Asymptotic behavior near transition fronts for equations of generalized Cahn–Hilliard form

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Abstract

We consider the asymptotic behavior of perturbations of standing wave solutions arising in evolutionary PDE of generalized Cahn–Hilliard form in one space dimension. Such equations are well known to arise in the study of spinodal decomposition, a phenomenon in which the rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into its two components with their concentrations separated by sharp transition layers. Motivated by work of Bricmont, Kupiainen, and Taskinen [5], we regard the study of standing waves as an interesting step toward understanding the dynamics of these transitions. A critical feature of the Cahn–Hilliard equation is that the linear operator that arises upon linearization of the equation about a standing wave solution has essential spectrum extending onto the imaginary axis, a feature that is known to complicate the step from spectral to nonlinear stability. Under the assumption of spectral stability, described in terms of an appropriate Evans function, we develop detailed asymptotics for perturbations from standing wave solutions, establishing phase-asymptotic orbital stability for initial perturbations decaying with appropriate algebraic rate.

1 Introduction

We consider the asymptotic behavior of perturbations of standing-wave solutions $\bar{u}(x), \bar{u}(\pm\infty) = u_\pm$ arising as equilibrium solutions in equations of generalized Cahn–Hilliard form

$$u_t = (b(u)u_x)_x - (c(u)u_{xxx})_x, \quad x, u, b, c \in \mathbb{R}, t > 0, \tag{1.1}$$

for which we assume

(H0) $b, c \in C^2(\mathbb{R})$.

(H1) $b(u_\pm) > 0$, $c(\bar{u}(x)) \geq c_0 > 0$, $x \in \mathbb{R}$.

Our analysis is motivated by the distinguished role the Cahn–Hilliard model plays in the study of spinodal decomposition, a phenomenon in which the rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into its two components with their concentrations separated by sharp transition layers [9, 11]. In this context, and for the case of incompressible fluids, the Cahn–Hilliard equation is

$$\begin{aligned} u_t &= \nabla \cdot \{M(u)\nabla(F'(u) - \kappa\triangle u)\} \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \triangle u}{\partial \nu} = 0; \quad x \in \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω denotes a bounded open subset of $\mathbb{R}^n$ (n typically 2 or 3), ν denotes the unit normal vector to Ω, u denotes the concentration of one component of the binary alloy (or possibly a difference between this concentration and the concentration at homogenous mixing), $F(u)$ denotes the bulk free energy of the alloy, and the typically small parameter κ is a measure of the strength of interfacial energy. (In the case that κ depends on u, a new term appears in (1.2)) of the form $-\frac{1}{2}\kappa'(u)|\nabla u|^2$; see [9]. We take this natural form of the equation from [12].).

An intesting feature of equations of form (1.1) is that the second and fourth order regularizations can balance, and a wide variety of stationary solutions can arise, including standing waves. For example, in the case of (1.1) with

$$b(u) = \frac{3}{2}u^2 - \frac{1}{2}; \quad c(u) \equiv 1 \tag{1.3}$$

(the case studied in [5], corresponding with $F(u) = (1/8)u^4 - (1/4)u^2$, $M(u) \equiv 1$, and $\kappa = 1$ in (1.2)), one readily verifies that the standing wave

$$\bar{u}(x) = \tanh(\frac{x}{2}) \tag{1.4}$$

is such a solution, often referred to as the *kink* solution. Motivated by the elegant result of Bricmont, Kupiainen, and Taskinen [5] (which we state below for comparison with the current analysis), we regard the study of standing waves as an interesting first step toward understanding the dynamics of these transitions. (See also [36] and [38] for related results in the case of multiple space dimensions). We stress that our focus is on standing waves only, and remark that linearization of (1.1) about a traveling wave leads to a linearized problem with non-vanishing convection and consequently an entirely different flavor than the problem that arises upon linearization about a standing wave. Indeed, upon shifting to a coordinate system moving along with the shock, a flux function is introduced $f(u) = -su$, and the wave can be regarded as a regularized undercompressive shock profile. Such problems have been considered in [16], and are quite similar to the problems considered in [25, 26].

A critical feature of equations of form (1.1) is that the linear operator that arises upon linearization of the equation about a standing wave solution has essential spectrum extending onto the imaginary axis, a feature that is known to complicate the step from spectral to nonlinear stability. The purpose of this paper is to study precisely this step from spectral to nonlinear stability. In particular, under the assumption of spectral stability (described below in terms of an appropriate Evans function and verified in the particular case (1.1)–(1.3) with wave (1.4)), we develop detailed asymptotics for perturbations from standing wave solutions, establishing phase-asymptotic orbital stability for initial perturbations decaying with algebraic rate $(1+|x|)^{-3/2}$.

Our approach to this problem will be to extend to this setting methods developed previously in the context of conservation laws with diffusive and/or dispersive regularity,

$$u_t + f(u)_x = (b(u)u_x)_x + (c(u)u_{xx})_x + ..., \tag{1.5}$$

which also have no spectral gap. More precisely, we proceed by computing pointwise estimates on the Green's function for the linear equation that arises upon linearization of (1.1) about the wave $\bar{u}(x)$ (employing a contour-shifting approach introduced in [22, 52]; see also [23, 45]). Such estimates are dependent upon the spectrum of the linear operator, which we understand here in terms of an appropriate Evans function (see, for example, [14, 21, 35, 52] and the discussion and references below). Finally, we employ the local tracking method developed in [34], an approach through which Green's function estimates on the linearized operator can be used to approximately locate shifts from the standing wave $\bar{u}$. Our general approach is similar to the analysis of [5], in which case the authors also employ Green's function estimates on the linear operator in order to close an iteration on the perturbation in some appropriately weighted space. A fundamental difference between the two analyses is that in [5], this iteration is carried out by the renormalization group method, a theory that has its origins in particle physics (see [8]) and was introduced in the context of time-asymptotic behavior for nonlinear PDE in [17, 18, 20], and further developed in [3, 4] (it pre-dates the current approach by about eight years). Briefly, the renormalization group method is an approach toward understanding asymptotic behavior of PDE that makes use of certain natural scalings in the PDE. This method is ideally suited for cases such as (1.1), for which the equation that arises upon linearization about the wave $\bar{u}(x)$ has the same natural scaling as one would use for the heat equation. Indeed, though the analysis of [5] is carried out in the context of a self-adjoint linearized equation (the *integrated equation*), the method could in principle be extended to the current setting for which the linearized operator is not generally self-adjoint even after integration. Nonetheless, the Green's function estimates required for any such extension would almost certainly be obtained by methods very similar to those employed here, which have been developed particularly for the case of non-self-adjoint problems. In this way, the current approach seems to represent a change of thinking from the point of view of classical mathematical physics to that of modern Evans function techniques. We emphasize, however, that the spectral analysis of both [5] and the current analysis rely on the self-adjoint nature of the equation, and in the non-self-adjoint case spectral stability is left as a separate analysis (see the discussion following Remark 1.1). A more fundamental difficulty with the use of such a scaling technique as the renormalization group method in the context of the Cahn–Hilliard equation arises in the extension to multiple space dimensions, in which case it is known that the leading eigenvalue of the linearized operator about a planar wave $\bar{u}(x_1)$ (leading in the case of stability) scales as $\lambda \sim |\xi|^3$ (see [50]), and consequently introduces a new cubic scale into the problem, which competes with the natural heat-equation scaling of (1.1). (Here, $\xi \in \mathbb{R}^{d-1}$ denotes a Fourier variable associated with spatial components transverse to the planar wave.) A different approach is taken in this setting in [36, 38], quite similar to the method employed here, and the authors conclude stability in space dimensions $d \geq 3$ for the planar wave

$$\bar{u}(x_1) = \tanh \frac{x_1}{2},$$

arising in (1.2) with $F(u) = (1/8)u^4 - (1/4)u^2$, $M(u) \equiv 1$ and $\kappa = 1$. The primary difference between the approach of [36, 38] and the current analysis is the local tracking function $\delta(t)$ employed here, which can be generalized to the case of multiple space dimensions as a function of t and the transverse variable $\tilde{x} = (x_2, x_3, ..., x_d)$ (see [25, 26, 27, 28]). It is precisely this local tracking, which does not seem to fit in any straightforward fashion into the renormalization group framework, that allows us here to obtain stability for more slowly decaying data than considered in [5], and that allows for the extension of this method to the case of space dimensions $d \geq 2$ (an analysis we carry out in a separate paper). Finally, we note that in adddition to the standing waves considered in the current analysis and the planar waves discussed above, the Cahn–Hillard equation admits a wide range of periodic solutions, and the methods here can be extended to that setting in a manner similar to the analyses of Oh and Zumbrun in the case of periodic solutions arising in the context of viscous conservation laws [46, 47].

Before setting up the analysis, we mention that equations of form (1.1) have been shown to arise in the area of mathematical biology in the context of cell formation and aggregation [44], and also that equations similar to (1.1) arise naturally in the modeling of thin film flows, for which under certain circumstances, the height $h(t, x)$ of a film moving along an inclined plane can be modeled by fourth order equations

$$h_t + (h^2 - h^3)_x = \beta(u^3 u_x)_x - \gamma(u^3 u_{xxx})_x$$

(see [6] and the references therein).

As in the case of conservation laws, it is readily seen that solutions $u(t, x)$ initially near $\bar{u}(x)$, will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined uniquely by the amount of perturbation mass (measured as $\int_{\mathbb{R}}(u(0, x) - \bar{u}(x))dx$) carried into the shock layer. We proceed, then, by defining the perturbation

$$v(t, x) = u(t, x + \delta(t)) - \bar{u}(x), \tag{1.6}$$

for which $\delta(t)$ will be chosen by the analysis to track the location of the perturbed solution in time. In this way, we compare the shapes of u and $\bar{u}$ rather than their locations. (The type of stability we will conclude is *orbital*.) Substituting v into (1.1), we obtain the perturbation equation

$$v_t + (a(x)v)_x = (b(x)v_x)_x - (c(x)v_{xxx})_x + \dot{\delta}(t)(\bar{u}_x(x) + v_x) + Q(x, v, v_x, v_{xxx})_x, \tag{1.7}$$

with

$$\begin{aligned} a(x) = -b'(\bar{u})\bar{u}_x + c'(\bar{u})\bar{u}_{xxx}; \quad b(x) = b(\bar{u}(x)); \quad c(x) := c(\bar{u}(x)) \\ Q(x, v, v_x, v_{xxx}) = \mathbf{O}(|vv_x|) + \mathbf{O}(|vv_{xxx}|) + \mathbf{O}(e^{-\eta|x|}v^2), \end{aligned} \tag{1.8}$$

where $\eta > 0$ and Q is a continuously differentiable function of its arguments. According to hypotheses (H0)–(H1), we have that $a \in C^1(\mathbb{R})$, $b, c \in C^2(\mathbb{R})$ and for $k = 0, 1$

$$|\partial_x^k a(x)| = \mathbf{O}(e^{-\alpha|x|}); \quad |\partial_x^k (b(x) - b_\pm)| = \mathbf{O}(e^{-\alpha|x|}); \quad |\partial_x^k (c(x) - c_\pm)| = \mathbf{O}(e^{-\alpha|x|}), \tag{1.9}$$

where $\alpha > 0$ and $\pm$ represent the asymptotic limits as $x \to \pm\infty$. Regarding the convection coefficient $a(x)$, we note that it decays to 0 at both $\pm\infty$ (see Section 2 for more details). In comparison with conservation laws

$$u_t + f(u)_x = (b(u)u_x)_x,$$

this corresponds with the *degenerate* case, for which one of the asymptotic convection coefficients $a_\pm = f'(u_\pm)$ for a traveling wave $\bar{u}(x-st)$ is equal to the wave speed s, and both values can be taken without loss of generality as 0 (see [23, 24]). In the case (1.1)–(1.3) the wave (1.4) satisfies

$$a(x) = -3\bar{u}(x)\bar{u}_x(x),$$

from which we see that for $x<0$, we have $a(x)>0$, while for $x>0$, we have $a(x)<0$. In this way, the wave (1.4) is similar to Lax waves in the sense that characteristic speeds (values of the convection coefficient $a(x)$) impinge on the transition layer from both sides. The critical difference, again, is that in the case of equations of form (1.1) these characteristic speeds approach 0 asymptotically. Finally, we mention that whereas this decay of $a(x)$ is algebraic in the case of degenerate viscous shock waves, it is exponential in the current setting, a difference critical in the Evans function analysis of Section 2.

Another important feature of equations of form (1.1) is that the diffusion coefficient $b(\bar{u}(x))$ may not always be positive. In the case (1.1)–(1.3) with wave (1.4), we have

$$b(\bar{u}) = \frac{3}{2}\bar{u}^2 - \frac{1}{2}, \tag{1.10}$$

which is asymptotically positive at both $\pm\infty$, but negative near the transition layer at $x=0$. Our assumptions (H0)–(H1) clearly do not give us control over the possible destabilizing effect of this feature of the equations. The effect is, however, encoded in the spectrum of the linear operator

$$Lv = -(c(x)v_{xxx})_x + (b(x)v_x)_x - (a(x)v)_x, \tag{1.11}$$

which we discuss further below. For now, we suffice to remark that the essential spectrum of L consists of the negative real axis, up to the imaginary axis, and so as in the case of regularized conservation laws, the step from spectral stability to nonlinear stability is problematic.

Integrating (1.7), we have (after integration by parts and upon observing that $e^{Lt}\bar{u}_x = \bar{u}_x$) the integral equation

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x(x) \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(y,v,v_y,v_{yyy}) + \dot{\delta}(s)v\Big]dyds, \end{aligned} \tag{1.12}$$

where $G(t,x;y)$ denotes a Green's function for the linear part of (1.7):

$$G_t = LG; \quad G(0,x;y) = \delta_y(x). \tag{1.13}$$

We divide $G(t,x;y)$ into terms for which the x dependence is exactly $\bar{u}_x(x)$ (the *excited* terms) and the remainder, $\tilde{G}(t,x;y)$. In general, the excited terms do not decay for large time and correspond with mass accumulating at the origin, shifting the asymptotic location of the waves. Writing

$$G(t,x;y) = \tilde{G}(t,x;y) + \bar{u}_x(x)e(t,y),$$

we have

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy + \bar{u}_x(x)\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy + \delta(t)\bar{u}_x(x) \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[Q(y,v,v_y,v_{yyy}) + \dot{\delta}(s)v\Big]dyds \\ &\quad - \bar{u}_x(x)\int_0^t \int_{-\infty}^{+\infty} e_y(t-s,y)\Big[Q(y,v,v_y,v_{yyy}) + \dot{\delta}(s)v\Big]dyds. \end{aligned}$$

Choosing $\delta(t)$ to eliminate precisely the excited term, we have

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s,y)\Big[Q(y,v,v_y,v_{yyy}) + \dot{\delta}(s)v\Big]dyds, \tag{1.14}$$

after which, we have

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y) v_0(y) dy \\ &- \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y) \Big[Q(y,v,v_y,v_{yyy}) + \dot{\delta}(s) v \Big] dy ds. \end{aligned} \tag{1.15}$$

Integral equations for v_x and $\dot{\delta}$ can immediately be obtained by differentiation of both sides of (1.14) and (1.15). Our approach will be to determine estimates on $\tilde{G}(t,x;y)$ and $e(t,y)$ sufficient for closing simultaneous iterations on $v(t,x)$, $v_x(t,x)$, $\delta(t)$, and $\dot{\delta}(t)$. To this end, we analyze the Green's function $G(t,x;y)$ through its Laplace transform $G_\lambda(x,y)$, which satisfies the ODE (t transformed to λ)

$$LG_\lambda - \lambda G_\lambda = -\delta_y(x), \tag{1.16}$$

and can be estimated by standard methods. Letting $\phi_1^+(x;\lambda)$ and $\phi_2^+(x;\lambda)$ denote the (necessarily) two linearly independent asymptotically decaying solutions at $+\infty$ of the eigenvalue ODE

$$L\phi = \lambda\phi, \tag{1.17}$$

and $\phi_1^-(x,\lambda)$ and $\phi_2^-(x,\lambda)$ similarly the two linearly independent asymptotically decaying solutions at $-\infty$, we write $G_\lambda(x,y)$ as a linear combination

$$G_\lambda(x,y) = \begin{cases} \phi_1^-(x;\lambda) N_1^+(y;\lambda) + \phi_2^-(x;\lambda) N_2^+(y;\lambda), & x < y \\ \phi_1^+(x;\lambda) N_1^-(y;\lambda) + \phi_2^+(x;\lambda) N_2^-(y;\lambda), & x > y. \end{cases} \tag{1.18}$$

Insisting, as usual, on the continuity of $G_\lambda(x,y)$ and its first two x-derivatives in x, and on the jump in $\partial_x^3 G_\lambda(x,y)$ at $x = y$, we have

$$\begin{aligned} N_1^+(y;\lambda) &= \frac{W(\phi_1^+,\phi_2^+,\phi_2^-)}{c(y)W_\lambda(y)}; \quad N_2^+(y;\lambda) = -\frac{W(\phi_1^+,\phi_2^+,\phi_1^-)}{c(y)W_\lambda(y)} \\ N_1^-(y;\lambda) &= -\frac{W(\phi_1^-,\phi_2^-,\phi_2^+)}{c(y)W_\lambda(y)} \quad N_2^-(y;\lambda) = \frac{W(\phi_1^-,\phi_2^-,\phi_1^+)}{c(y)W_\lambda(y)}, \end{aligned} \tag{1.19}$$

where $W(\phi_1,\phi_2,...,\phi_n)$ denotes a square determinant of column vectors created by augmentation with an appropriate number of x-derivatives and $W_\lambda(y) := W(\phi_1^-,\phi_2^-,\phi_1^+,\phi_2^+)$. We see immediately from (1.18) and (1.19) that, away from essential spectrum, $G_\lambda(x,y)$ is bounded so long as $W_\lambda(y)$ is bounded away from 0. Since $W_\lambda(y)$ is a Wronskian for (1.17), for each fixed λ its sign does not change as y varies. In light of this, we define the *Evans function* as

$$D(\lambda) = W_\lambda(0). \tag{1.20}$$

Introduced by Evans in the context of nerve impulse equations [14] (see also the early analysis of Jones for pulse solutions to the FitzHugh–Nagumo equation [35]), the Evans function serves as a characteristic function for the operator L. More precisely, away from essential spectrum, zeros of the Evans function correspond in location and multiplicity with eigenvalues of the operator L, an observation that has been made precise in [1] in the case—pertaining to reaction–diffusion equations—of isolated eigenvalues, and in [21, 52] and [40] in the cases—pertaining respectively to conservation laws and the nonlinear Schrödinger equation—of nonstandard "effective" eigenvalues embedded in essential spectrum. (The latter correspond with resonant poles of L, as also examined in [48].)

