Pointwise Green's function estimates toward stability for multidimensional fourth order viscous shock fronts

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Abstract

For the case of multidimensional viscous conservation laws with fourth order smoothing only, we develop detailed pointwise estimates on the Green's function for the linear fourth order convection equation that arises upon linearization of the conservation law about a viscous planar wave solution. As in previous analyses in the case of second order smoothing, our estimates are sufficient to establish that spectral stability implies nonlinear stability, though the full development of this result will be considered in a companion paper.

1 Introduction

We consider the multidimensional viscous conservation law

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = -\sum_{jklm} \Big(b^{jklm}(u) u_{x_j x_k x_l} \Big)_{x_m}, \tag{1.1}$$
$$u(0,x) = u_0(x); \quad u_0(\pm\infty) = u_\pm,$$

where u, f^j, $b^{jklm} \in \mathbb{R}$, $x \in \mathbb{R}^d$, for some dimension $d \geq 2$ and for $t > 0$. In particular, we develop detailed pointwise estimates on the Green's function for the linear fourth order convection equation that arises upon linearization of (1.1) about the planar viscous shock front $\bar{u}(x_1)$, $\bar{u}(\pm\infty) = u_\pm$. (Due to the generality of f^1, we can choose a moving coordinate system along which the traveling viscous shock front $\bar{u}(x_1 - st)$ becomes stationary.) Our estimates are sufficient to establish that spectral stability (defined below) implies nonlinear stability, though we leave the full development of this result to a companion paper [HH].

Throughout the paper, we will refer to the following fundamental assumptions on (1.1) and the planar wave solution $\bar{u}(x_1 - st)$:

(H0) (regularity) f^j, $b^{jklm} \in C^2(\mathbb{R})$, $\quad b^{1111}(\bar{u}(x_1)) \geq b_0 > 0$.

(H1) (non-sonicity) $\partial_u f^1(u_\pm) \neq s$.

(H2) $\sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_j \xi_k \xi_l \xi_m \geq \theta |\xi|^4$ for all $\xi \in \mathbb{R}^d$ and some $\theta > 0$.

Conservation laws of form (1.1) that satisfy hypotheses (H0)–(H2) arise, for example, in the study of thin film flow, in which the height $h(t,x)$ of a film moving along an inclined plane can, under certain circumstances, be modeled by equations with fourth order smoothing only, such as

$$h_t + (h^2 - h^3)_{x_1} = -\nabla \cdot (h^3 \nabla \triangle h), \quad x \in \mathbb{R}^2 \tag{1.2}$$

(see [BMS] and the references therein). In this setting, the onset of fingering instabilities is a critical issue, and spectral stability has been considered in [BMSZ]. To date, however, no results on nonlinear stability for such equations have been established.

Equations of form (1.1) are often studied through their inviscid approximation

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = 0. \tag{1.3}$$

A simple change of coordinates, $\bar{x} = \epsilon x$, $\bar{t} = \epsilon t$, scales (1.1) to the small viscosity equation

$$u^\epsilon_t + \sum_{j=1}^{d} f^j(u^\epsilon)_{x_j} = -\epsilon^3 \sum_{jklm} \Big(b^{jklm}(u^\epsilon) u^\epsilon_{x_j x_k x_l} \Big)_{x_m}, \tag{1.4}$$

from which we can see that large time behavior in (1.1) is closely tied to bounded time behavior in (1.3). In many respects, (1.3) is the preferable equation to focus on. In certain cases it can be solved explicitly, or its solutions carefully approximated, and perhaps more important, the physical regularity terms required for (1.1) are often small and difficult to gain precise knowledge of. It is well known, however, that equations of form (1.3) must typically be solved in the space of discontinuous solutions, where non-physical solutions arise. Numerous admissibility criteria have been developed in order to select the physically relevant solutions, but developments over the last fifteen years indicate that even in the scalar case some understanding of the regularization is crucial (see, for example, [BMS, JMS, Wu]). In particular, the physically relevant solutions of a system modeled by a conservation law with one regularization may differ significantly from the physically relevant solutions of the same system under a different regularization.

Such considerations indicate the need for a detailed understanding of the dynamics involved with solutions of (1.1). Of particular importance are those solutions that are stable, and hence typically correspond with observable phenomena. Unfortunately, the stability analysis of solutions of regularized conservation laws such as (1.1) has proven to be a quite difficult problem. The pointwise Green's function method, however, introduced by Liu [L.1], and developed by Liu and his collaborators [L.2, LY, LZ.1, LZ.2], has proven to be quite robust: in applications to viscous shock waves arising in second order multidimensional systems [HoffZ.1, HoffZ.2, Z.1, ZS], viscous shock waves arising in scalar conservation laws of arbitrary order [HowardZ.1], degenerate viscous shock waves [H.1, H.2, HowardZ.2], viscous rarefaction waves [SZ], systems with physical (non-Laplacian) viscosity [MZ.2, Z.2], and relaxation systems [MZ.2]. In this paper, we extend the pointwise Green's function development to the case of fourth order regularization only. The critical new issue here is an absence of the highly regularizing second order viscosity, which has been assumed present in all of the fully regularized analyses mentioned above (which omits the physical viscosity analysis), including the high-order analysis [HowardZ.1]. On a technical level, this absence of second order viscosity means that our linear decay rate is $t^{-1/4}$ rather than $t^{-1/2}$, and consequently that our interaction analysis is considerably more delicate than those of previous cases. Most notably, in the case $d = 3$, we must proceed as in the detailed development of [HoffZ.1, HoffZ.2], but in the case of undercompressive viscous shocks, which did not arise in that setting.

It is well known that for $d = 1$ solutions $u(t, x)$ of (1.1) initialized by $u(0, x)$ near a standing wave solution $\bar{u}(x)$ will not generally approach $\bar{u}(x)$ time asymptotically, but rather will approach a translate of $\bar{u}(x)$ determined by the amount of mass (measured by $\int_{\mathbb{R}} \Big(u(0, x) - \bar{u}(x) \Big) dx$) carried into the shock as well as the amount of mass convected along outgoing characteristics to the far field. For $d = 1$, a local tracking function $\delta(t)$ will serve to approximate this shift at each time t. In particular, we define our perturbation by the relation $v(t, x) = u(t, x) - \bar{u}(x - \delta(t))$, and choose $\delta(t)$ so that at each time t, we are comparing the shapes of $u(t, x)$ and $\bar{u}(x)$ rather than their locations. (See, for example, [HowardZ.1].)

In the case $d \geq 2$, the shift along the planar shock front $\bar{u}(x_1)$ depends additionally on the transverse variable $\tilde{x} = (x_2, x_3, ..., x_d)$. In this case, $u(t,x)$ does not approach a shifted wave asymptotically (the shift goes to 0 as $t \to \infty$), but these *ripples* along the shock layer slow asymptotic convergence, hindering our analysis. We proceed, then, by introducing the perturbation $v(t,x)$, defined through

$$v(t,x) = u(t,x) - \bar{u}(x_1 - \delta(t,\tilde{x})),$$

to arrive at the perturbation equation (closely following the notation of [HoffZ.2])

$$(\partial_t - L)v = (\partial_t - L)(\bar{u}_{x_1}(x_1)\delta(t,\tilde{x})) + \sum_{m=1}^{d}(Q^m + R^m + S^m)_{x_m}, \tag{1.5}$$

where Q, R, S are smooth function of their arguments, and

$$\begin{aligned}
Lv &:= -\sum_{jklm}(b^{jklm}(x_1)v_{x_jx_kx_l})_{x_m} - \sum_{j=1}^{d}(a_j(x_1)v)_{x_j} \\
a_j(x_1) &:= \partial_u f^j(\bar{u}(x_1)) + \partial_u b^{111j}(\bar{u}(x_1))\bar{u}_{x_1x_1x_1} && (1.6)\\
b^{jklm}(x_1) &:= b^{jklm}(\bar{u}(x_1)), && (1.7)
\end{aligned}$$

with also

$$\begin{aligned}
Q^m &= \mathbf{O}(v^2) + \sum_{jkl}\mathbf{O}(|vv_{x_jx_kx_l}|), \quad \text{for each } m = 1, ..., d \\
R^1 &= \mathbf{O}(e^{-\bar{\eta}|x_1|})\Big[\mathbf{O}(|\delta\delta_t|) + \mathbf{O}(|\delta|)\mathbf{O}(\sum_{j\neq 1}|\delta_{x_j}| + \sum_{jk\neq 1}|\delta_{x_jx_k}| + \sum_{jkl\neq 1}|\delta_{x_jx_kx_l}|) \\
&\quad + \mathbf{O}(\sum_{j\neq 1}|\delta_{x_j}|)\mathbf{O}(\sum_{k\neq 1}|\delta_{x_k}| + \sum_{kl\neq 1}|\delta_{x_kx_l}|)\Big] && (1.8)\\
R^m &= \mathbf{O}(e^{-\bar{\eta}|x_1|})\Big[\mathbf{O}(|\delta|)\mathbf{O}(\sum_{j\neq 1}|\delta_{x_j}| + \sum_{jk\neq 1}|\delta_{x_jx_k}| + \sum_{jkl\neq 1}|\delta_{x_jx_kx_l}|) \\
&\quad + \mathbf{O}(\sum_{j\neq 1}|\delta_{x_j}|)\mathbf{O}(\sum_{kl\neq 1}|\delta_{x_kx_l}|)\Big], \quad m \neq 1 \\
S^m &= \mathbf{O}(e^{-\bar{\eta}|x_1|})\Big[\mathbf{O}(|\delta|)\mathbf{O}(|v|) + \mathbf{O}(|\delta|)\mathbf{O}(\sum_{jkl}|v_{x_jx_kx_l}|) \\
&\quad + \mathbf{O}(|v|)\mathbf{O}(\sum_{jkl}|\delta| + |\delta_{x_j}| + |\delta_{x_jx_k}| + |\delta_{x_jx_kx_l}|)\Big], \quad \text{for each } m.
\end{aligned}$$

According to (H0)–(H2), we can make the following conclusions about the coefficients of (1.5):

(C0) (regularity) $a_j(x_1) \in C^1(\mathbb{R})$, $b^{jklm}(x_1) \in C^2(\mathbb{R})$, $b^{1111}(x_1) \geq b_0 > 0$.

(C0') (asymptotic decay) $|\frac{\partial^k}{\partial x_1^k}(a_j(x_1) - a_j^{\pm})| = \mathbf{O}(e^{-\alpha|x_1|})$, $k = 0, 1$, $|\frac{\partial^k}{\partial x_1^k}(b^{jklm}(x_1) - b_{\pm}^{jklm})| = \mathbf{O}(e^{-\alpha|x|})$, $k = 0, 1, 2$ some $\alpha > 0$.

(C1) (non-sonicity) Either $a_1^+ < 0 < a_1^-$ (Lax case) or $\mathrm{sgn}(a_1^+ a_1^-) = 1$ (undercompressive case).

(C2) $\sum_{jklm} b^{jklm}(x_1)\xi_j\xi_k\xi_l\xi_m \geq \theta|\xi|^4$ for all $\xi \in \mathbb{R}^d$ and some $\theta > 0$.

