(x;\lambda)$ with $\mathbf{X}(x;\lambda)e^{\mathcal{D}(\lambda)x}$ and $\tilde{\mathbf{X}}(x;\lambda)$ with $\tilde{\mathbf{X}}(x;\lambda)e^{-\mathcal{D}(\lambda)x}$, we readily see that the asymptotic Lagrangian subspaces $\ell_-(\lambda)$ and $\tilde{\ell}_+(\lambda)$ are well defined with respective frames

$$\mathbf{X}_-(\lambda) = \begin{pmatrix} R & R \\ RD(\lambda)^2 & R(D(\lambda)^*)^2 \\ RD(\lambda)^3 & R(D(\lambda)^*)^3 \\ RD(\lambda) & RD(\lambda)^* \end{pmatrix}; \quad \tilde{\mathbf{X}}_+(\lambda) = \begin{pmatrix} R & R \\ RD(\lambda)^2 & R(D(\lambda)^*)^2 \\ -RD(\lambda)^3 & -R(D(\lambda)^*)^3 \\ -RD(\lambda) & -RD(\lambda)^* \end{pmatrix}. \tag{6.8}$$

Likewise, we obtain asymptotic frames associated with solutions that do not lie left (respectively right) in $\mathbb{R}$, and we see from Lemma 6.1 that these will be $\mathbf{X}_-^g(\lambda) = \tilde{\mathbf{X}}_+(\lambda)$ and $\tilde{\mathbf{X}}_+^g(\lambda) = \mathbf{X}_-(\lambda)$.

We need to check directly that $\mathbf{X}_-(\lambda)$, $\tilde{\mathbf{X}}_+(\lambda)$, $\mathbf{X}_-^g(\lambda)$, and $\tilde{\mathbf{X}}_+^g(\lambda)$ are frames for Lagrangian subspaces. The calculation is the same for each case, so we provide details only for the first. If we compute $\mathbf{X}_-(\lambda)^* J \mathbf{X}_-(\lambda)$, and use the orthogonality relation $R^*R = I$, we obtain a diagonal $2n \times 2n$ matrix with upper left $n \times n$ submatrix

$$-D(\lambda)^3 + (D(\lambda)^*)^3 - (D(\lambda)^*)^2 D(\lambda) + D(\lambda)^* D(\lambda)^2$$

and lower right $n \times n$ submatrix

$$-(D(\lambda)^*)^3 + D(\lambda)^3 - D(\lambda)^2 D(\lambda)^* + D(\lambda)(D(\lambda)^*)^2.$$

The entries of $D(\lambda)^2$ are purely imaginary, so that $D(\lambda)^2 = -(D(\lambda)^*)^2$. If follows immediately that the two matrix expressions above are both 0, and we can conclude that $\mathbf{X}_-(\lambda)^* J \mathbf{X}_-(\lambda) = 0$.

For the second part of Assumption **(B2)**, we need to verify that the matrix $\mathbf{X}_-(\lambda)^* J \tilde{\mathbf{X}}_+(\lambda)$ is non-singular for all $\lambda < \kappa$. Computing directly as with the calculation of $\mathbf{X}_-(\lambda)^* J \mathbf{X}_-(\lambda)$ just above, we find that

$$\mathbf{X}_-(\lambda)^* J \tilde{\mathbf{X}}_+(\lambda) = \begin{pmatrix} 0 & 4(D(\lambda)^*)^3 \\ 4D(\lambda)^3 & 0 \end{pmatrix},$$

and since the matrix $D(\lambda)$ is diagonal with non-zero entries (for $\lambda < \kappa$), we can conclude that $\mathbf{X}_-(\lambda)^* J \tilde{\mathbf{X}}_+(\lambda)$ is non-singular.

In order to verify Assumption **(B3)** in this case, we begin by observing that

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that

$$\mathbf{X}(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}(x;\lambda) = X_1(x;\lambda)^* X_1(x;\lambda),$$

where the $n \times 2n$ matrix $X_1(x;\lambda)$ comprises the first n rows of each column of the $4n \times 2n$ matrix $\mathbf{X}(x;\lambda)$. We compute

$$\int_{-\infty}^{c} \mathbf{X}(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}(x;\lambda) dx = \int_{-\infty}^{c} X_1(x;\lambda)^* X_1(x;\lambda) dx.$$

The columns of $X_1(x;\lambda)$ are $2n$ linearly independent solutions of (6.1), and so this matrix is positive definite by precisely the same considerations as discussed in Section 4.