Briefly, we remark on the connection between our form of the Evans function (1.20) and the definition of Evans et al. [1, 14, 35]. Letting $\Phi_k^\pm := (\phi_k^\pm, \phi_k^{\pm\prime}, \phi_k^{\pm\prime\prime}, \phi_k^{\pm\prime\prime\prime})^{\mathrm{tr}}$, define

$$Y_\pm(x;\lambda) = \Phi_1^\pm(x;\lambda) \wedge \Phi_2^\pm(x;\lambda), \tag{1.21}$$

where $\wedge$ represents a wedge product on the space of vectors in $\mathbb{R}^4$ (i.e., $\wedge$ denotes an associative, bilinear operation on vectors $v \in \mathbb{R}^4$, characterized by the relation between standard $\mathbb{R}^4$ basis vectors e_j and e_k, $e_j \wedge e_k = -e_k \wedge e_j$). In this case, we obtain through straightforward calculation that

$$D(\lambda) = e^{-\int_0^x \mathrm{tr}\mathbb{A}(y;\lambda)dy} Y_-(x;\lambda) \wedge Y_+(x;\lambda), \tag{1.22}$$

which is the form of [1, 14, 35]. (As described below, the matrix $\mathbb{A}$ is simply the matrix that arises when (1.17) is written as a first order system.) One notable advantage of the formulation (1.22) is that the 2-forms $Y_\pm(x;\lambda)$ are the asymptotically decaying solutions of the equations

$$Y_\pm' = \mathcal{A}_\pm(x;\lambda)Y_\pm,$$

where $\mathcal{A}$ is defined by

$$\begin{aligned} Y_\pm' &= (\Phi_1^\pm \wedge \Phi_2^\pm)' = \Phi_1^{\pm\prime} \wedge \Phi_2^\pm + \Phi_1^\pm \wedge \Phi_2^{\pm\prime} \\ &= (\mathbb{A}\Phi_1^\pm) \wedge \Phi_2^\pm + \Phi_1^\pm \wedge \mathbb{A}\Phi_2^\pm = \mathcal{A}_\pm \Phi_1^\pm \wedge \Phi_2^\pm = \mathcal{A}_\pm Y_\pm. \end{aligned}$$

In this way, the wedge formulation can remain analytically valid in cases for which the wedge products $Y_\pm$ cannot be uniquely resolved into components. In the current setting, the $\Phi_k^\pm$ remain sufficiently distinct (away from essential spectrum) that the two formulations are equivalent.

We will observe in Section 2 that $D(\lambda)$ is not analytic at $\lambda = 0$, but can be extended analytically on the Riemann surface $\rho = \sqrt{\lambda}$. (This is quite similar to the analysis of Kapitula and Rubin in the case of the complex Ginzburg–Landau equation [39], and the analyses of Guès et al. and of Zumbrun in the case of multidimensional shock stability in the vicinity of glancing points [19, 51], but differs markedly from the case of degenerate conservation laws, for which the Evans function has a dependence of the form $\sqrt{\lambda}\ln\lambda$ [23, 24].) In light of this, we define the function $D_a(\rho) := D(\lambda)$. The primary concern of this analysis is to show that under assumptions (H0)–(H1), nonlinear stability of standing wave solutions $\bar{u}(x)$ of (1.1) is implied by the spectral condition $(\mathcal{D})$:

$(\mathcal{D})$: The Evans function $D(\lambda)$ has precisely one zero in $\{\text{Re}\lambda \geq 0\}$, necessarily at 0, and $\partial_{\rho\rho}D_a(0) \neq 0$.

Remark 1.1. *Under the assumption of condition $(\mathcal{D})$, the point spectrum of L, aside from the distinguished point at $\lambda = 0$, lies to the left of a parabolic contour, which passes through the negative real axis and opens into the negative real half plane,*

$$\lambda_s(k) = -\tilde{c}_2 k^2 + i\tilde{c}_1 k + \tilde{c}_0. \tag{1.23}$$

We will denote this contour Γ_s. The essential spectrum of L consists of the negative real axis.

While condition $(\mathcal{D})$ is generally quite difficult to verify analytically (see, for example, [13, 15, 35]), it can be analyzed numerically [2, 33, 31]. In the case of equations (1.1)–(1.3) and the standing wave (1.4), $(\mathcal{D})$ is straightforward to verify (see [5] or Lemma 2.6 of the current analysis, in which we review the analysis of [5] for the sake of completeness).

We are now in a position to state the first theorem of the paper.

Theorem 1.1. *Suppose $\bar{u}(x)$ is a standing wave solution to (1.1) and suppose (H0)–(H1) hold, as well as stability criterion $(\mathcal{D})$. Then for some fixed C and C_E, and for positive contants M and K sufficiently large, and for $\eta > 0$, depending only on the spectrum and coefficients of L, the Green's function $G(t,x;y)$ described in (1.13) satisfies the following estimates for $y \leq 0$ (with symmetric estimates in the case $y \geq 0$).*

(I) For either $|x-y| \geq Kt$ or $t \leq 1$, and for α a multi-index in the variables x and y,

$$|\partial^\alpha G(t,x;y)| \leq Ct^{-\frac{1+|\alpha|}{4}} e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}}, \quad |\alpha| \leq 3.$$

(II) For $|x-y| \leq Kt$ and $t \geq 1$

$$G(t,x;y) = \tilde{G}(t,x;y) + E(t,x;y),$$

where

(i) $y \leq x \leq 0$

$$\begin{aligned}
\tilde{G}(t,x;y) &= \frac{C}{\sqrt{4b_- t}}\Big[e^{-\frac{(x-y)^2}{4b_- t}} - e^{-\frac{(x+y)^2}{4b_- t}}\Big] + \mathbf{O}(t^{-1})e^{-\frac{(x-y)^2}{Mt}} \\
\tilde{G}_y(t,x;y) &= \frac{C}{\sqrt{4b_- t}}\partial_y\Big[e^{-\frac{(x-y)^2}{4b_- t}} - e^{-\frac{(x+y)^2}{4b_- t}}\Big] \\
&\quad + \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(\frac{|x+y|}{t^{3/2}})e^{-\frac{(x+y)^2}{Mt}} \\
\tilde{G}_x(t,x;y) &= \mathbf{O}(\frac{|x-y|}{t^{3/2}})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1/2}e^{-\eta|x|})e^{-\frac{y^2}{Mt}} + \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}} \\
\tilde{G}_{xy}(t,x;y) &= \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(\frac{|y|}{t^{3/2}}e^{-\eta|x|})e^{-\frac{y^2}{Mt}}.
\end{aligned}$$

(ii) $x \leq y \leq 0$

$$\begin{aligned}
\tilde{G}(t,x;y) &= \frac{C}{\sqrt{4b_- t}}\Big[e^{-\frac{(x-y)^2}{4b_- t}} - e^{-\frac{(x+y)^2}{4b_- t}}\Big] + \mathbf{O}(t^{-1})e^{-\frac{(x-y)^2}{Mt}} \\
\tilde{G}_y(t,x;y) &= \mathbf{O}(\frac{|x-y|}{t^{3/2}})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}} \\
\tilde{G}_x(t,x;y) &= \frac{C}{\sqrt{4b_- t}}\partial_x\Big[e^{-\frac{(x-y)^2}{4b_- t}} - e^{-\frac{(x+y)^2}{4b_- t}}\Big] \\
&\quad + \mathbf{O}(t^{-1/2}e^{-\eta|x|})e^{-\frac{y^2}{Mt}} + \mathbf{O}(\frac{|x+y|}{t^{3/2}})e^{-\frac{(x+y)^2}{Mt}} + \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}} \\
\tilde{G}_{xy}(t,x;y) &= \mathbf{O}(t^{-3/2})e^{-\frac{(x-y)^2}{Mt}}
\end{aligned}$$

(iii) $y \leq 0 \leq x$

$$\begin{aligned}
\tilde{G}(t,x;y) &= \mathbf{O}(t^{-1/2})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} \\
\tilde{G}_y(t,x;y) &= \mathbf{O}(\frac{|x-\sqrt{\frac{b_+}{b_-}}y|}{t^{3/2}})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} + \mathbf{O}(\frac{|y|}{t^{3/2}}e^{-\eta|x|})e^{-\frac{y^2}{Mt}} + \mathbf{O}(t^{-3/2})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} \\
\tilde{G}_x(t,x;y) &= \mathbf{O}(\frac{|x-\sqrt{\frac{b_+}{b_-}}y|}{t^{3/2}})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} + \mathbf{O}(t^{-1/2}e^{-\eta|x|})e^{-\frac{y^2}{Mt}} + \mathbf{O}(t^{-3/2})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} \\
\tilde{G}_{xy}(t,x;y) &= \mathbf{O}(t^{-3/2})e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{Mt}} + \mathbf{O}(\frac{|y|}{t^{3/2}}e^{-\eta|x|})e^{-\frac{y^2}{Mt}}
\end{aligned}$$

and in all cases

$$\begin{aligned}
E(t,x;y) &= C_E\bar{u}_x(x)\int_{-\infty}^{\frac{y}{\sqrt{4b_- t}}} e^{-\zeta^2}d\zeta\Big(1+\mathbf{O}(e^{-\eta|y|})\Big) \\
E_y(t,x;y) &= \frac{C_E}{\sqrt{4b_- t}}\bar{u}_x(x)e^{-\frac{y^2}{4b_- t}}\Big(1+\mathbf{O}(e^{-\eta|y|})\Big).
\end{aligned}$$

The estimates of Theorem 1.1 are quite similar to those of Lemma 2.1 in [38], in which the authors are considering a multidimensional generalization of (1.1)–(1.3). Similar estimates are also obtained in [5] during the course of the analysis, though in that case the authors work with the integrated equation and consequently the term $E(t,x;y)$, which does not decay as t grows, does not appear. In both of these cases, the analysis is restricted to the case of (1.2), with $F(u) = (1/8)u^4 - (1/4)u^2$, $M(u) \equiv 1$, and $\kappa = 1$, and the wave (1.4).

The estimates of Theorem 1.1 are sufficient for establishing the following theorem on the perturbations $v(t,x)$.

Theorem 1.2. *Suppose $\bar{u}(x)$ is a standing wave solution to (1.1), and suppose (H0)–(H1) hold, as well as stability criterion $(\mathcal{D})$. Then for Hölder continuous initial perturbations $(u(0,x)-\bar{u}(x)) \in C^{0+\gamma}(\mathbb{R})$, $\gamma > 0$, with*

$$\begin{aligned}\int_{-\infty}^{+\infty} (u(0,x)-\bar{u}(x))dx = 0 \\ |u(0,x)-\bar{u}(x)| \le E_0(1+|x|)^{-3/2},\end{aligned} \tag{1.24}$$

for some E_0 sufficiently small, and for $\delta(t)$ as defined in (1.14), there holds

$$\begin{aligned}|u(t,x+\delta(t))-\bar{u}(x)| &\le CE_0(1+|x|+\sqrt{t})^{-3/2} \\ |\partial_x(u(t,x+\delta(t))-\bar{u}(x))| &\le CE_0\Big[t^{-1/4}(1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2}+(1+t)^{-3/4}e^{-\eta|x|}\Big] \\ |\delta(t)| &\le CE_0(1+t)^{-1/4} \\ |\dot{\delta}(t)| &\le CE_0(1+t)^{-5/4}.\end{aligned} \tag{1.25}$$

In the estimates of Theorem 1.2, we have included $v_x(t,x)$ in order to take advantage of the particular form of the nonlinearity Q, in which the term $\mathbf{O}(|vv_x|)$ is better than $\mathbf{O}(v^2)$, which arises in a similar analysis in the case of regularized conservation laws.

Remark 1.2. *We note that in the current setting the assumption of zero mass initial data can be taken without loss of generality. As in the case of Lax waves arising in regularized conservation laws, the asymptotic shift of the wave can be located here entirely from the initial perturbation. In this way, the shift to a zero mass perturbation is simply a change of perspective, in which we consider the stability of the asymptotically selected wave rather than of the original wave. The validity of this a priori selection of the asymptotic translate is justified precisely by our decay estimate on $\delta(t)$, from which we verify that perturbations of the zero-mass wave do indeed approach the zero-mass wave. Of course, this means the stability we conclude is orbital.*

In light of Remark 1.2, we can state the following theorem regarding initial perturbations with non-zero mass.

Corollary 1.1. *Suppose $\bar{u}(x)$ is a standing wave solution to (1.1), and suppose (H0)–(H1) hold, as well as stability criterion $(\mathcal{D})$. Then for Hölder continuous initial perturbations $(u(0,x)-\bar{u}(x)) \in C^{0+\gamma}(\mathbb{R})$, $\gamma > 0$, with*

$$|u(0,x)-\bar{u}(x)| \le E_0(1+|x|)^{-3/2}, \tag{1.26}$$

for some E_0 sufficiently small, there holds

$$|u(t,x+x_0)-\bar{u}(x)| \le CE_0\Big[(1+|x|+\sqrt{t})^{-3/2}+(1+t)^{-1/4}e^{-\eta|x|}\Big], \tag{1.27}$$

where

$$x_0 = -\frac{1}{u_+-u_-}\int_{-\infty}^{+\infty} u(0,x)-\bar{u}(x)dx.$$

We note specifically the relationship between Corollary 1.1 and the result of [5]. In [5], the authors employ the renormalization group method to establish the following result on (1.1)–(1.3):

For any $p > 2$, there exists $\delta > 0$ such that for $\|v_0\|_{X_p} \le \delta$,

$$X_p := \Big\{v_0 \in C^0(\mathbb{R}) : \|v_0\|_X := \sup_{x\in\mathbb{R}}(1+|x|^{p+1})|v_0(x)| < \infty\Big\},$$

the Cauchy problem (1.1)–(1.3) with

$$u(1,x) = \bar{u}(x) + v_0(x),$$

where $\bar{u}(x)$ is as in (1.4), has a unique smooth solution $u(t,x)$, $t \in (1,\infty)$ satisfying

$$\|u(t,\cdot)-\bar{u}(\cdot - x_0)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } t \to \infty,$$

for some $x_0 \in \mathbb{R}$ depending only on v_0.

Moreover, for some constants A and B depending only on v_0, there holds

$$u(t, x + x_0) = \bar{u}(x) + \frac{1}{\sqrt{4\pi t}} \frac{d}{dx}\Big(e^{-\frac{x^2}{4t}}(A\bar{u}(x) + B)\Big) + \mathbf{o}(t^{-1}).$$

The difference between Corollary 1.1 and the result of [5] (for the equation (1.1)–(1.3), with wave (1.4)) lies predominately in the difference in assumptions on the initial perturbation. In taking more rapidly decaying initial perturbations, we could increase the decay rate of our perturbation in time. More precisely, in the estimate of Theorem 1.2, for $1 < r < 2$ initial perturbations of size $(1 + |x|)^{-r}$ give temporal decay on $u(t, x + \delta(t)) - \bar{u}(x)$ at rate $(1 + t)^{-r/2}$ and $\delta(t)$ decay at rate $(1 + t)^{(1-r)/2}$. For $r > 2$, $\delta(t)$ decays at the maximal rate $(1 + t)^{-1/2}$, and the estimate of Corollary 1.1 becomes

$$|u(t, x + x_0) - \bar{u}(x)| \le CE_0\Big[(1 + t)^{-1/2}e^{-\eta|x|} + (1 + t)^{-1}\Big], \tag{1.28}$$

which almost recovers the detail of [5] Theorem 1.1. Finally, we note that the second order term in [5] Theorem 1.1 is the analogue here of the diffusion waves of Liu [41], and could be recovered in the current setting by an analysis similar to that of [29, 30, 49].

Outline of the paper. In Section 2, we carry out a general analysis of the Evans function for the eigenvalue problem (1.17), providing full details in the case (1.1)–(1.3) with wave (1.4). In addition, we use this analysis to develop estimates on the Laplace-transformed Green's function $G_\lambda(x, y)$. In Section 3, we prove Theorem 1.1, while in Section 4 we prove Theorem 1.2. Finally, some representative estimates on the integrations arising from our integral equations are provided in Section 5.

2 Analysis of the Evans Function

In this section we analyze the Evans function, as defined in (1.20), for our linear eigenvalue problem (1.17). We begin with a remark on the structure of standing waves $\bar{u}(x)$ arising in the context of (1.1). Upon substitution of such a wave into (1.1), we have the ODE

$$(c(\bar{u})\bar{u}_{xxx})_x - (b(\bar{u})\bar{u}_x)_x = 0. \tag{2.1}$$

Observing that $\bar{u}_x(x) \to 0$ as $x \to \pm\infty$, we can integrate (2.1) to obtain the third order equation

$$c(\bar{u})\bar{u}_{xxx} - b(\bar{u})\bar{u}_x = 0. \tag{2.2}$$

Asymptotically, solutions to (2.2) behave like solutions to the constant coefficient equations

$$c_\pm\bar{u}_{xxx} - b_\pm\bar{u}_x = 0, \tag{2.3}$$

whose solutions grow and decay with rates $\pm\sqrt{b_\pm/c_\pm}$ and 0. Since the solutions with rate 0 correspond with constant solutions to (2.1), we see that any solution that does not blow up must approach its endstates at exponential rate.