Here,

$$a_j^{\pm} := \lim_{x_1 \to \pm\infty} a_j(x_1); \quad \text{and} \quad b_{\pm}^{jklm} := \lim_{x_1 \to \pm\infty} b^{jklm}(x_1).$$

Integrating (1.5) (and after one application of integration by parts on the nonlinear interaction), we arrive at the integral equation

$$\begin{aligned} v(t,x) &= \int_{\mathbb{R}^d} G(t,x;y)v_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)\Big[(\partial_s - L_y)(\bar{u}_{y_1}\delta) + \sum_{m=1}^d (Q^m + R^m + S^m)_{y_m}\Big]dyds, \end{aligned} \tag{1.9}$$

where $G(t,x;y)$ represents the Green's function for the linear part of (1.5):

$$G_t + \sum_{j=1}^d (a_j(x_1)G)_{x_j} = -\sum_{jklm}(b^{jklm}(x_1)G_{x_jx_kx_l})_{x_m}; \quad G(0,x;y) = \delta_y(x). \tag{1.10}$$

The idea behind the pointwise Green's function approach to stability is to obtain estimates on $G(t,x;y)$ sufficiently sharp so that an iteration on (1.9) can be closed. (See, for example, [HoffZ.2, HowardZ.1, Z.1] for complete nonlinear analyses in similar situations.) In the current analysis, we develop estimates on the Green's function $G(t,x;y)$. We carry out the nonlinear iteration in a companion paper [HH].

Observing that the coefficients of our linear equation (1.10) depend only on x_1, we take a Fourier transform in the transverse variable $\tilde{x} = (x_2, x_3, ..., x_d)$ (scaling the Fourier transform as $(2\pi)^{\frac{1-d}{2}} \int_{\mathbb{R}^{d-1}} e^{-i\xi\cdot\tilde{x}}v(t,x_1,\tilde{x})d\tilde{x}$, $\xi = (\xi_2, \xi_3, ..., \xi_d)$) to obtain

$$\begin{aligned} \hat{G}_t :=& L_\xi \hat{G} = -(b^{1111}(x_1)\hat{G}_{x_1x_1x_1})_{x_1} - (a_1(x_1)\hat{G})_{x_1} - i\sum_{j\neq 1} a_j(x_1)\xi_j\hat{G} - \sum_{jklm\neq 1} b^{jklm}(x_1)\xi_j\xi_k\xi_l\xi_m\hat{G} \\ &-i\sum_{j\neq 1} b^{j111}(x_1)\xi_j\hat{G}_{x_1x_1x_1} - i\sum_{j\neq 1}(b^{1\overline{j11}}(x_1)\xi_j\hat{G}_{x_1x_1})_{x_1} + \sum_{jk\neq 1} b^{j\overline{k11}}(x_1)\xi_j\xi_k\hat{G}_{x_1x_1} \\ &+ \sum_{jk\neq 1}(b^{1\overline{jk1}}(x_1)\xi_j\xi_k\hat{G}_{x_1})_{x_1} + \sum_{jkl\neq 1} b^{j\overline{kl1}}(x_1)\xi_j\xi_k\xi_l\hat{G}_{x_1} + i\sum_{jkl\neq 1}(b^{1jkl}(x_1)\xi_j\xi_k\xi_l\hat{G})_{x_1}, \\ &\hat{G}(0,x_1,\xi,y) = (2\pi)^{\frac{1-d}{2}} e^{-i\xi\cdot\tilde{y}}\delta_{y_1}(x_1), \end{aligned}$$

where the notation $\overline{\cdots}$ indicates summation over a permutation of indices, for instance,

$$b^{1\overline{j11}} = b^{1j11} + b^{11j1} + b^{111j}.$$

(We note that L_0 is the linearized operator for the scalar case, so that $d = 1$ estimates can be obtained from our analysis by setting $\xi = 0$.) Typically, we analyze $\hat{G}(t,x_1,\xi,y)$ through its Laplace transform ($t \to \lambda$), $G_{\lambda,\xi}(x_1,y)$, which satisfies the ODE,

$$L_\xi G_{\lambda,\xi} - \lambda G_{\lambda,\xi} = -(2\pi)^{\frac{1-d}{2}} e^{-i\xi\cdot\tilde{y}}\delta_{y_1}(x_1), \tag{1.11}$$

and can be estimated by standard methods. Letting $\phi_1^-(x_1;\lambda,\xi)$ and $\phi_2^-(x_1;\lambda,\xi)$ denote the (necessarily) two linearly independent asymptotically decaying solutions at $-\infty$ of the eigenvalue ODE

$$L_\xi\phi = \lambda\phi, \tag{1.12}$$

and $\phi_1^+(x_1;\lambda,\xi)$ and $\phi_2^+(x_1;\lambda,\xi)$ similarly the two linearly independent asymptotically decaying solutions at $-\infty$, we construct the ODE Green's function as

$$G_{\lambda,\xi}(x_1,y) = \begin{cases} \phi_1^-(x_1;\lambda,\xi)N_1^+(y;\lambda,\xi) + \phi_2^-(x_1;\lambda,\xi)N_2^+(y;\lambda,\xi), & x_1 < y_1 \\ \phi_1^+(x_1;\lambda,\xi)N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi)N_2^-(y;\lambda,\xi), & x_1 > y_1. \end{cases} \tag{1.13}$$

Insisting, as usual, on the continuity of $G_{\lambda,\xi}(x_1, y)$ and its first two derivatives in x_1 with respect to x_1, and on the jump in $\partial^3_{x_1} G_{\lambda,\xi}(x_1, y)$ defined through (1.11), we arrive at a linear system of algebraic equations for the $N_k^\pm$ that can be solved by Cramer's rule. We have

$$N_1^+(y) = (2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^+, \phi_2^+, \phi_2^-)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)}, \quad N_2^+(y) = -(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^-, \phi_1^+, \phi_2^+)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)}$$

$$N_1^-(y) = -(2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^-, \phi_2^-, \phi_2^+)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)}, \quad N_2^-(y) = (2\pi)^{\frac{1-d}{2}} e^{-i\tilde{y}\cdot\xi} \frac{W(\phi_1^-, \phi_2^-, \phi_1^+)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)},$$

where $W_{\lambda,\xi}(y_1) = W(\phi_1^+, \phi_2^+, \phi_1^-, \phi_2^-)$, and following [HowardZ.1], our notation $W(\phi_1, \phi_2, ..., \phi_n)$ indicates a square Wronskian or determinant of column vectors created by augmentation with an appropriate number of derivatives. For instance, the Evans function in this case is defined through

$$D(\lambda, \xi) = W(\phi_1^+, \phi_2^+, \phi_1^-, \phi_2^-)\Big|_{y_1=0} = \det \begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_1^- & \phi_2^- \\ \phi_1^{+\prime} & \phi_2^{+\prime} & \phi_1^{-\prime} & \phi_2^{-\prime} \\ \phi_1^{+\prime\prime} & \phi_2^{+\prime\prime} & \phi_1^{-\prime\prime} & \phi_2^{-\prime\prime} \\ \phi_1^{+\prime\prime\prime} & \phi_2^{+\prime\prime\prime} & \phi_1^{-\prime\prime\prime} & \phi_2^{-\prime\prime\prime} \end{pmatrix}.$$

Critically, we see by construction that $G_{\lambda,\xi}(x_1, y)$ is well behaved except at zeros of the Evans function, which away from essential spectrum correspond exactly with point spectrum of the operator L_ξ (see [AGJ, GZ, ZH]). In certain cases, the Evans function can be studied analytically (see, for example [BMSZ, D]), while more generally its zeros are determined numberically (see, for example, [B, OZ]). Our approach will be 1. to determine conditions on the Evans function necessary for stability, and 2. to impose conditions on zeros of the Evans function sufficient for linear stability, and develop pointwise Green's function estimates under these conditions sufficient for establishing nonlinear stability. We leave the full development of nonlinear stability (an iteration on the integral equation (1.9)) to a companion paper [HH].

Following [Z.1, ZS], we will analyze the Evans function with respect to a radial coordinate ρ, defined through

$$(\lambda, \xi) = (\rho\lambda_0, \rho\xi_0), \quad \text{where } |(\lambda_0, \xi_0)| = 1. \tag{1.14}$$

Clearly, $\rho = |(\lambda, \xi)| = \sqrt{|\lambda|^2 + |\xi|^2}$. In particular, we will analyze

$$D_{\lambda_0,\xi_0}(\rho) := D(\rho\lambda_0, \rho\xi_0), \tag{1.15}$$

and the *reduced Evans functions*

$$\bar{\triangle}(\lambda_0, \xi_0) := \lim_{\rho\to 0} \rho^{-1} D_{\lambda_0,\xi_0}(\rho).$$

In terms of these definitions, our stability conditions take the forms $(\mathcal{D}_n)$ (necessity) and $(\mathcal{D}_s)$ (sufficiency).

$(\mathcal{D}_n)$ **Necessary conditions for linear stability**.

Condition (1).

$$D(\lambda, \xi) \neq 0, \quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \mathrm{Re}\lambda > 0\},$$
$$\bar{\triangle}(\lambda_0, \xi_0) \neq 0, \quad \{(\lambda_0, \xi_0) : \xi_0 \in \mathbb{R}^{d-1}, \mathrm{Re}\lambda_0 > 0\}.$$

Condition (2). There is a neighborhood V of zero in (complex) ξ-space so that L_ξ has a unique L^2 eigenvalue, $\lambda_*(\xi)$, defined through $D(\lambda_*, \xi) = 0$, $\lambda_*(0) = 0$, satisfying

$$\lambda_*(\xi) = -i\frac{[\tilde{f}]}{[u]} \cdot \xi - \lambda_2^{jk}\xi_j\xi_k + i\lambda_3^{jkl}\xi_j\xi_k\xi_l - \lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m + \mathbf{O}(|\xi|^5),$$

where summation is assumed over repeated indices, and we use the standard jump notation $[u] = u_+ - u_-$ and

$$[\tilde{f}] = (f^2(u_+) - f^2(u_-), f^3(u_+) - f^3(u_-), ..., f^d(u_+) - f^d(u_-)),$$

and for which we assume

$$\lambda_2^{jk}\xi_j\xi_k \geq \lambda_2^0|\xi|^2, \quad \xi \in \mathbb{R}^{d-1}, \lambda_2^0 \geq 0,$$

and

$$\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m \geq \lambda_4^0|\xi|^4, \quad \xi \in \mathbb{R}^{d-1}, \lambda_4^0 \geq 0.$$

($\mathcal{D}_s$) **Sufficient conditions for linear (and therefore nonlinear) stability**.

Condition (1).

$$D(\lambda,\xi) \neq 0, \quad \{(\lambda,\xi) : \xi \in \mathbb{R}^{d-1}, \text{Re}\lambda \geq 0, (\lambda,\xi) \neq (0,0)\},$$
$$\bar{\Delta}(\lambda_0,\xi_0) \neq 0, \quad \{(\lambda_0,\xi_0) : \xi_0 \in \mathbb{R}^{d-1}, \text{Re}\lambda_0 > 0\},$$

Condition (2). The curve $\lambda_*(\xi)$ from ($\mathcal{D}_n$) satisfies one of the following (Condition (2a) or condition (2b)):

Condition (2a).

$$\lambda_2^{jk}\xi_k\xi_j \geq \lambda_2^0|\xi|^2, \quad \xi \in \mathbb{R}^{d-1}, \lambda_2^0 > 0.$$

Condition (2b).

$$\lambda_2^{jk}\xi_j\xi_k = \lambda_3^{jkl}\xi_j\xi_k\xi_l = 0, \quad \xi \in \mathbb{R}^{d-1}$$

and

$$\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m \geq \lambda_4^0|\xi|^4, \quad \xi \in \mathbb{R}^{d-1}, \lambda_4^0 > 0.$$

Condition (3). For $\rho \geq \rho_0 > 0$, there exist constants c_1 and C_2 so that the spectrum of L_ξ lies entirely to the left of a contour defined through the relation

$$\text{Re }\lambda = -c_1(|\text{Re }\xi|^4 - C_2|\text{Im }\xi|^4 + |\text{Im }\lambda|).$$

We will refer to the contour defined by this relation as Γ_{bound}.

