# Nonlinear stability of degenerate shock profiles

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January 9, 2006

**Abstract**

We consider degenerate viscous shock profiles arising in systems of two regularized conservation laws, where degeneracy here describes viscous shock profiles for which the asymptotic endstates are sonic to the associated hyperbolic system (the shock speed is equal to one of the characteristic speeds). Proceeding with pointwise estimates on the Green's function for the linear system of equations that arises upon linearization of the conservation law about a degenerate viscous shock profile, we establish that spectral stability, defined in terms of an Evans function, implies nonlinear stability. The asymptotic rate of decay for the perturbation is found both pointwise and in all $L^p$ norms, $p \geq 1$.

## 1 Introduction

We consider degenerate viscous shock profiles arising in the regularized system of conservation laws,

$$\begin{aligned} u_t + f(u)_x = u_{xx}, \quad u, f \in \mathbb{R}^2, \\ u(0,x) = u_0(x); \quad u_0(\pm\infty) = u_\pm; \end{aligned} \tag{1.1}$$

that is, solutions of the form $\bar{u}(x-st) = (\bar{u}_1(x-st), \bar{u}_2(x-st))^{\text{tr}}$ whose endpoints satisfy the Rankine–Hugoniot condition

$$s = \frac{f_k(u_1^+, u_2^+) - f_k(u_1^-, u_2^-)}{u_k^+ - u_k^-}, \quad k = 1, 2$$

and for which $s \in \text{Spectrum}(df(u_\pm)) =: \{a_k^\pm\}_{k=1}^2$. Throughout the analysis, we will make the following assumptions on the structure of (1.1) and the profile $\bar{u}(x-st)$:

(H0) $f \in C^2(\mathbb{R})$

(H1) (Lax degeneracy) Either $a_1^- < s < a_2^-$ and $a_1^+ < s = a_2^+$ (right side degenerate) or $a_1^- = s < a_2^-$ and $a_1^+ < s < a_2^+$ (left side degenerate).

(H2) (First order degeneracy) For either case $a_k^\pm = s$, we assume there holds

$$l_k^\pm d^2 f(u_\pm)(r_k^\pm, r_k^\pm) \neq 0,$$

where $l_k^\pm$ and $r_k^\pm$ denote the left and right eigenvectors of $df(u_\pm)$ respectively, and $d^2f(u_\pm)$ denotes the operator

$$d^2 f(u_\pm)(v,v) = \begin{pmatrix} \frac{1}{2}\partial_{u_1u_1} f_1(u_1^\pm, u_2^\pm)v_1^2 + \partial_{u_1u_2} f_1(u_1^\pm, u_2^\pm)v_1v_2 + \frac{1}{2}\partial_{u_1u_1} f_1(u_1^\pm, u_2^\pm)v_2^2 \\ \frac{1}{2}\partial_{u_1u_1} f_2(u_1^\pm, u_2^\pm)v_1^2 + \partial_{u_1u_2} f_2(u_1^\pm, u_2^\pm)v_1v_2 + \frac{1}{2}\partial_{u_1u_1} f_2(u_1^\pm, u_2^\pm)v_2^2. \end{pmatrix}$$

Under assumption (H2), both $\bar{u}_1(x-st)$ and $\bar{u}_2(x-st)$ decay to the degenerate side endstate with rate $|x-st|^{-1}$, a critical feature of the degenerate case (see [15]).

We note that assumptions (H1) and (H2) describe the most generic degenerate case. In particular, (H1) asserts that there is only degeneracy on one side, and not associated with both characteristics, while (H2) is analogous to the condition for single equations $f''(u_\pm) \neq 0$ (see [10, 11]). Our restriction to the case of identity viscosity and two equations follows from technical restrictions in [12, 15], and we regard the cases of general viscosity (including partial regularization) and an arbitrary number of equations as interesting directions for future work (see Remark 1 at the end of this section for a discussion of related issues). Finally, we mention that for the non-degenerate case, for which $s \notin \text{Spectrum}(df(u_\pm)) =: \{a_k^\pm\}_{k=1}^2$, the equations under consideration here, as well as a much broader class of equations, have been analyzed in [23] and in [16, 34].

Under the assumption of conditions (H0)–(H2), we use the detailed pointwise Green's function estimates developed in [12] for the linear system of convection–diffusion equations that arises upon linearization of (1.1) about $\bar{u}$ to establish that spectral stability as defined in [15] (and reviewed below) is sufficient for nonlinear stability. This extends the scalar case analysis of [10, 11] to the case of systems. We remark that the only systems analysis to date in the case of degenerate viscous shock profiles regards the partially regularized p-system,

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - p(v)_x &= u_{xx}, \end{aligned}$$

which can be reduced through coupling to a scalar analysis and analyzed by energy methods (see [27]).

Our analysis is motivated by the critical role degenerate viscous shock waves play in the theory of combustion: they correspond with the Chapman–Jouguet detonations and deflagrations for which wave speed is respectively minimal or maximal [6, 24, 25]. In particular, under certain conditions, Chapman–Jouguet detonations are the waves expected to be time-asymptotically selected in the *ignition problem*, for which initial data is taken as a large initializing pulse [6]. We are also motivated by the role degenerate waves play as a critical boundary case between Lax and undercompressive waves, and by the interest in near-linear systems for which approach to endstate is non-integrable (see also [32]).

Shifting without loss of generality to a moving coordinate system for which $s = 0$, we observe that one critical feature of degenerate profiles $\bar{u}(x)$ arising in (1.1) is that when (1.1) is linearized about $\bar{u}(x)$, the linearized eigenvalue problem $L(\bar{u}(x))v = \lambda v$ has the property that zero lies not only in both the point spectrum and the essential spectrum of $L(\bar{u}(x))$ (as is generically the case for viscous shock profiles), but is also a branch point of the Evans function (see [1, 5, 9, 18, 22, 15] and below). In the case of non-degenerate profiles, Gardner and Zumbrun have shown that for branch points near the origin (within the gap of their Gap Lemma), the Evans function can be analytically extended through the branch on an appropriate Riemann manifold [9]. Kapitula and Rubin have more recently employed a similar extension in the cases of the cubic nonlinear Schrödinger equation and the Ginzburg–Landau equation [21]. The algebraic (and non-integrable) approach to endstates of coefficients of $L(\bar{u}(x))$ in the case of degenerate profiles (see (H2)), however, seems to preclude the possibility of a similar analysis. Without analyticity of the Evans function, even in this extended sense, the usual manner of analysis near the origin by Taylor expansion cannot apply. Rather, the Evans function must be understood here as the sum of an analytic term and a lower order correction

$$D(\lambda) = D_a(\lambda) + \mathbf{O}(|\lambda^{3/2} \ln \lambda|),$$

for $|\lambda|$ sufficiently small.

It is well known that solutions $u(t,x)$, initially near $\bar{u}(x)$, will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined uniquely by the amount of perturbation mass (measured as $\int_{\mathbb{R}}(u(0,x) - \bar{u}(x))dx$) carried into the shock layer and the amount carried out to the far field along outgoing characteristics. We proceed, then, by defining the perturbation

$$v(t,x) = u(t, x + \delta(t)) - \bar{u}(x),$$

for which $\delta(t)$ will be chosen by the analysis to track the location of our perturbed wave in time. In this way, we compare the shapes of $u$ and $\bar{u}$, not their locations. Substituting $v$ into (1.1), we obtain the perturbation equation

$$v_t = Lv + Q(v)_x + \dot{\delta}(\bar{u}_x + v_x), \tag{1.2}$$

where $Lv = v_{xx} - (A(x)v)_x$, $A(x) = df(\bar{u}(x))$, and $Q(v) = \mathbf{O}(v^2)$ is a smooth function of $v$. Restricting the discussion without loss of generality to the case of right-side degeneracy, we observe that according to hypotheses (H0)–(H2), we have, for $A_\pm := \lim_{x\to\infty} A(x)$, that $A(x) \in C^1(\mathbb{R})$, and

$$\begin{aligned} |\partial_x^k(A(x) - A_+)| &= \mathbf{O}(|x|^{-k-1}), k = 0, 1, \quad \text{(degenerate side)} \\ |\partial_x^k(A(x) - A_-)| &= \mathbf{O}(e^{-\alpha|x|}), k = 0, 1, \quad \text{(non-degenerate side)}, \end{aligned}$$

for some $\alpha > 0$. We will denote the eigenvalues of $A(x)$ by $a_1(x)$ and $a_2(x)$ and take the convention that $a_2(x)$ is the eigenvalue that approaches 0 as $x \to +\infty$. Setting, then,

$$a_j^{\pm} = \lim_{x \to \pm\infty} a_j(x)$$

(with $a_2^+ = 0$), we have

$$|\partial_x^k (a_j(x) - a_j^+)| = \mathbf{O}(|x|^{-k-1}), k = 0, 1, \quad \text{(degenerate side)}$$
$$|\partial_x^k (a_j(x) - a_j^-)| = \mathbf{O}(e^{-\alpha|x|}), k = 0, 1, \quad \text{(non-degenerate side).}$$

On the non-degenerate side, we take the convention $a_1^- < a_2^-$.

Integrating (1.2), we have (after integration by parts on the second integral and upon observing that $e^{Lt}\bar{u}_x = \bar{u}_x$)

$$\begin{aligned} v(t,x) = &\int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds, \end{aligned} \tag{1.3}$$

where $G(t,x;y)$ represents a Green's function for the linear part of (1.2):

$$G_t + (A(x)G)_x = G_{xx}; \quad G(0,x;y) = \delta_y(x)I. \tag{1.4}$$

The main result of this paper is a demonstration that the estimates of [12] on the Green's function of (1.4) are sufficient for closing an iteration on (1.3), yielding estimates on $v(t,x)$.

The Green's function estimates of [12] are divided into those terms for which the $x$ dependence is exactly $\bar{u}_x(x)$ (referred to as the *excited* terms and denoted $\bar{u}_x(x)e(t,y)$) and the remaining terms, denoted $\tilde{G}$. Typically, the excited terms do not decay in $t$ and represent mass that accumulates in the shock layer, shifting the shock. Our approach will be to choose the shift $\delta(t)$ to annihilate this mass. Writing

$$G(t,x;y) = \tilde{G}(t,x;y) + \bar{u}_x(x)e(t,y),$$

we have

$$\begin{aligned} v(t,x) = &\int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy + \bar{u}_x(x)\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy + \delta(t)\bar{u}_x(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds. \end{aligned}$$

Choosing, then, $\delta(t)$ to eliminate the linear excited terms, we have

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy, \tag{1.5}$$

and

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds. \tag{1.6}$$

The Green's function $G(t,x;y)$ is analyzed in [12] through its Laplace transform $G_\lambda(x,y)$, which satisfies the ODE ($t \to \lambda$)

$$G_{\lambda xx} - (A(x)G_\lambda)_x - \lambda G_\lambda = -\delta_y(x)I,$$

and can be estimated by standard methods. Letting $\varphi_1^+$ and $\varphi_2^+$ represent the (necessarily) two linearly independent asymptotically decaying solutions at $+\infty$ of the eigenvalue ODE

$$L\varphi = \lambda\varphi, \tag{1.7}$$

and $\varphi_1^-$ and $\varphi_2^-$ similarly the two linearly independent asymptotically decaying solutions at $-\infty$, we write $G_\lambda(x,y)$ as a linear combination

$$G_\lambda(x,y) = \begin{cases} \varphi_1^+(x;\lambda)N_1^-(y;\lambda) + \varphi_2^+(x;\lambda)N_2^-(y;\lambda) & x > y \\ \varphi_1^-(x;\lambda)N_1^+(y;\lambda) + \varphi_2^-(x;\lambda)N_2^+(y;\lambda) & x < y. \end{cases}$$

Insisting on the continuity of $G_\lambda(x,y)$ across $y = x$, and a unit jump in $\partial_x G_\lambda(x,y)$ at the same point, we have (suppressing $\lambda$ dependence for notational brevity)

$$\begin{aligned} \varphi_1^+(y)N_1^-(y) + \varphi_2^+(y)N_2^-(y) - \varphi_1^-(y)N_1^+(y) - \varphi_2^-(y)N_2^+(y) &= 0 \\ {\varphi_1^+}'(y)N_1^-(y) + {\varphi_2^+}'(y)N_2^-(y) - {\varphi_1^-}'(y)N_1^+(y) - {\varphi_2^-}'(y)N_2^+(y) &= -I. \end{aligned} \tag{1.8}$$

Equations (1.8) represent a system of eight equations and eight unknowns, which decouple into two sets of four equations and four unknowns. Solving by Cramer's rule, we have, for example,

$$N_{11}^-(y;\lambda) = -\frac{\det \begin{pmatrix} \varphi_{21}^+ & \varphi_{11}^- & \varphi_{21}^- \\ {\varphi_{22}^+}' & {\varphi_{12}^-}' & {\varphi_{22}^-}' \\ {\varphi_{22}^+}' & {\varphi_{12}^-}' & {\varphi_{22}^-}' \end{pmatrix}}{\det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ {\varphi_1^+}' & {\varphi_2^+}' & {\varphi_1^-}' & {\varphi_2^-}' \end{pmatrix}}.$$

Clearly, then, $G_\lambda(x,y)$ will have no singularities so long as

$$W(x;\lambda) := \det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ {\varphi_1^+}' & {\varphi_2^+}' & {\varphi_1^-}' & {\varphi_2^-}' \end{pmatrix} \neq 0.$$

Following Jones et al. [1, 5, 9, 18, 22], we define the *Evans function* as $D(\lambda) = W(0;\lambda)$. In general, zeros of the Evans function correspond with eigenvalues of the operator $L$, an observation that has been made precise in [1] in the case—pertaining to reaction–diffusion equations—of isolated eigenvalues and in [9, 34] in the case—pertaining to conservation laws—of nonstandard "effective" eigenvalues embedded in essential spectrum. (The latter correspond with resonant poles of $L$, as examined in the scalar context in [29]).