We now begin our analysis of the Evans function by writing the eigenvalue problem (1.17) as a first order system

$$W' = \mathbb{A}(x; \lambda)W, \tag{2.4}$$

where

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda}{c(x)} - \frac{a'(x)}{c(x)} & -\frac{a(x)}{c(x)} + \frac{b'(x)}{c(x)} & \frac{b(x)}{c(x)} & -\frac{c'(x)}{c(x)} \end{pmatrix}.$$

Under assumptions (H0)–(H1), $\mathbb{A}(x; \lambda)$ has the asymptotic behavior

$$\mathbb{A}(x;\lambda) = \begin{cases} \mathbb{A}_-(\lambda) + E(x;\lambda), & x < 0 \\ \mathbb{A}_+(\lambda) + E(x;\lambda), & x > 0, \end{cases}$$

where

$$\lim_{x\to\pm\infty}\mathbb{A}(x;\lambda)=\mathbb{A}_\pm(\lambda)=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\frac{\lambda}{c_\pm}&0&\frac{b_\pm}{c_\pm}&0\end{pmatrix},\tag{2.5}$$

and for $|\lambda|$ bounded $E(x;\lambda)=\mathbf{O}(e^{-\alpha|x|})$. The eigenvalues of $\mathbb{A}_\pm(\lambda)$ satisfy the relation,

$$\mu^2=\frac{-b_\pm\pm\sqrt{b_\pm^2-4\lambda c_\pm}}{-2c_\pm}.$$

Ordering these so that $j<k\Rightarrow \mathrm{Re}\mu_j(\lambda)\le\mathrm{Re}\mu_k(\lambda)$, we have

$$\mu_1^\pm(\lambda)=-\sqrt{\frac{b_\pm}{c_\pm}}+\mathbf{O}(|\lambda|);\quad \mu_2^\pm(\lambda)=-\frac{1}{\sqrt{b_\pm}}\sqrt{\lambda}+\mathbf{O}(|\lambda|^{3/2})$$
$$\mu_3^\pm(\lambda)=+\frac{1}{\sqrt{b_\pm}}\sqrt{\lambda}+\mathbf{O}(|\lambda|^{3/2});\quad \mu_4^\pm(\lambda)=+\sqrt{\frac{b_\pm}{c_\pm}}+\mathbf{O}(|\lambda|),$$

with associated eigenvectors $V_k^\pm=(1,\mu_k^\pm,(\mu_k^\pm)^2,(\mu_k^\pm)^3)^{\mathrm{tr}}$. A straightforward way in which to understand these expressions is through the relationship

$$\sqrt{b_\pm-\sqrt{b_\pm^2-4\lambda c_\pm}}=\frac{\sqrt{4c_\pm}\sqrt{\lambda}}{\sqrt{b_\pm+\sqrt{b_\pm^2-4\lambda c_\pm}}},$$

where the right-hand side is now in the form of $\sqrt{\lambda}$ multiplied by a function that for $|\lambda|$ sufficiently small is analytic in λ. We note that for $|\lambda|$ sufficiently small, and away from essential spectrum (the negative real axis), these eigenvalues remain distinct. Also, we clearly have $\mu_1^\pm=-\mu_4^\pm$ and $\mu_2^\pm=-\mu_3^\pm$.

Lemma 2.1. *For the eigenvalue problem (1.17), assume $a\in C^1(\mathbb{R})$, $b,c\in C^2(\mathbb{R})$, with $b_\pm>0$ and $c_\pm>0$, and additionally that (1.9) holds. Then for some $\bar\alpha>0$ and $k=0,1,2,3$, we have the following estimates on a choice of linearly independent solutions of (1.17). For $|\lambda|\le r$, some $r>0$ sufficiently small, there holds:*

(i) For $x\le 0$

$$\partial_x^k\phi_1^-(x;\lambda)=e^{\mu_3^-(\lambda)x}(\mu_3^-(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\phi_2^-(x;\lambda)=e^{\mu_4^-(\lambda)x}(\mu_4^-(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\psi_1^-(x;\lambda)=e^{\mu_1^-(\lambda)x}(\mu_1^-(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\psi_2^-(x;\lambda)=\frac{1}{\mu_2^-(\lambda)}\Big(\mu_2^-(\lambda)^k e^{\mu_2^-(\lambda)x}-\mu_3^-(\lambda)^k e^{\mu_3^-(\lambda)x}\Big)+\mathbf{O}(e^{-\bar\alpha|x|}).$$

(ii) For $x\ge 0$

$$\partial_x^k\phi_1^+(x;\lambda)=e^{\mu_1^+(\lambda)x}(\mu_1^+(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\phi_2^+(x;\lambda)=e^{\mu_2^+(\lambda)x}(\mu_2^+(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\psi_1^+(x;\lambda)=\frac{1}{\mu_3^+(\lambda)}\Big(\mu_3^+(\lambda)^k e^{\mu_3^+(\lambda)x}-\mu_2^+(\lambda)^k e^{\mu_2^+(\lambda)x}\Big)+\mathbf{O}(e^{-\bar\alpha|x|})$$
$$\partial_x^k\psi_2^+(x;\lambda)=e^{\mu_4^+(\lambda)x}(\mu_4^+(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|})).$$

Proof. Lemma 2.1 can be proved by standard methods such as those of [52], pp. 779–781. We remark here only on the choice of our slow growth solutions $\psi_2^-(x;\lambda)$ and $\psi_1^+(x;\lambda)$. We could alternatively make the choice

$$\partial_x^k\tilde\psi_2^-(x;\lambda)=e^{\mu_2^-(\lambda)x}(\mu_2^-(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|}))$$
$$\partial_x^k\tilde\psi_1^+(x;\lambda)=e^{\mu_3^+(\lambda)x}(\mu_3^+(\lambda)^k+\mathbf{O}(e^{-\bar\alpha|x|})).$$

The difficulty with this choice is that these growth solutions coalesce with the slow decay solutions ϕ_1^- and ϕ_2^+ as $\lambda \to 0$. (See [23], in which this latter convention is taken.) Following the analysis of [5], we take the linear combination of solutions stated in order to have a set of solutions of (1.17) that remains linearly independent as $\lambda \to 0$.

In order to see that $\psi_2^-(x;\lambda)$ and $\psi_1^+(x;\lambda)$ are valid choices, we note that their derivation depends precisely on this coalescence with the slow decay solutions ϕ_1^- and ϕ_2^+ as $\lambda \to 0$. In the case, for example, of $\psi_2^-(x;\lambda)$, we proceed by letting $\phi_0^-(x)$ be any $\lambda = 0$ solution of (1.17) that neither grows nor decays as $x \to -\infty$. (The existence of such a solution follows immmediately from the slow decay solution ϕ_1^-.) We search for solutions of the form $\psi_2^-(x;\lambda) = \phi_0^-(x)w(x;\lambda)$, for which direct substitution reveals

$$\begin{aligned}-c(x)\phi_0^{-\prime\prime\prime}w' - (c(x)(3\phi_0^{-\prime\prime}w' + 3\phi_0^{-\prime}w'' + \phi_0^- w'''))' \\ + b(x)\phi_0^{-\prime}w' + (b(x)\phi_0^- w')' - a(x)\phi_0^- w' = \lambda\phi_0^- w,\end{aligned} \tag{2.6}$$

and we make the critical observation that the only coefficient of undifferentiated w is $\lambda\phi_0^-$. In this way, the first order system associated with (2.6) has the form ($W_k = \partial^k w$)

$$W' = \mathbb{A}_-(\lambda)W + \mathcal{E}(x;\lambda)W,$$

where $\mathbb{A}_-(\lambda)$ is precisely as in (2.5), while $\mathcal{E}(x;\lambda)$ has the particular form

$$\mathcal{E}(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda\mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) & \mathbf{O}(e^{-\alpha|x|}) \end{pmatrix}, \tag{2.7}$$

for some $\alpha > 0$. Searching for solutions $W = e^{\mu_3^- x}Z$, we have the equation

$$Z' + \mu_3^- IZ = \mathbb{A}_-(\lambda)Z + \mathcal{E}Z,$$

which can be integrated as

$$\begin{aligned}Z(x) = V_3^-(\lambda) + \int_{-\infty}^{x} e^{(\mathbb{A}-\mu_3^-)(x-y)}P\mathcal{E}(y;\lambda)Z(y;\lambda)dy \\ + \int_{x}^{-M} e^{(\mathbb{A}-\mu_3^-)(x-y)}Q\mathcal{E}(y;\lambda)Z(y;\lambda)dy,\end{aligned}$$

where similarly as in [52], P is a projection operator projecting onto the eigenspace associated with μ_1^-, μ_2^-, and μ_3^-, while Q is the projection operator projecting onto the eigenspace associated with μ_4^-. In this way, the first integral decays at exponential rate due to the exponential decay of $\mathcal{E}$ and the second integral decays at exponential rate due to the exponential decay of the combination $e^{(\mu_4^- - \mu_3^-)(x-y)}\mathcal{E}(y;\lambda)$. Consequently, a standard contraction mapping closes for all $|x|$ sufficiently large. This in turn justifies a direct iteration, in which $V_3(\lambda)$ is taken as a first approximation for $Z(x)$. Upon substitution for a second iterate, we observe

$$\mathcal{E}(y;\lambda)V_3^+(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda\mathbf{O}(e^{-\alpha|y|}) & \mathbf{O}(e^{-\alpha|y|}) & \mathbf{O}(e^{-\alpha|y|}) & \mathbf{O}(e^{-\alpha|y|}) \end{pmatrix} \begin{pmatrix} 1 \\ \mu_3^-(\lambda) \\ \mu_3^-(\lambda)^2 \\ \mu_3^-(\lambda)^3 \end{pmatrix} = \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|y|}).$$

Continuing the iteration, we conclude

$$\phi_1^-(x;\lambda) = e^{\mu_3^-(\lambda)x}\phi_0^-(x)(1 + \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|})).$$

Similarly, a slow growth solution can be constructed from $\phi_0^-(x)$, as

$$\tilde{\psi}_2^-(x;\lambda) = e^{\mu_2^-(\lambda)x}\phi_0^-(x)(1 + \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|})).$$

Any linear combination of $\phi_1^-(x;\lambda)$ and $\tilde\psi_2^-(x;\lambda)$ is also a solution of (1.17), and so we are justified in defining

$$\psi_2^-(x;\lambda) := \frac{1}{\mu_2^-}\Big(\tilde\psi_2^-(x;\lambda) - \phi_1^-(x;\lambda)\Big).$$

We have, then,

$$\begin{aligned}\frac{1}{\mu_2^-}&\Big(\tilde\psi_2^-(x;\lambda) - \phi_1^-(x;\lambda)\Big)\\ &= \frac{1}{\mu_2^-}\Big(e^{\mu_2^-(\lambda)x}\phi_0^-(x)(1+\mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|})) - e^{\mu_3^-(\lambda)x}\phi_0^-(x)(1+\mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|})).\Big)\\ &= \frac{\phi_0^-(x)}{\mu_2^-}\Big(e^{\mu_2^-(\lambda)x} - e^{\mu_3^-(\lambda)x} + \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|})\Big).\end{aligned}$$

The estimate we state is obtained by scaling $\phi_0^-(x)$ as

$$\phi_0^-(x) = 1 + \mathbf{O}(e^{-\alpha|x|}).$$

The case of $\psi_1^+(x;\lambda)$ can be analyzed similarly. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, and for $\phi_k^\pm$, $\psi_k^\pm$ as in Lemma 2.1, with $k=0,1$ and for $D(\lambda)$ as in (1.20) we have the following estimates. For some $\tilde\alpha>0$,*

(i) For $x\le 0$

$$\begin{aligned}\partial_x^k\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_2^-(\lambda)x}\Big((-\mu_2^-)^k(\mu_1^- - \mu_4^-)(\mu_1^- - \mu_3^-)(\mu_4^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_1^-(\lambda)x}\Big((-\mu_1^-)^k(\mu_2^- - \mu_4^-)(\frac{\mu_2^- - \mu_3^-}{\mu_2^-})(\mu_4^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_1^-,\psi_1^-,\psi_2^-)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_4^-(\lambda)x}\Big((-\mu_4^-)(\mu_2^- - \mu_1^-)(\frac{\mu_2^- - \mu_3^-}{\mu_2^-})(\mu_1^- - \mu_3^-) + \mathbf{O}(e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_2^-,\psi_1^-,\psi_2^-)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}\Big(\frac{(\mu_2^- - \mu_1^-)(\mu_2^- - \mu_4^-)(\mu_1^- - \mu_4^-)}{\mu_2^-}\Big)\Big(e^{-\mu_3^-(\lambda)x}(-\mu_3^-)^k - e^{-\mu_2^-(\lambda)x}(-\mu_2^-)^k\\ &\quad + \mathbf{O}(|\lambda^{1/2}|e^{-\tilde\alpha|x|})\Big).\end{aligned}$$

(ii) For $x\ge 0$

$$\begin{aligned}\partial_x^k\frac{W(\phi_1^+,\phi_2^+,\psi_1^+)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_4^+(\lambda)x}\Big((-\mu_4^+)^k(\frac{\mu_3^+ - \mu_2^+}{\mu_3^+})(\mu_3^+ - \mu_1^+)(\mu_2^+ - \mu_1^+) + \mathbf{O}(e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_1^+,\phi_2^+,\psi_2^+)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_3^+(\lambda)x}\Big((-\mu_3^+)^k(\mu_4^+ - \mu_2^+)(\mu_4^+ - \mu_1^+)(\mu_2^+ - \mu_1^+) + \mathbf{O}(e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_1^+,\psi_1^+,\psi_2^+)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}\Big(\frac{(\mu_4^+ - \mu_3^+)(\mu_4^+ - \mu_1^+)(\mu_3^+ - \mu_1^+)}{\mu_3^+}\Big)\Big(e^{-\mu_2^+(\lambda)x}(-\mu_2^+)^k - e^{-\mu_3^+(\lambda)x}(-\mu_3^+)^k\\ &\quad + \mathbf{O}(|\lambda^{1/2}|e^{-\tilde\alpha|x|})\Big)\\ \partial_x^k\frac{W(\phi_2^+,\psi_1^+,\psi_2^+)}{c(x)W_\lambda(x)} &= \frac{1}{c(0)D(\lambda)}e^{-\mu_1^+(\lambda)x}\Big((-\mu_1^+)^k(\mu_4^+ - \mu_3^+)(\mu_4^+ - \mu_2^+)(\frac{\mu_3^+ - \mu_2^+}{\mu_3^+}) + \mathbf{O}(e^{-\tilde\alpha|x|})).\end{aligned}$$

Proof. The estimates of Lemma 2.2 are each carried out in a similar straightforward manner, so we proceed only in the case of

$$\partial_x\frac{W(\phi_2^-,\psi_1^-,\psi_2^-)}{c(x)W_\lambda(x)}.$$

Computing directly, and observing $(c(x)W_\lambda(x))' = 0$, we have

$$\partial_x \frac{W(\phi_2^-, \psi_1^-, \psi_2^-)}{c(x)W_\lambda(x)} = \frac{\partial_x W(\phi_2^-, \psi_1^-, \psi_2^-)}{c(x)W_\lambda(x)}.$$

For the denominator, we note Abel's relation,

$$\begin{aligned} W_\lambda'(x) &= W_\lambda(0) e^{\int_0^x \mathrm{tr}\mathbb{A}(y;\lambda)dy} = D(\lambda) e^{-\int_0^x \frac{c'(y)}{c(y)}} dy \\ &= D(\lambda)\frac{c(0)}{c(x)}, \end{aligned}$$

or

$$c(x)W_\lambda(x) = c(0)D(\lambda). \tag{2.8}$$

For the numerator, we compute

$$\begin{aligned} \partial_x W(\phi_2^-, \psi_1^-, \psi_2^-) &= \det \begin{pmatrix} \phi_2^- & \psi_1^- & \psi_2^- \\ \phi_2^{-\prime} & \psi_1^{-\prime} & \psi_2^{-\prime} \\ \phi_2^{-\prime\prime\prime} & \psi_1^{-\prime\prime\prime} & \psi_2^{-\prime\prime\prime} \end{pmatrix} \\ &= \frac{1}{\mu_2^-} \det \begin{pmatrix} 1 & 1 & 1 \\ \mu_4^- & \mu_1^- & \mu_2^- \\ (\mu_4^-)^3 & (\mu_1^-)^3 & (\mu_2^-)^3 \end{pmatrix} e^{(\mu_1^- + \mu_2^- + \mu_4^-)x} \\ &- \frac{1}{\mu_2^-} \det \begin{pmatrix} 1 & 1 & 1 \\ \mu_4^- & \mu_1^- & \mu_3^- \\ (\mu_4^-)^3 & (\mu_1^-)^3 & (\mu_3^-)^3 \end{pmatrix} e^{(\mu_1^- + \mu_3^- + \mu_4^-)x} + \mathbf{O}(e^{-\tilde{\alpha}|x|}). \end{aligned} \tag{2.9}$$

We mention particularly that in the exponentially decaying terms of this last calculation, we have observed the estimate

$$\begin{aligned} \Big| \frac{1}{\mu_2^-(\lambda)} (e^{\mu_2^-(\lambda)x} - e^{\mu_3^-(\lambda)x}) \mathbf{O}(e^{-\bar{\alpha}|x|}) \Big| &\le C_1 \frac{1}{\mu_2^-(\lambda)} |\sqrt{\lambda} x| \mathbf{O}(e^{-\bar{\alpha}|x|}) \\ &= \mathbf{O}(e^{-\tilde{\alpha}|x|}), \end{aligned}$$

for some $0 < \tilde{\alpha} < \bar{\alpha}$. Computing the determinants, we find

$$\begin{aligned} \partial_x W(\phi_2^-, \psi_1^-, \psi_2^-) &= \frac{1}{\mu_2^-} \Big[(-\mu_3^-)(\mu_2^- - \mu_1^-)(\mu_2^- - \mu_4^-)(\mu_3^- - \mu_4^-) e^{(\mu_1^- + \mu_2^- + \mu_4^-)x} \\ &- (-\mu_2^-)(\mu_3^- - \mu_1^-)(\mu_3^- - \mu_4^-)(\mu_1^- - \mu_4^-) \Big] e^{(\mu_1^- + \mu_3^- + \mu_4^-)x} + \mathbf{O}(e^{-\tilde{\alpha}|x|}), \end{aligned}$$

for which we observe that with $\mu_1^- = -\mu_4^-$ and $\mu_2^- = -\mu_3^-$, we have

$$(\mu_2^- - \mu_1^-)(\mu_2^- - \mu_4^-)(\mu_3^- - \mu_4^-) = (\mu_3^- - \mu_1^-)(\mu_3^- - \mu_4^-)(\mu_1^- - \mu_4^-),$$

and

$$\begin{aligned} e^{(\mu_1^- + \mu_2^- + \mu_4^-)x} &= e^{-\mu_3^- x} \\ e^{(\mu_1^- + \mu_3^- + \mu_4^-)x} &= e^{-\mu_2^- x}. \end{aligned}$$

Combining these last observations with (2.8), we conclude our argument for the case of interest. The remaining estimates follow similarly. □

In addition to Lemmas 2.1 and 2.2, we have the following estimates on the $N_k^\pm$.