Remark on thin film equations. Undercompressive viscous shock waves arising in the thin films equation (1.2) have been shown through numerical calculations to satisfy Condition (2a) (see [BMSZ], especially the discussion around Figure 6). In the case of compressive waves arising in (1.2) (and generalizations; see (3.1) of [BMSZ]), the authors of [BMSZ] established the exact representation

$$\lambda_2^0 = \int_{-\infty}^{+\infty} \frac{f(u_-) - f(\bar{u})}{u_+ - u_1} dx.$$

Remark on Conditions (2a), (2b), and (3). In the case of incoming characteristics, signals propagate into the shock layer and then convect and diffuse along the shock layer with rates depending on Conditions (2a) and (2b). In the case of Condition (2a), the signal propagates along the shock layer similarly as the solution to a second-order convection–diffusion equation. In the case of Condition (2b), the signal propagates along the shock layer similarly as a fourth order convection–regularity equation. Condition (3) insures that the small t behavior in the shock layer is fourth order.

Remark on Condition (1). The condition $\bar{\Delta}(\lambda_0,\xi_0) \neq 0$ is a *transversality* condition. In the case of compressive waves,

$$\bar{\Delta}(\lambda_0,\xi_0) = \gamma(i\xi_0 \cdot [\tilde{f}] + \lambda_0[u]),$$

from which $\bar{\Delta}(\lambda_0,\xi_0)\neq 0 \Rightarrow \gamma\neq 0$, and additionally $\bar{\Delta}(\lambda_0,\xi_0)$ is only 0 for λ_0 purely complex.

We note finally that a fundamental difference between the current analysis and the analysis of [HoffZ.1, HoffZ.2] is that we take $(\mathcal{D}_n)$ and $(\mathcal{D}_s)$ as assumptions, while in [HoffZ.1, HoffZ.2] the authors establish similar conditions analytically. In certain cases, such spectral conditions can be verified analytically [D], but more generally they must be verified numerically [B, BMSZ].

We are now in a position to state the main result of the paper.

Theorem 1.1. Under assumptions (H0)–(H2) and $(\mathcal{D}_s)$, we have the following estimates on solutions $G(t,x;y)$ to the Green's function equation (1.10). For some constants M and η and for d-dimensional multi-index α, with $|\alpha|\leq 3$, $\alpha_1\leq 1$

(I) Lax case $(a_1^+<0<a_1^-)$

(i) $y_1,x_1\leq 0$

$$\begin{aligned}\partial_y^\alpha G(t,x;y) &= \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})e^{-\frac{|x-y-a_-t|^{4/3}}{Mt^{1/3}}} + \bar{u}_{x_1}(x_1)\partial_y^\alpha e(t,\tilde{x},y)\\ &+\mathbf{O}(e^{-\eta|x_1|})R(t,\tilde{x},y;d+|\alpha|) + +\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde{x}-\tilde{y}|+t)})I_{\{|y_1|\leq|a_1^-|t\}}.\end{aligned}$$

(ii) $x_1\leq 0\leq y_1$

$$\begin{aligned}\partial_y^\alpha G(t,x;y) &= \bar{u}_{x_1}(x_1)\partial_y^\alpha e(t,\tilde{x},y) + \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})\mathbf{O}(e^{-\eta|x_1|})e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^+t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1+a_1^+t)^{4/3}}{Mt^{1/3}}}\\ &+\mathbf{O}(e^{-\eta|x_1|})R(t,\tilde{x},y;d+|\alpha|) + +\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde{x}-\tilde{y}|+t)})I_{\{|y_1|\leq|a_1^+|t\}}.\end{aligned}$$

where for $y_1\gtrless 0$

$$\partial_y^\alpha e(t,\tilde{x},y) = \mathbf{O}(t^{-\frac{d-1+|\alpha|}{4}})e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^\pm t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1+a_1^\pm t)^{4/3}}{Mt^{1/3}}} + R(t,\tilde{x},y;d-1+|\alpha|)$$

with

$$\tilde{a}^\pm_{\text{eff}}(t,y_1) := (1+\frac{y_1}{a_1^\pm t})\tilde{a}_{\text{ave}} - \frac{y_1}{a_1^\pm t}\tilde{a}^\pm,$$

and for the case of Condition (2a), we have (again for $y_1\gtrless 0$)

$$R(t,\tilde{x},y;\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}}\wedge|y_1\mp a_1^\pm t|^{-\frac{\kappa}{2}})\Big[e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^\pm_{\text{eff}}t|^{4/3}}{Mt^{1/3}}} + e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^\pm_{\text{eff}}t|^{2}}{M|y_1\mp a_1^\pm t|}}\Big]I_{\{|y_1|\leq|a_1^\pm|t\}}$$

while in the case of Condition (2b),

$$R(t,\tilde{x},y;\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}})e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^-_{\text{eff}}t|^{4/3}}{Mt^{1/3}}}I_{\{|y_1|\leq|a_1^-|t\}},$$

In the Lax case, estimates for $x_1\geq 0$ are symmetric. Moreover, due to non-sonicity condition (H1), the first time derivative of $e(t,\tilde{x};y)$ admits the same estimate as the first space derivatives.

(II) Undercompressive case $(a_1^-,a_1^+>0)$.

(i) $y_1,x_1\leq 0$

$$\begin{aligned}\partial_y^\alpha G(t,x;y) &= \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})e^{-\frac{|x-y-a_-t|^{4/3}}{Mt^{1/3}}} + \bar{u}_{x_1}(x_1)\partial_y^\alpha e(t,\tilde{x},y)\\ &+\mathbf{O}(t^{-\frac{d}{4}})\mathbf{O}(e^{-\eta|x_1|})\Big(\mathbf{O}(t^{-\frac{|\alpha|}{4}})+\alpha_1\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}})\Big)e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^-t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1+a_1^-t)^{4/3}}{Mt^{1/3}}}\\ &+\mathbf{O}(e^{-\eta|x_1|})R^-_\alpha(t,\tilde{x},y;d) + \mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde{x}-\tilde{y}|+t)})I_{\{|y_1|\leq|a_1^-|t\}}.\end{aligned}$$

(ii) $x_1 \le 0 \le y_1$

$$\partial_y^\alpha G(t,x,y) = \bar u_{x_1}(x_1)\partial_y^\alpha e(t,\tilde x,y) + \mathbf{O}(t^{-\frac{d+|\alpha|-\alpha_1}{4}})\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})e^{-\frac{|\tilde x-\tilde y-\tilde a^+ t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1-a_1^+ t)^{4/3}}{Mt^{1/3}}}$$
$$+\mathbf{O}(e^{-\eta|x_1|})R^+(t,\tilde x,y;d+|\alpha|-\alpha_1) + \mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde x-\tilde y|+t)})I_{\{|y_1|\le|a_1^+ t|\}}.$$

(iii) $y_1 \le 0 \le x_1$

$$\partial_y^\alpha G(t,x;y) = \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})e^{-\frac{(x_1-\frac{a_1^+}{a_1^-}y_1-a_1^+ t)^{4/3}}{Mt^{1/3}}}e^{-\frac{|\tilde x-\tilde y-(\frac{\tilde a^+-\tilde a^-}{a_1^- t}y_1+\tilde a^+)t|^{4/3}}{Mt^{1/3}}} + \bar u_{x_1}(x_1)\partial_y^\alpha e(t,\tilde x,y)$$
$$+\mathbf{O}(e^{-\eta|x_1|})R_\alpha^-(t,\tilde x,y;d) + \mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde x-\tilde y|+t)})I_{\{|y_1|\le|a_1^- t|\}}.$$

(iv) $x_1, y_1 \ge 0$

$$\partial_y^\alpha G(t,x;y) = \mathbf{O}(t^{-\frac{d+|\alpha|}{4}})e^{-\frac{|x-y-a_+ t|^{4/3}}{Mt^{1/3}}} + \bar u_{x_1}(x_1)\partial_y^\alpha e(t,\tilde x,y)$$
$$+\mathbf{O}(e^{-\eta|x_1|})R^+(t,\tilde x,y;d+|\alpha|-\alpha_1) + \mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta(|\tilde x-\tilde y|+t)})I_{\{|y_1|\le|a_1^+ t|\}},$$

where for $y_1 < 0$,

$$\partial_y^\alpha e(t,\tilde x;y) = \mathbf{O}(t^{-\frac{d-1}{4}})\Big[\mathbf{O}(t^{-\frac{|\alpha|}{4}}) + \alpha_1\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}})\Big]e^{-\frac{|\tilde x-\tilde y-\tilde a^- t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1+a_1^- t)^{4/3}}{Mt^{1/3}}}$$
$$+R_\alpha^-(t,\tilde x;y,d-1),$$

while for $y_1 > 0$ we have,

$$\partial_y^\alpha e(t,\tilde x;y) = \mathbf{O}(t^{-\frac{d-1}{4}})\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}})e^{-\frac{|\tilde x-\tilde y-\tilde a^+ t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1-a_1^+ t)^{4/3}}{Mt^{1/3}}}$$
$$+R_\alpha^+(t,\tilde x;y,d-1).$$

In the case of Condition (2a), we have

$$R_\alpha^-(t,\tilde x,y;\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}}\wedge|y_1+a_1^- t|^{-\frac{\kappa}{2}})\Big(\mathbf{O}(t^{-\frac{|\alpha|}{4}}\wedge|y_1+a_1^- t|^{-\frac{|\alpha|}{2}})$$
$$+\alpha_1\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}}\wedge|y_1+a_1^- t|^{-\frac{|\alpha|-\alpha_1}{2}})\Big[e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^- t|^{4/3}}{Mt^{1/3}}} + e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^- t|^2}{M|y_1+a_1^- t|}}\Big]I_{\{|y_1|\le|a_1^- t|\}}.$$
$$R^+(t,\tilde x,y;\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}}\wedge|y_1-a_1^+ t|^{-\frac{\kappa}{2}})\mathbf{O}(e^{-\eta|y_1|})\Big[e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^+ t|^{4/3}}{Mt^{1/3}}} + e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^+ t|^2}{M|y_1-a_1^+ t|}}\Big]I_{\{|y_1|\le|a_1^+ t|\}}.$$

while in the case of Condition (2b), we have

$$R_\alpha^-(t,\tilde x;y,\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}})\Big(\mathbf{O}(t^{-\frac{|\alpha|}{4}}) + \alpha_1\mathbf{O}(e^{-\eta|y_1|})\mathbf{O}(t^{-\frac{|\alpha|-\alpha_1}{4}})\Big)e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^- t|^{4/3}}{Mt^{1/3}}}I_{\{|y_1|\le|a_1^- t|\}}$$
$$R^+(t,\tilde x;y,\kappa) = \mathbf{O}(t^{-\frac{\kappa}{4}})\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})e^{-\frac{|\tilde x-\tilde y-\tilde a_{\text{eff}}^+ t|^{4/3}}{Mt^{1/3}}}I_{\{|y_1|\le|a_1^+ t|\}}.$$

Moreover, due to non-sonicity condition (H1), the first time derivative of $e(t,\tilde x;y)$ admits the same estimate as a single derivative with respect to any transverse coordinate.