In [15], the authors established that under assumptions (H0)–(H2), the Evans function $D(\lambda)$ can be constructed as a function analytic away from the negative real axis and to the right of a parabola opening into the negative-real half plane. In addition, near the critical point $\lambda = 0$, the authors showed that the Evans function can be constructed as an analytic function plus a smaller error,

$$D(\lambda) = D_a(\lambda) + \mathbf{O}(|\lambda^{3/2}\ln\lambda|), \quad \text{as } \lambda \to 0,$$

where $D_a(\lambda) = \mathbf{O}(|\lambda|)$ is analytic in a neighborhood of $\lambda = 0$. Following [15], we introduce the following stability condition $(\mathcal{D})$:

$(\mathcal{D})$: $D(\lambda)$ has precisely one zero in $\{\text{Re}\,\lambda \geq 0\}$, necessarily at $\lambda = 0$, and $D_a'(0) \neq 0$.

While Condition $(\mathcal{D})$ is generally quite difficult to verify analytically (see, for example, [4, 7, 8, 17, 19, 20, 26, 30]), it can be checked numerically (see [2, 3, 28]). A condition that lends itself more readily to exact study is the *stability index*, typically defined as

$$\Gamma := \text{sgn} D_a'(0) \times \text{sgn} \lim_{\mathbb{R}\ni\lambda\to\infty} D(\lambda).$$

For $\lambda \in \mathbb{R}_+$, we have $D(\lambda) \in \mathbb{R}$, so that in the event that $\Gamma = -1$, $D(\lambda)$ must have a positive real root, which guarantees instability. In the case that $\Gamma = 1$, the question of stability remains undecided.

We are now in a position to state the fundamental result of [12].

**Theorem 1.** *Suppose $\bar{u}(x)$ is a standing wave solution to (1.1) and suppose (H0)–(H2) hold, as well as stability criterion $(\mathcal{D})$. Then for some positive constants $M$, $K$, $P_+$, $\epsilon_0$ and $\eta$, depending only on $df(\bar{u}(x))$ and the spectrum of the operator $L$, the Green's function $G(t,x;y)$ described through (1.4) satisfies the following estimates.*

*(i)* $y \leq x \leq 0$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|x-\frac{a_1^-}{a_2^-}y - a_1^- t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|x-\frac{a_1^-}{a_2^-}y|\leq|a_1^-|t\}}.$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|x-\frac{a_1^-}{a_2^-}y - a_1^- t| + t^{1/2})^{-5/2}\ln(e+t)\Big)I_{\{|x-\frac{a_1^-}{a_2^-}y|\leq|a_1^-|t\}}.$$

*where*

$$e_-(t,y) = \mathbf{O}(1)e^{-\frac{(y+a_2^- t)^2}{Mt}} + \mathbf{O}(1)I_{\{|y|\leq|a_2^-|t\}}$$
$$\partial_y e_-(t,y) = \mathbf{O}(t^{-1/2})e^{-\frac{(y+a_2^- t)^2}{Mt}} + \mathbf{O}\Big((1+|y+a_2^- t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|y|\leq|a_2^-|t\}}$$
$$\partial_t e_-(t,y) = \mathbf{O}(t^{-1/2})e^{-\frac{(y+a_2^- t)^2}{Mt}} + \mathbf{O}\Big((1+|y+a_2^- t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|y|\leq|a_2^-|t\}}.$$

*(ii)* $x \leq y \leq 0$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_1^- t)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|x-\frac{a_1^-}{a_2^-}y - a_1^- t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|x-\frac{a_1^-}{a_2^-}y|\leq|a_1^-|t\}}.$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_1^- t)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|x-\frac{a_1^-}{a_2^-}y - a_1^- t| + t^{1/2})^{-5/2}\ln(e+t)\Big)I_{\{|x-\frac{a_1^-}{a_2^-}y|\leq|a_1^-|t\}}.$$

*(iii)* $x \leq 0 < K \leq y$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})e^{-\frac{(x-a_1^-\int_K^y \frac{ds}{a_1(s)} - a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_+(t,y) + \mathbf{O}(t^{-1/2})e^{-\frac{(x-a_1^- t)^2}{Mt}}I_{\{|x-a_1^-\int_K^y \frac{ds}{a_1(s)}|\leq|a_1^-|t\}}$$
$$+ \mathbf{O}\Big((1+|x-a_1^-\int_K^y \frac{ds}{a_1(s)} - a_1^- t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|x-a_1^-\int_K^y \frac{ds}{a_1(s)}|\leq|a_1^-|t\}}$$
$$+ \mathbf{O}(1+|x-a_1^- t| + t^{1/2})^{-3/2}\mathbf{O}(1+y)I_{\{|x-a_1^-\int_K^y \frac{ds}{a_1(s)}|\leq|a_1^-|t\}}.$$

$$G_y(t,x;y) = [\mathbf{O}(t^{-1}(1+t)^{1/4}) \wedge \mathbf{O}(t^{-1}(1+y))]e^{-\frac{(x-a_1^-\int_K^y \frac{ds}{a_1(s)} - a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_+(t,y)$$
$$+ \mathbf{O}\Big((1+|x-a_1^-\int_K^y \frac{ds}{a_1(s)} - a_1^- t| + t^{1/2})^{-5/2}\ln(e+t)\Big)I_{\{|x-a_1^-\int_K^y \frac{ds}{a_1(s)}|\leq|a_1^-|t\}}$$
$$+ \mathbf{O}(1+|x-a_1^- t| + t^{1/2})^{-3/2}I_{\{|x-a_1^-\int_K^y \frac{ds}{a_1(s)}|\leq|a_1^-|t\}},$$

*where*

$$e_+(t,y) = \mathbf{O}(1)e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(1)I_{\{|\int_K^y \frac{ds}{a_1(s)}|\le t\}} + \mathbf{O}(1)e^{-\frac{y^2}{Mt}}$$

$$\partial_y e_+(t,y) = \mathbf{O}(t^{-1/2})e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(t^{-1/2}(1+t)^{-1/2}(1+y))e^{-\frac{y^2}{Mt}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)}+t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|\int_K^y \frac{ds}{a_1(s)}|\le t\}},$$

$$\partial_t e_+(t,y) = \mathbf{O}(t^{-1/2})e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(t^{-1/2}(1+t)^{-1/2})e^{-\frac{y^2}{Mt}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)}+t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|\int_K^y \frac{ds}{a_1(s)}|\le t\}}.$$

*(iv)* $y \le 0 \le x$

$$G(t,x;y) = \mathbf{O}(t^{-1/2}(1+t)^{1/4})\mathbf{O}((1+x)^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}}I_{\{|y|\ge|a_2^-|t\}} + \bar{u}_x(x)e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|y+a_2^- t|)^{-1/2}\Big)\mathbf{O}((1+x)^{-1})e^{-\frac{x^2}{Mt}}I_{\{|y|\le|a_2^-|t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4}\ln(e+t))\mathbf{O}((1+x)^{-2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}}I_{\{|y|\ge|a_2^-|t\}}$$
$$+ \mathbf{O}\Big((1+|y+a_2^- t|)^{-1/2}\ln(e+t)\Big)\mathbf{O}((1+x)^{-2})e^{-\frac{x^2}{Mt}}I_{\{|y|\le|a_2^-|t\}}$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1}(1+t)^{1/4})\mathbf{O}((1+x)^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}}I_{\{|y|\ge|a_2^-|t\}} + \bar{u}_x(x)\partial_y e_-(t,y)$$
$$+ \mathbf{O}\Big((1+|y+a_2^- t|)^{-3/2}\Big)\mathbf{O}((1+x)^{-1})e^{-\frac{x^2}{Mt}}I_{\{|y|\le|a_2^-|t\}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4}\ln(e+t))\mathbf{O}((1+x)^{-2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}}I_{\{|y|\ge|a_2^-|t\}}$$
$$+ \mathbf{O}\Big((1+|y+a_2^- t|)^{-3/2}\ln(e+t)\Big)\mathbf{O}((1+x)^{-2})I_{\{|y|\le|a_2^-|t\}}.$$

*(v)* $0 < K \le y \le x$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}((1+|x|)^{-1})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar{u}_x(x)\Big(I_{\{|x-y|\le\epsilon_0\sqrt{t}\}} - I_{\{|y|\le\epsilon_0\sqrt{t}\}}\Big)$$
$$+ \bar{u}_x(x)e_+(t,y) + \mathbf{O}(t^{-1/2}[\ln(e+t)]^2)\mathbf{O}((1+|x|)^{-2})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4})\mathbf{O}((1+|x|)^{-1})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}}I_{\{|\int_K^y \frac{ds}{a_1(s)}|\ge t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4}\ln(e+t))\mathbf{O}((1+|x|)^{-2})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}}I_{\{|\int_K^y \frac{ds}{a_1(s)}|\ge t\}},$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}((1+|x|)^{-2})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}} + \bar{u}_x(x)\partial_y e_+(t,y)$$
$$+ \mathbf{O}(t^{-1}(1+t)^{-1/2}\ln(e+t))\mathbf{O}((1+|x|)^{-1})e^{-\frac{(x-y)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{-1/2})\mathbf{O}((1+|x|)^{-1})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}}$$
$$+ \mathbf{O}(t^{-1}\ln(e+t))\mathbf{O}((1+|x|)^{-2})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1}(1+t)^{1/4}\ln(e+t))\mathbf{O}(e^{-\eta|x-y|})$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4})\mathbf{O}((1+|x|)^{-1})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}}I_{\{|\int_K^y \frac{ds}{a_1(s)}|\ge t\}}.$$

*(vi)* $0 < K \leq x \leq y$

$$G(t,x;y) = \mathbf{O}(t^{-1/2})\mathbf{O}((1+|x|)^{-1})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar{u}_x(x)\Big(I_{\{|x-y|\leq\epsilon_0\sqrt{t}\}} - I_{\{|y|\leq\epsilon_0\sqrt{t}\}}\Big)$$
$$+ \bar{u}_x(x)e_+(t,y) + \mathbf{O}(t^{-1/2}[\ln(e+t)]^2)\mathbf{O}((1+|x|)^{-1})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}}$$
$$+ \mathbf{O}\Big((1+|\int_x^y \frac{ds}{a_1(s)} + t| + t^{1/2})^{-3/2}\ln(e+t)\Big)I_{\{|\int_x^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4})\mathbf{O}((1+|x|)^{-1})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\geq t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4}\ln(e+t))\mathbf{O}((1+|x|)^{-2})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\geq t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4})\mathbf{O}(1+|x|)^{-1}e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}(t^{-1/2}(1+t)^{1/4}\ln(e+t))\mathbf{O}(1+|x|)^{-2}e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)} + t|^{-1/2})\Big)\mathbf{O}\Big((1+|x|)^{-1}\Big)e^{-\frac{x^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)} + t|)^{-1/2}\ln(e+t)\Big)\mathbf{O}\Big((1+|x|)^{-2}\Big)e^{-\frac{x^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}.$$

$$G_y(t,x;y) = \mathbf{O}(t^{-1})\mathbf{O}((1+|x|)^{-1})\mathbf{O}(1+|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}}$$
$$+ \bar{u}_x(x)\partial_y e_+(t,y) + \mathbf{O}\Big((1+|\int_x^y \frac{ds}{a_1(s)} + t| + t^{1/2})^{-5/2}\ln(e+t)\Big)I_{\{|\int_x^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4})\mathbf{O}((1+|x|)^{-1})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\geq t\}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4}\ln(e+t))\mathbf{O}((1+|x|)^{-2})e^{-\frac{(\int_K^x \frac{ds}{a_1(s)}+\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\geq t\}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4})\mathbf{O}(1+|x|)^{-1}e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}(t^{-1}(1+t)^{1/4}\ln(e+t))\mathbf{O}(1+|x|)^{-2}e^{-\frac{(\int_K^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)} + t|)^{-3/2}\Big)\mathbf{O}\Big((1+|x|)^{-1}\Big)e^{-\frac{x^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}$$
$$+ \mathbf{O}\Big((1+|\int_K^y \frac{ds}{a_1(s)} + t|)^{-3/2}\ln(e+t)\Big)\mathbf{O}\Big((1+|x|)^{-2}\Big)e^{-\frac{x^2}{Mt}} I_{\{|\int_K^y \frac{ds}{a_1(s)}|\leq t\}}.$$

*In the case $0 \leq y \leq K$, the integrals $\int_K^y \frac{ds}{a_1(s)}$ can be replaced with 0, and similarly for $0 \leq x \leq K$. Here, capital $I$ denotes a characteristic function on the indicated interval and $\mathbf{O}(\cdot)$ denotes a function that is bounded by a constant multiple of the argument.*

For a discussion of the estimates of Theorem 1, we refer to [12], in which they are derived by contour-shifting arguments developed in [34] and extended to the case of degenerate viscous shock waves by the techniques of [10, 11]. We mention here only that the extensive number of summands in the cases $x, y \geq 0$ arises from the various interactions between the degenerate and non-degenerate characteristic speeds, and also that the behavior as $t \to 0$ (in particular, for $|x-y| \geq \bar{K}t$, $\bar{K}$ sufficiently large), is of the form of heat kernels,

$$\mathbf{O}(t^{-1/2})e^{-\frac{(x-y)^2}{Mt}},$$

which can be subsumed into the estimates of Theorem 1. Regarding the excited terms, we note the relation

$$e(t,y) = e_-(t,y)I_{\{y\leq 0\}} + e_+(t,y)I_{\{y>0\}}.$$

Also, we observe that in the case $x \le 0 < K \le y$, the term

$$P_+ \bar{u}_x(x) I_{\{|y| \le \epsilon_0 \sqrt{t}\}}$$

arises naturally in the analysis and is bounded by the expression

$$\bar{u}_x(x) \mathbf{O}(1) e^{-\frac{y^2}{Lt}}.$$

In the cases $x, y \ge K > 0$, the term that arises naturally in the analysis is

$$P_+ \bar{u}_x(x) I_{\{|x-y| \le \epsilon_0 \sqrt{t}\}},$$

and we replace this with

$$P_+ \bar{u}_x(x) I_{\{|x-y| \le \epsilon_0 \sqrt{t}\}} = P_+ \bar{u}_x(x) I_{\{|y| \le \epsilon_0 \sqrt{t}\}} + P_+ \bar{u}_x(x) \Big( I_{\{|x-y| \le \epsilon_0 \sqrt{t}\}} - I_{\{|y| \le \epsilon_0 \sqrt{t}\}} \Big), \tag{1.9}$$

where the first expression on the right-hand side of (1.9) is an expression of $e_+(t, y)$ and the second expression on the right-hand side of (1.9) appears in $\tilde{G}$.