Lemma 2.3. *Under the assumptions of Theorem 1.1, we have the following estimates on the $N_k^\pm$ of (1.19) (with similar estimates for $x \ge 0$).*

For $x \le 0$, $k = 0, 1$, and for $|\lambda| \le r$, some r sufficiently small,

$$\begin{aligned} \partial_x^k N_1^-(x;\lambda) &= \mathbf{O}(|\lambda^{-1+\frac{k}{2}}|) e^{-\mu_2^-(\lambda)x} \\ \partial_x^k N_2^-(x;\lambda) &= \mathbf{O}(|\lambda^{-\frac{1}{2}+\frac{k}{2}}|) e^{-\mu_2^-(\lambda)x} \\ \partial_x^k N_1^+(x;\lambda) &= \mathbf{O}(|\lambda^{-\frac{1}{2}}|) \Big(e^{\mu_2^-(\lambda)x} \mu_2^-(\lambda)^k - e^{\mu_3^-(\lambda)x} \mu_3^-(\lambda)^k \Big) + \mathbf{O}(|\lambda^{-\frac{1}{2}+\frac{k}{2}}|) e^{\mu_3^-(\lambda)x} \\ \partial_x^k N_2^+(x;\lambda) &= \mathbf{O}(|\lambda^{-1+\frac{k}{2}}|) e^{-\mu_2^-(\lambda)x} + \mathbf{O}(1) e^{-\mu_4^-(\lambda)x}. \end{aligned}$$

Proof. We note at the outset that for $x \le 0$, we will take the expansions

$$\phi_k^+(x;\lambda) = A_k^+(\lambda)\phi_1^-(x;\lambda) + B_k^+(\lambda)\phi_2^-(x;\lambda) + C_k^+(\lambda)\psi_1^-(x;\lambda) + D_k^+(\lambda)\psi_2^-(x;\lambda). \tag{2.10}$$

Without loss of generality, we take the convention

$$\phi_1^+(x;0) = \bar{u}_x(x) = \phi_2^-(x;0), \tag{2.11}$$

according to which there must hold

$$\begin{aligned} &A_1^+(\lambda) = \mathbf{O}(|\lambda^{1/2}|); \quad B_1^+(\lambda) = 1 + \mathbf{O}(|\lambda^{1/2}|); \quad C_1^+(\lambda) = \mathbf{O}(|\lambda^{1/2}|); \quad D_1^+(\lambda) = \mathbf{O}(|\lambda^{1/2}|); \\ &A_2^+(\lambda) = \mathbf{O}(1); \quad B_2^+(\lambda) = \mathbf{O}(1); \quad C_2^+(\lambda) = \mathbf{O}(1); \quad D_2^+(\lambda) = \mathbf{O}(1). \end{aligned} \tag{2.12}$$

As each estimate of Lemma 2.3 is proven similarly, we provide details only in the case of $\partial_x N_1^-$. We have, similarly as in the proof of Lemma 2.2

$$\partial_x N_1^-(x;\lambda) = -\frac{\partial_x W(\phi_1^-, \phi_2^-, \phi_2^+)}{c(x)W_\lambda(x)},$$

for which we compute

$$\partial_x W(\phi_1^-, \phi_2^-, \phi_2^+) = \det \begin{pmatrix} \phi_1^- & \phi_2^- & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime\prime} & \phi_2^{-\prime\prime\prime} & \phi_2^{+\prime\prime\prime} \end{pmatrix}.$$

In order to derive expressions for the terms $\phi_k^{\pm\prime\prime\prime}$, we consider the eigenvalue problem (1.17). In general, we have the relation

$$-(c(x)\phi_k^{\pm\prime\prime\prime})' + (b(x)\phi_k^{\pm\prime})' - (a(x)\phi_k^{\pm})' = \lambda\phi_k^{\pm}. \tag{2.13}$$

For ϕ_1^- and ϕ_2^-, upon integration from $-\infty$ up to x, and keeping in mind that $a(x)$ decays at exponential rate as $x \to \pm\infty$, we have

$$-c(x)\phi_k^{-\prime\prime\prime} + b(x)\phi_k^{-\prime} - a(x)\phi_k^- = \lambda \int_{-\infty}^x \phi_k^-(y;\lambda)dy.$$

The most straightforward case is $\phi_2^-(x;\lambda)$, which for $x < 0$ decays at exponential rate even for $\lambda = 0$. In this case, we define

$$\mathcal{W}_2^-(x;\lambda) = \int_{-\infty}^x \phi_2^-(y;\lambda)dy = \mathbf{O}(1)e^{\mu_4^-(\lambda)x}. \tag{2.14}$$

Proceeding similarly for $\phi_1^-(x;\lambda)$, which fails to decay for $\lambda = 0$, we define

$$\begin{aligned} \mathcal{W}_1^-(x;\lambda) &= \sqrt{\lambda}\int_{-\infty}^x \phi_1^-(y;\lambda)dy = \sqrt{\lambda}\int_{-\infty}^x e^{\mu_3^-(\lambda)y}(1 + \mathbf{O}(e^{-\bar{\alpha}|y|}))dy \\ &= \sqrt{\lambda}\frac{1}{\mu_3^-(\lambda)}e^{\mu_3^-(\lambda)x} + \sqrt{\lambda}\mathbf{O}(e^{-\alpha|x|}), \end{aligned}$$

so that

$$-c(x)\phi_1^{-\prime\prime\prime} + b(x)\phi_1^{-\prime} - a(x)\phi_1^- = \sqrt{\lambda}\mathcal{W}_1^-(x;\lambda),$$

with

$$\mathcal{W}_1^-(x;\lambda) = e^{\mu_3^-(\lambda)x}(\frac{\sqrt{\lambda}}{\mu_3^-(\lambda)} + \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|}).$$

Finally, for $\phi_2^+(x;\lambda)$ in the case $x < 0$, we must expand in terms of the linearly independent solutions ϕ_k^- and ψ_k^-,

$$\phi_2^+(x;\lambda) = A_2^+(\lambda)\phi_1^-(x;\lambda) + B_2^+(\lambda)\phi_2^-(x;\lambda) + C_2^+(\lambda)\psi_1^-(x;\lambda) + D_2^+(\lambda)\psi_2^-(x,\lambda).$$

In this case, we integrate (2.13) for $y \in [x, +\infty)$ to obtain

$$c(x)\phi_2^{+\prime\prime\prime}(x;\lambda) - b(x)\phi_2^{+\prime}(x;\lambda) + a(x)\phi_2^+(x;\lambda) = \lambda \int_x^{+\infty} \phi_2^+(y;\lambda)dy =: \sqrt{\lambda}\mathcal{W}_2^+(x;\lambda),$$

where we divide this final integration up as

$$\int_x^{+\infty} \phi_2^+(y;\lambda)dy = \int_x^{-M} \phi_2^+(y;\lambda)dy + \int_{-M}^{+\infty} \phi_2^+(y;\lambda)dy,$$

where $M > 0$ is chosen sufficiently large so that the estimates of Lemma 2.1 hold for $x \le -M$. For the integration over $y > -M$, we have

$$\int_{-M}^{+\infty} \phi_2^+(y;\lambda)dy = \int_{-M}^{0} \mathbf{O}(1)dy + \int_0^{+\infty} e^{\mu_2^+(\lambda)y}(1 + \mathbf{O}(e^{-\bar{\alpha}|y|}))dy = \mathbf{O}(|\lambda^{-1/2}|).$$

On the other hand, for integration over $y < -M$, we have

$$\begin{aligned}\int_x^{-M} \phi_2^+(y;\lambda)dy = A_2^+(\lambda)\int_x^{-M} \phi_1^-(y;\lambda)dy + B_2^+(\lambda)\int_x^{-M} \phi_2^-(y;\lambda)dy \\ + C_2^+(\lambda)\int_x^{-M} \psi_1^-(y;\lambda)dy + D_2^+(\lambda)\int_x^{-M} \psi_2^-(y,\lambda)dy.\end{aligned}$$

Combining these last observations, we conclude

$$\begin{aligned}\mathcal{W}_2^+(x;\lambda) = \mathbf{O}(1) + \sqrt{\lambda}\Big[A_2^+(\lambda)\int_x^{-M} \phi_1^-(y;\lambda)dy + B_2^+(\lambda)\int_x^{-M} \phi_2^-(y;\lambda)dy \\ + C_2^+(\lambda)\int_x^{-M} \psi_1^-(y;\lambda)dy + D_2^+(\lambda)\int_x^{-M} \psi_2^-(y,\lambda)dy\Big] = \mathbf{O}(1)e^{\mu_1^-(\lambda)x}\end{aligned}$$

Omitting coefficient dependence on y for brevity of notation, we have

$$\begin{aligned}&\det\begin{pmatrix}\phi_1^- & \phi_2^- & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_2^{+\prime} \\ \phi_1^{-\prime\prime\prime} & \phi_2^{-\prime\prime\prime} & \phi_2^{+\prime\prime\prime}\end{pmatrix} \\ &= \det\begin{pmatrix}\phi_1^- & \phi_2^- & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_2^{+\prime} \\ \frac{b}{c}\phi_1^{-\prime} - \frac{a}{c}\phi_1^- - \frac{\sqrt{\lambda}}{c}\mathcal{W}_1^- & \frac{b}{c}\phi_2^{-\prime} - \frac{a}{c}\phi_2^- - \frac{\lambda}{c}\mathcal{W}_2^- & \frac{b}{c}\phi_2^{+\prime} - \frac{a}{c}\phi_2^+ - \frac{\sqrt{\lambda}}{c}\mathcal{W}_2^+\end{pmatrix} \\ &= \det\left[\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a}{c} & \frac{b}{c} & 1\end{pmatrix}\begin{pmatrix}\phi_1^- & \phi_2^- & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_2^{+\prime} \\ -\frac{\sqrt{\lambda}}{c}\mathcal{W}_1^- & -\frac{\lambda}{c}\mathcal{W}_2^- & -\frac{\sqrt{\lambda}}{c}\mathcal{W}_2^+\end{pmatrix}\right] \\ &= \det\begin{pmatrix}\phi_1^- & \phi_2^- & \phi_2^+ \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \phi_2^{+\prime} \\ -\frac{\sqrt{\lambda}}{c}\mathcal{W}_1^- & -\frac{\lambda}{c}\mathcal{W}_2^- & -\frac{\sqrt{\lambda}}{c}\mathcal{W}_2^+\end{pmatrix}.\end{aligned} \tag{2.15}$$

We proceed by expanding this last determinant along its bottom row, giving

$$-\frac{\sqrt{\lambda}}{c(x)}\mathcal{W}_1^-(x;\lambda)W(\phi_2^-,\phi_2^+) + \frac{\lambda}{c(x)}\mathcal{W}_2^-(x;\lambda)W(\phi_1^-,\phi_2^+) + \frac{\sqrt{\lambda}}{c(x)}\mathcal{W}_2^+(x;\lambda)W(\phi_1^-,\phi_2^-). \tag{2.16}$$

For the first expression in (2.16), we observe

$$\begin{aligned}W(\phi_2^-,\phi_2^+) &= A_2^+(\lambda)W(\phi_2^-,\phi_1^-) + C_2^+(\lambda)W(\phi_2^-,\psi_1^-) + D_2^+(\lambda)W(\phi_2^-,\psi_2^-) \\ &= \mathbf{O}(1)e^{(\mu_1^- + \mu_4^-)y} = \mathbf{O}(1),\end{aligned}$$

while similarly

$$W(\phi_1^-, \phi_2^+) = \mathbf{O}(1)e^{(\mu_1^- + \mu_3^-)y}.$$

In this way, the terms in (2.16) give respectively

$$\mathbf{O}(|\lambda^{1/2}|)e^{(\mu_1^- + \mu_3^- + \mu_4^-)y} + \mathbf{O}(|\lambda|)e^{(\mu_1^- + \mu_3^- + \mu_4^-)y} + \mathbf{O}(|\lambda^{1/2}|)e^{(\mu_1^- + \mu_3^- + \mu_4^-)y}.$$

We now recall the relationship $\mu_3^- + \mu_2^- = 0$, and note that according to spectral criterion $(\mathcal{D})$, there holds $D(\lambda) = c_1\lambda + \mathbf{O}(|\lambda^{3/2}|)$. Upon division, then, by $c(y)W_\lambda(y) = c(0)D(\lambda)$, we conclude the claimed estimate.

The remaining estimates of Lemma 2.3 follow similarly. □

Lemma 2.4. *Under the assumptions of Theorem 1.1, and for $C_k^+(\lambda)$ and $D_k^+(\lambda)$ as defined in (2.10), there holds*

$$C_1^+(\lambda)D_2^+(\lambda) - C_2^+(\lambda)D_1^+(\lambda) = \mathbf{O}(|\lambda|).$$

Proof. We first observe that by augmenting (2.10) with y-derivatives up to third order, we obtain the matrix equations

$$\begin{pmatrix} \phi_1^- & \phi_2^- & \psi_1^- & \psi_2^- \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \psi_1^{-\prime} & \psi_2^{-\prime} \\ \phi_1^{-\prime\prime} & \phi_2^{-\prime\prime} & \psi_1^{-\prime\prime} & \psi_2^{-\prime\prime} \\ \phi_1^{-\prime\prime\prime} & \phi_2^{-\prime\prime\prime} & \psi_1^{-\prime\prime\prime} & \psi_2^{-\prime\prime\prime} \end{pmatrix} \begin{pmatrix} A_k^+ \\ B_k^+ \\ C_k^+ \\ D_k^+ \end{pmatrix} = \begin{pmatrix} \phi_k^+ \\ \phi_k^{+\prime} \\ \phi_k^{+\prime\prime} \\ \phi_k^{+\prime\prime\prime} \end{pmatrix}. \tag{2.17}$$

Proceeding by Cramer's rule, we have, for example,

$$C_1^+(\lambda) = \frac{W(\phi_1^-, \phi_2^-, \phi_1^+, \psi_2^-)}{W(\phi_1^-, \phi_2^-, \psi_1^-, \psi_2^-)},$$

where the W notation is defined immediately following (1.19). By linear independence, the denominator of this last expression is non-zero, while for the numerator, we compute similarly as in (2.15)

$$\begin{aligned} W(\phi_1^-, \phi_2^-, \phi_1^+, \psi_2^-) &= \det \begin{pmatrix} \phi_1^- & \phi_2^- & \phi_1^+ & \psi_2^- \\ \phi_1^{-\prime} & \phi_2^{-\prime} & \psi_1^{+\prime} & \psi_2^{-\prime} \\ \phi_1^{-\prime\prime} & \phi_2^{-\prime\prime} & \psi_1^{+\prime\prime} & \psi_2^{-\prime\prime} \\ -\frac{\sqrt{\lambda}}{c(x)}\mathcal{W}_1^- & -\frac{\lambda}{c(x)}\mathcal{W}_2^+ & -\frac{\lambda}{c(x)}\mathcal{W}_1^+ & \psi_2^{-\prime\prime\prime} - \frac{b(x)}{c(x)}\psi_2^{-\prime} + \frac{a(x)}{b(x)}\psi_2^- \end{pmatrix} \\ &= \Big(\psi_2^{-\prime\prime\prime} - \frac{b(x)}{c(x)}\psi_2^{-\prime} + \frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-, \phi_2^-, \phi_1^+) + \mathbf{O}(|\lambda|), \end{aligned}$$

where ψ_2^- is treated in a distinguished manner because it cannot be integrated to an asymptotic limit. Proceeding similarly, we also have

$$\begin{aligned} W(\phi_1^-, \phi_2^-, \phi_1^-, \psi_2^-)C_2^+(\lambda) &= W(\phi_1^-, \phi_2^-, \phi_2^+, \psi_2^-) \\ &= \Big(\psi_2^{-\prime\prime\prime} - \frac{b(x)}{c(x)}\psi_2^{-\prime} + \frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-, \phi_2^-, \phi_2^+) + \mathbf{O}(|\lambda^{1/2}|) \\ W(\phi_1^-, \phi_2^-, \phi_1^-, \psi_2^-)D_1^+(\lambda) &= W(\phi_1^-, \phi_2^-, \psi_1^-, \phi_1^+) \\ &= -\Big(\psi_1^{-\prime\prime\prime} - \frac{b(x)}{c(x)}\psi_1^{-\prime} + \frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-, \phi_2^-, \phi_1^+) + \mathbf{O}(|\lambda|) \\ W(\phi_1^-, \phi_2^-, \phi_1^-, \psi_2^-)D_2^+(\lambda) &= W(\phi_1^-, \phi_2^-, \psi_1^-, \phi_2^+) \\ &= -\Big(\psi_1^{-\prime\prime\prime} - \frac{b(x)}{c(x)}\psi_1^{-\prime} + \frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-, \phi_2^-, \phi_2^+) + \mathbf{O}(|\lambda^{1/2}|). \end{aligned}$$