The estimates of Theorem 1.1 should be compared with those of Theorem 1.1 in [HowardZ.1] and with those of Theorem 1.2 in [HoffZ.1]. In particular, the only differences between our Lax case estimates and the estimates of [HoffZ.1] are: 1. the different exponential scaling for fourth order (as opposed to second order) regularization, 2. the designation of our the excited terms as $\bar{u}_{x_1}(x_1)e(t,\tilde{x};y)$, as in the refined analysis of Mascia and Zumbrun [MZ.2], and 3. in the case of Condition (2b), the change of diffusion rate of the signal once it enters the shock layer. This last difference, in particular, warrants discussion. The behavior of the Green's function in the shock layer is determined by the form of $\lambda_*(\xi)$. In particular, for $|y_1| \le |a_1^{\pm}|t$ (i.e., once a signal starting at y_1 reaches the shock layer), we have a contribution to the Green's function of form,

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})+\lambda_* t-\mu_2^-(\lambda_*,\xi)y_1} d\xi$$
$$\cong \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})+(1+\frac{y_1}{a_1^{\pm}})(-i\frac{[\tilde{f}]}{[u]}\cdot\xi-\lambda_2^{jk}\xi_j\xi_k+i\lambda_3^{jkl}\xi_j\xi_k\xi_l)t-\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m t} d\xi,$$

which corresponds (roughly) with a Green's function in dimension $d-1$ for a convection–regularity equation with r^{th} order regularity governed by λ_r. We observe that in the case of Condition (2b) for which $\lambda_2^{jk}\xi_j\xi_k \ge \lambda_2^0|\xi|^2$ for some $\lambda_2^0 > 0$, the long-time behavior of the Green's function after entering the shock layer will be second order rather than fourth. For example, in the compressive case, a signal beginning at some point $(y_1,\tilde{y})$, with $y_1 < 0$, will travel like a fourth order kernel until it strikes the shock layer (when $y_1 = -a_1^- t$), at which time it will move transversally through the shock layer, propagating as a second order kernel with time replaced by $|y_1 + a_1^- t|$. (Spectral condition (3) insures that small time behavior is always governed by fourth order dynamics.)

Another fundamentally new term arises in the undercompressive case, which did not arise in the analysis of [HoffZ.1]. (Undercompressive shocks arise in the general systems analysis of Zumbrun [Z.1], but the Green's function estimates employed there are not as detailed as those of [HoffZ.1] or those of the current analysis.) The interesting behavior consists of transmission of mass through the shock layer, as signified by the expression from the undercompressive case $y_1 \le 0 \le x_1$,

$$S(t,x;y) = \mathbf{O}(t^{-\frac{d}{4}})e^{-\frac{(x_1-\frac{a_1^+}{a_1^-}y_1-a_1^+t)^{4/3}}{Mt^{1/3}}} e^{-\frac{|\tilde{x}-\tilde{y}-(\frac{\tilde{a}^+-\tilde{a}^-}{a_1^- t}y_1+\tilde{a}^+)t|^{4/3}}{Mt^{1/3}}}.$$

Geometrically, this scaling is straighforward to see in the case of two dimensions, in which we observe that a signal beginning at point (y_1,y_2) and moving with convection $a^- = (a_1^-,a_2^-)$ $(a_1^- > 0)$ will strike the shock layer at time $t_{\text{SL}} = |y_1|/a_1^-$ (see Figure 1). Asymptotically, the convection switches in the shock layer, and the signal emerges with convection $a = (a_1^+,a_2^+)$. The position of a signal at time t that has passed through the shock layer becomes (x_1,x_2), where

$$\begin{aligned} x_1 &= a_1^+(t-t_{\text{SL}}) = a_1^+(t-\frac{|y_1|}{a_1^-}) \\ x_2 &= y_2 + \frac{a_2^-}{a_1^-}|y_1| + a_2^+(1-\frac{|y_1|}{a_1^-}), \end{aligned}$$

which correspond respectively with the exponents in $S(t,x;y)$.

We remark finally that for a detailed discussion of the physicality of the *effective convection* $\tilde{a}_{\text{eff}}^{\pm}$, the reader is referred to [HoffZ.1] pp. 373–375, under the heading *Geometric interpretation*.

2 Analysis of the Evans function

In this section, we analyze the Evans function for our linear equation (1.12). We begin by establishing estimates on the linearly indpendent growth and decay solutions ($\phi_k^{\pm}$ and $\psi_k^{\pm}$ respectively) to the eigenvalue

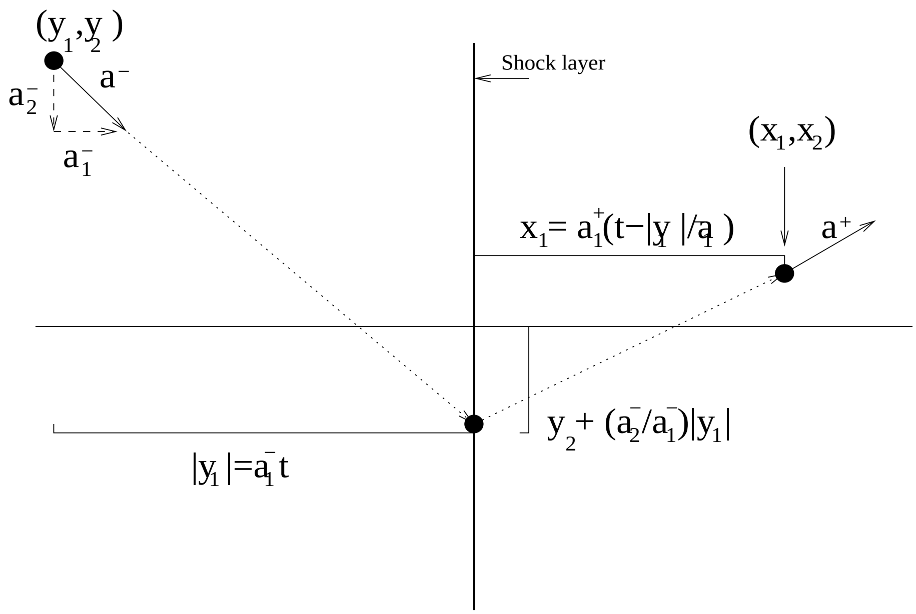

Figure 1: Signal convection through the shock layer, undercompressive case.

ODE (1.12). Following the analysis of [ZH], we proceed by writing our eigenvalue equation (1.12) as a first order system. Setting $W_1 = \phi$, $W_2 = \phi'$, $W_3 = \phi''$, and $W_4 = \phi'''$, we have

$$W' = \mathbb{A}_\pm(\lambda, \xi)W + \mathbf{O}(e^{-\alpha|x_1|})W, \tag{2.1}$$

where

$$\mathbb{A}_\pm(\lambda,\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-b_\pm^{1111})^{-1}\Lambda_\pm & -\frac{a_1^\pm}{b_\pm^{1111}} + i(b_\pm^{1111})^{-1}B_1^\pm(\xi) & (b_\pm^{1111})^{-1}B_2^\pm(\xi) & -i(b_\pm^{1111})^{-1}B_3^\pm(\xi) \end{pmatrix},$$

with

$$\begin{aligned} \Lambda_\pm(\lambda,\xi) :=& \lambda + i\sum_{j\neq 1} a_j^\pm \xi_j + \sum_{jklm\neq 1} b_\pm^{jklm}\xi_j\xi_k\xi_l\xi_m \\ B_0^\pm(\xi) :=& \sum_{jklm\neq 1} b_\pm^{jklm}\xi_j\xi_k\xi_l\xi_m \\ B_1^\pm(\xi) :=& \sum_{jkl\neq 1} b_\pm^{\overline{jkl1}}\xi_j\xi_k\xi_l \\ B_2^\pm(\xi) :=& \sum_{jk\neq 1} b_\pm^{\overline{jk11}}\xi_j\xi_k \\ B_3^\pm(\xi) :=& \sum_{j\neq 1} b^{\overline{j111}}\xi_j, \end{aligned} \tag{2.2}$$

where again $b^{\overline{jklm}}$ represents a sum over a permutation of indices. The growth and decay rates for solutions of (2.1) are simply the eigenvalues, $\mu_k^\pm$, of the asymptotic matrix $\mathbb{A}_\pm(\lambda,\xi)$, which satisfy

$$b_\pm^{1111}\mu^4 + iB_3^\pm(\xi)\mu^3 - B_2^\pm(\xi)\mu^2 - iB_1^\pm(\xi)\mu + a_1^\pm\mu + \Lambda_\pm = 0. \tag{2.3}$$

For ρ sufficiently small (and consequently $|\Lambda|$ sufficiently small), we have

$$\begin{aligned}
\mu_j^\pm &= -\sqrt[3]{\frac{a_1^\pm}{b_\pm^{1111}}} + \mathbf{O}(\rho)\\
\mu_k^\pm &= -\frac{1}{a_1^\pm}\Lambda_\pm - i\frac{B_1^\pm(\xi)}{(a_1^\pm)^2}\Lambda_\pm + \frac{B_2^\pm(\xi)}{(a_1^\pm)^3}\Lambda_\pm^2 + i\frac{B_3^\pm(\xi)}{(a_1^\pm)^4}\Lambda_\pm^3 - \frac{b_\pm^{1111}}{(a_1^\pm)^5}\Lambda_\pm^4 + \mathbf{O}(\rho^5)\\
\mu_l^\pm &= \sqrt[3]{\frac{a_1^\pm}{b_\pm^{1111}}}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + \mathbf{O}(\rho)\\
\mu_m^\pm &= \sqrt[3]{\frac{a_1^\pm}{b_\pm^{1111}}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \mathbf{O}(\rho).
\end{aligned}$$

We will choose j,k,l,m, depending on the sign of $a_1^\pm$, so that for ρ sufficiently small

$$k > j \Rightarrow \text{Re } \mu_k \geq \text{Re } \mu_j.$$

The essential spectrum boundary of L_ξ can be computed directly from (2.3) by setting $\mu = ik$, for which we obtain the curve along which the real part of μ changes sign. We find that the essential spectrum is bounded to the left of both contours

$$\begin{aligned}
\lambda_\pm(k,\xi) = &-i\sum_{j\neq 1} a_j^\pm \xi_j - B_0^\pm(\xi) - (ia_1^\pm + B_1^\pm(\xi))k - B_2^\pm(\xi)k^2\\
&- B_3^\pm(\xi)k^3 - b_\pm^{1111}k^4.
\end{aligned}$$

In particular, away from a ball around the origin, the essential spectrum is bounded to the left of our boundary contour Γ_{bound}.

We have the following lemma.

Lemma 2.1. Under conditions **(C0)**–**(C2)** and for some $\rho \leq \delta$, where δ is a constant sufficiently small (ρ defined in (1.14)), we have the following estimates on solutions to the eigenvalue equation (1.12). For some $\bar{\alpha} > 0$, and for $n = 0,1,2,3$,

(i) (Decay solutions) For $\text{Re}\mu_j^\pm \lessgtr 0$,

$$\frac{\partial^n}{\partial x_1^n}\phi_j^\pm(x_1;\lambda,\xi) = e^{\mu_j^\pm(\lambda,\xi)x_1}((\mu_j^\pm)^n + \mathbf{O}(e^{-\bar{\alpha}|x_1|})).$$

(ii) (Growth solutions) For $\text{Re}\mu_j^\pm \gtrless 0$,

$$\frac{\partial^n}{\partial x_1^n}\psi_j^\pm(x_1;\lambda,\xi) = e^{\mu_j^\pm(\lambda,\xi)x_1}((\mu_j^\pm)^n + \mathbf{O}(e^{-\bar{\alpha}|x_1|})).$$

(iii) (Dual estimates) For j, k, l, and m all different indicies,

$$\frac{\partial^n}{\partial y_1^n}\frac{W(\theta_j^\pm,\theta_k^\pm,\theta_l^\pm)}{W_{\lambda,\xi}(y_1)b^{1111}(y_1)} = \frac{\mathbf{O}(1)}{D(\lambda,\xi)}e^{-\mu_m^\pm y_1}\Big[(\mu_m^\pm)^n(\mu_k^\pm - \mu_j^\pm)(\mu_l^\pm - \mu_k^\pm)(\mu_l^\pm - \mu_j^\pm) + \mathbf{O}(e^{-\bar{\alpha}|y_1|})\Big],$$

where each of the $\theta_j^\pm$, $\theta_k^\pm$, and $\theta_l^\pm$ represent either $\phi_j^\pm$ or $\psi_j^\pm$.
(iv) If θ_j, θ_k, and θ_l all decay at exponential rate for $\rho = 0$, then

$$\frac{\partial}{\partial y_1} \frac{W(\theta_j, \theta_k, \theta_l)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big|_{\rho=0} = \mathbf{O}(1).$$