The primary result of this paper is an estimate on the perturbation $v(t, x)$ in terms of the functions listed below, in which $L > 0$ and $\eta > 0$ are constants fixed throughout the analysis.
For $x \le 0$

$$\begin{aligned}
\theta^-(t, x) &= (1 + t)^{-1/2} e^{-\frac{(x - a_1^- t)^2}{Lt}} \\
\psi_1^-(t, x) &= (1 + |x| + t)^{-1/2} (1 + |x - a_1^- t|)^{-1/2} I_{\{a_1^- t \le x \le 0\}} \\
\psi_2^-(t, x) &= (1 + |x - a_1^- t| + t^{1/2})^{-3/2} \ln(e + t) \\
\alpha_1^-(t, x) &= (1 + |x|)^{-1/2} (1 + t)^{-3/4} I_{\{a_1^- t \le x \le 0\}} \\
\alpha_2^-(t, x) &= e^{-\eta |x|} (1 + t)^{-1/2},
\end{aligned}$$

and for $x \ge 0$

$$\begin{aligned}
\theta_1^+(t, x) &= (1 + t)^{-1/2} (1 + x)^{-1} e^{-\frac{x^2}{Lt}} \\
\theta_2^+(t, x) &= (1 + t)^{-1/2} \ln(e + t) (1 + x)^{-2} e^{-\frac{x^2}{Lt}} \\
\psi_1^+(t, x) &= (1 + x + t^{1/2})^{-3/2} \\
\psi_2^+(t, x) &= (1 + x + t)^{-3/2} \ln(e + t).
\end{aligned}$$

Here, $I$ denotes an indicator function on the specified interval.

We are now in a position to state the main result of the paper.

**Theorem 2.** *Suppose $\bar{u}(x)$ is a viscous profile solution to (1.1), where $f$ has possibly been redefined so that the profile speed can be taken as $s = 0$. Under conditions (H0)–(H2), and under the assumption of spectral condition $(\mathcal{D})$, we have the following: for initial perturbations*

$$|u(0, x) - \bar{u}(x)| \le E_0 (1 + |x|)^{-r}, \quad r > 2,$$

*some $E_0$ sufficiently small, there holds*

*For $x \le 0$,*

$$|u(t, x + \delta(t)) - \bar{u}(x)| \le C E_0 \Big[ \theta^- + \psi_1^- + \psi_2^- + \alpha_1^- + \alpha_2^- \Big](t, x),$$

*and for $x \ge 0$,*

$$|u(t, x + \delta(t)) - \bar{u}(x)| \le C E_0 \Big[ \theta_1^+ + \theta_2^+ + \psi_1^+ + \psi_2^+ \Big](t, x),$$

*with $\delta(t)$ defined in (1.5) satisfying*

$$\begin{aligned}
\delta(\infty) &= \int_{-\infty}^{+\infty} (u(0, x) - \bar{u}(x)) dx \\
|\dot{\delta}(t)| &\le C E_0 (1 + t)^{-1}.
\end{aligned}$$

We have the immediate corollary on $L^p$ asymptotic behavior.

**Corollary 1.** *Suppose $\bar{u}(x)$ is a viscous profile solution to (1.1), where $f$ has possibly been redefined so that the profile speed can be taken as $s = 0$. Under conditions (H0)–(H2), and under the assumption of spectral condition $(\mathcal{D})$, we have the following: for initial perturbations*

$$|u(0,x) - \bar{u}(x)| \leq E_0(1+|x|)^{-r}, \quad r > 2,$$

*some $E_0$ sufficiently small, there holds*

$$\|u(t, x + \delta(t)) - \bar{u}(x)\|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})},$$

*where $1 \leq p \leq \infty$ and $\delta(t)$ is as in Theorem 2.*

**Remark 1.** *Before proceeding with the proof of Theorem 2, we note several directions in which further analysis is warranted. First, we would like to extend the current analysis to the case of an arbitrary number of equations. We view the primary obstacle in that direction as a full development of a framework, similar as in [9], for the Evans function analysis. We are also interested in extension to generalized viscosities, both strictly parabolic type, as considered (for the case of non-degenerate waves) in [9, 34], and mixed hyperbolic–parabolic type as considered in [14, 31, 33]. The latter case is of particular interest, as it includes the reacting Navier–Stokes equations in which degenerate shock profiles arise as the Chapman–Jouguet solutions [24, 25]. Finally, we note that the Chapman–Jouguet waves typically arise in combination with rarefaction waves, and so we regard the analysis of two-wave degenerate–rarefaction patterns as a case of fundamental importance.*

# 2 Estimates on the Perturbation

In this section, we employ estimates on the linear and nonlinear integrals in (1.6) to establish Theorem 2.

**Lemma 1.** *Under the assumptions of Theorem 1, and for $v_0(y)$ satisfying*

$$|v_0(y)| \leq E_0(1+|y|)^{-r}, \quad r > 2,$$

*there holds*

*For $x \leq 0$,*

$$\Big|\int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy\Big| \leq CE_0\Big[\theta^-(t,x) + \psi_2^-(t,x)\Big],$$

$$\Big|\int_{-\infty}^{+\infty} e_t(t,y)v_0(y)dy\Big| \leq CE_0(1+t)^{-1},$$

*and for $x \geq 0$,*

$$\Big|\int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy\Big| \leq CE_0\Big[\theta_1^+(t,x) + \theta_2^+(t,x) + \psi_1^+(t,x) + \psi_2^+(t,x)\Big],$$

$$\Big|\int_{-\infty}^{+\infty} e_t(t,y)v_0(y)dy\Big| \leq CE_0(1+t)^{-1}.$$

*Here, the constant $M$ appearing in $G(t,x;y)$, and the constant $L$ appearing in the $\theta_k^\pm$ satisfy $L > M$.*

**Lemma 2.** *For the Green's function $G(t,x;y)$ of Theorem 1 there holds*

*For $x \leq 0$,*

$$\int_0^t \int_{-\infty}^{+\infty} |G_y(t-s,x;y)|\Psi(s,y)dyds \leq C\Big[\theta^-(t,x) + \psi_1^-(t,x) + \psi_2^-(t,x) + \alpha_1^-(t,x) + \alpha_2^-(t,x)\Big],$$

*and for $x \geq 0$,*

$$\int_0^t \int_{-\infty}^{+\infty} |G_y(t-s,x;y)|\Psi(s,y)dyds \leq C\Big[\theta_1^+(t,x) + \theta_2^+(t,x) + \psi_1^+(t,x) + \psi_2^+(t,x)\Big],$$

*where*

$$\Psi(s,y) = \Psi_-(s,y) I_{\{y\leq 0\}} + \Psi_+(s,y) I_{\{y\geq 0\}},$$

*and*

$$\begin{aligned}\Psi_-(s,y) &= \Big(\theta^-(s,y) + \psi_1^-(s,y) + \psi_2^-(s,y) + \alpha_1^-(s,y) + \alpha_2^-(s,y)\Big)^2 \\ &+ (1+s)^{-1}\Big(\theta^-(s,y) + \psi_1^-(s,y) + \psi_2^-(s,y) + \alpha_1^-(s,y) + \alpha_2^-(s,y)\Big) \\ \Psi_+(s,y) &= \Big(\theta_1^+(s,y) + \theta_2^+(s,y) + \psi_1^+(s,y) + \psi_2^+(s,y)\Big)^2 \\ &+ (1+s)^{-1}\Big(\theta_1^+(s,y) + \theta_2^+(s,y) + \psi_1^+(s,y) + \psi_2^+(s,y)\Big).\end{aligned}$$

*Here, the constant $M$ appearing in $G(t,x;y)$, and the constant $L$ appearing in the $\theta_k^\pm$ satisfy $L > M$.*

The proofs of Lemma 1 and 2 are given in Section 3.

**Proof of Theorem 2.** We proceed now by defining the iteration variable

$$\begin{aligned}\zeta(t) := &\sup_{y\leq 0, 0\leq s\leq t}\Big[|v(s,y)|(\theta^-(s,y) + \psi_1^-(s,y) + \psi_2^-(s,y) + \alpha_1^-(s,y) + \alpha_2^-(s,y))^{-1}\Big] \\ + &\sup_{y\geq 0, 0\leq s\leq t}\Big[|v(s,y)|(\theta_1^+(s,y) + \theta_2^+(s,y) + \psi_1^+(s,y) + \psi_2^+(s,y))^{-1}\Big].\end{aligned}$$

Clearly, then, for $x \leq 0$,

$$|v(t,x)| \leq \zeta(t)\Big(\theta^-(t,x) + \psi_1^-(t,x) + \psi_2^-(t,x) + \alpha_1^-(t,x) + \alpha_2^-(t,x)\Big),$$

while for $x \geq 0$,

$$|v(t,x)| \leq \zeta(t)\Big(\theta_1^+(t,x) + \theta_2^+(t,x) + \psi_1^+(t,x) + \psi_2^+(t,x)\Big).$$

We have the following claim.

**Claim 1.** *Suppose there exists some constant $C$ so that*

$$\zeta(t) \leq C(E_0 + \zeta(t)^2),$$

*where $E_0$ is as in Theorem 2. Then for $E_0$ sufficiently small, $\zeta(t) < 2CE_0$.*

**Proof of Claim 1.** We first observe that we have control over $\zeta(0)$ directly from the defininition of $\zeta(t)$. Recalling the relation

$$|v(0,y)| \leq E_0(1+|y|)^{-r},$$

we have that $\zeta(0) \leq C_1 E_0$, for some constant $C_1$. We proceed now by choosing $E_0$ sufficiently small so that $C_1^2 E_0 < 1$ and $4C^2 E_0 < 1$. First, this insures

$$\zeta(0) \leq C(E_0 + \zeta(0)^2) \leq CE_0 + CC_1^2 E_0^2 < CE_0 + CE_0 = 2CE_0.$$

Next, we let $T$ denote first time for which we have equality, $\zeta(T) = 2CE_0$. We have, then,

$$\zeta(T) \leq C(E_0 + \zeta(T)^2) = C(E_0 + 4C^2 E_0^2) < C(E_0 + E_0) = 2CE_0,$$

a contradiction. □

In the case $x \leq 0$, according to (1.6), we have

$$\begin{aligned}|v(t,x)| &\leq \int_{-\infty}^{+\infty} |\tilde{G}(t,x;y)||v_0(y)|dy + C\int_0^t\int_{-\infty}^{+\infty} |G_y(t-s,x;y)|\Big[v(s,y)^2 + |\dot{\delta}(s)||v(s,y)|\Big]dyds \\ &\leq C_1 E_0\Big(\theta^-(t,x) + \psi_2^-(t,x)\Big) + C_2\zeta(t)^2\int_0^t\int_{-\infty}^{+\infty}|G_y(t-s,x;y)|\Psi(s,y)dyds \\ &\leq C(E_0 + \zeta(t)^2)\Big(\theta^-(t,x) + \psi_1^-(t,x) + \psi_2^-(t,x) + \alpha_1^-(t,x) + \alpha_2^-(t,x)\Big),\end{aligned}$$

and similarly for $x \geq 0$. In this way, we have

$$\Big(\theta^-(t,x) + \psi_1^-(t,x) + \psi_2^-(t,x) + \alpha_1^-(t,x) + \alpha_2^-(t,x)\Big)^{-1} |v(t,x)| \leq C(E_0 + \zeta(t)^2),$$

and again similarly for $x \leq 0$. Since this last expression is valid for each time $t$, and since $\zeta(t)$ is nondecreasing (i.e., $\sup_{0\leq s\leq t} \zeta(s) = \zeta(t)$), we have

$$\sup_{y\leq 0, 0\leq s\leq t} \Big[\Big(\theta^-(s,x) + \psi_1^-(s,x) + \psi_2^-(s,x) + \alpha_1^-(s,x) + \alpha_2^-(s,x)\Big)^{-1} |v(s,x)|\Big] \leq C(E_0 + \zeta(t)^2).$$

Proceeding similarly for $x \geq 0$, we conclude

$$\zeta(t) \leq C(E_0 + \zeta(t)^2).$$

Theorem 2 can now be established by Claim 1. □

**Proof of Corollary 1.** Corollary 1 is proven by direct integration of the estimates from Theorem 2.

# 3 Integral Estimates

In this section, we establish the integral estimates for Lemma 1 and Lemma 2. As these estimates will be a cornerstone in the extensions discussed in Remark 1, we carry them out in some detail. Even so, many cases are similar to cases already examined in detail (either here or in referenced work) or straightforward by direct calculation, and to the extent that completeness will allow, these are omitted.