We have now verified Assumptions **(A)**, **(B1)**, **(B2)**, and **(B3)** for this case, and so we can apply Theorem 1.1 to obtain the first claim in Theorem 1.4. For the second claim in Theorem 1.4, we will proceed as in the previous sections, using Hörmander's index to replace $\tilde{\ell}_+(\lambda_1)$ and $\tilde{\ell}_+(\lambda_2)$ with a target frame $\mathbf{X}_T$ with respect to which the calculations of the Maslov indices are monotonic. As discussed in [32], a natural frame to work with is

$$\mathbf{X}_T = \begin{pmatrix} 0 & 0 \\ 0 & I \\ I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $\mathbf{X}_T$ is the frame for a Lagrangian subspace of $\mathbb{C}^{4n}$, and we denote this subspace ℓ_T. Recalling our conventions for the components of y (just above (6.4)), we see that crossing points for this target will consist of pairs (x_*, λ_*) so that $\phi(x_*;\lambda_*) = 0$ and $\phi'(x_*;\lambda_*) = 0$.

Focusing on the case $\lambda = \lambda_2$, we recall from Section 4.1 that the difference

$$\operatorname{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) - \operatorname{Mas}(\ell(\cdot;\lambda_2), \ell_T; (-\infty, +\infty])$$

depends only on the fixed Lagrangian subspaces ℓ_T, $\tilde{\ell}_+(\lambda_2)$, $\ell_-(\lambda_2)$ and $\ell_+(\lambda_2)$, and corresponds with Hörmander's index

$$s(\ell_T, \tilde{\ell}_+(\lambda_2); \ell_-(\lambda_2), \ell_+(\lambda_2)). \tag{6.9}$$

In order to evaluate Hörmander's index, we will again use the interpolation-space approach of [32], and for this we need to work with a frame for $\tilde{\ell}_+(\lambda_2)$ for which that analysis holds. To this end, we introduce the inverse of the matrix

$$\begin{pmatrix} R & R \\ -RD(\lambda_2) & -RD(\lambda_2)^* \end{pmatrix},$$

which we find by inspection is

$$\tilde{M}(\lambda_2) := \begin{pmatrix} (D(\lambda_2) - D(\lambda_2)^*)^{-1} & 0 \\ 0 & (D(\lambda_2) - D(\lambda_2)^*)^{-1} \end{pmatrix} \begin{pmatrix} -D(\lambda_2)^* R^* & -R^* \\ D(\lambda_2) R^* & R^* \end{pmatrix}. \tag{6.10}$$

We will replace the frame $\tilde{\mathbf{X}}_+(\lambda_2)$ with the frame $\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)$. For notational purposes, we can express this new frame as

$$\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2) = \begin{pmatrix} I & 0 \\ \tilde{X}_{21} & \tilde{X}_{22} \\ \tilde{X}_{31} & \tilde{X}_{32} \\ 0 & I \end{pmatrix}; \quad \begin{pmatrix} \tilde{X}_{21} & \tilde{X}_{22} \\ \tilde{X}_{31} & \tilde{X}_{32} \end{pmatrix} = \begin{pmatrix} RD(\lambda_2)^2 & R(D(\lambda_2)^*)^2 \\ -RD(\lambda_2)^3 & -R(D(\lambda_2)^*)^3 \end{pmatrix} \tilde{M}(\lambda_2).$$

In order to apply the development of [32], we need to check two conditions on the frames $\mathbf{X}_T$ and $\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)$. First, we need to verify that

$$\mathbf{X}_T^* J(\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)) + (\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2))^* J \mathbf{X}_T = 0. \tag{6.11}$$

To see this, we compute directly to find

$$\mathbf{X}_T^* J(\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad (\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2))^* J \mathbf{X}_T = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

from which (6.11) is immediate. The second condition we need to check is that the matrix $\mathbf{X}_T^* J(\tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2))$ is non-singular, and this is immediately clear from the previous calculations. Recalling (4.18), we can now write

$$\begin{aligned} s(\ell_T, \tilde{\ell}_+(\lambda_2); \ell_-(\lambda_2), \ell_+(\lambda_2)) &= \mathcal{I}(\ell_+(\lambda_2); \mathbf{X}_T, \tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)) \\ &\quad - \mathcal{I}(\ell_-(\lambda_2); \mathbf{X}_T, \tilde{\mathbf{X}}_+(\lambda_2)\tilde{M}(\lambda_2)). \end{aligned} \tag{6.12}$$