In addition, if two of the θ_j, θ_k, θ_l coalesce at $\rho = 0$, then

$$\frac{1}{\rho} \frac{\partial}{\partial y_1} \frac{W(\theta_j, \theta_k, \theta_l)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big|_{\rho=0} = \mathbf{O}(1).$$

Proof. The proof of (i) and (ii) is standard and can be carried out as in [ZH] through iteration on an integral equation representation of (2.1). For estimates (iii), we proceed from (i) and (ii) by direct calculation. According to Abel's representation of the Wronskian, we have

$$\partial_{y_1} W_{\lambda,\xi}(y_1) = \Big(-i \frac{B_3^\pm(\xi)}{b_\pm^{1111}} + \mathbf{O}(e^{-\bar{\alpha}|y_1|}) \Big),$$

so that

$$W_{\lambda,\xi}(y_1) = D(\lambda, \xi) e^{-i \frac{B_3(\xi)}{b_\pm^{1111}} y_1 + \mathbf{O}(1)}.$$

Similarly, since the $\mu_k^\pm$ are all roots of the polynomial equation (2.3), we must have

$$\mu_1^\pm + \mu_2^\pm + \mu_3^\pm + \mu_4^\pm = -i \frac{B_3(\xi)}{b_\pm^{1111}}.$$

Observing directly from (i) and (ii) that

$$W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)(y_1) = \mathbf{O}(1) e^{(\mu_j^\pm + \mu_k^\pm + \mu_l^\pm) y_1},$$

we conclude the estimate

$$\frac{W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} = \frac{\mathbf{O}(1)}{D(\lambda, \xi)} e^{-\mu_m^\pm y_1}.$$

For $n = 1$, we compute

$$\begin{aligned}
\partial_{y_1} \frac{W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} =& \frac{b^{1111}(y_1) W_{\lambda,\xi}(y_1) \partial_{y_1} W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)}{W_{\lambda,\xi}(y_1)^2 b^{1111}(y_1)^2} \\
&- \frac{W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)(b^{1111}(y_1) \partial_{y_1} W_{\lambda,\xi} + W_{\lambda,\xi} \partial_{y_1} b^{1111}(y_1))}{W_{\lambda,\xi}(y_1)^2 b^{1111}(y_1)^2} \\
=& \frac{\partial_{y_1} W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm) - (-i \frac{B_3(\xi)}{b_\pm^{1111}} + \mathbf{O}(e^{-\alpha|y_1|})) W(\theta_j^\pm, \theta_k^\pm, \theta_l^\pm)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
= \frac{1}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} & \det \begin{pmatrix} \theta_j^\pm & \theta_k^\pm & \theta_l^\pm \\ \theta_j^{\pm\prime} & \theta_k^{\pm\prime} & \theta_l^{\pm\prime} \\ \theta_j^{\pm\prime\prime\prime} + i \frac{B_3(\xi)}{b_\pm^{1111}} \theta_j^\pm & \theta_k^{\pm\prime\prime\prime} + i \frac{B_3(\xi)}{b_\pm^{1111}} \theta_k^\pm & \theta_l^{\pm\prime\prime\prime} + i \frac{B_3(\xi)}{b_\pm^{1111}} \theta_l^\pm \end{pmatrix} + \frac{\mathbf{O}(e^{-\bar{\alpha}|y_1|})}{D(\lambda, \xi)} \\
=& \frac{e^{-\mu_m^\pm y_1}}{D(\lambda, \xi)} \Big[(-\mu_m^\pm)(\mu_k - \mu_j)(\mu_l - \mu_k)(\mu_l - \mu_j) + \mathbf{O}(e^{-\bar{\alpha}|y_1|}) \Big].
\end{aligned}$$

Estimates on higher order derivatives are similar. In order to establish (iv), we proceed as for the case $n = 1$ above, except that we track $\mathbf{O}(|\xi|)$ behavior rather than $\mathbf{O}(e^{-\bar{\alpha}|y|})$ behavior. First, a more precise statement of Abel's representation of the Wronskian takes the form

$$\partial_{y_1} W_{\lambda,\xi}(y_1) = -i\Big(\frac{\partial_{y_1} b^{1111}(y_1) + \sum_{j\neq 1} b^{\overline{j111}}(y_1)\xi_j}{b^{1111}(y_1)}\Big)W_{\lambda,\xi}(y_1),$$

for which we have

$$\partial_{y_1} W_{\lambda,\xi}(y_1) = -i\Big(\frac{\partial_{y_1} b^{1111}(y_1)}{b^{1111}(y_1)} + \mathbf{O}(\rho)\Big)W_{\lambda,\xi}(y_1).$$

Computing directly, we find

$$\partial_{y_1}\frac{W(\theta_j,\theta_k,\theta_l)}{W_{\lambda,\xi}(y_1)b^{1111}(y_1)} = \frac{b^{1111}(y_1)\partial_{y_1}W(\theta_j,\theta_k,\theta_l) + (\mathbf{O}(|\xi|)W(\theta_j,\theta_k,\theta_l)}{W_{\lambda,\xi}(y_1)b^{1111}(y_1)}.$$

For the numerator,

$$\partial_{y_1}W(\theta_j,\theta_k,\theta_l) + \mathbf{O}(|\xi|)W(\theta_j,\theta_k,\theta_l) =$$

$$\det\begin{pmatrix} \theta_j & \theta_k & \theta_l \\ \theta_j' & \theta_k' & \theta_l' \\ \theta_j''' + \mathbf{O}(|\xi|)\theta_j'' & \theta_k''' + \mathbf{O}(|\xi|)\theta_k'' & \theta_l''' + \mathbf{O}(|\xi|)\theta_l'' \end{pmatrix}.$$

In the event that $\theta_k(y_1)\Big|_{\rho=0}$ decays at exponential rate as $x_1 \to -\infty$, we have (upon setting $\rho = 0$ in $L_\xi\theta_k = \lambda\theta_k$)

$$-(b^{1111}(y_1)\theta_k''')' - (a_1(x_1)\theta_k)' = 0,$$

for which we integrate over $(-\infty, x_1]$ to obtain

$$\theta_k'''(y_1)\Big|_{\rho=0} = -\frac{a_1(x_1)}{b^{1111}(x_1)}\theta_k(y_1)\Big|_{\rho=0}.$$

We have, then,

$$\det\begin{pmatrix} \theta_j & \theta_k & \theta_l \\ \theta_j' & \theta_k' & \theta_l' \\ \theta_j''' + \mathbf{O}(|\xi|)\theta_j'' & \theta_k''' + \mathbf{O}(|\xi|)\theta_k'' & \theta_l''' + \mathbf{O}(|\xi|)\theta_l'' \end{pmatrix}\Big|_{\rho=0}$$
$$= \det\begin{pmatrix} \theta_j & \theta_k & \theta_l \\ \theta_j' & \theta_k' & \theta_l' \\ -\frac{a_1(x_1)}{b^{1111}(x_1)}\theta_j & -\frac{a_1(x_1)}{b^{1111}(x_1)}\theta_k & -\frac{a_1(x_1)}{b^{1111}(x_1)}\theta_l \end{pmatrix} = 0.$$

The final claim of Lemma 2.1 follows similarly from linear dependence of the column vectors. □

We are now in a position to derive conditions on the Evans function equivalent to $(\mathcal{D}_n)$ and $(\mathcal{D}_s)$. We begin with an observation similar to Lemma 1.1 from [HoffZ.1].

Lemma 2.2. Under conditions (C0)–(C2) (in particular for both the Lax case and the undercompressive case) and for $\lambda_*(\xi)$ defined as the continous curve satisfiying

$$D(\lambda_*, \xi) = 0, \quad \lambda_*(0) = 0,$$

we have

$$\lambda_*(\xi) = -i\frac{[\tilde{f}]}{[u]}\cdot\xi + \mathbf{O}(|\xi|^2).$$

Proof. Following [HoffZ.1], we proceed by expanding the eigenvalues and eigenfunctions in ξ (summation assumed over double indices)

$$\begin{aligned}\lambda_*(\xi) =& \lambda_k\xi_k + \lambda_{kj}\xi_j\xi_k + \mathbf{O}(|\xi|^3)\\ \phi_*(x_1,\xi) =& \bar{u}_{x_1}(x_1) + \phi_k(x_1)\xi_k + \phi_{kj}(x_1)\xi_k\xi_j + \mathbf{O}(|\xi|^3),\end{aligned}$$

where $L_\xi\phi_* = \lambda_*\phi_*$. Substituting these expansions into (1.12), and equating first order coefficients, we have

$$\begin{aligned}-\sum_{j\neq 1}(b^{1111}(x_1)\phi_j'''(x_1)\xi_j)_{x_1} - \sum_{j\neq 1}(a_1(x_1)\phi_j\xi_j)_{x_1} - i\sum_{j\neq 1}a_j(x_1)\xi_j\bar{u}_{x_1} - i\sum_{j\neq 1}b^{j111}(x_1)\xi_j\bar{u}_{x_1x_1x_1x_1}\\ -i\sum_{j\neq 1}(b^{1\overline{j11}}(x_1)\xi_j\bar{u}_{x_1x_1x_1})_{x_1} = \sum_{j\neq 1}\lambda_j\xi_j\bar{u}_{x_1}. \qquad (2.4)\end{aligned}$$

Integrating now on $(-\infty,+\infty)$ and recalling that ϕ_* must decay at each asymptotic limit, we determine

$$-i\int_{-\infty}^{+\infty}\sum_{j\neq 1}\xi_j\Big(a_j(x_1)\bar{u}_{x_1} + b^{j111}(x_1)\bar{u}_{x_1x_1x_1x_2}\Big)dx_1 = \sum_{j\neq 1}\lambda_j\xi_j(u_+ - u_-).$$

Recalling (1.7) and matching coefficients of the ξ_j, we arrive at the claim. □

In [HoffZ.1], the authors carried this argument to the next order in ξ and rigorously determined the second order behavior of $\lambda_*(\xi)$. For completeness, we add a remark here regarding the difficulty in proceeding similarly in the current setting. Equating second order coefficients in our expansion representation of (1.12), we have

$$\begin{aligned}-\sum_{jk\neq 1}(b^{1111}(x_1)\phi_{kj}'''(x_1)\xi_k\xi_j)_{x_1} - \sum_{jk\neq 1}(a_1(x_1)\phi_{kj}\xi_k\xi_j)_{x_1} - i\sum_{jk\neq 1}a_j(x_1)\xi_j\xi_k\phi_k\\ -i\sum_{jk\neq 1}b^{j111}(x_1)\xi_j\xi_k\phi_k''' - i\sum_{jk\neq 1}(b^{1\overline{j11}}(x_1)\xi_j\xi_k\phi_k'')_{x_1} + \sum_{jk\neq 1}b^{j\overline{k11}}(x_1)\xi_j\xi_k\bar{u}_{x_1x_1x_1}\\ + \sum_{jk\neq 1}(b^{1\overline{jk1}}(x_1)\xi_j\xi_k\bar{u}_{x_1x_1})_{x_1} = \sum_{jk\neq 1}\lambda_{kj}\xi_k\xi_j\bar{u}_{x_1} + \sum_{jk\neq 1}\lambda_j\phi_k\xi_k\xi_j.\end{aligned}$$

Integrating on $(-\infty,+\infty)$, we determine

$$\begin{aligned}\sum_{jk\neq 1}\xi_j\xi_k\int_{-\infty}^{+\infty}\Big(-ia_j(x_1)\phi_k(x_1) - ib^{j111}(x_1)\phi_k''' + b^{j\overline{k11}}(x_1)\bar{u}_{x_1x_1x_1}\Big)dx_1\\ = \sum_{jk\neq 1}\xi_j\xi_k\int_{-\infty}^{+\infty}\Big(\lambda_{jk}\bar{u}_{x_1} + \lambda_j\phi_k(x_1)\Big)dx_1.\end{aligned}$$

Equating coefficients, we find

$$\lambda_{jk} = [u]^{-1}\int_{-\infty}^{+\infty}\Big(-ia_j(x_1)\phi_k(x_1) + i\frac{[f_j]}{[u]}\phi_k(x_1) - ib^{j111}(x_1)\phi_k'''(x_1) + b^{j\overline{k11}}(x_1)\bar{u}_{x_1x_1x_1}\Big)dx_1.$$

(This last representation should be compared with equation 5.10 in [HoffZ.1].) Finally, we can obtain a representation for the ϕ_k in terms of $\bar{u}_{x_1}$ by integrating (2.4) on $(-\infty,x]$. The determination, then, of the coefficients λ_{jk} can be reduced to an understanding of the standing wave $\bar{u}(x_1)$. In the case of [HoffZ.1], in which the authors consider second order diffusion, the standing wave $\bar{u}$ is necessarily monotonic, and the authors make use of the observation that $[u]$ and $\bar{u}_{x_1}$ have the same sign for all x_1. In the case of fourth order diffusion (even in the presence of second order diffusion), and also in the case of second (or higher) order systems, $\bar{u}$ is typically not monotonic, and information sufficient for determining behavior of the coefficients λ_{jk} is prohibitively difficult to determine in this manner. Consequently, for the remainder of this section, we follow the methods of [Z.1, ZS], developed in the context of systems.