Combining these observations, we compute

$$\begin{aligned}
&W(\phi_1^-,\phi_2^-,\phi_1^-,\psi_2^-)\Big(C_1^+(\lambda)D_2^+(\lambda)-C_2^+(\lambda)D_1^+(\lambda)\Big)\\
&=\Big[\Big({\psi_2^-}'''-\frac{b(x)}{c(x)}{\psi_2^-}'+\frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-,\phi_2^-,\phi_1^+)+\mathbf{O}(|\lambda|)\Big]\\
&\times\Big[-\Big({\psi_1^-}'''-\frac{b(x)}{c(x)}{\psi_1^-}'+\frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-,\phi_2^-,\phi_2^+)+\mathbf{O}(|\lambda^{1/2}|)\Big]\\
&-\Big[\Big({\psi_2^-}'''-\frac{b(x)}{c(x)}{\psi_2^-}'+\frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-,\phi_2^-,\phi_2^+)+\mathbf{O}(|\lambda^{1/2}|)\Big]\\
&\times\Big[-\Big({\psi_1^-}'''-\frac{b(x)}{c(x)}{\psi_1^-}'+\frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-,\phi_2^-,\phi_1^+)+\mathbf{O}(|\lambda|)\Big]\\
&=\Big({\psi_2^-}'''-\frac{b(x)}{c(x)}{\psi_2^-}'+\frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-,\phi_2^-,\phi_1^+)\mathbf{O}(|\lambda^{1/2}|)\\
&-\Big({\psi_1^-}'''-\frac{b(x)}{c(x)}{\psi_1^-}'+\frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-,\phi_2^-,\phi_2^+)\mathbf{O}(|\lambda|)\\
&-\Big({\psi_2^-}'''-\frac{b(x)}{c(x)}{\psi_2^-}'+\frac{a(x)}{b(x)}\psi_2^-\Big)W(\phi_1^-,\phi_2^-,\phi_2^+)\mathbf{O}(|\lambda|)\\
&+\Big({\psi_1^-}'''-\frac{b(x)}{c(x)}{\psi_1^-}'+\frac{a(x)}{b(x)}\psi_1^-\Big)W(\phi_1^-,\phi_2^-,\phi_1^+)\mathbf{O}(|\lambda^{1/2}|)\\
&+\mathbf{O}(|\lambda^{3/2}|).
\end{aligned}$$

Recalling that $\phi_2^-(x;0)=\bar{u}_x(x)=\phi_1^+(x,0)$, we have that determinants involving both of these functions must vanish as $\lambda\to 0$, necessarily at rate $|\lambda^{1/2}|$ or better by analyticity in ρ. We conclude the claimed estimate. □

We next state our main theorem on behavior of the Evans function for small values of λ, which we note is simply the extension to the current setting of Proposition 2.7 of [7].

Lemma 2.5. *Under the assumptions of Theorem 1.1, and for $D(\lambda)$ as defined in (1.20), we have*

$$D(\lambda)=\gamma(u_+-u_-)\lambda+\mathbf{O}(|\lambda^{3/2}|),$$

with

$$\gamma=-\frac{1}{c(0)}\det\begin{pmatrix}\phi_1^-(0;\lambda) & \bar{u}_x(0) & \phi_2^+(0;\lambda)\\ {\phi_1^-}'(0;\lambda) & \bar{u}_{xx}(0) & {\phi_2^+}'(0;\lambda)\\ {\phi_1^-}''(0;\lambda) & \bar{u}_{xxx}(0) & {\phi_2^+}''(0;\lambda)\end{pmatrix}.$$

Proof. Observing that the solutions $\phi_k^\pm(x;\lambda)$ are analytic in the variable $\rho:=\sqrt{\lambda}$, we set

$$D_a(\rho):=D(\lambda).$$

We have, then,

$$\begin{aligned}
\partial_\rho D_a(0)&=W(\partial_\rho\phi_1^-,\phi_2^-,\phi_1^+,\phi_2^+)+W(\phi_1^-,\partial_\rho\phi_2^-,\phi_1^+,\phi_2^+)\\
&+W(\phi_1^-,\phi_2^-,\partial_\rho\phi_1^+,\phi_2^+)+W(\phi_1^-,\phi_2^-,\phi_1^+,\partial_\rho\phi_2^+)\\
&=W(\phi_1^-,\partial_\rho(\phi_2^--\phi_1^+),\bar{u}_x(0),\phi_2^+),
\end{aligned}$$

where the terms for which neither ϕ_1^+ nor ϕ_2^- are differentiated are 0 due to the choice of scaling (2.11). We observe here that the fast-decaying solutions ϕ_2^- and ϕ_1^+ have been derived analytically in $\lambda=\rho^2$, and thus

$$\begin{aligned}
\partial_\rho\phi_2^-(x;0)&=0\\
\partial_\rho\phi_1^+(x;0)&=0.
\end{aligned}\tag{2.18}$$

In this way, we see immediately that $\partial_\rho D_a(0)=0$.

Proceeding similarly for $\partial_{\rho\rho} D_a(\rho)$, we find

$$\begin{aligned}\partial_{\rho\rho} D_a(0) &= 2W(\partial_\rho \phi_1^-(0;0), \partial_\rho(\phi_2^- - \phi_1^+)(0;0), \bar{u}_x(0), \phi_2^+(0;0)) \\ &+ 2W(\phi_1^-(0;0), \partial_\rho(\phi_2^- - \phi_1^+)(0;0), \bar{u}_x(0), \partial_\rho \phi_2^+(0;0)) \\ &+ 2W(\phi_1^-(0;0), \partial_\rho \phi_2^-(0;0), \partial_\rho \phi_1^+(0;0), \phi_2^+(0;0)) \\ &+ W(\phi_1^-(0;0), \partial_{\rho\rho}(\phi_2^- - \phi_1^+)(0;0), \bar{u}_x(0), \phi_2^+(0;0)).\end{aligned}$$

Upon substitution of (2.18), all terms with only single partials are eliminated, and we have

$$\partial_{\rho\rho} D_a(0) = W(\phi_1^-(0;0), \partial_{\rho\rho}(\phi_2^- - \phi_1^+)(0;0), \bar{u}_x(0), \phi_2^+(0;0)).$$

In order to understand ρ derivatives of ϕ_2^- and ϕ_1^+, we proceed similarly as in (2.13), setting

$$-(c(x)\phi_k^{\pm\prime\prime\prime})' + (b(x)\phi_k^{\pm\prime})' - (a(x)\phi_k^{\pm})' = \rho^2 \phi_k^{\pm}, \tag{2.19}$$

where prime here denotes differentiation with respect to x. Keeping in mind that ϕ_2^- and ϕ_1^+ both decay at exponential rate even for $\lambda = 0$, we have, upon integration on $(-\infty, x]$,

$$-c(x)\phi_2^{-\prime\prime\prime} + b(x)\phi_2^{-\prime} - a(x)\phi_2^- = \rho^2 \mathcal{W}_2^-,$$

where $\mathcal{W}_2^-$ is as in (2.14), and similarly for ϕ_1^+. Upon differentiation with respect to ρ, we find

$$-c(x)(\partial_\rho \phi_2^-)''' + b(x)(\partial_\rho \phi_2^-)' - a(x)(\partial_\rho \phi_2^-) = 2\rho \mathcal{W}_2^- + \rho^2(\partial_\rho \mathcal{W}_2^-).$$

with again a similar relationship for ϕ_1^+. We next take a second ρ derivative of (2.19) to obtain, upon integration and evaluation at $\rho = 0$,

$$-c(x)(\partial_{\rho\rho} \phi_2^-)''' + b(x)(\partial_{\rho\rho} \phi_2^-)' - a(x)(\partial_{\rho\rho} \phi_2^-) = 2\mathcal{W}_2^-(x;0).$$

Recalling that

$$\mathcal{W}_2^-(x;\rho) := \int_{-\infty}^x \phi_2^-(y;\rho)dy,$$

we have

$$\mathcal{W}_2^-(x;0) = \int_{-\infty}^x \bar{u}_y(y)dy = \bar{u}(x) - u_-,$$

and similarly

$$\mathcal{W}_1^+(x;0) = \int_{-\infty}^x \bar{u}_y(y)dy = \bar{u}(x) - u_+.$$

We conclude

$$-c(x)(\partial_{\rho\rho}(\phi_2^- - \phi_1^+))''' + b(x)(\partial_{\rho\rho}(\phi_2^- - \phi_1^+))' - a(x)(\partial_{\rho\rho}(\phi_2^- - \phi_1^+)) = 2(u_+ - u_-).$$

Finally, we have

$$\begin{aligned}\partial_{\rho\rho} D_a(0) &= \det \begin{pmatrix} \phi_1^-(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)(0;0) & \bar{u}_x(0) & \phi_2^+(0;0) \\ \phi_1^{-\prime}(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)'(0;0) & \bar{u}_{xx}(0) & \phi_2^{+\prime}(0;0) \\ \phi_1^{-\prime\prime}(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)''(0;0) & \bar{u}_{xxx}(0) & \phi_2^{+\prime\prime}(0;0) \\ \phi_1^{-\prime\prime\prime}(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)'''(0;0) & \bar{u}_{xxxx}(0) & \phi_2^{+\prime\prime\prime}(0;0) \end{pmatrix} \\ &= \det \begin{pmatrix} \phi_1^-(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)(0;0) & \bar{u}_x(0) & \phi_2^+(0;0) \\ \phi_1^{-\prime}(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)'(0;0) & \bar{u}_{xx}(0) & \phi_2^{+\prime}(0;0) \\ \phi_1^{-\prime\prime}(0;0) & \partial_{\rho\rho}(\phi_2^- - \phi_1^+)''(0;0) & \bar{u}_{xxx}(0) & \phi_2^{+\prime\prime}(0;0) \\ 0 & -\frac{2}{c(0)}(u_+ - u_-) & 0 & 0 \end{pmatrix},\end{aligned}$$

from which the claimed relationship is immediate. □

We now state for completeness a lemma asserting that $(\mathcal{D})$ is known to hold in the case (1.1)–(1.3) with wave (1.4).

Lemma 2.6. *For the case (1.1)–(1.3), and for the standing front (1.4), spectral condition ($\mathcal{D}$) holds.*

Proof. In order to verify the first assertion of ($\mathcal{D}$)—that the only zero that $D(\lambda)$ has with non-negative real part is $\lambda = 0$—we first note that the integrated variable $w(x) = \int_{-\infty}^x \phi(y;\lambda)dy$ satisfies the integrated eigenvalue problem

$$\mathcal{L}w = \lambda w; \quad \mathcal{L}w := \partial_x\Big[(-\partial_{xx} + 1 + V(x))w_x\Big],$$

where $V(x) = -(3/2)\cosh^{-2}(x/2)$, and $\mathcal{L}$ is clearly a self-adjoint operator. Away from essential spectrum, the eigenvalues of $\mathcal{L}$ correspond with those of L, and so aside from the point $\lambda = 0$ (which is an eigenvalue of L but not of $\mathcal{L}$), our study reduces to that of $\mathcal{L}$, which has been considered in [5, 36, 38] (see additionally the alternative analysis of [10]). For completeness, we briefly review the observations of [5, 36, 38]. As pointed out in [36, 38], the middle operator

$$Mw := -w_{xx} + (1 + V(x))w$$

has been shown in [43] to have two isolated eigenvalues at 0 and $\frac{3}{4}$, and essential spectrum on the line $[1,\infty)$. It is an immediate consequence of the Spectral Theorem (see e.g. [37], p. 360) that M is a (non-strictly) positive operator. In the event that λ is an eigenvalue of $\mathcal{L}$ (necessarily real), we have that for some eigenfunction w associated with λ that $(Mw_x)_x = \lambda w$. Upon multiplication by $\bar{w}$ and integration over $\mathbb{R}$, we have

$$-\int_{-\infty}^{+\infty} \bar{w}_x M w_x dx = \lambda \|w\|_{L^2}^2.$$

We can conclude by the positivity of M that $\lambda \le 0$.

According to Lemma 2.5, the second assertion in stability criterion ($\mathcal{D}$) is verified if $\gamma \neq 0$. This can be checked by a careful study of solutions to the eigenvalue problem (1.17). In the case of (1.1)–(1.3), and profile (1.4), (1.17) becomes

$$-\phi_{xxxx} + (b(x)\phi)_{xx} = \lambda\phi, \tag{2.20}$$

with

$$b(x) = \frac{3}{2}\bar{u}(x)^2 - \frac{1}{2}; \quad \bar{u}(x) = \tanh\frac{x}{2}.$$

In the case of $\phi_1^-(x;\lambda)$, (2.20) can be integrated on $x \in (-\infty, x]$, and we obtain

$$-\phi_{1\ xxx}^- + (b(x)\phi_1^-)_x = \sqrt{\lambda}\mathcal{W}_1^-(x;\lambda),$$

where as in the proof of Lemma 2.3

$$\mathcal{W}_1^-(x;\lambda) = e^{\mu_3^-(\lambda)x}\Big(\frac{\sqrt{\lambda}}{\mu_3^-(\lambda)} + \mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|}\Big).$$

Setting $\lambda = 0$, we have

$$-\phi_1^-(x;0)_{xxx} + (b(x)\phi_1^-(x;0))_x = 0. \tag{2.21}$$

Proceeding similarly, we see that $\phi_2^+(x;0)$ solves precisely the same equation. We can solve (2.21) exactly by integrating once, and solving the equation

$$-\phi_{xx} + b(x)\phi = c_1.$$

Taking $c_1 = 1$, we find the three linearly independent solutions

$$\begin{aligned}
\phi_1^0(x) &= \bar{u}_x(x)\\
\phi_2^0(x) &= \bar{u}_x(x)\Big[2\sinh\frac{x}{2}\cosh^3\frac{x}{2} + 3\sinh\frac{x}{2}\cosh\frac{x}{2} + \frac{3}{2}x\Big]\\
\phi_3^0(x) &= \cosh^2\frac{x}{2}.
\end{aligned}$$

It follows that $\phi_1^-(x;0)$ and $\phi_2^+(x;0)$ must both be linear combinations of ϕ_2^0 and ϕ_3^0, where without loss of generality we can choose to remove a constant multiple of the exponentially decaying function $\bar{u}_x$ without affecting the estimates of Lemma 2.1. More precisely, we obtain a scaling that agrees with that of Lemma 2.1 by choosing

$$\phi_1^-(x;0) = -(\phi_2^0(x) + \phi_3^0(x))$$
$$\phi_2^+(x;0) = \phi_2^0(x) - \phi_3^0(x).$$

Computing directly, we find $\gamma = 2$.

This concludes the proof of Lemma 2.6. □

We conclude this section by combining the estimates of Lemmas 2.1–2.3 to obtain estimates on $G_\lambda(x,y)$ for values of $|\lambda|$ sufficiently small. We have the following lemma.

Lemma 2.7. *Under the assumptions of Theorem 1.1, and for $G_\lambda(x,y)$ as defined in (1.18), we have the following estimates. For*

$$G_\lambda(x,y) = \tilde{G}_\lambda(x,y) + E_\lambda(x,y),$$

and $|\lambda| < r$, for some suitably small constant r, there holds

(i) For $y \le x \le 0$

$$\begin{aligned}
\tilde{G}_\lambda(x,y) &= c_1 \lambda^{-1/2}(e^{\mu_2^-(\lambda)x} - e^{\mu_3^-(\lambda)x})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|)e^{\mu_3^-(\lambda)(x+y)} \\
&\quad + \mathbf{O}(1)e^{\mu_2^-(\lambda)(x-y)} \\
\partial_y \tilde{G}_\lambda(x,y) &= c_1 \frac{-\mu_2^-(\lambda)}{\lambda^{1/2}}(e^{\mu_2^-(\lambda)x} - e^{\mu_3^-(\lambda)x})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(1)e^{\mu_3^-(\lambda)(x+y)} \\
&\quad + \mathbf{O}(|\lambda^{1/2}|)e^{\mu_2^-(\lambda)(x-y)} + \mathbf{O}(|\lambda^{-1/2}|e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}) \\
\partial_x \tilde{G}_\lambda(x,y) &= \mathbf{O}(1)(e^{\mu_2^-(\lambda)x} + e^{\mu_3^-(\lambda)x})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} \\
&\quad + \mathbf{O}(e^{-\eta|x-y|}) \\
\partial_{xy} \tilde{G}_\lambda(x,y) &= \mathbf{O}(|\lambda^{1/2}|)(e^{\mu_2^-(\lambda)x} + e^{\mu_3^-(\lambda)x})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} \\
&\quad + \mathbf{O}(e^{-\eta|x-y|})
\end{aligned}$$

(ii) For $x \le y \le 0$

$$\begin{aligned}
\tilde{G}_\lambda(x,y) &= c_2 \lambda^{-1/2} e^{-\mu_2^-(\lambda)x}(e^{\mu_2^-(\lambda)y} - e^{\mu_3^-(\lambda)y}) + \mathbf{O}(|\lambda^{-1/2}|)e^{\mu_3^-(\lambda)(x+y)} \\
&\quad + \mathbf{O}(1)e^{\mu_3^-(\lambda)(x-y)} \\
\partial_y \tilde{G}_\lambda(x,y) &= \mathbf{O}(1)e^{-\mu_2^-(\lambda)x}(e^{\mu_2^-(\lambda)y} + e^{\mu_3^-(\lambda)y}) + \mathbf{O}(|\lambda^{-1/2}e^{-\eta|x|}|)e^{-\mu_2^-(\lambda)y} \\
&\quad + \mathbf{O}(e^{-\eta|x-y|}) \\
\partial_x \tilde{G}_\lambda(x,y) &= c_2 \frac{-\mu_2^-(\lambda)}{\lambda^{1/2}} e^{-\mu_2^-(\lambda)x}(e^{\mu_2^-(\lambda)y} - e^{\mu_3^-(\lambda)y}) + \mathbf{O}(1)e^{\mu_3^-(\lambda)(x+y)} \\
&\quad + \mathbf{O}(|\lambda^{1/2}|)e^{\mu_3^-(\lambda)(x-y)} + \mathbf{O}(|\lambda^{-1/2}e^{-\eta|x|}|)e^{-\mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}) \\
\partial_{xy} \tilde{G}_\lambda(x,y) &= \mathbf{O}(|\lambda^{1/2}|)e^{-\mu_2^-(\lambda)x}(e^{\mu_2^-(\lambda)y} + e^{\mu_3^-(\lambda)y}) + \mathbf{O}(|\lambda^{1/2}|e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} \\
&\quad + \mathbf{O}(e^{-\eta|x-y|}).
\end{aligned}$$