If λ_2 is not an eigenvalue for (6.1), then $\ell_+(\lambda_2)$ is the Lagrangian subspace with frame $\tilde{\mathbf{X}}_+^g(\lambda_2)$, which, as noted above, is equal to $\mathbf{X}_-(\lambda_2)$. I.e., $\ell_+(\lambda_2) = \ell_-(\lambda_2)$, and so clearly Hörmander's index is 0. We can conclude that if λ_2 is not an eigenvalue for (6.1), then

$$\text{Mas}(\ell(\cdot; \lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) = \text{Mas}(\ell(\cdot; \lambda_2), \ell_T; (-\infty, +\infty]).$$

As a transition to the case in which λ_2 is an eigenvalue of (6.1), we claim that in either case (i.e., whether or not λ_2 is an eigenvalue of (6.1)), the crossing points arising in the calculation of $\text{Mas}(\ell(\cdot;\lambda_2), \ell_T; (-\infty, +\infty])$ all have the same direction (negative). To see this, we employ again Lemma 4.4. Assumptions **($\mathcal{A}$)** and **($\mathcal{B}$1)** from that lemma follow immediately from **(FP1)** and **(FP2)**, and for **($\mathcal{B}$2)** we need to check two things: (1) If P_T denotes projection onto the Lagrangian subspace ℓ_T, then the matrix $P_T \mathbb{B}(x;\lambda_2) P_T$ is non-negative for a.e. $x \in \mathbb{R}$; and (2) if $y(x;\lambda_2)$ is any non-trivial solution of (6.5) with $y(x;\lambda_2) \in \ell_T$ for all x in some interval $[a,b]$, $a < b$, then

$$\int_a^b (\mathbb{B}(x;\lambda_2) y(x;\lambda_2), y(x;\lambda_2)) dx > 0.$$

For (1), we observe that

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \implies P_T v = \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ 0 \end{pmatrix},$$

and consequently

$$v^* P_T \mathbb{B}(x;\lambda_1) P_T v = (0 \;\; v_2^* \;\; v_3^* \;\; 0) \begin{pmatrix} \lambda I - V(x) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} = |v_2|^2 \geq 0.$$

For (2), suppose $y(x;\lambda_2)$ is any non-trivial solution of (6.5) so that $y(x;\lambda_2) \in \ell_T$ for all x in some interval $[a,b]$, $a < b$. Then, in particular, $\phi(x;\lambda_2) = 0$ for all such x, and since $\phi(x;\lambda_2)$ and its first two derivatives are absolutely continuous on $\mathbb{R}$ we can conclude that $\phi^{(k)}(x;\lambda_2) = 0$, $k = 1,2,3$, for a.e. $x \in (a,b)$. But then $y(x;\lambda_2) = 0$ for a.e. $x \in (a,b)$, contradicting our assumption that $y(x;\lambda_2)$ is non-trivial. We conclude that Items (1) and (2) both hold, and from Lemma 4.4 we can conclude that crossing points arising in the calculation of $\text{Mas}(\ell(\cdot;\lambda_2), \ell_T; (-\infty, +\infty])$ all have the same sign (negative).

We can now use this monotonicity to argue precisely as in Section 4 that whether or not λ_2 is an eigenvalue of (6.1) we must have

$$\text{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) = \text{Mas}(\ell(\cdot;\lambda_2), \ell_T; (-\infty, +\infty)).$$

The same considerations hold for λ_1, allowing us to write

$$\begin{aligned} \mathcal{N}([\lambda_1, \lambda_2)) &= -\text{Mas}(\ell(\cdot;\lambda_2), \tilde{\ell}_+(\lambda_2); (-\infty, +\infty]) + \text{Mas}(\ell(\cdot;\lambda_1), \tilde{\ell}_+(\lambda_1); (-\infty, +\infty]) \\ &= -\text{Mas}(\ell(\cdot;\lambda_2), \ell_T; (-\infty, +\infty)) + \text{Mas}(\ell(\cdot;\lambda_1), \ell_T; (-\infty, +\infty)) \\ &= \sum_{x \in \mathbb{R}} \dim(\ell(x;\lambda_2) \cap \ell_T) - \sum_{x \in \mathbb{R}} \dim(\ell(x;\lambda_1) \cap \ell_T) \\ &= \sum_{x \in \mathbb{R}} \dim \ker(\mathbf{X}(x;\lambda_2)^* J \mathbf{X}_T) - \sum_{x \in \mathbb{R}} \dim \ker(\mathbf{X}(x;\lambda_1)^* J \mathbf{X}_T). \end{aligned}$$