Lemma 2.3. Under conditions (C0)–(C2) (in particular for both the Lax case and the undercompressive case) and for $D_{\lambda_0,\xi_0}(\rho)$ defined as in (1.15), the limit

$$\lim_{\rho\to 0}\rho^{-1}D_{\lambda_0,\xi_0}(\rho) = \bar{\triangle}(\lambda_0,\xi_0)$$

exists and is analytic in λ_0 and ξ_0 in a neighborhood of $(0,0)$. Moreover, $\bar{\triangle}(\lambda_0.\xi_0)$ is homogeneous of degree 1.

Proof. Though the proof of Lemma 2.3 follows closely along the lines of the proof of Theorem 7.1 in [ZS], we include it here for completeness.

Lax case. In the Lax case, each ODE decay solution $\phi_k^\pm$ is *fast* (asymptotically decays at exponential rate for $\rho = 0$). By linear independence of the modes, and since $\bar{u}_{x_1}$ is a solution of (1.12) that decays at both asymptotic limits, we can choose without loss of generality

$$\phi_1^+(x_1;0,0) = \phi_2^-(x_1;0,0) = \bar{u}_{x_1}(x_1). \tag{2.5}$$

We have, then, clearly

$$D_{\lambda_0,\xi_0}(0) = W(\bar{u}_{x_1},\phi_2^+,\phi_1^-,\bar{u}_{x_1})\Big|_{\rho=0} = 0.$$

Additionally, we have

$$\begin{aligned}\frac{\partial}{\partial\rho}D_{\lambda_0,\xi_0}(0) =& W(\frac{\partial\phi_1^+}{\partial\rho},\phi_2^+,\phi_1^-,\bar{u}_{x_1}) + W(\bar{u}_{x_1},\phi_2^+,\phi_1^-,\frac{\partial\phi_2^-}{\partial\rho})\\ =& W(\frac{\partial\phi_1^+}{\partial\rho}-\frac{\partial\phi_2^-}{\partial\rho},\phi_2^+,\phi_1^-,\bar{u}_{x_1}).\end{aligned}$$

Re-writing (1.12) in terms of ρ (and hence in terms of λ_0 and $\xi_0 = (\xi_2^0,\xi_3^0,...,\xi_d^0)$), we have

$$\begin{aligned}&-\rho^4\sum_{jklm\neq 1} b^{jklm}(x_1)\xi_j^0\xi_k^0\xi_l^0\xi_m^0\phi+\rho^3\sum_{jkl\neq 1} b^{jk\overline{l1}}(x_1)\xi_j^0\xi_k^0\xi_l^0\phi_{x_1} + i\rho^3\sum_{jkl\neq 1}(b^{1jkl}(x_1)\xi_j^0\xi_k^0\xi_l^0\phi)_{x_1}\\ &+\rho^2\sum_{jk\neq 1}(b^{1\overline{jk1}}(x_1)\xi_j^0\xi_k^0\phi_{x_1})_{x_1}+\rho^2\sum_{jk\neq 1} b^{j\overline{k11}}(x_1)\xi_j^0\xi_k^0\phi_{x_1x_1} - i\rho\sum_{j\neq 1}(b^{1\overline{j11}}(x_1)\xi_j^0\phi_{x_1x_1})_{x_1}\\ &-i\rho\sum_{j\neq 1} b^{j111}(x_1)\xi_j^0\phi_{x_1x_1x_1}-(b^{1111}(x_1)\phi_{x_1x_1x_1})_{x_1} - (a_1(x_1)\phi)_{x_1} - i\rho\sum_{j\neq 1} a_j(x_1)\xi_j^0\phi = \rho\lambda_0\phi.\end{aligned}$$

Setting $\rho = 0$, we have

$$-(b^{1111}(x_1)\phi_{x_1x_1x_1})_{x_1} - (a_1(x_1)\phi)_{x_1} = 0.$$

Since in the Lax case each $\phi_k^\pm$ is fast, we integrate on either $(-\infty,x_1)$ or $(x_1,+\infty)$ to obtain

$$-b^{1111}(x_1)\phi_{x_1x_1x_1} - a_1(x_1)\phi = 0.$$

Writing $z_+ = \frac{\partial \phi_1^+}{\partial \rho}$ and $z_- = \frac{\partial \phi_2^-}{\partial \rho}$, we take a ρ derivative of (1.12) and set $\rho = 0$ to find (with $' := \partial_{x_1}$)

$$\begin{aligned} -(b^{1111}(x_1) z_\pm''')' - (a_1(x_1) z_\pm)' =& i \sum_{j \neq 1} (b^{1\overline{j11}}(x_1) \xi_j^0 \bar{u}_{x_1 x_1 x_1})' \\ &+ i \sum_{j \neq 1} b^{j111}(x_1) \xi_j^0 \bar{u}_{x_1 x_1 x_1 x_1} + i \sum_{j \neq 1} a_j(x_1) \xi_j^0 \bar{u}_{x_1} + \lambda_0 \bar{u}_{x_1}. \end{aligned}$$

Integrating the equation with z_+ over $(x_1, +\infty)$, and recalling the definitions (1.7), we observe

$$\begin{aligned} -b^{1111}(x_1) z_+''' - a_1(x_1) z_+ =& i \sum_{j \neq 1} b^{1\overline{j11}}(x_1) \xi_j^0 \bar{u}_{x_1 x_1 x_1} \\ -i \sum_{j \neq 1} \xi_j^0 \int_{x_1}^{+\infty} (b^{j111}(\bar{u}(x_1)) \bar{u}_{x_1 x_1 x_1})_{x_1} dx_1 &- i \sum_{j \neq 1} \int_{x_1}^{+\infty} f^j(\bar{u}(x_1))_{x_1} dx_1 + \lambda_0 (\bar{u}(x_1) - u_+), \end{aligned}$$

for which

$$\begin{aligned} -b^{1111}(x_1) z_+''' - a_1(x_1) z_+ =& i \sum_{j \neq 1} b^{1\overline{j11}}(x_1) \xi_j^0 \bar{u}_{x_1 x_1 x_1} \\ i \sum_{j \neq 1} \xi_j^0 \Big[b^{j111}(\bar{u}(x_1)) \bar{u}_{x_1 x_1 x_1} &+ (f^j(\bar{u}(x_1) - f^j(u_+)) \Big] + \lambda_0 (\bar{u}(x_1) - u_+). \end{aligned}$$

Similarly,

$$\begin{aligned} -b^{1111}(x_1) z_-''' - a_1(x_1) z_- =& i \sum_{j \neq 1} b^{1\overline{j11}}(x_1) \xi_j^0 \bar{u}_{x_1 x_1 x_1} \\ i \sum_{j \neq 1} \xi_j^0 \Big[b^{j111}(\bar{u}(x_1)) \bar{u}_{x_1 x_1 x_1} &+ (f^j(\bar{u}(x_1) - f^j(u_-)) \Big] + \lambda_0 (\bar{u}(x_1) - u_-), \end{aligned}$$

and we conclude

$$-b^{1111}(x_1)(z_+ - z_-)''' - a_1(x_1)(z_+ - z_-) = -i \xi_0 \cdot [\tilde{f}] - \lambda_0 [u], \tag{2.6}$$

where

$$[\tilde{f}] := (f^2(u_+) - f^2(u_-), f^3(u_+) - f^3(u_-), ..., f^d(u_+) - f^d(u_-)), \quad [u] := u_+ - u_-.$$

We have, then,

$$
\begin{aligned}
\partial_\rho D_{\lambda_0,\xi_0}(0) &= \det \begin{pmatrix} (z_+ - z_-) & \phi_2^+ & \phi_1^- & \bar{u}_{x_1} \\ (z_+ - z_-)' & {\phi_2^+}' & {\phi_1^-}' & \bar{u}_{x_1x_1} \\ (z_+ - z_-)'' & {\phi_2^+}'' & {\phi_1^-}'' & \bar{u}_{x_1x_1x_1} \\ (z_+ - z_-)''' & {\phi_2^+}''' & {\phi_1^-}''' & \bar{u}_{x_1x_1x_1x_1} \end{pmatrix} \\
&= \det \begin{pmatrix} (z_+ - z_-) & \phi_2^+ & \phi_1^- & \bar{u}_{x_1} \\ (z_+ - z_-)' & {\phi_2^+}' & {\phi_1^-}' & \bar{u}_{x_1x_1} \\ (z_+ - z_-)'' & {\phi_2^+}'' & {\phi_1^-}'' & \bar{u}_{x_1x_1x_1} \\ -\frac{a_1(0)}{b^{1111}(0)}(z_+ - z_-) + \frac{i\xi_0\cdot[\tilde{f}]+\lambda_0[u]}{b^{1111}(0)} & -\frac{a_1(0)}{b^{1111}(0)}\phi_2^+ & -\frac{a_1(0)}{b^{1111}(0)}\phi_1^- & -\frac{a_1(0)}{b^{1111}(0)}\bar{u}_{x_1} \end{pmatrix} \\
&= \det \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a_1(0)}{b^{1111}(0)} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (z_+ - z_-) & \phi_2^+ & \phi_1^- & \bar{u}_{x_1} \\ (z_+ - z_-)' & {\phi_2^+}' & {\phi_1^-}' & \bar{u}_{x_1x_1} \\ (z_+ - z_-)'' & {\phi_2^+}'' & {\phi_1^-}'' & \bar{u}_{x_1x_1x_1} \\ \frac{i\xi_0\cdot[\tilde{f}]+\lambda_0[u]}{b^{1111}(0)} & 0 & 0 & 0 \end{pmatrix} \right] \\
&= -\Big(i\xi_0\cdot[\tilde{f}] + \lambda_0[u]\Big)\Big[b^{1111}(0)^{-1} \det \begin{pmatrix} \phi_2^+ & \phi_1^- & \bar{u}_{x_1} \\ {\phi_2^+}' & {\phi_1^-}' & \bar{u}_{x_1x_1} \\ {\phi_2^+}'' & {\phi_1^-}'' & \bar{u}_{x_1x_1x_1} \end{pmatrix}\Big].
\end{aligned}
$$

Following [Z.1, ZS], we define

$$\triangle(\lambda_0, \xi_0) = i\xi_0 \cdot [\tilde{f}] + \lambda_0[u]$$

and

$$\gamma = -b^{1111}(0)^{-1} \det \begin{pmatrix} \phi_2^+ & \phi_1^- & \bar{u}_{x_1} \\ {\phi_2^+}' & {\phi_1^-}' & \bar{u}_{x_1x_1} \\ {\phi_2^+}'' & {\phi_1^-}'' & \bar{u}_{x_1x_1x_1} \end{pmatrix}.$$