**Proof of Lemma 1.** For the case $x, y \leq 0$, we consider two critical cases, from which the remaining cases follow. First, we estimate

$$\begin{aligned}
&\Big|\int_{-\infty}^0 t^{-1/2} e^{-\frac{(x - \frac{a_1^-}{a_2^-}y - a_1^- t)^2}{Mt}} v_0(y)dy\Big| \\
&\leq \Big|\int_{\{|y|\leq |a_2^-|/(2|a_1^-|)|x-a_1^-t|\}} t^{-1/2} e^{-\frac{(x - \frac{a_1^-}{a_2^-}y - a_1^- t)^2}{Mt}} v_0(y)dy\Big| \\
&+ \Big|\int_{\{|y|\geq |a_2^-|/(2|a_1^-|)|x-a_1^-t|\}} t^{-1/2} e^{-\frac{(x - \frac{a_1^-}{a_2^-}y - a_1^- t)^2}{Mt}} v_0(y)dy\Big| \\
&\leq C\Big[(1+t)^{-1/2} e^{-\frac{(x-a_1^-t)^2}{Lt}} + (1 + |x - a_1^- t| + t^{1/2})^{-r}\Big],
\end{aligned} \tag{3.1}$$

where $L > M$. We observe that the appearance of $t^{1/2}$ in the second term is justified by the observation that in the case $|x - a_1^- t| \leq \sqrt{t}$, we immediately have an estimate of the form of the first term. We next consider the algebraic term in $G(t,x;y)$, for which we proceed almost exactly as above to obtain an estimate by

$$\begin{aligned}
&\Big|\int_{-\infty}^0 \Big((1 + |x - \frac{a_1^-}{a_2^-}y - a_1^- t| + t^{1/2})^{-3/2} \ln(e+t)\Big) v_0(y)dy\Big| \\
&\leq C(1 + |x - a_1^- t| + t^{1/2})^{-3/2} \ln(e+t).
\end{aligned}$$

For the case $x \leq 0 \leq K \leq y$, we again consider two cases, beginning with

$$\int_0^{+\infty} t^{-1/2} e^{-\frac{(x - a_1^- \int_K^y \frac{ds}{a_1(s)} ds - a_1^- t)^2}{Mt}} v_0(y)dy.$$

We observe here that $a_1(s)^{-1} = (a_1^-)^{-1} + \mathbf{O}((1+|s|)^{-1})$ (see the discussion following (1.2)), and consequently for $|y| \geq K$

$$\int_K^y \frac{ds}{a_1(s)} ds = (a_1^-)^{-1} y + \mathbf{O}(\ln(1+|y|)). \tag{3.2}$$

In this way, we can proceed as in (3.1) to obtain an estimate on this term by

$$C\Big[(1+t)^{-1/2}e^{-\frac{(x-a_1^- t)^2}{Lt}} + (1+|x-a_1^- t|+t^{1/2})^{-r}\Big].$$

For the second estimate in this case, we consider integrals of the form

$$\begin{aligned}\int_K^{+\infty} &(1+|x-a_1^- t|+t^{1/2})^{-3/2}(1+y)v_0(y)dy\\ &\le CE_0\int_K^{+\infty}(1+|x-a_1^- t|+t^{1/2})^{-3/2}(1+y)^{1-r}dy\\ &\le CE_0\psi_2^-(t,x),\end{aligned}$$

so long as $r>2$.

For the case $y\le 0\le x$, we consider two cases, beginning with

$$\begin{aligned}\Big|\int_{-\infty}^{-|a_2^-|t} &t^{-1/2}(1+t)^{1/4}(1+x)^{-1}e^{-\frac{(x-y-a_2^- t)^2}{Mt}}v_0(y)dy\Big|\\ &\le C(1+x)^{-1}(1+t)^{1/4}(1+t)^{-r}e^{-\frac{x^2}{Mt}},\end{aligned}$$

where we have observed that for $|y|\ge a_2^- t$, this kernel always decays like a heat kernel in $x$. In this case, we additionally consider the integral

$$\begin{aligned}\Big|\int_{-a_2^- t}^{0} &(1+|y+a_2^- t|)^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{Mt}}v_0(y)\Big|\\ &\le C(1+t)^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{Mt}}.\end{aligned}$$

For the case $x,\, y\ge K\ge 0$, we consider two cases, beginning with

$$\begin{aligned}\int_0^{+\infty} &t^{-1/2}(1+x)^{-1}(1+y)e^{-\frac{(x-y)^2}{Mt}}|v_0(y)|dy\\ &\le E_0\int_0^{+\infty}t^{-1/2}(1+x)^{-1}e^{-\frac{(x-y)^2}{Mt}}(1+y)^{1-r}dy\\ &\le CE_0\Big[t^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{Lt}}+(1+x)^{-r}\Big],\end{aligned}$$

which is sufficient for $r>2$. Second, we consider integrals of the form

$$P_+\bar{u}_x(x)\int_0^{+\infty}\Big(I_{\{|x-y|\le\epsilon_0\sqrt{t}\}}-I_{|y|\le\epsilon_0\sqrt{t}}\Big)dy. \tag{3.3}$$

Since $\bar{u}_x(x)=\mathbf{O}(1+|x|)^{-2}$, in the event that $|x|\ge\epsilon_0\sqrt{t}$, we have decay of the form $(1+|x|+\sqrt{t})^{-2}$, which is sufficient. In the alternative case that $|x|\le\epsilon_0\sqrt{t}$, we have

$$\begin{aligned}\Big|P_+\bar{u}_x(x)\Big[&\int_{x-\epsilon_0\sqrt{t}}^{x+\epsilon_0\sqrt{t}}v_0(y)dy-\int_0^{\epsilon_0\sqrt{t}}v_0(y)dy\Big]\Big|\\ &\le\Big|P_+\bar{u}_x(x)\int_{\epsilon_0\sqrt{t}}^{x+\epsilon_0\sqrt{t}}v_0(y)dy\Big|\le C(1+|x|)^{-1}(1+\sqrt{t})^{-r},\end{aligned}$$

which is sufficient for $r>2$. The remaining estimates of Lemma 1 can be proved similarly. We note here only that the defining estimate regards $e_+(t,y)$,

$$\Big|\int_{-\infty}^{+\infty}t^{-1/2}(1+t)^{-1/2}e^{-\frac{y^2}{M(t-s)}}v_0(y)dy\Big|\le C(1+t)^{-1}.$$

□

**Proof of Lemma 2.** We observe at the outset that in the case $x \le 0$, we have in principle ten nonlinearities to consider, ${\theta^-}^2$, ${\psi_1^-}^2$, ${\psi_2^-}^2$, ${\alpha_1^-}^2$, ${\alpha_2^-}^2$, $\dot{\delta}\theta^-$, $\dot{\delta}\psi_1^-$, $\dot{\delta}\psi_2^-$, $\dot{\delta}\alpha_1^-$, and $\dot{\delta}\alpha_2^-$. We observe, however, that, aside from a reduced rate of exponential decay, $\dot{\delta}\theta^-$ is bounded by ${\theta^-}^2$, and that $\dot{\delta}\psi_1^-$ is bounded by ${\psi_1^-}^2$, while $\dot{\delta}\alpha_2^-$ is bounded by ${\alpha_2^-}^2$—again, aside from a reduced rate of exponential decay. In this way, we reduce the number of nonlinearities in the analysis to seven.

**Case 1, $x$, $y \le 0$.** For the case $x$, $y \le 0$, we consider three integrals, beginning with

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} \Psi(s,y)dyds.$$

Integration against the nonlinearities $\theta^-(s,y)^2$, $\psi_1^-(s,y)^2$, and $|\dot{\delta}|(\theta^-(s,y) + \psi_1^-(s,y) + \psi_2^-(s,y))$ have been considered in [16], while integration against the nonlinearities $\alpha_1^-(s,y)^2$, $\alpha_2^-(s,y)^2$ and $|\dot{\delta}|(\alpha_1^-(s,y) + \alpha_2^-(s,y))$ have been considered in [14]. We focus our attention on the nonlinearity $\psi_2^-(s,y)^2$, for which we have integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} (1 + |y - a_1^- s| + s^{1/2})^{-3} [\ln(e+s)]^2 dyds. \tag{3.4}$$

Here, we have a factor $[\ln(e+s)]^2$ that did not appear in the analysis of [16]. In the event that $|x| \ge |a_1^-|t$, we write

$$x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s) = (x - a_1^- t) - (\frac{a_1^-}{a_2^-}y - a_1^- s), \tag{3.5}$$

for which we observe that there is no cancellation between summands. In this case, we have an estimate on (3.4) by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_1^- t)^2}{Mt}} \int_0^{t/2} (1+s^{1/2})^{-2} [\ln(e+s)]^2 ds \\ &+ C_2 (1+t^{1/2})^{-3} [\ln(e+t)]^2 e^{-\frac{(x-a_1^- t)^2}{Mt}} \int_{t/2}^t (t-s)^{-1/2} ds \\ &\le C t^{-1} [\ln(e+t)]^3 e^{-\frac{(x-a_1^- t)^2}{Mt}}. \end{aligned} \tag{3.6}$$

We note that the seeming blow-up as $t \to 0$ can be eliminated by integrating the kernel in each case to get an alternative estimate for $t$ bounded. In the event that $|x| \le |a_1^-|t$, we write

$$x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s) = \Big(x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s\Big) - \frac{a_1^-}{a_2^-}\Big(y - a_1^- s\Big), \tag{3.7}$$

from which we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} (1 + |y - a_1^- s| + s^{1/2})^{-3} \\ &\le C\Big[e^{-\epsilon\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}} (1 + |y - a_1^- s| + s^{1/2})^{-3} \\ &+ e^{-\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} (1 + |y - a_1^- s| + s^{1/2} + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s|)^{-3}\Big], \end{aligned} \tag{3.8}$$

for $L > M_1 > M$. For the first estimate in (3.8), we have integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\epsilon\frac{(x-\frac{a_1^-}{a_2^-}y - a_1^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}} (1 + |y - a_1^- s| + s^{1/2})^{-3} [\ln(e+s)]^2 dyds. \tag{3.9}$$

At this point, we subdivide the analysis into cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we write

$$x - a_1^-(t-s) - \frac{a_1^-}{a_2^-} a_1^- s = (x - a_1^- t) - \frac{a_1^-}{a_2^-}(a_1^- - a_2^-)s, \tag{3.10}$$

through which we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}}(1+|y-a_1^- s|+s^{1/2})^{-3} \\ &\le C\Big[e^{-\frac{(x-a_1^- t)^2}{Lt}}(1+|y-a_1^- s|+s^{1/2})^{-3} + e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}}(1+|y-a_1^- s|+s^{1/2}+|x-a_1^- t|)^{-3}\Big]. \end{aligned} \tag{3.11}$$

For the first estimate in (3.11), we proceed similarly as in (3.6), while for the second, we have an estimate by

$$C_1 t^{-1}(1+|x-a_1^- t|)^{-1/2}\int_0^{t/2}(1+s^{1/2})^{-3/2}[\ln(e+s)]^2 ds \le C\psi_1^-(t,x).$$

For $s \in [t/2, t]$, we write

$$x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s = (x - \frac{a_1^-}{a_2^-}a_1^- s) - a_1^-(t-s), \tag{3.12}$$

for which we observe that for $x \le 0$, the first summand in parentheses is always negative, and for $s \in [t/2, t]$, always of size $-(|x|+t)$. For $s \in [t/2, t]$, we have the inequality

$$\begin{aligned} &(t-s)^{-1/2}e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}} \\ &\le C\Big[(|x|+t)^{-1/2}e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}} + (t-s)^{-1/2}e^{-\frac{(|x|+t)^2}{Lt}}\Big]. \end{aligned} \tag{3.13}$$

For the second estimate in (3.13), we proceed similarly as in (3.6), while for the first, we have an estimate by

$$C_2(1+t^{1/2})^{-3}[\ln(e+t)]^2(|x|+t)^{-1/2}\int_{t/2}^t e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1(t-s)}} ds \le C\psi_1^-(t,x). \tag{3.14}$$

For the second estimate in (3.8), we have integrals of the form

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s))^2}{M(t-s)}}(1+|y-a_1^- s|+s^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3}[\ln(e+s)]^2 dyds. \tag{3.15}$$

Again, we subdivide the analysis into the cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (3.10) the estimate

$$\begin{aligned} &(1+|y-a_1^- s|+s^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3} \\ &\le C(1+|y-a_1^- s|+s^{1/2}+|x-a_1^- t|^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3}. \end{aligned} \tag{3.16}$$

We can estimate (3.15) for $s \in [0, t/2]$ by

$$\begin{aligned} &C_1 t^{-1}(1+|x-a_1^- t|^{1/2})^{-1}\int_0^{t/2}(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-1}[\ln(e+s)]^2 ds \\ &\le Ct^{-1}[\ln(e+t)]^3(1+|x-a_1^- t|^{1/2})^{-1}. \end{aligned}$$

For $s \in [t/2, t]$, we observe through (3.12) the estimate

$$\begin{aligned}(t-s)^{-1/2}&(1+|y-a_1^- s|+s^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3}\\ &\le C\Big[(|x|+t)^{-1/2}(1+|y-a_1^- s|+s^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3}\\ &+(t-s)^{-1/2}(1+|y-a_1^- s|+s^{1/2}+(|x|+t))^{-3}\Big].\end{aligned}$$

For the second, we have an estimate by

$$\begin{aligned}C_2(1+(|x|+t))^{-3}[\ln(e+t)]^2&\int_{t/2}^t (t-s)^{-1/2}ds\\ &\le C\psi_1^-(t,x),\end{aligned}$$

while for the first we have an estimate on (3.15) by

$$\begin{aligned}C(|x|+t)^{-1/2}&\int_{t/2}^t (1+t^{1/2}+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|)^{-3}[\ln(e+s)]^2 ds\\ &\le C(|x|+t)^{-1/2}(1+t^{1/2})^{-2}[\ln(e+t)]^2\le C\psi_1^-(t,x).\end{aligned}$$

We next consider integrals of the form

$$\int_0^t\int_{-\infty}^0 \Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big)\Psi(s,y)dyds, \tag{3.17}$$

beginning with the nonlinearity $\theta^-(s,y)^2$, for which we have integrals of the form

$$\int_0^t\int_{-\infty}^0 \Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big)(1+s)^{-1}e^{-\frac{(y-a_1^- s)^2}{Ms}}dyds. \tag{3.18}$$

In the event that $|x|\ge|a_1^-|t$, we observe that there is no cancellation between summands in (3.5), and consequently that we obtain an estimate by

$$\begin{aligned}C(1+|x-a_1^- t|)^{-3/2}&\int_0^t (1+(t-s)^{1/2})^{-1}\ln(e+(t-s))(1+s)^{-1}s^{1/2}\\ &\le C(1+|x-a_1^- t|)^{-3/2}\ln(e+t),\end{aligned} \tag{3.19}$$

which is sufficient in the case $|x-a_1^- t|\ge\sqrt{t}$ (i.e., it is bounded by $\psi_2^-(t,x)$). In the case $|x-a_1^- t|\le\sqrt{t}$, we require only $t^{-1/2}$ decay, which is immediate. In the event that $|x|\le|a_1^-|t$, we observe through (3.7) the inequality

$$\begin{aligned}(1+|x-&\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}e^{-\frac{(y-a_1^- s)^2}{Ms}}\\ &\le C\Big[(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|+(t-s)^{1/2})^{-5/2}e^{-\frac{(y-a_1^- s)^2}{Ms}}\\ &+(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1 t}}e^{-\epsilon\frac{(y-a_1^- s)^2}{Ms}}\Big].\end{aligned} \tag{3.20}$$