For these latter calculations, if we write

$$\mathbf{X}(x;\lambda) = \begin{pmatrix} X_{11}(x;\lambda) & X_{12}(x;\lambda) \\ X_{21}(x;\lambda) & X_{22}(x;\lambda) \\ X_{31}(x;\lambda) & X_{32}(x;\lambda) \\ X_{41}(x;\lambda) & X_{42}(x;\lambda), \end{pmatrix}$$

then we have

$$\mathbf{X}(x;\lambda_1)^* J \mathbf{X}_T = \begin{pmatrix} X_{11}(x;\lambda) & X_{12}(x;\lambda) \\ -X_{41}(x;\lambda) & -X_{42}(x;\lambda) \end{pmatrix}.$$

If we recall our specifications for the components of y in terms of ϕ, ϕ', ϕ'', and ϕ''', we see that we can write

$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in\mathbb{R}} \dim\ker\Phi(x;\lambda_2) - \sum_{x\in\mathbb{R}} \dim\ker\Phi(x;\lambda_1), \tag{6.13}$$

where (for $i=1,2$)

$$\Phi(x;\lambda_i) = \begin{pmatrix} \phi_1(x;\lambda_i) & \phi_2(x;\lambda_i) & \dots & \phi_{2n}(x;\lambda_i) \\ \phi_1'(x;\lambda_i) & \phi_2'(x;\lambda_i) & \dots & \phi_{2n}'(x;\lambda_i) \end{pmatrix},$$

with $\{\phi_j(x;\lambda_1)\}_{j=1}^{2n}$ comprising a collection of $2n$ linearly independent solutions of (6.1) that lie left in $\mathbb{R}$.

Last, we check that we can take λ_1 sufficiently negative so that there are no crossing points along the left shelf. For this, we begin by observing that $(s,\lambda_1)\in\mathbb{R}\times(-\infty,\kappa)$ will be a crossing point for $\ell(\cdot;\lambda_1)$ and $\tilde{\ell}_+(\lambda_1)$ if and only if λ_1 is an eigenvalue for

$$\phi'''' + V(x)\phi = \lambda\phi;$$

$$\tilde{\mathbf{X}}_+(\lambda_1)^* J \begin{pmatrix} \phi(s) \\ \phi''(s) \\ -\phi'''(s) \\ -\phi'(s) \end{pmatrix} = 0.$$

Suppose λ is an eigenvalue for this system, and let ϕ denote an associated eigenfunction. If we take an $L^2((-\infty,s),\mathbb{C}^n)$ inner product of the system with ϕ, we obtain the integral relation

$$\int_{-\infty}^s (\phi'''',\phi)dx + \int_{-\infty}^s (V\phi,\phi)dx = \lambda\|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)}. \tag{6.14}$$

For the first integral, we integrate by parts twice to obtain the relation

$$\int_{-\infty}^s (\phi'''',\phi)dx = \|\phi''\|^2_{L^2((-\infty,s),\mathbb{C}^n)} + \Big(\begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix}, \begin{pmatrix} \phi'''(s) \\ \phi''(s) \end{pmatrix}\Big).$$

Recalling (6.8), we can express the boundary condition as

$$\begin{pmatrix} R^* & (D(\lambda)^*)^2 R^* & -(D(\lambda)^*)^3 R^* & -D(\lambda)^* R^* \\ R^* & D(\lambda)^2 R^* & -D(\lambda)^3 R^* & -D(\lambda) R^* \end{pmatrix} \begin{pmatrix} \phi'''(s) \\ \phi'(s) \\ \phi(s) \\ \phi''(s) \end{pmatrix} = 0,$$

or equivalently

$$\begin{pmatrix} R^* & -D(\lambda)^*R^* \\ R^* & -D(\lambda)R^* \end{pmatrix} \begin{pmatrix} \phi'''(s) \\ \phi''(s) \end{pmatrix} = \begin{pmatrix} (D(\lambda)^*)^3R^* & (D(\lambda)^*)^2R^* \\ D(\lambda)^3R^* & D(\lambda)^2R^* \end{pmatrix} \begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix}.$$

Recalling (6.10), we see that

$$\begin{pmatrix} \phi'''(s) \\ \phi''(s) \end{pmatrix} = \tilde{M}(\lambda)^* \begin{pmatrix} (D(\lambda)^*)^3R^* & (D(\lambda)^*)^2R^* \\ D(\lambda)^3R^* & D(\lambda)^2R^* \end{pmatrix} \begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix},$$

where we can write

$$\begin{aligned} \tilde{M}(\lambda)^* \begin{pmatrix} (D(\lambda)^*)^3R^* & (D(\lambda)^*)^2R^* \\ D(\lambda)^3R^* & D(\lambda)^2R^* \end{pmatrix} &= \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} -D(\lambda) & D(\lambda)^* \\ -I & I \end{pmatrix} \\ \times \begin{pmatrix} (D(\lambda)^* - D(\lambda))^{-1} & 0 \\ 0 & (D(\lambda)^* - D(\lambda))^{-1} \end{pmatrix} & \begin{pmatrix} (D(\lambda)^*)^3 & (D(\lambda)^*)^2 \\ D(\lambda)^3 & D(\lambda)^2 \end{pmatrix} \begin{pmatrix} R^* & 0 \\ 0 & R^* \end{pmatrix}. \end{aligned} \tag{6.15}$$