(iii) For $y \le 0 \le x$

$$\begin{aligned}
\tilde{G}_\lambda(x,y) &= \mathbf{O}(|\lambda^{-1/2}|)e^{\mu_2^+(\lambda)x - \mu_2^-(\lambda)y} \\
\partial_y \tilde{G}_\lambda(x,y) &= \mathbf{O}(1)e^{\mu_2^+(\lambda)x - \mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} \\
\partial_x \tilde{G}_\lambda(x,y) &= \mathbf{O}(1)e^{\mu_2^+(\lambda)x - \mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} \\
\partial_{xy} \tilde{G}_\lambda(x,y) &= \mathbf{O}(|\lambda^{1/2}|)e^{\mu_2^+(\lambda)x - \mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x|})e^{-\mu_2^-(\lambda)y}
\end{aligned}$$

and in all cases

$$
\begin{aligned}
E_\lambda(x,y) &= \frac{c_E}{\lambda}\bar{u}_x(x)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\eta|y|})) \\
\partial_y E_\lambda(x,y) &= c_E\frac{-\mu_2^-(\lambda)}{\lambda}\bar{u}_x(x)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\eta|y|})).
\end{aligned}
\tag{2.22}
$$

Remark 2.1. *We note that the difference terms*

$$c_1\lambda^{-1/2}(e^{\mu_2^-(\lambda)x}-e^{\mu_3^-(\lambda)x}),$$

which clearly vanish at $x=0$, arise naturally from our choice of $\psi_2^-(x;\lambda)$ and moreover are expected since the mass acculating near $x=0$ is recorded in the excited terms. However, since we additionally have a term

$$\mathbf{O}(|\lambda^{-1/2}|)e^{\mu_3^-(\lambda)(x+y)},$$

there is no real advantage to be taken from the cancellation. In this way, our estimates do not seem quite as sharp as those of [38], obtained for a multidimensional generalization of (1.1)–(1.3). We would also mention that we will point out in the proof of Lemma 2.7 precisely why it is that y-derivatives of $E_\lambda(x,y)$ do not have a term that would correspond with differentiation of $e^{-\eta y}$.

Proof. We observe at the outset the relations,

$$
\begin{aligned}
\partial_y N_1^-(y;\lambda) &= -C_2^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_1^-)(y;\lambda)}{c(y)W_\lambda(y)} - D_2^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)(y;\lambda)}{c(y)W_\lambda(y)} \\
\partial_y N_2^-(y;\lambda) &= C_1^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_1^-)(y;\lambda)}{c(y)W_\lambda(y)} + D_1^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)(y;\lambda)}{c(y)W_\lambda(y)}.
\end{aligned}
\tag{2.23}
$$

Since proofs of the four estimates in each case are similar, we provide details only for the estimates on y-derivatives. The case of y-derivatives is chosen because there is a subtle point in this case in which the estimates of Lemma 2.2 do not quite suffice, and we must use (2.23) and the estimates of Lemma 2.3.

Case (i): $y\le x\le 0$. For $y\le x\le 0$, we have

$$
\begin{aligned}
\partial_y G_\lambda(x,y) &= \phi_1^+(x;\lambda)\partial_y N_1^-(y;\lambda)+\phi_2^+(x;\lambda)\partial_y N_2^-(y;\lambda) \\
&= \Big[(A_2^+C_1^+ - A_1^+C_2^+)\phi_1^-(x) + (B_2^+C_1^+ - B_1^+C_2^+)\phi_2^-(x) \\
&\quad + (D_2^+C_1^+ - D_1^+C_2^+)\psi_2^-(x)\Big]\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_1^-)(y)}{c(y)W_\lambda(y)} \\
&+ \Big[(A_2^+D_1^+ - A_1^+D_2^+)\phi_1^-(x) + (B_2^+D_1^+ - B_1^+D_2^+)\phi_2^-(x) \\
&\quad + (D_1^+C_2^+ - D_2^+C_1^+)\psi_1^-(x)\Big]\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)(y)}{c(y)W_\lambda(y)}.
\end{aligned}
\tag{2.24}
$$

We group these into three sets of terms, beginning with the leading order $\tilde{G}_\lambda$ expression

$$
\begin{aligned}
&(D_2^+C_1^+ - D_1^+C_2^+)\psi_2^-(x)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_1^-)(y)}{c(y)W_\lambda(y)} \\
&= (c_1\lambda^{-1/2}+\mathbf{O}(1))(e^{\mu_2^-(\lambda)x}-e^{\mu_3^-(\lambda)x}+\mathbf{O}(|\lambda^{1/2}|e^{-\alpha|x|}))e^{-\mu_2^-(\lambda)y}(-\mu_2^-(\lambda)+\mathbf{O}(e^{-\tilde{\alpha}|y|})) \\
&= c_1\frac{-\mu_2^-(\lambda)}{\lambda^{1/2}}(e^{\mu_2^-(\lambda)x}-e^{\mu_3^-(\lambda)x})e^{-\mu_2^-(\lambda)} + \mathbf{O}(|\lambda^{1/2}|)e^{\mu_2^-(\lambda)(x-y)} \\
&+ \mathbf{O}(|\lambda^{1/2}|e^{-\eta|x|})e^{-\mu_2^-(\lambda)y}\mathbf{O}(e^{-\eta(|x|+|y|)}).
\end{aligned}
$$

We note in particular that we have made critical use here of Lemma 2.4; otherwise, we have employed the estimates of Lemma 2.2 in straightfoward fashion.

Next, we consider contributions to the excited term $\partial_y E_\lambda(x,y)$. In this case, we take

$$\begin{aligned}
&- B_1^+(\lambda)\phi_2^-(x;\lambda)\Big[C_2^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_1^-)(y;\lambda)}{c(y)W_\lambda(y)} + D_2^+(\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)(y;\lambda)}{c(y)W_\lambda(y)}\Big]\\
&= B_1^+(\lambda)\phi_2^-(x;\lambda)\partial_y N_1^-(y;\lambda)\\
&= \Big(c_1\lambda^{-1/2} + \mathbf{O}(1)\Big)\Big(\bar{u}_x(x) + \mathbf{O}(|\lambda^{1/2}|e^{-\eta|x|})\Big)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\eta|y|}))\\
&= c_1\lambda^{-1/2}\bar{u}_x(x)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\eta|y|})) + \mathbf{O}(e^{-\eta|x|})e^{-\mu_2^-(\lambda)x}.
\end{aligned}$$

It is of particular importance to note that the undifferentiated excited term is precisely the same thing with y-derivatives omitted. In this way, we see that y-differentiation improves the singularity of $E_\lambda(x,y)$ as $\lambda \to 0$.

For the remaining terms, we group according to (2.23) in order to obtain

$$\begin{aligned}
&A_2^+(\lambda)\phi_1^-(x;\lambda)\partial_y N_2^-(y;\lambda) + A_1^+(\lambda)\phi_1^-(x;\lambda)\partial_y N_1^-(y;\lambda)\\
&+ B_2^+(\lambda)\phi_2^-(x;\lambda)\partial_y N_2^-(y;\lambda) + (D_2^+(\lambda)C_1^+(\lambda) - D_1^+(\lambda)C_2^+(\lambda))\psi_1^-(x;\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)}{c(y)W_\lambda(y)}
\end{aligned} \tag{2.25}$$

For the first three terms on the right hand side of (2.25), we have an estimate by

$$\begin{aligned}
&\Big(c_1 + \mathbf{O}(|\lambda^{1/2}|)\Big)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\eta|x|}))\\
&= \mathbf{O}(1)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}.
\end{aligned}$$

For the third term on the right hand side of (2.25), keeping in mind Lemma 2.4, we have

$$(D_2^+(\lambda)C_1^+(\lambda) - D_1^+(\lambda)C_2^+(\lambda))\psi_1^-(x;\lambda)\frac{\partial_y W(\phi_1^-,\phi_2^-,\psi_2^-)}{c(y)W_\lambda(y)} = \mathbf{O}(1)e^{\mu_1^-(\lambda)(x-y)}. \tag{2.26}$$

Case (ii): $x \le y \le 0$. For $x \le y \le 0$, we have

$$\partial_y G_\lambda(x,y) = \phi_1^-(x;\lambda)\partial_y N_1^+(y;\lambda) + \phi_2^-(x;\lambda)\partial_y N_2^+(y;\lambda). \tag{2.27}$$

Beginning again with the leading order $\tilde{G}_\lambda$ estimate, we have

$$\phi_1^-(x;\lambda)\partial_y N_1^+(y;\lambda) = \mathbf{O}(1)e^{\mu_3^-(\lambda)(x-y)}, \tag{2.28}$$

while for the excited and correction terms, we compute

$$\begin{aligned}
&\phi_2^-(x;\lambda)\partial_y N_2^+(y;\lambda)\\
&= \Big(\bar{u}_x(x) + \mathbf{O}(|\lambda^{1/2}|e^{-\eta|x|})\Big)\Big[(k_1\lambda^{-1/2} + \mathbf{O}(1))e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\alpha|y|})) + \mathbf{O}(1)e^{-\mu_4^-(\lambda)y}\Big]\\
&= k_1\lambda^{-1/2}\bar{u}_x(x)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\alpha|y|})) + \mathbf{O}(e^{-\eta|x|})e^{-\mu_2^-(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}).
\end{aligned}$$

For the excited term, we take

$$\partial_y E_\lambda(x,y) = k_1\lambda^{-1/2}\bar{u}_x(x)e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\alpha|y|})).$$

Case (iii): $y \le 0 \le x$. For $y \le 0 \le x$, we have

$$\partial_y G_\lambda(x,y) = \phi_1^+(x;\lambda)\partial_y N_1^-(y;\lambda) + \phi_2^+(x;\lambda)\partial_y N_2^-(y;\lambda). \tag{2.29}$$

For the leading order $\tilde{G}_\lambda$ term, we take

$$\phi_2^+(x;\lambda)\partial_y N_2^-(y;\lambda) = \mathbf{O}(1)e^{\mu_2^+(\lambda)x-\mu_2^-(\lambda)y},$$

while for the excited and correction terms, we compute

$$\phi_1^+(x;\lambda)\partial_y N_1^-(y;\lambda) = \Big(\bar{u}_x(x) + \mathbf{O}(|\lambda^{1/2}|e^{-\eta|x|})\Big)(n_1\lambda^{-1/2} + \mathbf{O}(1))e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\tilde{\alpha}|y|})).$$

For the excited term, we take

$$\partial_y E_\lambda(x,y) = n_1 \bar{u}_x(x)\lambda^{-1/2} e^{-\mu_2^-(\lambda)y}(1+\mathbf{O}(e^{-\tilde{\alpha}|y|})).$$

The remaining cases can be established similarly. □

Lemma 2.8. *Suppose the conditions of Theorem 1.1 hold, and $G_\lambda(x,y)$ is as defined in (1.18). Then for $|\lambda| \geq R$, some R sufficiently large, and for λ to the right of the contour Γ_s defined in Remark 1.1, we have the following estimates. For some $\beta > 0$ and for multi-index α in x and y,*

$$|\partial^\alpha G_\lambda(x,y)| \leq C|\lambda^{-\frac{3-k}{4}}|e^{-\beta|\lambda^{1/4}||x-y|}, \quad |\alpha| \leq 3.$$

Remark 2.2. *Regarding the proof of Lemma 2.8, we note that large $|\lambda|$ behavior corresponds with small t behavior, for which the fourth order effects dominate. Consequently, the proof of Lemma 2.8 is almost precisely the same as that of the corresponding Lemma 3.2 of [25], carried out in the case of multidimensional equations*

$$u_t + \sum_{j=1}^d f^j(u)_{x_j} = -\sum_{jklm}\Big(b^{jklm}(u)u_{x_j x_k x_l}\Big)_{x_m}.$$

We omit it here.

Lemma 2.9. *Suppose the conditions of Theorem 1.1 hold, and $G_\lambda(x,y)$ is as defined in (1.18). Then for $r \leq |\lambda| \leq R$, r as in Lemma 2.7 and R as in Lemma 2.8, and for λ to the right of the contour Γ_s defined in Remark 1.1, we have the following estimates. For some constant C sufficiently large, and for multi-index α in x and y,*

$$|\partial^\alpha G_\lambda(x,y)| \leq C, \quad |\alpha| \leq 3.$$

3 Proof of Theorem 1.1

In this section, we obtain estimates on the Green's function $G(t,x;y)$ through the inverse Laplace transform representation

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)d\lambda, \tag{3.1}$$

where Γ denotes a contour that lies to the right of the point spectrum of L and loops through the point at ∞. The validity of (3.1) can be established in a straightforward manner from the estimates of Lemmas 2.7, 2.8, and 2.9. See, in particular, Corollary 7.4 of [52].

3.1 Small Time Estimates

In the case $|x-y| \geq Kt$, for some K sufficiently large, and also in the case of $t \leq 1$ (in fact, for any fixed finite bound) behavior of the Green's function is entirely dominated by fourth order effects. In this case, the analysis of [25], carried out in the case of multidimensional equations

$$u_t + \sum_{j=1}^d f^j(u)_{x_j} = -\sum_{jklm}\Big(b^{jklm}(u)u_{x_j x_k x_l}\Big)_{x_m},$$

extends immediately to the current setting and we can conclude, as there,

$$|\partial^\alpha G(t,x;y)| \leq Ct^{-\frac{1+|\alpha|}{4}} e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}},$$

where α denotes a standard multi-index in x and y with $|\alpha| \leq 3$. The proof is carried out through the estimates of Lemma 2.8 and representation (3.1).

3.2 Large Time Estimates

In the case $|x-y| \le Kt$, K as in Section 3.1, we proceed predominately through the estimates of Lemma 2.7, employing Lemmas 2.8 and 2.9 only for integration over $|\lambda| \ge r$, which corresponds roughly with the non-critical cases of small and medium range times. As the analyses of several of the terms in Lemma 2.7 are quite similar, we will proceed by giving full details only for the most critical cases and pointing out the salient points for the remainder.

Case (i), $y \le x \le 0$. For the case $y \le x \le 0$, we begin with the leading term

$$c_1 \lambda^{-1/2} (e^{\mu_2^-(\lambda)x} - e^{\mu_3^-(\lambda)x}) e^{-\mu_2^-(\lambda)y},$$

for which we first evaluate

$$\frac{c_1}{2\pi i} \int_\Gamma \frac{e^{\lambda t + \mu_2^-(\lambda)(x-y)}}{\sqrt{\lambda}} d\lambda.$$

Recalling the expansion

$$\mu_2^-(\lambda) = -\sqrt{\frac{\lambda}{b_-}} + \mathbf{O}(|\lambda^{3/2}|),$$

we have that for $|\lambda^{3/2}(x-y)| \le \epsilon$, $\epsilon > 0$ sufficiently small, there holds

$$e^{\lambda t + \mu_2^-(\lambda)(x-y)} = e^{\lambda t - \sqrt{\frac{\lambda}{b_-}}|x-y|} (1 + \mathbf{O}(|\lambda^{3/2}(x-y)|)).$$

We focus first, then, on integrals

$$\frac{c_1}{2\pi i} \int_\Gamma \frac{e^{\lambda t - \sqrt{\frac{\lambda}{b_-}}(x-y)}}{\sqrt{\lambda}} d\lambda,$$

and treat the higher order terms later as small corrections. For $|x-y| \le \epsilon_1 t$, $\epsilon_1 > 0$ sufficiently small, we take the contour

$$\sqrt{\frac{\lambda}{b_-}} = \frac{|x-y|}{2tb_-} + ik. \tag{3.2}$$

Our approach will be to follow the contour (3.2) until we strike the contour Γ_s of Remark 1.1, and from that point to follow Γ_s out to the point at ∞, employing the estimates of Lemma 2.8 and 2.9. We denote the truncated contour taken prior to intersection with Γ_s as Γ^*, and let $\pm k^*$ denote the values of k at which Γ^* strikes Γ_s. We remark before proceeding that in the case $\epsilon_1 \le |x-y| \le Kt$, we take the similar contour

$$\sqrt{\frac{\lambda}{b_-}} = \frac{(x-y)}{Lt} + ik, \tag{3.3}$$

where L is chosen sufficiently large, and we have

$$\lambda(k) = b_- \frac{(x-y)^2}{L^2 t^2} + 2ikb_- \frac{x-y}{Lt} - b_- k^2; \quad d\lambda = 2i\sqrt{b_-}\sqrt{\lambda}.$$

In this case, we lose the precise kernel scaling—as stated for the leading order $\tilde{G}$ term in Theorem 1.1—but with $|x-y| \ge \epsilon_1 t$, we have exponential decay in t, and the term can be subsumed into the higher order estimates.