For the first estimate in (3.20), we consider integrals of the form

$$\int_0^t\int_{-\infty}^0 (1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^- s|+(t-s)^{1/2})^{-5/2}(1+s)^{-1}e^{-\frac{(y-a_1^- s)^2}{Ms}},$$

for which we divide the analysis into cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (3.10) the estimate

$$\begin{aligned}
&(1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + (t-s)^{1/2})^{-5/2}(1+s)^{-1/2} \\
&\le C\Big[(1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + |x - a_1^- t| + (t-s)^{1/2})^{-5/2}(1+s)^{-1/2} \\
&+ (1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + (t-s)^{1/2})^{-5/2}(1 + |x - a_1^- t|)^{-1/2}\Big].
\end{aligned} \tag{3.21}$$

For the first integral in (3.21), we proceed similarly as in (3.19), while for the second we have an estimate by

$$\begin{aligned}
&C_1(1 + t^{1/2})^{-1}(1 + |x - a_1^- t|)^{-1/2} \int_0^{t/2} (1 + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s|)^{-3/2} \\
&\le C\psi_1^-(t,x).
\end{aligned}$$

For $s \in [t/2, t]$, we observe through (3.12) the estimate

$$\begin{aligned}
&\Big(1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + (t-s)^{1/2}\Big)^{-5/2} \\
&\le C\Big(1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + (t-s)^{1/2} + (|x| + t)^{1/2}\Big)^{-5/2}.
\end{aligned}$$

Accordingly, we have an estimate for $s \in [t/2, t]$ by

$$\begin{aligned}
&C(1+t)^{-1}\ln(e+t) \int_{t/2}^t \Big((1 + |x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s| + (|x| + t)^{1/2}\Big)^{-3/2} ds \\
&\le C(1+t)^{-1}(1 + |x| + t)^{-1/2}\ln(e+t) \le \psi_1^-(t,x).
\end{aligned}$$

For the second estimate in (3.20), we have integrals of the form

$$\begin{aligned}
&\int_0^t \int_{-\infty}^0 (1 + |x - \frac{a_1^-}{a_2^-}y - a_1^-(t-s)| + (t-s)^{1/2})^{-5/2} e^{-\frac{(x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1 t}} \\
&\quad \times e^{-\epsilon\frac{(y - a_1^- s)^2}{Ms}}(1+s)^{-1}\ln(e + (t-s))dyds.
\end{aligned}$$

Recalling that the case $|x| \ge |a_1^-|t$ has already been considered, we take $|x| \le |a_1^-|t$, for which we divide the analysis into the cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (3.10) the inequality

$$\begin{aligned}
&e^{-\frac{(x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1 t}}(1+s)^{-1} \\
&\le C\Big[e^{-\frac{(x - a_1^- t)^2}{Lt}}(1+s)^{-1} + e^{-\frac{(x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1 t}}(1 + s + |x - a_1^- t|)^{-1}\Big].
\end{aligned} \tag{3.22}$$

For the first estimate in (3.22), we proceed similarly as in (3.6), while for the second estimate in (3.22), we have an estimate by

$$\begin{aligned}
&C_1(1+t)^{-5/4}\ln(e+t)(1 + |x - a_1^- t|)^{-1/2} \int_0^{t/2} (1+s)^{-1/2} s^{1/2} e^{-\frac{(x - a_1^-(t-s) - \frac{a_1^-}{a_2^-}a_1^- s)^2}{M_1 t}} ds \\
&\le C(1+t)^{-3/4}[\ln(e+t)](1 + |x - a_1^- t|)^{-1/2} \le C\psi_1^-(t,x).
\end{aligned}$$

For $s \in [t/2, t]$, we observe through (3.12) the inequality

$$(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s)^2}{M_1 t}}$$
$$\le C\Big[(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}+(|x|+t)^{1/2})^{-5/2}e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s)^2}{M_1 t}}$$
$$+(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}e^{-\frac{(|x|+t)^2}{Lt}}\Big]. \tag{3.23}$$

For the second estimate in (3.23), we have exponential decay in both $|x|$ and $t$, while for the first we have an estimate by

$$C_2(1+|x|+t)^{-3/4}(1+t)^{-1}\ln(e+t)\int_{t/2}^{t} e^{-\frac{(x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s)^2}{M_1 t}}ds$$
$$\le C(1+|x|+t)^{-3/4}(1+t)^{-1/2}\ln(e+t)\le C\psi_1^-(t,x).$$

We next consider integrals (3.17) with nonlinearity $\psi_1^-(s,y)^2$, for which we have integrals of the form

$$\int_0^t\int_{-|a_1^-|s}^0\Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big)(1+|y|+s)^{-1}(1+|y-a_1^-s|)^{-1}dyds. \tag{3.24}$$

In the event that $|x|\ge|a_1^-|t$, we observe that there is no cancellation between summands in (3.5), and consequently that we obtain an estimate by

$$C_1(1+|x-a_1^-t|)^{-3/2}\int_0^t(1+(t-s)^{1/2})^{-1}\ln(e+(t-s))(1+s)^{-1}\ln(e+s)ds$$
$$\le C(1+|x-a_1^-t|)^{-3/2}, \tag{3.25}$$

which is sufficient for $|x-a_1^-t|\ge\sqrt{t}$. In the case $|x-a_1^-t|\le\sqrt{t}$, we require only decay at rate $t^{-1/2}$, which is immediate. In the event that $|x|\le|a_1^-|t$, we observe through (3.7) the estimate

$$\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}\Big)^{-5/2}\Big(1+|y-a_1^-s|\Big)^{-1}$$
$$\le C\Big[\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}\Big)^{-5/2}\Big(1+|y-a_1^-s|\Big)^{-1}$$
$$+\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}\Big)^{-5/2}\Big(1+|y-a_1^-s|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|\Big)^{-1}\Big]. \tag{3.26}$$

For the first estimate in (3.26), we consider integrals of the form

$$\int_0^t\int_{-|a_1^-|s}^0\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}\Big)^{-5/2}$$
$$\times\ln(e+(t-s))(1+|y|+s)^{-1}\Big(1+|y-a_1^-s|\Big)^{-1}dyds.$$

For $s\in[0,t/2]$, we observe through (3.10), the inequality

$$\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}\Big)^{-5/2}(1+|y|+s)^{-1}$$
$$\le C\Big[\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|x-a_1^-t|+(t-s)^{1/2}\Big)^{-5/2}(1+|y|+s)^{-1}$$
$$+\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}\Big)^{-5/2}(1+|y|+s+|x-a_1^-t|)^{-1}\Big]. \tag{3.27}$$

For the first estimate in (3.27), we can proceed similarly as in (3.25), while for the second we have an estimate by

$$C_1(1+t)^{-3/4}(1+|x-a_1^-t|)^{-1/2}\ln(e+t)\int_0^{t/2}\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|\Big)^{-1}(1+s)^{-1/2}\ln(e+s)ds$$
$$\le C\psi_1^-(t,x).$$

For $s\in[t/2,t]$, we observe through (3.12) the inequality

$$\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}\Big)^{-5/2}$$
$$\le C\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+(t-s)^{1/2}+(|x|+t)^{1/2}\Big)^{-5/2}.$$

In this case, we obtain an estimate by

$$C(1+|x|+t)^{-5/4}\int_{t/2}^t(1+s)^{-1}\ln(e+(t-s))\ln(e+s)ds\le C\psi_1^-(t,x).$$

For the second estimate in (3.26), we consider integrals of the form

$$\int_0^t\int_{-|a_1^-|s}^0\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}\Big)^{-5/2}\ln(e+(t-s))$$
$$\times(1+|y|+s)^{-1}\Big(1+|y-a_1^-s|+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|\Big)^{-1}dyds.$$

For $s\in[0,t/2]$, we observe through (3.10) the inequality

$$\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|y-a_1^-s|\Big)^{-1}\Big(1+|y|+s\Big)^{-1}$$
$$\le C\Big[\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|y-a_1^-s|+|x-a_1^-t|\Big)^{-1}\Big(1+|y|+s\Big)^{-1}\tag{3.28}$$
$$+\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|y-a_1^-s|\Big)^{-1}\Big(1+|y|+s+|x-a_1^-t|\Big)^{-1}\Big].$$

For integration over the first estimate in (3.28), we have an estimate by

$$C_1(1+t^{1/2})^{-5/2}(1+|x-a_1^-t|)^{-1/2}\int_0^{t/2}\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|\Big)^{-1/2}\ln(e+s)\ln(e+(t-s))ds$$
$$\le C(1+t)^{-3/4}[\ln(e+t)]^2(1+|x-a_1^-t|)^{-1/2}\le C\psi_1^-(t,x),$$

and similarly for the second. For $s\in[t/2,t]$, we observe through (3.12) the inequality

$$\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}\Big)^{-5/2}\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|y-a_1^-s|\Big)^{-1}$$
$$\le C\Big[\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(|x|+t)^{1/2}\Big)^{-5/2}\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|+|y-a_1^-s|\Big)^{-1}\tag{3.29}$$
$$+\Big(1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2}\Big)^{-5/2}\Big(1+|x|+t\Big)^{-1}\Big].$$

For the first estimate in (3.29), we have an estimate by

$$C_2(1+t)^{-1}(|x|+t)^{-3/4}\int_{t/2}^t\Big(1+|x-a_1^-(t-s)-\frac{a_1^-}{a_2^-}a_1^-s|\Big)^{-1}\ln(e+(t-s))ds$$
$$\le C(1+t)^{-1}(|x|+t)^{-3/4}[\ln(e+t)]^2\le C\psi_1^-(t,x),$$

and similarly for the second.

For integrals (3.17) with nonlinearity $\alpha_1^-(s,y)^2$, we have integrals of the form

$$\int_0^t \int_{-|a_1^-|s}^0 \Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big)(1+|y|)^{-1}(1+s)^{-3/2}dyds. \quad (3.30)$$

In the event that $|x| \geq |a_1^-|t$, we observe that there is no cancellation between summands in (3.5), and consequently that we obtain an estimate by

$$\begin{aligned} &C_1(1+|x-a_1^-t|)^{-3/2}\int_0^t \Big((1+(t-s)^{1/2})^{-1}\ln(e+(t-s))\Big)\ln(e+s)(1+s)^{-3/2}ds \\ &\leq C(1+|x-a_1^-t|)^{-3/2}, \end{aligned} \quad (3.31)$$

which is sufficient for $|x-a_1^-t| \geq \sqrt{t}$. For the case $|x-a_1^-t| \leq \sqrt{t}$, we require only $t^{-1/2}$ decay, which is immediate. In the event that $|x| \leq |a_1^-|t$, we have an estimate by

$$\begin{aligned} &C_1(1+t^{1/2})^{-5/2}\ln(e+t)\int_0^{t/2}(1+s)^{-3/2}\ln(e+s)ds \\ &+ C_2(1+t)^{-3/2}\ln(e+s)\int_{t/2}^t (1+(t-s)^{1/2})^{-5/4}ds \\ &\leq C(1+t)^{-5/2}\ln(e+t), \end{aligned}$$

which is bounded by $\psi_1^-(t,x)$ since here $|x| \leq |a_1^-|t$.

For integrals (3.17) with nonlinearity $\alpha_2^-(s,y)^2$, we have integrals of the form

$$\int_0^t \int_{-|a_1^-|s}^0 \Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big)e^{-2\eta|y|}(1+s)^{-1}dyds. \quad (3.32)$$

In the event that $|x| \geq |a_1^-|t$, we observe that there is no cancellation between summands in (3.5), and consequently that we obtain an estimate by

$$\begin{aligned} &C_1(1+|x-a_1^-t|)^{-3/2}\int_0^t (1+(t-s)^{1/2})^{-1}\ln(e+(t-s))(1+s)^{-1}ds \\ &\leq C(1+|x-a_1^-t|)^{-3/2}, \end{aligned}$$

which for $|x-a_1^-t| \geq \sqrt{t}$ is bounded by $\psi_2^-(t,x)$. In the case $|x-a_1^-t| \leq \sqrt{t}$, $t^{-1/2}$ decay is sufficient. In the event that $|x| \leq |a_1^-|t$, we have an estimate by

$$\begin{aligned} &C_1(1+t^{1/2})^{-5/2}\ln(e+t)\int_0^{t/2}(1+s)^{-1}ds + C_2(1+t)^{-1}\int_{t/2}^t (1+(t-s)^{1/2})^{-5/4}\ln(e+(t-s))ds \\ &\leq C(1+t)^{-1}, \end{aligned}$$

which is bounded by $\psi_1^-(t,x)$ since here $|x| \leq |a_1^-|t$.