This matrix is clearly similar to the product of the middle three matrices, and so has the same eigenvalues as that matrix product. In order to compute these eigenvalues, we set

$$\Lambda(\lambda) := \text{diag}(\sqrt[4]{\nu_1 - \lambda} \;\; \sqrt[4]{\nu_2 - \lambda} \;\; \dots \;\; \sqrt[4]{\nu_n - \lambda}),$$

so that

$$\begin{aligned} D(\lambda) &= (-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\Lambda(\lambda); \quad D(\lambda)^* - D(\lambda) = (i\sqrt{2})\Lambda(\lambda); \\ D(\lambda)^2 &= i\Lambda(\lambda)^2; \quad D(\lambda)^3 = (\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\Lambda(\lambda), \end{aligned}$$

with corresponding adjoints. These relations allow us to express the product of the middle three matrices in (6.15) as

$$\begin{aligned} &\begin{pmatrix} (\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})\Lambda & (-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})\Lambda \\ -I & I \end{pmatrix} \begin{pmatrix} \frac{1}{i\sqrt{2}}\Lambda^{-1} & 0 \\ 0 & \frac{1}{i\sqrt{2}}\Lambda^{-1} \end{pmatrix} \begin{pmatrix} (\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})\Lambda^3 & -i\Lambda^2 \\ (\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\Lambda^3 & i\Lambda^2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}\Lambda^3 & -\Lambda^2 \\ -\Lambda^2 & \sqrt{2}\Lambda \end{pmatrix}. \end{aligned}$$

We observe that this matrix is self-adjoint, and it follows that the full matrix in (6.15) is self-adjoint. In addition, we can compute the eigenvalues of this matrix by computing the roots of the characteristic equation

$$\begin{aligned} \det \begin{pmatrix} \sqrt{2}\Lambda^3 - \sigma I & -\Lambda^2 \\ -\Lambda^2 & \sqrt{2}\Lambda - \sigma I \end{pmatrix} &= \det((\sqrt{2}\Lambda^3 - \sigma I)(\sqrt{2}\Lambda - \sigma I) - \Lambda^4) \\ &= \det(\sigma^2 I - \sqrt{2}(\Lambda + \Lambda^3)\sigma + \Lambda^4). \end{aligned}$$

Here, since Λ is a diagonal matrix, this determinant is a product

$$\prod_{j=1}^{n} (\sigma^2 - \sqrt{2}(\sqrt[4]{\nu_j - \lambda} + (\sqrt[4]{\nu_j - \lambda})^3)\sigma + \nu_j - \lambda),$$

which clearly can have no roots for $\sigma \leq 0$. We conclude that the matrix

$$\tilde{M}(\lambda)^* \begin{pmatrix} (D(\lambda)^*)^3 R^* & (D(\lambda)^*)^2 R^* \\ D(\lambda)^3 R^* & D(\lambda)^2 R^* \end{pmatrix}$$

is positive definite, and so

$$\left(\begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix}, \begin{pmatrix} \phi'''(s) \\ \phi''(s) \end{pmatrix} \right) = \left(\begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix}, \tilde{M}(\lambda)^* \begin{pmatrix} (D(\lambda)^*)^3 R^* & (D(\lambda)^*)^2 R^* \\ D(\lambda)^3 R^* & D(\lambda)^2 R^* \end{pmatrix} \begin{pmatrix} \phi(s) \\ -\phi'(s) \end{pmatrix} \right) \geq 0$$

for all $s \in \mathbb{R}$. Returning to (6.14), we see that

$$\lambda \|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)} \geq \int_{-\infty}^{s} (V\phi, \phi) dx \geq -\|V\|_{L^\infty(\mathbb{R},\mathbb{C}^{n\times n})} \|\phi\|^2_{L^2((-\infty,s),\mathbb{C}^n)},$$

and consequently

$$\lambda \geq -\|V\|_{L^\infty(\mathbb{R},\mathbb{C}^{n\times n})}.$$

In this way, we see that if we take $\lambda_1 < -\|V\|_{L^\infty(\mathbb{R},\mathbb{C}^{n\times n})}$ then there will be no crossing points along the vertical shelf at $\lambda = \lambda_1$. This gives the final claim in Theorem 1.4. □

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