We now compute the reduced Evans function directly, as

$$\bar{\triangle}(\lambda_0, \xi_0) = \lim_{\rho\to 0} \rho^{-1} D_{\lambda_0,\xi_0}(\rho) = \partial_\rho D_{\lambda_0,\xi_0}(0) = \gamma\triangle(\lambda_0, \xi_0).$$

Undercompressive case. For the case of undercompressive waves, we observe that in our labeling scheme ϕ_2^+ is a slow decay solution ($\mathbf{O}(1)$ for $\rho = 0$). We again have $D_{\lambda_0,\xi_0}(0) = 0$ immediately from (2.5). In order to compute $\partial_\rho D_{\lambda_0,\xi_0}(0)$, we observe that for $\rho = 0$, the slow mode ϕ_2^+ satisfies

$$-(b^{1111}(x_1)\phi_{x_1x_1x_1})_{x_1} - (a_1(x_1)\phi)_{x_1} = 0,$$

which upon integration on $(x_1, +\infty)$ (and with the scaling of ϕ_2^+ chosen in Lemma 2.1) becomes

$$b^{1111}(x_1){\phi_2^+}''' + a_1(x_1)\phi_2^+(x_1) = a_1^+.$$

Proceeding as in the Lax case, we have then

$$\begin{aligned}
\partial_\rho D_{\lambda_0,\xi_0}(0) =& \det\left[\begin{pmatrix} (z_+-z_-) & \phi_2^+ & \phi_1^- & \bar u_{x_1} \\ (z_+-z_-)' & {\phi_2^+}' & {\phi_1^-}' & \bar u_{x_1x_1} \\ (z_+-z_-)'' & {\phi_2^+}'' & {\phi_1^-}'' & \bar u_{x_1x_1x_1} \\ \frac{i\xi_0\cdot[\tilde f]+\lambda_0[u]}{b^{1111}(0)} & \frac{a_1^+}{b^{1111}(0)} & 0 & 0 \end{pmatrix}\right] \\
=& -(i\xi_0\cdot[\tilde f]+\lambda_0[u])b^{1111}(0)^{-1}\det\begin{pmatrix} \phi_2^+ & \phi_1^- & \bar u_{x_1} \\ {\phi_2^+}' & {\phi_1^-}' & \bar u_{x_1x_1} \\ {\phi_2^+}'' & {\phi_1^-}'' & \bar u_{x_1x_1x_1} \end{pmatrix} \\
&+a_1^+ b^{1111}(0)^{-1}\det\begin{pmatrix} (z_+-z_-) & \phi_1^- & \bar u_{x_1} \\ (z_+-z_-)' & {\phi_1^-}' & \bar u_{x_1x_1} \\ (z_+-z_-)'' & {\phi_1^-}'' & \bar u_{x_1x_1x_1} \end{pmatrix} \\
=& b^{1111}(0)^{-1}\det\begin{pmatrix} a_1^+(z_+-z_-)-(i\xi_0\cdot[\tilde f]+\lambda_0[u])\phi_2^+ & \phi_1^- & \bar u_{x_1} \\ a_1^+(z_+-z_-)'-(i\xi_0\cdot[\tilde f]+\lambda_0[u]){\phi_2^+}' & {\phi_1^-}' & \bar u_{x_1x_1} \\ a_1^+(z_+-z_-)''-(i\xi_0\cdot[\tilde f]+\lambda_0[u]){\phi_2^+}'' & {\phi_1^-}'' & \bar u_{x_1x_1x_1} \end{pmatrix}.
\end{aligned}$$

Observing now by direct substitution that $a_1^+(z_+-z_-)-(i\xi_0\cdot[\tilde f]+\lambda_0[u])\phi_2^+$ is a solution to

$$-b^{1111}(x_1)\phi_{x_1x_1x_1}-a_1(x_1)\phi=0, \tag{2.7}$$

we view this last determinant as a Wronskian of solutions of (2.7). Since we can construct solutions of (2.7) independently of λ_0 and ξ_0 we conclude by Abel's Wronskian formulation that we can write

$$b^{1111}(0)^{-1}\det\begin{pmatrix} a_1^+(z_+-z_-)-(i\xi_0\cdot[\tilde f]+\lambda_0[u])\phi_2^+ & \phi_1^- & \bar u_{x_1} \\ a_1^+(z_+-z_-)'-(i\xi_0\cdot[\tilde f]+\lambda_0[u]){\phi_2^+}' & {\phi_1^-}' & \bar u_{x_1x_1} \\ a_1^+(z_+-z_-)''-(i\xi_0\cdot[\tilde f]+\lambda_0[u]){\phi_2^+}'' & {\phi_1^-}'' & \bar u_{x_1x_1x_1} \end{pmatrix}=\gamma\triangle(\lambda_0,\xi_0),$$

where the *transversality constant* γ is a determinant of any three linearly independent solutions of (2.7). We conclude analyticity from a standard continuous dependence argument. Moreover, observing that for $(i\xi_0\cdot[\tilde f]+\lambda_0[u])=0$, (z_+-z_-) satisfies (2.7) (see (2.6)), we recover the assertion of Lemma 2.2. □

Our immediate goal now is to characterize the local behavior of zeros of the Evans function. In particular, our primary concern is the function $\lambda_*(\xi)$ defined through the relation

$$D(\lambda_*,\xi)=0, \quad \lambda_*(0)=0.$$

Following [ZS, Z.1] we define the function $g(\lambda_0,\xi_0,\rho)$ as

$$g(\lambda_0,\xi_0,\rho):=\rho^{-1}D_{\lambda_0,\xi_0}(\rho).$$

We will be interested in roots of the three functions $D(\lambda,\xi)$, $\bar\triangle(\lambda_0,\xi_0)$, and $g(\lambda_0,\xi_0,\rho)$, for which we will adhere to the following notation:

$$\begin{aligned}
D(\lambda_*(\xi),\xi)=&0, \quad \lambda_*(0)=0 \\
\bar\triangle(\lambda_0^*(\xi_0),\xi_0)=&0, \Rightarrow \lambda_0^*(\xi_0)=-i\frac{[\tilde f]}{[u]}\xi_0 \\
g(\bar\lambda_0(\xi_0,\rho),\xi_0,\rho)=&0, \quad \bar\lambda_0(\xi_0,0)=\lambda_0^*(\xi_0).
\end{aligned} \tag{2.8}$$

By direct substitution, we observe that given the curve $\bar\lambda_0(\xi_0,\rho)$, we can determine a solution curve $\lambda_*(\xi)$ through the correspondence,

$$\lambda_*(\xi)=|\xi|\bar\lambda_0(\frac{\xi}{|\xi|},|\xi|). \tag{2.9}$$

That is,

$$g(\bar\lambda_0(\frac{\xi}{|\xi|},|\xi|),\frac{\xi}{|\xi|},|\xi|)=|\xi|^{-1}D_{\bar\lambda_0,\frac{\xi}{|\xi|}}(|\xi|)=|\xi|^{-1}D(|\xi|\bar\lambda_0,\xi)=|\xi|^{-1}D(\lambda_*,\xi)=0.$$

We proceed, then, by developing an expansion

$$\bar\lambda_0(\xi_0,\rho)=\bar\lambda_0(\xi_0,0)+\partial_\rho\bar\lambda_0(\xi_0,0)\rho+\frac{1}{2}\partial_{\rho\rho}\bar\lambda_0(\xi_0,0)\rho^2+\frac{1}{3!}\partial_{\rho\rho\rho}\bar\lambda_0(\xi_0,0)\rho^3+\mathbf{O}(\rho^4),$$

and concluding that the leading eigenvalue satisfies

$$\begin{aligned}\lambda_*(\xi)=&|\xi|\bar\lambda_0(\frac{\xi}{|\xi|},|\xi|)=|\xi|\Big(\bar\lambda_0(\frac{\xi}{|\xi|},0)+\partial_\rho\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|+\frac{1}{2}\partial_{\rho\rho}\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|^2+\frac{1}{3!}\partial_{\rho\rho\rho}\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|^3+\mathbf{O}(|\xi|^4)\Big)\\ =&\bar\lambda_0(\xi,0)+\partial_\rho\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|^2+\frac{1}{2}\partial_{\rho\rho}\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|^3+\frac{1}{3!}\partial_{\rho\rho\rho}\bar\lambda_0(\frac{\xi}{|\xi|},0)|\xi|^4+\mathbf{O}(|\xi|^5).\end{aligned}$$

According to Lemma 2.2, in both the Lax case and the undercompressive case, we have, $\bar\lambda_0(\xi,0)=-i\frac{[\tilde f]}{[u]}\xi$. For higher order behavior, we follow [Z.1, ZS] and proceed by differentiating g with respect to ρ. Proceeding from (2.8), we have

$$g_{\lambda_0}\frac{\partial\bar\lambda_0}{\partial\rho}+g_\rho=0\Rightarrow\partial_\rho\bar\lambda_0(\xi_0,0)=-\frac{g_\rho(-i\frac{[\tilde f]}{[u]}\xi_0,\xi_0,0)}{g_{\lambda_0}(-i\frac{[\tilde f]}{[u]}\xi_0,\xi_0,0)}. \tag{2.10}$$

In the case of second order diffusion, behavior of $\partial_\rho\bar\lambda_0(\xi_0,0)$ is sufficient, because the $\mathbf{O}(|\xi|^2)$ term in our expansion of $\lambda_*(\xi)$ is always present. Even in the current setting of fourth order diffusion, this quadratic effect is typically present, but we find that the fourth order term dictates our asymptotic decay in time. Conseqently, we proceed to two higher order representations

$$\partial_{\rho\rho}\bar\lambda_0=-\frac{g_{\rho\rho}+2g_{\rho\lambda_0}\frac{\partial\bar\lambda_0}{\partial\rho}+g_{\lambda_0\lambda_0}(\frac{\partial\bar\lambda_0}{\partial\rho})^2}{g_{\lambda_0}},$$

and

$$\begin{aligned}\partial_{\rho\rho\rho}\bar\lambda_0=&-\frac{3g_{\lambda_0\lambda_0\lambda_0}(\frac{\partial\bar\lambda_0}{\partial\rho})^3+3g_{\lambda_0\lambda_0}(\frac{\partial\bar\lambda_0}{\partial\rho})(\frac{\partial^2\bar\lambda_0}{\partial\rho^2})+3g_{\lambda_0\lambda_0\rho}(\frac{\partial\bar\lambda_0}{\partial\rho})^2}{g_{\lambda_0}}\\ &-\frac{3g_{\lambda_0\rho\rho}(\frac{\partial\bar\lambda_0}{\partial\rho})+3g_{\lambda_0\rho}(\frac{\partial^2\bar\lambda_0}{\partial\rho^2})+g_{\rho\rho\rho}}{g_{\lambda_0}},\end{aligned}$$

where all evaluations are at the same points as in (2.10).