For integrals (3.17) with nonlinearity $|\dot{\delta}(s)|\psi_2^-(s,y)$, we have integrals of the form

$$\begin{aligned} &\int_0^t \int_{-\infty}^0 \Big((1+|x-\frac{a_1^-}{a_2^-}y-a_1^-(t-s)|+(t-s)^{1/2})^{-5/2}\ln(e+(t-s))\Big) \\ &\quad\times(1+s)^{-1}(1+|y-a_1^-s|+s^{1/2})^{-3/2}\ln(e+s)dyds. \end{aligned} \quad (3.33)$$

In the event that $|x| \geq |a_1^-|t$, we observe that there is no cancellation between summands in (3.5), and consequently that we obtain an estimate by

$$\begin{aligned} &C_1(1+|x-a_1^-t|)^{-3/2}\int_0^t (1+(t-s)^{1/2})^{-1}(1+s)^{-1}(1+s^{1/2})^{-1/2}\ln(e+s)ds \\ &\leq C(1+|x-a_1^-t|)^{-3/2}, \end{aligned}$$

which for $|x - a_1^- t| \geq \sqrt{t}$ is bounded by $\psi_2^-(t,x)$. In the case $|x - a_1^- t| \leq \sqrt{t}$, $t^{-1/2}$ decay is sufficient. In the event that $|x| \leq |a_1^-|t$, we have an estimate by

$$\begin{aligned} &C_1(1+t^{1/2})^{-5/2}\ln(e+t)\int_0^{t/2}(1+s)^{-1}(1+s^{1/2})^{-1/2}\ln(e+s)ds \\ &\quad + C_2(1+t)^{-1}(1+t^{1/2})^{-3/2}\ln(e+t)\int_{t/2}^t (1+(t-s)^{1/2})^{-3/2}ds \\ &\quad \leq C(1+t)^{-5/4}\ln(e+t), \end{aligned}$$

which is bounded by $\psi_1^-(t,x)$ since $t \geq (1/|a_1^-|)t$.

Integration against the final nonlinearity $|\dot{\delta}|\alpha_1^-(s,y)$ proceeds similarly as in the previous case.

We next consider the excited Green's function terms $\bar{u}_x(x)\partial_y e_-(t,y)$, for which we have integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1/2} e^{-\frac{(y+a_2^- t)^2}{Mt}} \Psi(s,y)dyds. \tag{3.34}$$

Integrals of this form have been analyzed in [14], in which an estimate by $C(1+t)^{-1/2}$ was determined. Since for $x \leq 0$, $\bar{u}_x(x)$ decays at exponential rate, this yields a term bounded by $\alpha_2^-(t,x)$, and similarly for the second term in $\partial_y e_-(t,y)$.

**Case 2,** $x \leq 0 < K \leq y$**.** In the cases for which $y \geq 0$, we carry out details only for $y \geq K$. For the interval $0 \leq y \leq K$, the bound on $y$, giving also a finite interval of integration, makes the analysis straightforward. For the case $x \leq 0 < K \leq y$, we consider integration against six Green's function estimates, beginning with integrals of the form

$$\int_0^t \int_K^\infty \Big(\Big[(t-s)^{-1}(1+(t-s))^{1/4}\Big] \wedge \Big[(t-s)^{-1}(1+y)\Big]\Big) e^{-\frac{(x-a_1^- \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s))^2}{M(t-s)}} \Psi(y,s)dyds, \tag{3.35}$$

where $\wedge$ denotes *minimum*. In the case $y > 0$, we have eight nonlinearities to consider, ${\theta_1^+}^2$, ${\theta_2^+}^2$, ${\psi_1^+}^2$, ${\psi_2^+}^2$, and the four terms $|\dot{\delta}|(\theta_1^+ + \theta_2^+ + \psi_1^+ + \psi_2^+)$. For the first, we have

$$\int_0^t \int_K^\infty (t-s)^{-1} e^{-\frac{(x-a_1 \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s))^2}{M(t-s)}} (1+s)^{-1}(1+y)^{-1} e^{-2\frac{y^2}{Ls}} dyds, \tag{3.36}$$

for which we observe the inequality

$$e^{-\frac{(x-a_1 \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s))^2}{M(t-s)}} e^{-2\frac{y^2}{Ls}} \leq C e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}, \tag{3.37}$$

where $M < M_1 < L$ (see (3.2)). In the case $|x| \geq |a_1^-|t$, there is no cancellation between $(x - a_1^- t)$ and $a_1^- s$, and we obtain an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_0^{t/2} (1+s)^{-1}\ln(e+s)ds + C_2(1+t)^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2}ds \\ &\quad \leq C(1+t)^{-1/2} e^{-\frac{(x-a_1^- t)^2}{Lt}} = C\theta^-(t,x). \end{aligned} \tag{3.38}$$

In the case $|x| \leq |a_1^-|t$, we divide the analysis into the subcases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}(1+s)^{-1} \\ &\quad \leq C\Big[e^{-\frac{(x-a_1^- t)^2}{Lt}}(1+s)^{-1} + e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}(1+s+|x-a_1^- t|)^{-1}\Big]. \end{aligned} \tag{3.39}$$

For the first, we proceed as in (3.38), while for the second we have an estimate by

$$C_1 t^{-1}(1+|x-a_1^- t|)^{-1/2}\int_0^{t/2}(1+s)^{-1/2}\ln(e+s)e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds$$
$$\le Ct^{-3/4}\ln(e+t)(1+|x-a_1^- t|)^{-1/2}.$$

For $s\in[t/2,t]$, we observe the inequality

$$(t-s)^{-1/2}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}\le C|x|^{-1/2}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}, \tag{3.40}$$

from which we obtain an estimate on (3.36) by

$$C_2|x|^{-1/2}(1+t)^{-1}\ln(e+t)\int_{t/2}^t (t-s)^{-1/2}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds$$
$$\le C|x|^{-1/2}(1+t)^{-1}\ln(e+t),$$

in which we have used the useful inequality

$$\int_{t/2}^t (t-s)^{-1/2}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds\le C. \tag{3.41}$$

For the nonlinearity $\theta_2^+(s,y)^2$, we can proceed exactly as in the case $\theta_1^+(s,y)^2$, except that in each instance when the term $(1+y)^{-1}$ integrated to $\ln(e+s)$, we employ the integrability of $(1+y)^{-3}$.

For the nonlinearity $\psi_1^+(s,y)^2$, we employ our alternative Green's function estimate and consider integrals of the form

$$\int_0^t\int_K^\infty (t-s)^{-3/4}e^{-\frac{(x-a_1^-\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3}dyds, \tag{3.42}$$

for which we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-a_1^-\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3}\\ &\le C\Big[e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}e^{-\epsilon\frac{(x-a_1\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3}\\ &+e^{-\frac{(x-a_1\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+|x-a_1^-(t-s)|+s^{1/2})^{-3}\Big]. \end{aligned} \tag{3.43}$$

For the first estimate in (3.43), we have integrals of the form

$$\int_0^t\int_K^{+\infty}(t-s)^{-3/4}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}e^{-\epsilon\frac{(x-a_1\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3}dyds. \tag{3.44}$$

In the event that $|x|\ge|a_1^-|t$, we have no cancellation between $(x-a_1^-t)$ and $a_1^-s$, and obtain an estimate by

$$\begin{aligned} &C_1t^{-3/4}e^{-\frac{(x-a_1^-t)^2}{Lt}}\int_0^{t/2}(1+s^{1/2})^{-2}+C_2(1+t^{1/2})^{-3}e^{-\frac{(x-a_1^-t)^2}{Lt}}\int_{t/2}^t(t-s)^{-1/4}ds\\ &\le C\theta^-(t,x). \end{aligned} \tag{3.45}$$

In the case $|x|\le|a_1^-|t$, we divide the analysis into intervals $s\in[0,t/2]$ and $s\in[t/2,t]$. For $s\in[0,t/2]$, we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}(1+y+s^{1/2})^{-3}\\ &\le C\Big[e^{-\frac{(x-a_1^-t)^2}{Lt}}(1+y+s^{1/2})^{-3}+e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}(1+y+s^{1/2}+|x-a_1^-t|^{1/2})^{-3}\Big]. \end{aligned} \tag{3.46}$$

For the first estimate in (3.46), we proceed as in (3.45), while for the second we have an estimate by

$$C_1 t^{-3/4}(1+|x-a_1^- t|^{1/2})^{-1}\int_0^{t/2}(1+s^{1/2})^{-1}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds \le C\psi_1^-(t,x),$$

in which we have observed the estimate

$$\int_0^{t/2}(1+s^{1/2})^{-1}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds \le Ct^{1/4}. \tag{3.47}$$

For $s\in[t/2,t]$, we observe through (3.40) from the previous case an estimate by

$$\begin{aligned}C_2(1+t^{1/2})^{-3}|x|^{-1/2}&\int_{t/2}^t (t-s)^{1/4}e^{-\frac{(x-a_1^-(t-s))^2}{M_1(t-s)}}ds\\ &\le C\alpha_1^-(t,x).\end{aligned}$$

For the second estimate in (3.43), we have integrals of the form

$$\int_0^t\int_K^{+\infty}(t-s)^{-3/4}e^{-\frac{(x-a_1\int_K^y\frac{d\tau}{a_1(\tau)}-a_1^-(t-s))^2}{M(t-s)}}(1+y+|x-a_1^-(t-s)|+s^{1/2})^{-3}dyds. \tag{3.48}$$

In the case $|x|\ge|a_1^-|t$, we have no cancellation between $x-a_1^-t$ and $a_1^-s$, and can consequently estimate (3.48) by

$$\begin{aligned}C_1t^{-1/4}(1+|x-a_1^-t|)^{-3/2}\int_0^{t/2}(1+s^{1/2})^{-3/2}ds+C_2(1+|x-a_1^-t|)^{-3/2}\int_{t/2}^t(t-s)^{-1/4}(1+s^{1/2})^{-3/2}\\ \le C(1+|x-a_1^-t|)^{-3/2},\end{aligned} \tag{3.49}$$

which is sufficient (bounded by $C\psi_2^-(t,x)$) in the case $|x-a_1^-t|\ge\sqrt{t}$. In the case $|x-a_1^-t|\ge\sqrt{t}$, we only require $t^{-1/2}$ decay to conclude an estimate by $\theta^-(t,x)$, and this is immediate. In the case $|x|\le|a_1^-|t$, we divide the analysis into the cases $s\in[0,t/2]$ and $s\in[t/2,t]$. For $s\in[0,t/2]$, we observe the estimate

$$\Big(1+|x-a_1^-(t-s)|+y+s^{1/2}\Big)^{-3}\le C\Big(1+|x-a_1^-(t-s)|+y+|x-a_1^-t|^{1/2}\Big)^{-3}, \tag{3.50}$$

from which we obtain an estimate on (3.48) by

$$C_1t^{-3/4}(1+|x-a_1^-t|^{1/2})^{-1}\int_0^{t/2}(1+|x-a_1^-(t-s)|)^{-1}ds\le C\psi_1^-(t,x).$$

For $s\in[t/2,t]$, we observe the estimate

$$\begin{aligned}(t-s)^{-1/2}&\Big(1+|x-a_1^-(t-s)|+y+s^{1/2}\Big)^{-3}\\ &\le C\Big[|x|^{-1/2}\Big(1+|x-a_1^-(t-s)|+y+s^{1/2}\Big)^{-3}\\ &+(t-s)^{-1/2}\Big(1+|x|+|x-a_1^-(t-s)|+y+s^{1/2}\Big)^{-3}\Big].\end{aligned} \tag{3.51}$$

For the first estimate in (3.51), we obtain an estimate by

$$C_2(1+t^{1/2})^{-3/2}|x|^{-1/2}\int_{t/2}^t(t-s)^{1/4}(1+|x-a_1^-(t-s)|+t^{1/2})^{-3/2}ds,$$

which is bounded by $\alpha_1^-(t,x)$ in the case that $|x|$ is bounded away from 0. For $|x|$ near 0, we proceed as in previous cases for small $t$. For the second estimate in (3.51), we obtain an estimate by

$$C_2(1+|x|+t^{1/2})^{-2}\int_{t/2}^t(t-s)^{-1/4}(1+|x-a_1^-(t-s)|)^{-1}ds\le C\alpha_1^-(t,x).$$

For the nonlinearity $\psi_2^+(s,y)^2$, the analysis proceeds as in the previous paragraph, except that the replacement of $t^{1/2}$ with $t$ more than corrects for the addtional $\ln(e+t)$.

We next consider integrals of the form

$$\int_0^t \int_K^\infty \Big(1 + |x - a_1^- \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} \ln(e+(t-s))\Psi(y,s)dyds. \tag{3.52}$$

For the nonlinearity $\theta_1^+(s,y)^2$, we have integrals of the form

$$\int_0^t \int_K^\infty \Big(1+|x-a_1^- \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} \ln(e+(t-s))(1+s)^{-1}(1+y)^{-2} e^{-\frac{y^2}{Ms}} dyds, \tag{3.53}$$

for which we observe the inequality

$$\begin{aligned} &\Big(1 + |x - a_1^- \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} e^{-\frac{y^2}{Ms}} \\ &\le C\Big[\Big(1 + |x - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} e^{-\frac{y^2}{Ms}} \\ &+ \Big(1 + |x - a_1^- \int_K^y \frac{d\tau}{a_1(\tau)} - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} e^{-\frac{(x-a_1^-(t-s))^2}{Ms}}\Big]. \end{aligned} \tag{3.54}$$

For the first estimate in (3.54), we consider integrals of the form

$$\int_0^t \int_K^\infty \Big(1 + |x - a_1^-(t-s)| + (t-s)^{1/2}\Big)^{-5/2} \ln(e+(t-s))(1+s)^{-1}(1+y)^{-2} e^{-\frac{y^2}{Ms}} dyds. \tag{3.55}$$

In the event that $|x| \ge |a_1^-|t$, we have no cancellation between $x - a_1^- t$ and $a_1^- s$, and consequently, we obtain an estimate by

$$C_1(1 + |x - a_1^- t|)^{-3/2} \int_0^t (1 + (t-s)^{1/2})^{-1} \ln(e+(t-s))(1+s)^{-1} ds, \tag{3.56}$$

which is sufficient for $|x - a_1^- t| \ge \sqrt{t}$, whereas in the case $|x - a_1^- t| \le \sqrt{t}$, we require only $t^{-1/2}$ decay. For the case $|x| \le |a_1^-|t$, we integrate $(1+y)^{-2}$ in (3.56) to obtain an estimate by

$$\begin{aligned} &C\int_0^t (1 + (t-s)^{1/2})^{-5/2} \ln(e+(t-s))(1+s)^{-1} ds \\ &\le C_1(1+t^{1/2})^{-5/2} \ln(e+t) \int_0^{t/2} (1+s)^{-1} ds \\ &+ C_2(1+t)^{-1} \int_{t/2}^t (1+(t-s)^{1/2})^{-5/2} \ln(e+(t-s)) ds \\ &\le \psi_1^-(t,x), \end{aligned}$$

where in this last inequality we have used in particular the observation that that $t \le |x|/|a_1^-|$. For the second estimate in (3.54), the analysis is almost identical.