Returning to our main contour (3.2), we observe the relationships

$$\lambda(k) = \frac{(x-y)^2}{4b_- t^2} + ik\frac{x-y}{t} - b_- k^2; \quad d\lambda = 2i\sqrt{b_-}\sqrt{\lambda} dk. \tag{3.4}$$

In this way,

$$\frac{c_1}{2\pi i} \int_{\Gamma^*} \frac{e^{\lambda t - \sqrt{\frac{\lambda}{b_-}}|x-y|}}{\sqrt{\lambda}} d\lambda = \frac{c_1\sqrt{b_-}}{\pi} e^{\frac{-(x-y)^2}{4b_- t}} \int_{-k^*}^{+k^*} e^{-b_- k^2 t} dk = \frac{c_1}{\sqrt{\pi t}} e^{\frac{-(x-y)^2}{4b_- t}}, \tag{3.5}$$

wherein we arrive at the sharp leading order estimate of Theorem 1.1. Along the contour Γ_s, we clearly have exponential decay in t, which in the case $|x-y| \le Kt$ gives an estimate which can be subsumed into the higher order terms (see the more detailed remarks in the paragraph labeled *higher order corrections*). Proceeding similarly in the case

$$\frac{c_1}{2\pi i}\int_\Gamma \frac{e^{\lambda t-\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}}{\sqrt{\lambda}}d\lambda,$$

and keeping in mind that $\mu_2^-(\lambda) = -\mu_3^-(\lambda)$, we determine as above an expression

$$\frac{c_1}{\sqrt{\pi t}}e^{-\frac{(x+y)^2}{4b_- t}},$$

plus higher order corrections.

Higher order corrections. We next consider in more detail the higher order corrections left over from the estimates above. First, we have the $\mathbf{O}(|\lambda^{3/2}|)$ correction, arising from integrals

$$\int_\Gamma \frac{e^{\lambda t-\sqrt{\frac{\lambda}{b_-}}(x-y)}}{\sqrt{\lambda}}\mathbf{O}(|\lambda^{3/2}(x-y)|)d\lambda.$$

Proceeding with the contour (3.3), we have

$$\begin{aligned}
\Big|\int_{\Gamma^*} &\frac{e^{\lambda t-\sqrt{\frac{\lambda}{b_-}}(x-y)}}{\sqrt{\lambda}}\mathbf{O}(|\lambda^{3/2}(x-y)|)d\lambda\Big| \\
&\le Ce^{-\frac{(x-y)^2}{2Lt}}\int_{-k^*}^{+k^*} e^{-b_-k^2t}\mathbf{O}(|\frac{|x-y|^3}{t^2}|+|k^2||x-y|)dk \\
&= \Big(\mathbf{O}(\frac{|x-y|^3}{t^2})+\mathbf{O}(\frac{|x-y|}{t^{3/2}})\Big)e^{-\frac{(x-y)^2}{2Lt}} = \mathbf{O}(\frac{|x-y|}{t^{3/2}})e^{-\frac{(x-y)^2}{4Lt}}.
\end{aligned}$$

Also, regarding the correction terms from the scattering estimates, we note that in the case $|x-y| \le Kt$, we have

$$t \ge \frac{|x-y|}{K} \ge \frac{|x-y|}{K}\frac{|x-y|}{Kt} = \frac{|x-y|^2}{K^2 t},$$

so that exponential decay in time, which is always the case along the contour Γ_s, gives the broadened kernel decay.

Second and third summands in $\tilde{G}_\lambda(x,y)$. The second estimate on $\tilde{G}_\lambda(x,y)$ in the case $y \le x \le 0$ can be analyzed precisely as above. Though the third can also be analyzed in a similar manner, we carry out a portion of the calculation in order to indicate one important feature of most of the derivative estimates. We have

$$\frac{1}{2\pi i}\int_\Gamma \mathbf{O}(1)e^{\lambda t+\mu_2^-(\lambda)(x-y)}d\lambda.$$

Proceeding as above along the contour (3.2) (for $|\lambda|$ suitably small and to first order in $|\lambda^{1/2}|$), we have

$$\frac{1}{2\pi i}e^{-\frac{(x-y)^2}{4b_- t}}\int_{-k^*}^{+k^*}\mathbf{O}(1)e^{-b_-k^2t}2i\Big[\frac{|x-y|}{2tb_-}+ik\Big]dk.$$

For the first term in the square brackets, we immediately obtain the sought estimate by

$$C\frac{|x-y|}{t^{3/2}}e^{\frac{(x-y)^2}{4b_- t}},$$

subject to higher order corrections as discussed above. For the second term in the square brackets, a normed estimate provides the slightly worse estimate by

$$Ct^{-1}e^{\frac{(x-y)^2}{4b_- t}}.$$

The point we would like to make here is that the analyticity of our $\mathbf{O}(1)$ term in $\rho = \sqrt{\lambda}$ allows us to take advantage of the cancellation

$$\int_{-k^*}^{+k^*} e^{-b_- k^2 t} k dk = 0.$$

In this way, the final estimate on this term takes the form

$$\mathbf{O}(\frac{|x-y|}{t^{3/2}}) e^{-\frac{|x-y|^2}{Mt}}.$$

Case (iii), $y \le 0 \le x$. In the case $y \le 0 \le x$, we have the leading term

$$\mathbf{O}(|\lambda^{-1/2}|) e^{\mu_2^+(\lambda)x - \mu_2^-(\lambda)y},$$

which differs from the case $x \le y \le 0$ only in the different forms of $\mu_2^+(\lambda)$ and $\mu_2^-(\lambda)$. Recalling the expansions

$$\mu_2^-(\lambda) = -\sqrt{\frac{\lambda}{b_-}} + \mathbf{O}(|\lambda^{3/2}|)$$

$$\mu_2^+(\lambda) = -\sqrt{\frac{\lambda}{b_+}} + \mathbf{O}(|\lambda^{3/2}|),$$

we write

$$\mu_2^+(\lambda)x - \mu_2^-(\lambda)y = -\sqrt{\frac{\lambda}{b_+}}x + \sqrt{\frac{\lambda}{b_-}}y + \mathbf{O}(|\lambda^{3/2}|(|x|+|y|)).$$

We focus on the primary contribution,

$$\frac{1}{2\pi i}\int_\Gamma \mathbf{O}(|\lambda^{-1/2}|) e^{\lambda t - \sqrt{\frac{\lambda}{b_+}}x + \sqrt{\frac{\lambda}{b_-}}y} d\lambda = \frac{1}{2\pi i}\int_\Gamma \mathbf{O}(|\lambda^{-1/2}|) e^{\lambda t - \sqrt{\frac{b_-}{b_+}}\left(\sqrt{\frac{\lambda}{b_-}}(x - \sqrt{\frac{b_+}{b_-}}y)\right)} d\lambda,$$

which can be analyzed precisely as in the previous cases if (for $|x - \sqrt{\frac{b_+}{b_-}}y| \le \epsilon t$, some ϵ suitably small) we take the contour

$$\sqrt{\frac{\lambda}{b_-}} = \frac{|x - \sqrt{\frac{b_+}{b_-}}y|}{2b_- t} + ik,$$

through which we obtain the estimate by

$$\mathbf{O}(t^{-1/2}) e^{-\frac{(x-\sqrt{\frac{b_+}{b_-}}y)^2}{4b_- t}}.$$

Similarly as in the previous cases, for $|x - \sqrt{\frac{b_+}{b_-}}y| \ge \epsilon t$, we take the crude contour

$$\sqrt{\frac{\lambda}{b_-}} = \frac{|x - \sqrt{\frac{b_+}{b_-}}y|}{Lt} + ik,$$

for a suitably large choice of L.

We finally remark that estimates on the excited terms can be obtained similarly, as can estimates on differentiated terms. □

4 Proof of Theorem 1.2

In this section, we combine the estimates of Theorem 1.1 with the integral representations (1.14) and (1.15)—and corresponding integral representations for v_x and $\dot{\delta}(t)$ found by direct differentiation—to determine estimates on the perturbation $v(t,x)$ and the local tracking function $\delta(t)$. To this end, we state two lemmas corresponding with estimates on the linear and nonlinear integrals in (1.14) and (1.15).

Lemma 4.1. *For $G(t,x;y)$ as described in Theorem 1.1, and for $v_0(y)$ as described in Theorem 1.2, there holds*

$$\Big|\int_{-\infty}^{+\infty}\tilde{G}(t,x;y)v_0(y)dy\Big| \le CE_0(1+|x|+\sqrt{t})^{-3/2}$$
$$\Big|\int_{-\infty}^{+\infty}\tilde{G}_x(t,x;y)v_0(y)dy\Big| \le CE_0 t^{-1/4}(1+t)^{-1/4}\Big[(1+|x|+\sqrt{t})^{-3/2}+(1+t)^{-1/4}e^{-\eta|x|}\Big]$$
$$\int_{-\infty}^{+\infty}|e_y(t,y)||V_0(y)|dy \le C(1+t)^{-1/4}$$
$$\int_{-\infty}^{+\infty}|e_{yt}(t,y)|V_0(y)dy \le C(1+t)^{-5/4}.$$

Lemma 4.2. *For $G(t,x;y)$ as in Theorem 1.1, and*

$$\Psi(s,y)=s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2}+s^{-3/4}(1+s)^{-3/4}e^{-\eta|y|},$$

there holds

$$\int_0^t\int_{-\infty}^{+\infty}|\tilde{G}_y(t-s,x;y)|\Psi(s,y)dyds \le C(1+|x|+\sqrt{t})^{-3/2}$$
$$\int_0^t\int_{-\infty}^{+\infty}|\tilde{G}_{yx}(t-s,x;y)|\Psi(s,y)dyds \le Ct^{-1/4}(1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2}$$
$$\int_0^t\int_{-\infty}^{+\infty}|e_y(t-s,y)|\Psi(s,y)dyds \le C(1+t)^{-1/2}$$
$$\int_0^t\int_{-\infty}^{+\infty}|e_{yt}(t-s,y)|\Psi(s,y)dyds \le C(1+t)^{-1}.$$

We note that the expressions in $\Psi(s,y)$ follow from the linear estimates of Lemma 4.1 and the nonlinearites

$$\mathbf{O}(|vv_x|), \quad \mathbf{O}(|vv_{xxx}|), \quad \mathbf{O}(|e^{-\eta|x|}v^2|), \quad \mathbf{O}(|\dot{\delta}(t)v|).$$

In addition to Lemma 4.1 and Lemma 4.2, we require the following lemma regarding small time behavior of the perturbation $v(t,x)$. The salient points are that (1) for small t, the behavior of $v(t,x)$ is dominated by fourth order effects, and (2) the behavior of derivatives v_x, v_{xx}, and v_{xxx} can be linked to the behavior of v.

Lemma 4.3. *Under the assumptions of Theorem 1.1, and under the additional restriction of Hölder continuity on the initial perturbation $v_0 \in C^{0+\gamma}$, $\gamma>0$, the integral equations (1.14) and (1.15) determine a unique local solution $v\in C^{0+\frac{\gamma}{4}}(t)\cap C^{0+\gamma}(x)$, $\delta\in C^{1+\frac{\gamma}{4}}(t)$, extending so long as $|v|_{C^{0+\gamma}}$ remains bounded. Moreover, on this time interval*

$$\sup_{x\in\mathbb{R}}|v(t,x)|(1+|x|+\sqrt{t})^{3/2}$$

remains continuous so long as both it and $\dot{\delta}(t+1)$ are uniformly bounded, and for $\tau>0$ sufficiently small and $t\ge\tau$,

$$\sup_{x\in\mathbb{R}}|\partial_x^k v(t,x)|(1+|x|+\sqrt{t})^{3/2} \le C\tau^{-k/4}\sup_{x\in\mathbb{R}}|v(t-\tau,x)|(1+|x|+\sqrt{t-\tau})^{3/2},$$

for $k=1,2,3$, and

$$\sup_{x\in\mathbb{R}}|\partial_x^{k+1}v(t,x)|\Big[t^{-1/4}(1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2}+(1+t)^{-3/4}e^{-\eta|x|}\Big]^{-1}$$
$$\le C\tau^{-k/4}\sup_{x\in\mathbb{R}}|\partial_x v(t-\tau,x)|\Big[(t-\tau)^{-1/4}(1+(t-\tau))^{-1/4}(1+|x|+\sqrt{t-\tau})^{-3/2}$$
$$+(1+(t-\tau))^{-3/4}e^{-\eta|x|}\Big]^{-1},$$

for $k=1,2$.

We proceed now by defining the iteration variable

$$\begin{aligned}\zeta(t) = \sup_{s\in[0,t],y\in\mathbb{R}} \Big[&\frac{|v(s,y)|}{(1+|y|+\sqrt{s})^{-3/2}} + \frac{|v_y(s,y)|}{s^{-1/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} + (1+s)^{-3/4}e^{-\eta|y|}} \\ &+ \frac{|\delta(s)|}{(1+s)^{-1/4}} + \frac{|\dot{\delta}(s)|}{(1+s)^{-5/4}}\Big].\end{aligned} \tag{4.1}$$

Claim 4.1. *Suppose there exists a constant C so that*

$$\zeta(t) \le C(E_0 + \zeta(t)^2), \tag{4.2}$$

where E_0 is as in Theorem 1.2. Then for E_0 sufficiently small, there holds

$$\zeta(t) < 2CE_0.$$

Proof of Claim 4.1. We first observe that we have control over $\zeta(0)$ directly from the definition of $\zeta(t)$. Recalling the relation

$$|v(0,y)| \le E_0(1+|y|)^{-3/2},$$

we see immediately that the first quotient in the definition of $\zeta(0)$ is bounded by E_0. In addition, according to Lemma 4.3 there holds

$$\sup_{y\in\mathbb{R}} \frac{|v_y(s,y)|}{(1+|y|+\sqrt{s})^{-3/2}} \le C\tau^{-1/4} \sup_{y\in\mathbb{R}} \frac{|v_y(s-\tau,y)|}{(1+|y|+\sqrt{s-\tau})^{-3/2}},$$

for $s \ge \tau$, τ suitably small, so that

$$\sup_{y\in\mathbb{R}} \lim_{s\to 0} \frac{|v_y(s,y)|}{s^{-1/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2}} \le C \sup_{y\in\mathbb{R}} \frac{|v(0,y)|}{(1+|y|)^{-3/2}} \le CE_0.$$

Finally, we can choose $\delta(t)$ in a smooth fashion so that $\delta(0) = 0$ and $\dot{\delta}(0) = 0$. In this way, we have

$$\zeta(0) \le C_1 E_0,$$

for a constant C_1. We proceed now by choosing E_0 sufficiently small so that $C_1^2 E_0 < 1$ and $4C^2 E_0 < 1$. First, this choice insures

$$\zeta(0) \le C(E_0 + \zeta(0)^2) \le C(E_0 + C_1^2 E_0^2) < 2CE_0.$$

Next, let T denote the first time (if it exists) for which we have equality $\zeta(T) = 2CE_0$. We have, then,

$$\zeta(T) \le C(E_0 + \zeta(T)^2) = C(E_0 + 4C^2 E_0^2) < 2CE_0,$$

a contradiction. We observe that continuity of $\zeta(t)$, critical for our argument, is clear from the short time estimates of Lemma 4.3.

This concludes the proof of Claim 4.1. □

We proceed now to verify the assumption of Claim 4.1 through a straightforward calculation involving the integral representations (1.14) and (1.15), combined with the small time estimates of Lemma 4.3. In particular, for the variable v_{xxx}, which is not carried through the iteration, we employ Lemma 4.3 to write

$$|v_{yyy}(s,y)| \le Cs^{-1/2}\Big[s^{-1/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} + (1+s)^{-3/4}e^{-\eta|y|}\Big]\zeta(0).$$

Focusing on the case $v(t,x)$, we have

$$\begin{aligned}|v(t,x)| &\le \Big|\int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy\Big| \\ &+ \int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(t-s,x;y)|\Big[|Q(y,v,v_y,v_{yyy})| + |\dot{\delta}(s)v|\Big]dyds \\ &\le CE_0(1+|x|+\sqrt{t})^{-3/2} \\ &+ \zeta(t)^2 \int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(t-s,x;y)|\Big[s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2} + s^{-3/4}(1+s)^{-3/4}e^{-\eta|y|}\Big]dyds \\ &\le C(E_0 + \zeta(t)^2)(1+|x|+\sqrt{t})^{-3/2}.\end{aligned}$$

In this way, we have

$$\frac{|v(t,x)|}{(1+|x|+\sqrt{t})^{-3/2}} \le C_1 E_0 (1+\zeta(t)^2).$$

Taking a supremum norm over both sides of this last expression, and noting that $\zeta(t)$ is a non-decreasing function, we conclude

$$\sup_{x\in[0,t],y\in\mathbb{R}} \frac{|v(t,x)|}{(1+|x|+\sqrt{t})^{-3/2}} \le C_1 E_0 (1+\zeta(t)^2).$$

Proceeding similarly for v_x, $\delta(t)$, and $\dot{\delta}(t)$, we conclude

$$\zeta(t) \le C E_0 (1+\zeta(t)^2),$$

from which Theorem 1.2 follows from Claim 4.1. □

5 Proof of Technical Lemmas

In this section, we prove Lemma 4.1 and Lemma 4.2. The proof of Lemma 4.3 is almost identical to that of Lemma 3.4 of [26] and is omitted. At the outset, we recall the notation

$$\begin{aligned} G(t,x;y) &= \tilde{G}(t,x;y) + E(t,x;y) \\ E(t,x;y) &= \bar{u}_x(x) e(t,y). \end{aligned}$$

Proof of Lemma 4.1. We divide the analysis up into the cases I and II from Theorem 1.1.

Case I, $|x-y| \ge Kt$ *or* $t \le 1$. For $|x-y| \ge Kt$ or for $t \le 1$, and for the first estimate of Lemma 4.1, we have integrals

$$\int_{-\infty}^{+\infty} t^{-1/4} e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}} (1+|y|)^{-3/2} dy,$$

where for simplicity we have extended the range of y over all of $\mathbb{R}$. We first observe that for $|x-y| \ge Kt$, the kernel decays at exponential rate in t, and we have

$$e^{-\frac{K^{4/3}}{2M}t} \int_{-\infty}^{+\infty} t^{-1/4} e^{-\frac{|x-y|^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy.$$

(For $t \le 1$, we need not keep track of t decay.) We now divide the integration over y into the subintervals $y \in [-|x|/2, |x|/2]$, and its complement, for which we compute

$$\begin{aligned} \int_{-\frac{|x|}{2}}^{+\frac{|x|}{2}} t^{-1/4} e^{-\frac{|x-y|^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy + \int_{[-\frac{|x|}{2},\frac{|x|}{2}]^c} t^{-1/4} e^{-\frac{|x-y|^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy \\ \le C_1 t^{-1/4} e^{-\frac{x^{4/3}}{M2^{7/3}t^{1/3}}} + C_2 (1+|x|)^{-3/2}, \end{aligned} \tag{5.1}$$

where the expressions on the final right hand side are respective estimates, the first obtained by integration of $(1+|y|)^{-3/2}$ and the second obtained through integration of the L^1 kernel. Finally, we note that for $t \ge |x|$, we have exponential decay in both t and $|x|$ from the term $\exp(-K^{4/3}/(2M)t)$, while for $|x| \ge t$, the kernel decay in (5.1) gives exponential decay in $|x|$. Combining these observations, we obtain a much better estimate than the one claimed; in fact, the estimate here is already sufficient to give the second estimate of Lemma 4.1 as well. (The analysis is clearly driven by the case $|x-y| \le Kt$.)