For the remaining six nonlinearities in this case, we proceed almost identically as in the previous paragraph. In particular, we observe that in each of these cases the critical region $|x| \le |a_1^- t|$ yields time decay $t^{-1}$, which gives an estimate by $\psi_1^-(t,x)$.

We next consider integrals of the form

$$\int_0^t \int_K^{\bar{C}(t-s)} \Big[(1 + |x - a_1^-(t-s)| + t^{1/2})^{-3/2}\Big] \wedge \Big[(1 + |x - a_1^-(t-s)| + t^{1/2})^{-2}(1+y)\Big] \Psi(s,y) dyds, \tag{3.57}$$

where $\wedge$ represents *minimum* and where $\bar{C}$ is large enough so that $y \ge \bar{C}(t-s)$ implies

$$|x - a_1^- \int_K^y \frac{d\tau}{a_1(\tau)}| > |a_1^-|(t-s).$$

For the nonlinearity $\theta_1^+(s,y)^2$, we have

$$\int_0^t \int_K^{\bar{C}(t-s)} (1+|x-a_1^-(t-s)|+t^{1/2})^{-3/2}(1+s)^{-1}(1+y)^{-2}e^{-2\frac{y^2}{Ls}}dyds, \tag{3.58}$$

for which we immediately observe a bound by

$$C\int_0^t (1+|x-a_1^-(t-s)|+t^{1/2})^{-3/2}(1+s)^{-1}e^{-2\frac{y^2}{Ls}}ds. \tag{3.59}$$

In the event that $|x|\geq|a_1^-|t$, we have no cancellation between $(x-a_1^-t)$ and $a_1^-s$, and consequently we have an estimate by

$$C_1(1+|x-a_1^-t|+t^{1/2})^{-3/2}\ln(e+t)\leq C\psi_2^-(t,x). \tag{3.60}$$

In the case $|x|\leq|a_1^-t|$, we observe the inequality

$$\begin{aligned}&(1+|x-a_1^-(t-s)|+t^{1/2})^{-3/2}(1+s)^{-1}\\ &\leq C\Big[(1+|x-a_1^-(t-s)|+|x-a_1^-t|+t^{1/2})^{-3/2}(1+s)^{-1}\\ &+(1+|x-a_1^-(t-s)|+t^{1/2})^{-3/2}(1+s+|x-a_1^-t|)^{-1}\Big].\end{aligned} \tag{3.61}$$

For the first estimate in (3.61), we proceed as in (3.60), while for the second we have an estimate by

$$C_1(1+t^{1/2})^{-1}(1+|x-a_1^-t|)^{-1/2}\int_0^t(1+|x-a_1^-(t-s)|)^{-1/2}(1+s)^{-1/2}ds\leq C\psi_1^-(t,x). \tag{3.62}$$

For the nonlinearity $\theta_2^+(s,y)^2$, we consider integrals

$$\int_0^t\int_K^{\bar{C}(t-s)}(1+|x-a_1^-(t-s)|+t^{1/2})^{-2}(1+s)^{-1}[\ln(e+s)]^2(1+y)^{-3}dyds, \tag{3.63}$$

for which we proceed as in the previous paragraph, observing that the additional $t^{-1/4}$ decay in this estimate on the kernel compensates for the term $[\ln(e+s)]^2$ in this nonlinearity.

For the remaining six nonlinearities, we proceed as in the previous two paragraphs.

We next begin our consideration of integrals associated with the excited term $\bar{u}_x\partial_y e_3(t,y)$, the first of which takes the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty (t-s)^{-1/2}e^{-\frac{(\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}\Psi(y,s)dyds. \tag{3.64}$$

For the nonlinearity $\theta_1^+(s,y)^2$, we have integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty (t-s)^{-1/2}e^{-\frac{(\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}(1+s)^{-1}(1+y)^{-2}e^{-\frac{y^2}{Ms}}dyds, \tag{3.65}$$

for which we observe the inequality

$$\begin{aligned}&e^{-\frac{(\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}e^{-\frac{y^2}{Ms}}\\ &\leq C\Big[e^{-\eta(t-s)}e^{-\frac{y^2}{Ms}}+e^{-\frac{(\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}e^{-\eta\frac{(t-s)^2}{Ms}}\Big].\end{aligned} \tag{3.66}$$

For the first estimate in (3.66), upon integration of $(1+y)^{-2}$, we observe an estimate by

$$C_1\bar{u}_x(x)\int_0^t(t-s)^{-1/2}e^{-\eta(t-s)}(1+s)^{-1}ds\leq C\alpha_2^-(t,x),$$

while for the second estimate in (3.66), we obtain an estimate by

$$C_1\bar{u}_x(x)e^{-\eta_1 t}\int_0^{t/2}(1+s)^{-1}ds + C_2\bar{u}_x(x)(1+t)^{-1}\int_{t/2}^t (t-s)^{-1/2}e^{-\eta\frac{(t-s)^2}{Ms}}ds \le C\alpha_2^-(t,x).$$

Analysis of the nonlinearity $\theta_2^+(s,y)^2$ proceeds as in the previous paragraph, wherein each estimate yielded enough decay in $t$ to absorb an additional $[\ln(e+t)]^2$.

For the nonlinearity $\psi_1^+(s,y)^2$, we have integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty (t-s)^{-1/2}e^{-\frac{(\int_K^y \frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3}dyds, \tag{3.67}$$

for which we observe the inequality

$$\begin{aligned} &e^{-\frac{(\int_K^y \frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}(1+y+s^{1/2})^{-3} \\ &\le C\Big[e^{-\epsilon\frac{(\int_K^y \frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}e^{-\eta(t-s)}(1+y+s^{1/2})^{-3} + e^{-\frac{(\int_K^y \frac{d\tau}{a_1(\tau)}+(t-s))^2}{M(t-s)}}(1+y+(t-s)+s^{1/2})^{-3}\Big]. \end{aligned} \tag{3.68}$$

For the first estimate in (3.68), we have an estimate by

$$C_1\bar{u}_x(x)\int_0^t e^{-\eta(t-s)}(1+s^{1/2})^{-3}ds \le C\alpha_2^-(t,x), \tag{3.69}$$

while for the second we have an estimate by

$$C_1\bar{u}_x(x)t^{-1/2}\int_0^t(1+(t-s))^{-2}ds + C_2\bar{u}_x(x)(1+t^{1/2})^{-1}\int_{t/2}^t(1+(t-s))^{-2}ds \le C\alpha_2^-(t,x). \tag{3.70}$$

Analysis of the remaining nonlinearities in this case is straightfoward from the observation that in each case, an integral over $y$ of the nonlinearity yields time decay in $s$ at an integrable rate.

We next consider integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty (t-s)^{-1}(1+y)e^{-\frac{y^2}{M(t-s)}}\Psi(s,y)dyds. \tag{3.71}$$

For the nonlinearity $\theta_1^+(s,y)^2$, we have integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty (t-s)^{-1}e^{-\frac{y^2}{M(t-s)}}(1+s)^{-1}(1+y)^{-1}e^{-\frac{y^2}{Ms}}dyds, \tag{3.72}$$

for which, upon integration of $(1+y)^{-1}$, we have an estimate by $C\bar{u}_x(x)(1+t)^{-1}[\ln(e+t)]^2$, which is bounded by $C\alpha_1^-(t,x)$. Estimates on the remaining nonlinearities for this case follow almost identically.

We next consider integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty \Big(1+|\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s)|+(t-s)^{1/2}\Big)^{-3/2}\ln(e+(t-s))\Psi(s,y)dyds. \tag{3.73}$$

For the nonlinearity $\theta_1^+(s,y)^2$, we have integrals of the form

$$\bar{u}_x(x)\int_0^t\int_K^\infty \Big(1+|\int_K^y\frac{d\tau}{a_1(\tau)}+(t-s)|+(t-s)^{1/2}\Big)^{-3/2}\ln(e+(t-s))(1+s)^{-1}(1+y)^{-2}e^{-\frac{y^2}{Ms}}dyds. \tag{3.74}$$

Integrating $(1+y)^{-2}$, we obtain an estimate by

$$C\bar{u}_x(x)\int_0^t(1+(t-s)^{1/2})^{-3/2}\ln(e+(t-s))(1+s)^{-1}ds \le C\alpha_2^-(t,x).$$

The remaining nonlinearities in this case follow similarly.

**Case 3, $y \le 0 \le x$.** For the case $y \le 0 \le x$, we first consider integrals of the form

$$\int_0^t \int_{-\infty}^{-a_2^-(t-s)} (t-s)^{-1}(1+(t-s))^{1/4}(1+x)^{-1} e^{-\frac{(x-y-a_2^-(t-s))^2}{M(t-s)}} \Psi(y,s) dyds. \tag{3.75}$$

For the nonlinearity $\theta^-(s,y)^2$, we have integrals of the form

$$\int_0^t \int_{-\infty}^{-a_2^-(t-s)} (t-s)^{-1}(1+(t-s))^{1/4}(1+x)^{-1} e^{-\frac{(x-y-a_2^-(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_1^- s)^2}{Ms}} dyds, \tag{3.76}$$

for which we observe that for $y \in (-\infty, -a_2^-(t-s)]$ and $x > 0$, there holds

$$x - y - a_2^-(t-s) \ge x,$$

through which we immediately have decay of heat kernel type $exp(-x^2/(Lt))$. Extracting this decay, which serves to increase the value of $M$ in what remains (to $M_1$ in our notation), we observe the equality,

$$e^{-\frac{(x-y-a_2^-(t-s))^2}{M_1(t-s)}} e^{-\frac{(y-a_1^- s)^2}{M_1 s}} = e^{-\frac{(x-a_2^-(t-s)-a_1^- s)^2}{M_1 t}} e^{-\frac{t}{M_1 s(t-s)}(y - \frac{xs-(a_1^-+a_2^-)(t-s)s}{t})^2}, \tag{3.77}$$

derived in straightforward fashion by completion of an appropriate square (see Lemma 6 of [16]). Integrating this final kernel in $y$, we obtain an estimate by

$$\begin{aligned} &Ct^{-1/2} e^{-\frac{x^2}{Lt}} (1+x)^{-1} \int_0^t (t-s)^{-1/2}(1+(t-s))^{1/4}(1+s)^{-1/2} e^{-\frac{(x-a_2^-(t-s)-a_1^- s)^2}{M_1 t}} ds \\ &\le Ct^{-1/2}(1+x)^{-1} e^{-\frac{x^2}{Lt}}. \end{aligned} \tag{3.78}$$

For the nonlinearity $\theta_1^-(s,y)^2$, we consider integrals of the form

$$\int_0^t \int_{(-|a_1^-|s)\wedge(-|a_2^-|(t-s))}^{-a_2^-(t-s)} (t-s)^{-1}(1+(t-s))^{1/4}(1+x)^{-1} e^{-\frac{(x-y-a_2^-(t-s))^2}{M(t-s)}} (1+|y|+s)^{-1}(1+|y-a_1^- s|)^{-1} dyds, \tag{3.79}$$

for which we again have immediate decay of type $exp(-x^2/(Lt))$. In this case, integrating directly, we determine an estimate by

$$\begin{aligned} &C_1 t^{-1}(1+t)^{1/4}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/2} (1+s)^{-1} \ln(e+t) ds \\ &+ C_2(1+t)^{-1}\ln(e+t)(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/2}^t (t-s)^{-1}(1+(t-s))^{1/4} ds \\ &\le C\theta_1^+(t,x). \end{aligned} \tag{3.80}$$

For the remaining nonlinearities, we observe that integration over $y$ gives $s$-decay at a minimum rate of $(1+s)^{-1}[\ln(e+s)]^2$ (the minimum rate occurs for nonlinearity $\psi_2^-(s,y)^2$), and consequently we can apply precisely the same argument as in the previous paragraph.

We next consider integrals of the form

$$\int_0^t \int_{-a_2^-(t-s)}^0 (1+|y+a_2^-(t-s)|)^{-3/2}(1+x)^{-1} e^{-\frac{x^2}{M(t-s)}} \Psi(s,y) dyds, \tag{3.81}$$

which for the nonlinearity $\theta^-(s,y)^2$ becomes

$$\int_0^t \int_{-a_2^-(t-s)}^0 (1+|y+a_2^-(t-s)|)^{-3/2}(1+x)^{-1} e^{-\frac{x^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_1^- s)^2}{Ms}} dyds. \tag{3.82}$$

Writing

$$-y - a_2^-(t-s) = (-a_2^-(t-s) - a_1^- s) - (y - a_1^- s), \tag{3.83}$$

we observe the inequality

$$\begin{aligned} &(1+|y+a_2^-(t-s)|)^{-3/2} e^{-\frac{(y-a_1^- s)^2}{Ms}} \\ &\le C\Big[(1+|a_2^-(t-s)+a_1^- s|)^{-3/2} e^{-\frac{(y-a_1^- s)^2}{Ms}} + (1+|y+a_2^-(t-s)|)^{-3/2} e^{-\frac{(a_2^-(t-s)+a_1^- s)^2}{Ms}}\Big]. \end{aligned} \tag{3.84}$$

For integration over the interval $s \in [0, t/\gamma]$, $\gamma$ sufficiently large, we have

$$|a_2^-(t-s) + a_1^- s| \ge \eta t,$$

where $\eta = (a_2^-(\gamma-1) + a_1^-)/\gamma > 0$. Respectively, the estimates of (3.84) lead to estimates by

$$\begin{aligned} &C_1(1+t)^{-3/2}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/\gamma} s^{1/2}(1+s)^{-1} ds \\ &+ C_2 e^{-\eta_1 t}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/\gamma} (1+s)^{-1} ds \\ &\le C\theta_1^+(t,x). \end{aligned}$$

In the case $s \in [t/\gamma, t]$, we obtain estimates, again respectively, by

$$\begin{aligned} &C_1(1+t)^{-1/2}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/\gamma}^t (1+|a_2^-(t-s)+a_1^- s|)^{-3/2} ds \\ &+ C_2(1+t)^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/\gamma}^t e^{-\frac{(a_2^-(t-s)+a_1^- s)^2}{Ms}} ds \\ &\le C\theta_1^+(t,x). \end{aligned}$$

For the nonlinearity $\psi_1^-(s,y)^2$, we consider integrals of the form

$$\int_0^t \int_{[-a_2^-(t-s)]\vee[-|a_1^-|s]}^0 (1+|y+a_2^-(t-s)|)^{-3/2}(1+x)^{-1} e^{-\frac{x^2}{M(t-s)}} (1+|y|+s)^{-1}(1+|y-a_1^- s|)^{-1} dyds. \tag{3.85}$$

for which we observe through (3.83) the inequality

$$\begin{aligned} &(1+|y+a_2^-(t-s)|)^{-3/2}(1+|y-a_1^- s|)^{-1} \\ &\le C\Big[(1+|a_2^-(t-s)+a_1^- s|)^{-3/2}(1+|y-a_1^- s|)^{-1} \\ &+ (1+|y+a_2^-(t-s)|)^{-3/2}(1+|y-a_1^- s|+|a_2^-(t-s)+a_1^- s|)^{-1}\Big]. \end{aligned} \tag{3.86}$$

Dividing our analysis into the cases $s \in [0, t/\gamma]$ and $s \in [t/\gamma, t]$, we obtain an estimate by $C\theta_1^+(t,x)$ as in the previous paragraph.

Integration against the nonlinearity $\psi_2^-(s,y)^2$ can be analyzed similarly as in the previous two paragraphs to obtain an estimate again by $C\theta_1^+(t,x)$.

For the nonlinearity $\alpha_1^-(s,y)^2$, we consider integrals of the form

$$\int_0^t \int_{[-a_2^-(t-s)]\vee[a_1^- s]}^0 (1+|y+a_2^-(t-s)|)^{-3/2}(1+x)^{-1} e^{-\frac{x^2}{M(t-s)}} (1+|y|)^{-1}(1+s)^{-3/2} dyds. \tag{3.87}$$

for which we observe the inequality

$$\begin{aligned} &(1+|y+a_2^-(t-s)|)^{-3/2}(1+|y|)^{-1} \\ &\le C\Big[(1+|a_2^-(t-s)|)^{-3/2}(1+|y|)^{-1} + (1+|y+a_2^-(t-s)|)^{-3/2}(1+|y|+(t-s))^{-1}\Big]. \end{aligned} \tag{3.88}$$

For the first estimate in (3.88), we obtain an estimate by

$$\begin{aligned}
&C_1(1+t)^{-3/2}(1+x)^{-1}e^{-\frac{x^2}{Lt}}\int_0^{t/2}(1+s)^{-3/2}\ln(e+s)ds\\
&\quad+C_2(1+t)^{-3/2}(1+x)^{-1}e^{-\frac{x^2}{Lt}}\int_{t/2}^{t}(1+|a_2^-(t-s)|)^{-3/2}\ln(e+s)ds\\
&\quad\le C\theta_1^+(t,x),
\end{aligned}$$

while for the second we have an estimate by

$$\begin{aligned}
&C_1(1+t)^{-1}(1+x)^{-1}e^{-\frac{x^2}{Lt}}\int_0^{t/2}(1+s)^{-3/2}ds+C_2(1+t)^{-3/2}(1+x)^{-1}e^{-\frac{x^2}{Lt}}\int_{t/2}^{t}(1+(t-s))^{-1}ds\\
&\quad\le C\theta_1^+(t,x).
\end{aligned}$$

Integration against the remaining nonlinearities $\alpha_2^-(s,y)^2$, $|\dot\delta(s)|\psi_2^-(s,y)$, and $\dot\delta(s)\alpha_2^-(s,y)$ can be analyzed similarly as in the previous cases.

Integration of all nonlinearities against the Green's function estimates involving $\ln t$ can be analyzed precisely as in the previous cases to get an estimate by $C\theta_2^+(t,x)$. Integration of all nonlinearities against the excited Green's function estimate have been analyzed in the case $x,y<0$.

**Case 4,** $0\le K\le y,x$**.** For the case $0\le K\le y,x$, we first consider integrals of the form

$$\int_0^t\int_K^\infty(t-s)^{-1}(1+x)^{-1}(1+y)e^{-\frac{(x-y)^2}{M(t-s)}}\Psi(s,y)dyds, \tag{3.89}$$

which for the nonlinearity $\theta_1^+(s,y)^2$ becomes

$$\int_0^t\int_K^\infty(t-s)^{-1}(1+x)^{-1}(1+y)e^{-\frac{(x-y)^2}{M(t-s)}}(1+s)^{-1}(1+y)^{-2}e^{-\frac{y^2}{Ms}}dyds. \tag{3.90}$$

According to Lemma 6 of [16], we have the equality

$$e^{-\frac{(x-y)^2}{M(t-s)}}e^{-\frac{y^2}{Ms}}=e^{-\frac{x^2}{Mt}}e^{-\frac{t}{Ms(t-s)}(y-\frac{xs}{t})^2}, \tag{3.91}$$

derived by straightforward completion of an appropriate square. We have, then, upon direct integration of $y$, an estimate on (3.90) by

$$\begin{aligned}
&Ct^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{Mt}}\int_0^t(t-s)^{-1/2}(1+s)^{-1/2}ds\\
&\quad\le C_1t^{-1/2}(1+x)^{-1}e^{-\frac{x^2}{Mt}},
\end{aligned}$$

which provides an estimate by $C\theta_1^+(t,x)$. (In fact, we can refine this estimate slightly by integrating $(1+y)^{-1}$.)

For the nonlinearity $\theta_2^+(s,y)^2$, we proceed as in the previous paragraph, except that we integrate the term $(1+y)^{-3}$ that remains after multiplication by $(1+y)$. We obtain an estimate by $C\theta_1^+(t,x)$.

For the nonlinearity $\psi_1^+(s,y)^2$, we consider integrals of the form

$$\int_0^t\int_K^\infty(t-s)^{-1}(1+x)^{-1}(1+y)e^{-\frac{(x-y)^2}{M(t-s)}}(1+y+s^{1/2})^{-3}dyds, \tag{3.92}$$

for which we observe the inequality

$$\begin{aligned}
&e^{-\frac{(x-y)^2}{M(t-s)}}(1+y+s^{1/2})^{-3}\\
&\le C\Big[e^{-\frac{x^2}{Lt}}(1+y+s^{1/2})^{-3}+e^{-\frac{(x-y)^2}{M(t-s)}}(1+y+x+s^{1/2})^{-3}\Big].
\end{aligned} \tag{3.93}$$

For the first estimate in (3.93), we obtain an estimate by

$$C_1 t^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/2} (1+s^{1/2})^{-1} ds + C_2 (1+t^{1/2})^{-2}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds \\ \le C\theta_1^+(t,x),$$

while for the second we have an estimate by

$$C_1 t^{-1/2}(1+x)^{-2} \int_0^{t/2} (1+s^{1/2})^{-1} ds + C_2(1+t^{1/2})^{-1}(1+x)^{-2} \int_{t/2}^t (t-s)^{-1/2} ds \\ \le C(1+x)^{-2}.$$

In the case $x \ge \sqrt{t}$, this last estimate is bounded by $\psi_1^+(t,x)$, while in the case $x \le \sqrt{t}$, we only require $t^{-1/2}(1+x)^{-1}$ decay, which is immediate.

The remaining nonlinearities in this case can be analyzed similarly as were the first three.

We next consider integrals of the form

$$\int_0^t \int_0^x (t-s)^{-1}(1+(t-s))^{1/4} \ln(e+(t-s)) e^{-\eta|x-y|} \Psi(s,y) dy ds, \tag{3.94}$$

which for the nonlinearity $\theta_1^+(s,y)^2$ becomes

$$\int_0^t \int_0^x (t-s)^{-1}(1+(t-s))^{1/4} \ln(e+(t-s)) e^{-\eta|x-y|} (1+s)^{-1}(1+y)^{-2} e^{-\frac{y^2}{Ms}} dy ds. \tag{3.95}$$

In this case, we observe the inequality

$$e^{-\eta|x-y|}(1+y)^{-2} e^{-\frac{y^2}{Ms}} \\ \le C\Big[e^{-\eta_1|x|}(1+y)^{-2} e^{-\frac{y^2}{Ms}} + e^{-\eta|x-y|}(1+x)^{-2} e^{-\frac{x^2}{Lt}}\Big]. \tag{3.96}$$

For the first estimate in (3.96), we obtain an estimate by

$$C_1 t^{-3/4} \ln(e+t) e^{-\eta_1|x|} \int_0^{t/2} (1+s)^{-1} ds \\ + C_2(1+t)^{-1} e^{-\eta_1|x|} \int_{t/2}^t (t-s)^{-1}(1+(t-s))^{1/4} \ln(e+(t-s)) ds \\ \le C t^{-3/4} [\ln(e+t)]^2 e^{-\eta_1|x|},$$

which is sufficient, since in the event that $|x| \le \sqrt{t}$, we only require decay of the form $t^{-1/2}(1+|x|)^{-1}$, while for $|x| \ge \sqrt{t}$, we get exponential decay in both $x$ and $t$. For the second estimate in (3.96), we obtain an estimate by

$$C_1 t^{-3/4} \ln(e+t)(1+x)^{-2} e^{-\frac{x^2}{Lt}} \int_0^{t/2} (1+s)^{-1} ds \\ + C_2(1+t)^{-1}(1+x)^{-2} e^{-\frac{x^2}{Lt}} \int_{t/2}^t (t-s)^{-1}(1+(t-s))^{1/4} \ln(e+(t-s)) ds \\ \le C(1+t)^{-3/4}[\ln(e+t)]^2 (1+x)^{-2} e^{-\frac{x^2}{Lt}} \le C\theta_1^+(t,x).$$

Integration against the remaining nonlinearities in this case can by analyzed similarly as in the previous paragraph.

We next consider integrals of the form

$$\int_0^t \int_K^{+\infty} (t-s)^{-1}(1+(t-s))^{1/4}(1+x)^{-1} e^{-\frac{(\int_K^x \frac{d\tau}{a_1(\tau)} + \int_K^y \frac{d\tau}{a_1(\tau)} + (t-s))^2}{M(t-s)}} I_{\{|\int_K^y \frac{d\tau}{a_1(\tau)}| \ge (t-s)\}} \Psi(y,s) dy ds, \tag{3.97}$$

which for the nonlinearity $\theta_1^+(s,y)^2$ becomes

$$\int_0^t \int_K^{+\infty} (t-s)^{-1}(1+(t-s))^{1/4}(1+x)^{-1} e^{-\frac{(\int_K^x \frac{d\tau}{a_1(\tau)} + \int_K^y \frac{d\tau}{a_1(\tau)} + (t-s))^2}{M(t-s)}} I_{\{|\int_K^y \frac{d\tau}{a_1(\tau)}| \geq (t-s)\}} \times (1+s)^{-1}(1+y)^{-2} e^{-\frac{y^2}{Ms}} dy ds. \tag{3.98}$$

In this case we observe that for $|\int_K^y \frac{d\tau}{a_1(\tau)}| \geq (t-s)$, we have

$$|\int_K^x \frac{d\tau}{a_1(\tau)} + \int_K^y \frac{d\tau}{a_1(\tau)} + (t-s)| \geq |\int_K^x \frac{d\tau}{a_1(\tau)}|,$$

from which we have decay of rate $\exp(-x^2/(Lt))$ in all cases. We have, then, an estimate by

$$\begin{aligned} &C_1 t^{-1}(1+t)^{1/4}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/2} (1+s)^{-1} ds \\ &+ C_2(1+t)^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/2}^{t-1} (t-s)^{-1}(1+(t-s))^{1/4} ds \\ &+ C_3(1+t)^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t-1}^{t} (t-s)^{-1/2}(1+(t-s))^{1/4} ds \leq C\theta_1^+(t,x). \end{aligned}$$

The remaining nonlinearities in this case can be analyzed similarly.

We next consider integrals of the form

$$\int_0^t \int_x^{+\infty} (t-s)^{-1} e^{-\frac{(\int_x^y \frac{d\tau}{a_1(\tau)} + (t-s))^2}{M(t-s)}} \Psi(y,s) dy ds, \tag{3.99}$$

which for the nonlinearity $\theta_1^+(s,y)^2$ becomes

$$\int_0^t \int_x^{+\infty} (t-s)^{-1} e^{-\frac{(\int_x^y \frac{d\tau}{a_1(\tau)} + (t-s))^2}{M(t-s)}} (1+s)^{-1}(1+y)^{-2} e^{-\frac{y^2}{Ms}} dy ds. \tag{3.100}$$

In this case, we observe that for $y \geq x$, the nonlinearity $(1+y)^{-1}\exp(-y^2/(Ms))$ yields decay $(1+x)^{-1}\exp(-x^2/(Lt))$ (with a part of the kernel remaining for integration). We immediately obtain an estimate by

$$\begin{aligned} &C_1 t^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_0^{t/2} (1+s)^{-1} \ln(e+s) ds + C_2(1+t)^{-1}(1+x)^{-1} e^{-\frac{x^2}{Lt}} \int_{t/2}^{t} (t-s)^{-1/2} ds \\ &\leq C\theta_1^+(t,x). \end{aligned}$$

In this case integration against the remaining nonlinearities can be analyzed as in the previous paragraph, in each case through the observation that we only have this kernel in the case $y \geq x$.

The remaining Green's function estimates occur only in the case $y \geq x$, for which we can proceed in the case of each nonlinearity as in the previous paragraphs. This concludes the proof of Lemma (2). □

*Acknowledgements.* The author gratefully acknowledges Kevin Zumbrun for several illuminating discussions on various aspects of the analysis. This research was partially supported by the National Science Foundation under Grant No. DMS–0230003.

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