Case II, $|x-y| \le Kt$ *and* $t \ge 1$. For the case $|x-y| \le Kt$ and $t \ge 1$, we first observe through integrating by parts and taking advantage of our zero-mass condition on v_0 that we have sufficient decay in t. In order to see this more precisely, we write

$$\Big| \int_{-\infty}^{+\infty} \tilde{G}(t,x;y) v_0(y) dy \Big| = \Big| \int_{-\infty}^{+\infty} \tilde{G}_y(t,x;y) V_0(y) dy \Big|,$$

where

$$V_0(y) = \int_{-\infty}^{y} v_0(z)dz,$$

and consider the primary $\tilde{G}_y$ contribution

$$\mathbf{O}(\frac{|x-y|}{t^{3/2}})e^{-\frac{(x-y)^2}{Mt}} = \mathbf{O}(t^{-1})e^{-\frac{(x-y)^2}{2Mt}}. \tag{5.2}$$

For this contribution, we estimate

$$\begin{aligned}&\int_{-\infty}^{+\infty} t^{-1}e^{-\frac{(x-y)^2}{2Mt}}(1+|y|)^{-1/2}dy \\ &= \int_{-\sqrt{t}}^{+\sqrt{t}} t^{-1}e^{-\frac{(x-y)^2}{2Mt}}(1+|y|)^{-1/2}dy + \int_{|y|\geq\sqrt{t}} t^{-1}e^{-\frac{(x-y)^2}{2Mt}}(1+|y|)^{-1/2}dy \leq Ct^{-3/4},\end{aligned}$$

which gives the required decay in time. (The calculations for other contributions to $\tilde{G}$ are almost identical.)

In order to establish the appropriate decay in $|x|$, we divide the integration up as

$$\begin{aligned}&\int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy \\ &\quad= \int_{-\frac{|x|}{2}}^{+\frac{|x|}{2}} \tilde{G}(t,x;y)v_0(y)dy + \int_{[-\frac{|x|}{2},+\frac{|x|}{2}]^c} \tilde{G}(t,x;y)v_0(y)dy.\end{aligned} \tag{5.3}$$

For the first integral on the right hand side of (5.3), we integrate by parts to obtain

$$\tilde{G}(t,x;\frac{|x|}{2})V_0(|x|/2) - \tilde{G}(t,x;-\frac{|x|}{2})V_0(-|x|/2) - \int_{-\frac{|x|}{2}}^{+\frac{|x|}{2}} \tilde{G}_y(t,x;y)V_0(y)dy. \tag{5.4}$$

In all cases, the first two (boundary) expressions in (5.4) provide the estimate

$$Ct^{-1/2}e^{-\frac{x^2}{Lt}}(1+|x|)^{-1/2},$$

for some constants C and L sufficiently large. We easily obtain the claimed decay in $|x|$ upon observing (for $|x|\geq 1$)

$$t^{-1/2}e^{-\frac{x^2}{Lt}} = t^{-1/2}\frac{|x|}{|x|}e^{-\frac{x^2}{Lt}} \leq C_1|x|^{-1}e^{-\frac{x^2}{2Lt}}. \tag{5.5}$$

For the final expression in (5.4), we again specialize to the case (5.2), for which we have

$$e^{-\frac{x^2}{4Mt}} \int_{-\frac{|x|}{2}}^{+\frac{|x|}{2}} t^{-1}e^{-\frac{(x-y)^2}{4Mt}}(1+|y|)^{-1/2}dy \leq Ct^{-1}(1+|x|)^{1/2}e^{-\frac{x^2}{4Mt}},$$

which can be shown by an argument similar to (5.5) to give decay $|x|^{-3/2}$. For the second integral on the right hand side of (5.3), we observe that in all cases integrability of $\tilde{G}$ immediately gives the spatial decay $(1+|x|)^{-3/2}$. These two decay rates $(1+t)^{-3/4}$ and $(1+|x|)^{-3/2}$ are sufficient to give the claimed estimate.

The analysis of the second integral in Lemma 4.1 is almost identical to that of the first. We note here only that it is the expression

$$\mathbf{O}(\frac{y}{t^{3/2}}e^{-\eta|x|})e^{-\frac{y^2}{Mt}}$$

that gives rise to the exponentially decaying estimate.

Excited estimates. For the third estimate in Lemma 4.1, we integrate by parts and estimate (specializing to the case $y \leq 0$)

$$\begin{aligned}&\int_{-\infty}^{0} \frac{C_E}{\sqrt{4b_- t}}e^{-\frac{y^2}{4b_- t}}(1+|y|)^{-1/2}dy \\ &= \int_{-\infty}^{-\sqrt{t}} \frac{C_E}{\sqrt{4b_- t}}e^{-\frac{y^2}{4b_- t}}(1+|y|)^{-1/2}dy + \int_{-\sqrt{t}}^{0} \frac{C_E}{\sqrt{4b_- t}}e^{-\frac{y^2}{4b_- t}}(1+|y|)^{-1/2}dy \leq Ct^{-1/4}.\end{aligned}$$

For the fourth estimate in Lemma 4.1, we explicitly compute the time derivative of $e_y(t,y)$ as

$$e_{yt}(t,y) = \Big[-\frac{2C_E b_-}{(4b_- t)^{3/2}} e^{-\frac{y^2}{4b_- t}} + \frac{C_E y^2}{(4b_-)^{3/2} t^{5/2}} e^{-\frac{y^2}{4b_- t}} \Big] (1 + \mathbf{O}(e^{-\eta|y|})),$$

from which the primary contribution is

$$\int_{-\infty}^{0} t^{-3/2} e^{-\frac{y^2}{4b_- t}} (1+|y|)^{-1/2} dy \leq Ct^{-5/4}.$$

By virtue of the alternative Case I estimate for $t \leq 1$, we can take $\dot{\delta}(t)$ bounded as $t \to 0$.

This concludes the proof of Lemma 4.1. □

Proof of Lemma 4.2. For Lemma 4.2, we have two nonlinearities to consider, the summands of $\Psi(s,y)$. For each estimate on $G(t,x;y)$, we will carry out detailed estimates only for the first of these summands, noting only that the analysis of the second is almost identical. Proceeding as in the proof of Lemma 4.1, we first divide the analysis into the cases I and II.

Case I, $|x-y| \geq K(t-s)$ or $t-s \leq 1$. For $|x-y| \geq K(t-s)$ or $t-s \leq 1$, and for the first estimate of Lemma 4.2, we have integrals

$$\int_0^t \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{|x-y|^{4/3}}{M(t-s)^{1/3}}} \Psi(s,y) dy ds,$$

where for simplicity we have extended the range of integration over all of $\mathbb{R}$. For the first nonlinearity in $\Psi(s,y)$, we have integrals

$$\int_0^t \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{|x-y|^{4/3}}{M(t-s)^{1/3}}} s^{-3/4} (1+s)^{-1/2} (1+|y|+\sqrt{s})^{-3/2} dy ds. \tag{5.6}$$

Proceeding similarly as in Case I of Lemma 4.1, we first observe that for $|x-y| \geq K(t-s)$, the kernel decays at exponential rate in $(t-s)$, and we have

$$\int_0^t e^{-\frac{K^{4/3}}{2M}(t-s)} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{|x-y|^{4/3}}{2M(t-s)^{1/3}}} s^{-3/4} (1+s)^{-1/2} (1+|y|+\sqrt{s})^{-3/2} dy ds.$$

We now divide the integration over y into the subintervals $y \in [-\frac{|x|}{2}, \frac{|x|}{2}]$ and its complement, for which we compute

$$\begin{aligned}
&\int_0^t e^{-\frac{K^{4/3}}{2M}(t-s)} \int_{-\frac{|x|}{2}}^{\frac{|x|}{2}} (t-s)^{-1/2} e^{-\frac{|x-y|^{4/3}}{2M(t-s)^{1/3}}} s^{-3/4} (1+s)^{-1/2} (1+|y|+\sqrt{s})^{-3/2} dy ds \\
&+ \int_0^t e^{-\frac{K^{4/3}}{2M}(t-s)} \int_{[-\frac{|x|}{2}, \frac{|x|}{2}]^c} (t-s)^{-1/2} e^{-\frac{|x-y|^{4/3}}{2M(t-s)^{1/3}}} s^{-3/4} (1+s)^{-1/2} (1+|y|+\sqrt{s})^{-3/2} dy ds \\
&\leq C_1 \int_0^t e^{-\frac{K^{4/3}}{2M}(t-s)} (t-s)^{-1/4} e^{-\frac{|x|^{4/3}}{2^{7/3} M(t-s)^{1/3}}} s^{-3/4} (1+s)^{-1/2} (1+\sqrt{s})^{-3/2} ds \\
&\leq C_2 \int_0^t e^{-\frac{K^{4/3}}{2M}(t-s)} (t-s)^{-1/4} s^{-3/4} (1+s)^{-1/2} (1+|x|+\sqrt{s})^{-3/2} ds.
\end{aligned} \tag{5.7}$$

In both of the last two estimates of (5.7), we can integrate the exponential decay in $(t-s)$. We observe that for $s \in [0, t/2]$, we have exponential decay in t, while for $s \in [t/2, t]$, we have algebraic decay at rate $(1+t)^{-2}$. In the event that $|x| \geq t$, we clearly have a sufficient estimate due to the $|x|$ decay, while for $|x| \leq t$, we have a sufficient estimate due to the t decay. We proceed in almost identical fashion for the remaining nonlinearity. Again, the analysis is driven by the case $|x-y| \leq K(t-s)$.

Case II, $|x-y| \leq K(t-s)$ and $(t-s) \geq 1$. For the case $|x-y| \leq K(t-s)$, and for $(t-s) \geq 1$, we proceed through the observation that in cases (i) and (ii) of Theorem 1.1, we have the crude estimate (for $t \geq 1$)

$$|\tilde{G}_y(t,x;y)| \leq C(1+t)^{-3/2} (1+|x-y|) e^{-\frac{(x-y)^2}{Mt}},$$

with additionally

$$|\tilde{G}_{xy}(t,x;y)| \le C(1+t)^{-3/2}e^{-\frac{(x-y)^2}{Mt}} + (1+t)^{-3/2}(1+|y|)e^{-\eta|x|}e^{-\frac{y^2}{Mt}},$$

while for case (iii) we have

$$|\tilde{G}_y(t,x;y)| \le C(1+t)^{-3/2}(1+|x-y|)e^{-\frac{(x-y)^2}{Mt}} + C(1+t)^{-3/2}(1+|y|)e^{-\eta|x|}e^{-\frac{y^2}{Mt}},$$

with additionally

$$|\tilde{G}_{xy}(t,x;y)| \le C(1+t)^{-3/2}e^{-\frac{(x-y)^2}{Mt}} + (1+t)^{-3/2}(1+|y|)e^{-\eta|x|}e^{-\frac{y^2}{Mt}}.$$

In cases (i) and (ii), and for the first summand of $\Psi(s,y)$, we consider integrals

$$\int_0^t \int_{-\infty}^0 (1+(t-s))^{-3/2}(1+|x-y|)e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2}dyds. \tag{5.8}$$

For t decay, we divide the integration over s into the subintervals $s\in[0,t/2]$ and $s\in[t/2,t]$. Observing the inequality

$$\frac{x-y}{(t-s)^{1/2}}e^{-\frac{(x-y)^2}{M(t-s)}} \le Ce^{-\frac{(x-y)^2}{2M(t-s)}}, \tag{5.9}$$

we determine an estimate by

$$\begin{aligned}
&C_1(1+t)^{-1}\int_0^{t/2}\int_{-\infty}^0 s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2}dyds\\
&\quad + C_2t^{-3/4}(1+t)^{-5/4}\int_{t/2}^t\int_{-\infty}^0 (1+(t-s))^{-1}e^{-\frac{(x-y)^2}{Mt}}dyds\\
&\quad \le C_3(1+t)^{-1}\int_0^{t/2} s^{-3/4}(1+s)^{-1/2}(1+\sqrt{s})^{-1/2}ds\\
&\quad + C_2t^{-3/4}(1+t)^{-5/4}\int_{t/2}^t\int_{-\infty}^0 (1+(t-s))^{-1/2}ds\\
&\quad \le C(1+t)^{-1}.
\end{aligned} \tag{5.10}$$

For $|x|$ decay, we divide the integration into subintervals of y, $y\in(-\infty,x/2]$ and $y\in[x/2,0]$. For $y\in(-\infty,x/2]$, we observe (5.9) and integrate the heat kernel to obtain an estimate by

$$C(1+|x|)^{-3/2}\int_0^t (1+(t-s))^{-1/2}s^{-3/4}(1+s)^{-1/2}ds \le C(1+|x|)^{-3/2}.$$

Alternatively, for $y\in[x/2,0]$, we observe that we have kernel decay

$$e^{-\frac{x^2}{Lt}},$$

and can otherwise proceed precisely as in (5.10) to obtain an estimate by

$$C(1+t)^{-1}e^{-\frac{x^2}{Lt}}.$$

This last estimate is sufficient by the argument of (5.5).

For the second (x-derivative) estimate of Lemma 4.2 in cases (i) and (ii), we have two Green's kernels to consider, of which we begin with integrals

$$\int_0^t \int_{-\infty}^0 (1+(t-s))^{-3/2}e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2}dyds. \tag{5.11}$$

For t decay, we divide the integration over s into subintervals $s \in [0, t/2]$ and $s \in [t/2, t]$ and obtain an estimate by

$$\begin{aligned} C_1(1+t)^{-3/2} &\int_0^{t/2} s^{-3/4}(1+s)^{-1/2}(1+\sqrt{s})^{-1/2} ds \\ &+ C_2 t^{-3/4}(1+t)^{-1/2}(1+\sqrt{t})^{-3/2} \int_{t/2}^t (1+(t-s))^{-1} ds \\ &\le C(1+t)^{-3/2}. \end{aligned}$$

For $|x|$ decay, we divide the integration into subintervals of y, $y \in (-\infty, x/2]$ and $y \in [x/2, 0]$. For $y \in (-\infty, x/2]$, we integrate the heat kernel to obtain an estimate by

$$C(1+|x|)^{-3/2} \int_0^t (1+(t-s))^{-1} s^{-3/4}(1+s)^{-1/2} ds \le C(1+t)^{-1}(1+|x|)^{-3/2}.$$

Alternatively, for $y \in [x/2, 0]$, our t-decay argument leads to an estimate by

$$C(1+t)^{-3/2} e^{-\frac{x^2}{Lt}},$$

which is sufficient by the argument of (5.5). For the second Green's kernel, we have integrals

$$\int_0^t \int_{-\infty}^0 (1+(t-s))^{-3/2}(1+|y|) e^{-\eta|x|} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2} dyds. \tag{5.12}$$

In this case, we always have exponential decay in $|x|$, and so we focus entirely on temporal decay. Dividing the integration over s into subintervals, we obtain an estimate by

$$\begin{aligned} C_1 e^{-\eta|x|}(1+t)^{-3/2} &\int_0^t \int_{-\infty}^0 (1+|y|)^{-1/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/2} dyds \\ + C_2 e^{-\eta|x|} t^{-3/4}(1+t)^{-1/2}(1+\sqrt{t})^{-3/2} &\int_0^t \int_{-\infty}^0 (1+(t-s))^{-1} e^{-\frac{y^2}{2M(t-s)}} dyds \\ \le C(1+t)^{-5/4} e^{-\eta|x|}, \end{aligned}$$

much better than required.

The case $y \le 0 \le x$ can be analyzed similarly as were the previous cases.

Excited estimates. We last consider the third and fourth estimates of Lemma 4.2. For the third, and for the first summand in $\Psi(s, y)$, we estimate

$$\begin{aligned} &\int_0^t \int_{-\infty}^0 \frac{C_E}{\sqrt{4b_-(t-s)}} e^{-\frac{y^2}{4b_-(t-s)}} s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2} dyds \\ &\le C_1 t^{-1/2} \int_0^{t/2} s^{-3/4}(1+s)^{-1/2}(1+\sqrt{s})^{-1/2} ds + C_2 t^{-3/4}(1+t)^{-5/4} \int_{t/2}^t ds \\ &\le C t^{-1/2}, \end{aligned} \tag{5.13}$$

where the seeming blow-up as $t \to 0$ can be eliminated by proceeding alternatively for t small.

For the fourth estimate in Lemma 4.2, and for the first summand of $\Psi(s, y)$, we estimate

$$\begin{aligned} &\int_0^{t-1} \int_{-\infty}^0 (t-s)^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/2}(1+|y|+\sqrt{s})^{-3/2} dyds \\ &\le C_1 t^{-3/2} \int_0^{t/2} s^{-3/4}(1+s)^{-3/4} + C_2 t^{-3/4}(1+t)^{-5/4} \int_{t/2}^{t-1} (t-s)^{-1} ds \\ &\le C t^{-3/2}, \end{aligned}$$

where we can eliminate the seeming blow-up as $t \to 0$ by proceeding alternatively for t small, and where the integration over $s \in [t-1, t]$ involves the small time estimates on $G(t, x; y)$ and can be carried out as in Case I.

This concludes the proof of Lemma 4.2. □

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