Finally, we write our expansion coefficients in terms of the Evans function $D_{\lambda_0,\xi_0}(\rho)$—in particular, in terms of its behavior at $\rho=0$. Beginning with the defining relation

$$\rho g(\lambda_0,\xi_0,\rho)=D_{\lambda_0,\xi_0}(\rho),$$

we compute

$$g(\lambda_0,\xi_0,\rho)+\rho g_\rho(\lambda_0,\xi_0,\rho)=\partial_\rho D_{\lambda_0,\xi_0}(\rho)\Rightarrow g(\lambda_0,\xi_0,0)=\partial_\rho D_{\lambda_0,\xi_0}(0).$$

Similarly,

$$\frac{\partial^k g}{\partial \rho^k}(\lambda_0, \xi_0, 0) = \frac{1}{k+1}\frac{\partial^{k+1}}{\partial \rho^{k+1}} D_{\lambda_0,\xi_0}(0).$$

On the other hand, taking a derivative with respect to λ_0, we observe

$$\frac{\partial^k}{\partial \lambda_0^k} g(\lambda_0, \xi_0, 0) = \frac{\partial^k}{\partial \lambda_0^k}(\lim_{\rho \to 0} \rho^{-1} D_{\lambda_0,\xi_0}(\rho)) = \frac{\partial^k}{\partial \lambda_0^k} \bar{\triangle}(\lambda_0, \xi_0).$$

Mixed partials follow similarly. We have, then,

$$\begin{aligned}
\beta_1 :=& \partial_\rho \bar{\lambda}_0(\xi_0, 0) = -\frac{g_\rho(-i\frac{[\tilde{f}]}{[u]}\xi_0, \xi_0, 0)}{g_{\lambda_0}(-i\frac{[\tilde{f}]}{[u]}\xi_0, \xi_0, 0)} = -\frac{\frac{1}{2}\partial_{\rho\rho} D_{\lambda_0,\xi_0}(0)}{\partial_{\lambda_0}\bar{\triangle}(\lambda_0, \xi_0)} \\
\beta_2 :=& \frac{1}{2}\partial_{\rho\rho}\bar{\lambda}_0(\xi_0, 0) = -\frac{g_{\rho\rho} + 2g_{\rho\lambda_0}\frac{\partial \bar{\lambda}_0}{\partial \rho} + g_{\lambda_0\lambda_0}(\frac{\partial \bar{\lambda}_0}{\partial \rho})^2}{g_{\lambda_0}} \\
=& -\frac{\frac{1}{3}\partial_{\rho\rho\rho} D_{\lambda_0,\xi_0}(0) + \frac{1}{2}\beta_1 \partial_{\lambda_0\rho\rho} D_{\lambda_0,\xi_0}(0) + \beta_1^2 \partial_{\lambda_0\lambda_0}\bar{\triangle}(\lambda_0, \xi_0)}{\partial_{\lambda_0}\bar{\triangle}(\lambda_0, \xi_0)} \\
\beta_3 :=& -\frac{3g_{\lambda_0\lambda_0\lambda_0}(\frac{\partial \bar{\lambda}_0}{\partial \rho})^3 + 3g_{\lambda_0\lambda_0}(\frac{\partial \bar{\lambda}_0}{\partial \rho})(\frac{\partial^2 \bar{\lambda}_0}{\partial \rho^2}) + 3g_{\lambda_0\lambda_0\rho}(\frac{\partial \bar{\lambda}_0}{\partial \rho})^2}{g_{\lambda_0}} \\
&- \frac{3g_{\lambda_0\rho\rho}(\frac{\partial \bar{\lambda}_0}{\partial \rho}) + 3g_{\lambda_0\rho}(\frac{\partial^2 \bar{\lambda}_0}{\partial \rho^2}) + g_{\rho\rho\rho}}{g_{\lambda_0}} \\
=& -\frac{3\beta_1^3 \partial_{\lambda_0\lambda_0\lambda_0}\bar{\triangle}(\lambda_0, \xi_0) + 3\beta_1(2\beta_2)\partial_{\lambda_0\lambda_0}\bar{\triangle}(\lambda_0, \xi_0) + 3\frac{1}{2}\beta_1^2 \partial_{\lambda_0\lambda_0\rho\rho} D_{\lambda_0,\xi_0}(0)}{\partial_{\lambda_0}\bar{\triangle}(\lambda_0, \xi_0)} \\
&- \frac{3\beta_1\frac{1}{3}\partial_{\lambda_0\rho\rho\rho} D_{\lambda_0,\xi_0}(0) + 3(2\beta_2)\frac{1}{2}\partial_{\lambda_0\rho\rho} D_{\lambda_0,\xi_0}(0) + \frac{1}{4}\partial_{\rho\rho\rho\rho} D_{\lambda_0,\xi_0}(0)}{\partial_{\lambda_0}\bar{\triangle}(\lambda_0, \xi_0)}.
\end{aligned}$$

We observe that β_1 corresponds with the value β from [Z.1, ZS]. Having specified these critical values, we re-state our necessary and sufficient conditions for nonlinear stability.

($\mathcal{D}_n$) **Necessary conditions for linear stability**.

$$\begin{aligned}
D(\lambda, \xi) \neq 0, &\quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \text{Re}\lambda > 0\}, \\
\bar{\triangle}(\lambda_0, \xi_0) \neq 0, &\quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \text{Re}\lambda > 0\}, \\
\text{Re}\beta_1 \leq 0, &\quad \text{Re }\beta_3 \leq 0.
\end{aligned}$$

($\mathcal{D}_s$) **Sufficient conditions for linear (and therefore nonlinear) stability**.

$$\begin{aligned}
D(\lambda, \xi) \neq 0, &\quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \text{Re}\lambda \geq 0, (\lambda, \xi) \neq (0, 0)\}, \\
\bar{\triangle}(\lambda_0, \xi_0) \neq 0, &\quad \{(\lambda, \xi) : \xi \in \mathbb{R}^{d-1}, \text{Re}\lambda > 0\}, \\
\text{Re}\beta_1 \leq 0, &\quad \text{Re }\beta_3 < 0.
\end{aligned}$$

3 Estimates on $G_\lambda(x_1, y)$

We now combine the asymptotic estimates of Section 2 with representation (1.11) to obtain estimates on $G_{\lambda,\xi}(x_1, y)$. We begin with the case ρ small.

Lemma 3.1. Under conditions **(C0)–(C2)** and for $\rho \le \delta$, some $\delta > 0$ sufficiently small, we have the following estimates on $G_{\lambda,\xi}(x_1, y)$, as constructed in (1.11).

(I) Lax case ($a_1^+ < 0 < a_1^-$)

(i) For $y_1 \le x_1 \le 0$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(1)e^{\mu_2^-(x_1-y_1)} + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(\rho)e^{\mu_2^-(x_1-y_1)} + \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1}. \end{aligned}$$

(ii) For $x_1 \le y_1 \le 0$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1}. \end{aligned}$$

(iii) For $x_1 \le 0 \le y_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1}. \end{aligned}$$

(iv) For $y_1 \le 0 \le x_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1}. \end{aligned}$$

(v) For $0 \le y_1 \le x_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1}. \end{aligned}$$

(vi) For $0 \le x_1 \le y_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(1)e^{\mu_3^+(x_1-y_1)} + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} \\ e^{i\xi\cdot\tilde{y}} \partial_{y_1} G_{\lambda,\xi}(x_1, y) =& \mathbf{O}(\rho)e^{\mu_3^+(x_1-y_1)} + \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_3^+ y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_3^+ y_1}. \end{aligned}$$

(II) Undercompressive case ($a_1^- > 0$, $a_1^+ > 0$)

(i) For $y_1 \le x_1 \le 0$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(1)e^{\mu_2^-(x_1-y_1)} + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =& \Big(\mathbf{O}(\rho)+\mathbf{O}(e^{-\eta|y_1|})\Big)e^{\mu_2^-(x_1-y_1)} + \frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}\Big(\mathbf{O}(\rho)e^{-\mu_2^- y_1} + \mathbf{O}(e^{-\eta|y_1|})\Big) \\ &+ \frac{\mathbf{O}(\rho)}{D(\lambda,\xi)}\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|}) + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1}. \end{aligned}$$

(ii) For $x_1 \le y_1 \le 0$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}\Big(\mathbf{O}(\rho)e^{-\mu_2^- y_1} + \mathbf{O}(e^{-\eta|y_1|})\Big) \\ &+ \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1}. \end{aligned}$$

(iii) For $x_1 \le 0 \le y_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =& \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =& \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)}. \end{aligned}$$

(iv) For $y_1 \le 0 \le x_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(1)e^{\mu_2^+ x_1-\mu_2^- y_1} + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(\rho)e^{\mu_2^+ x_1-\mu_2^- y_1} + \frac{\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}\Big(\mathbf{O}(\rho)e^{-\mu_2^- y_1} + \mathbf{O}(e^{-\eta|y_1|})\Big) \\ &+ \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^+ y_1}. \end{aligned}$$

(v) For $0 \le y_1 \le x_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(1)e^{\mu_2^+(x_1-y_1)} + \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =& \mathbf{O}(\rho)e^{\mu_2^+(x_1-y_1)} + \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} \\ &+ \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)}. \end{aligned}$$

(vi) For $0 \leq x_1 \leq y_1$

$$\begin{aligned} e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) =&\mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} \\ e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y) =&\mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)}{D(\lambda,\xi)}e^{-\eta|x_1-y_1|}. \end{aligned}$$

Proof. We proceed directly from the estimates of Lemma 2.1. In the case $x_1 \geq 0$, we will expand (suppressing the dependence of $\phi_k^{\pm}$ on λ and ξ for brevity of notation)

$$\begin{aligned} \phi_1^{\pm}(x_1) =&A_1^{\pm}(\lambda,\xi)\phi_1^{\mp}(x_1) + B_1^{\pm}(\lambda,\xi)\phi_2^{\mp}(x_1) + C_1^{\pm}(\lambda,\xi)\psi_1^{\mp}(x_1) + D_1^{\pm}(\lambda,\xi)\psi_2^{\mp}(x_1) \\ \phi_2^{\pm}(x_1) =&A_2^{\pm}(\lambda,\xi)\phi_1^{\mp}(x_1) + B_2^{\pm}(\lambda,\xi)\phi_1^{\mp}(x_1) + C_2^{\pm}(\lambda,\xi)\psi_1^{\mp}(x_1) + D_2^{\pm}(\lambda,\xi)\psi_2^{\mp}(x_1). \end{aligned} \tag{3.1}$$

Without loss of generality, in both the Lax case and the undercompressive case, we can label the $\phi_k^{\pm}$ so that

$$\phi_1^{-}(x_1;0,0) = \bar{u}_{x_1}(x_1) = \phi_1^{+}(x_1;0,0),$$

with the consequence that

$$\phi_1^{\pm}(x_1;\lambda,\xi) = \bar{u}_{x_1}(x_1) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|}).$$

(In the undercompressive case, ϕ_1^{+} is the unique fast mode, and consequently must correspond in this manner with $\bar{u}_{x_1}(x_1)$. For the remaining cases, this constitutes a rescaling from the estimates of Lemma 3.1. We note, however, that the slow-mode estimates can still be taken from Lemma 3.1, and that for fast modes we only require the estimate $\phi_k^{\pm}(x_1) = \mathbf{O}(e^{-\eta|x_1|})$.) In order for our representations to match, we must have

$$B_1^{\pm}(\lambda,\xi) = \mathbf{O}(\rho); \quad C_1^{\pm}(\lambda,\xi) = \mathbf{O}(\rho); \quad D_1^{\pm}(\lambda,\xi) = \mathbf{O}(\rho)$$

and additionally

$$C_1^{\pm}(\lambda_*,\xi)D_2^{\pm}(\lambda_*,\xi) - D_1^{\pm}(\lambda_*,\xi)C_2^{\pm}(\lambda_*,\xi) = 0,$$

where the curve $\lambda_*(\xi)$ is defined through $D(\lambda_*(\xi),\xi) = 0$. While the former of these last two expressions is clear, the latter can be observed through consideration of the eigenfunction $\varphi_*(x_1;\lambda_*,\xi)$ associated with $\lambda_*(\xi)$. In particular, since φ_* decays at exponential rate at both $\pm\infty$, we must have, for $\rho \leq r$,

$$\begin{aligned} \phi_*(x_1;\lambda_*,\xi) =&E(\lambda_*,\xi)\phi_1^{+}(x_1;\lambda_*,\xi) + F(\lambda_*,\xi)\phi_2^{+}(x_1;\lambda_*,\xi) \\ =&E(\lambda_*,\xi)\Big(A_1^{+}(\lambda_*,\xi)\phi_1^{-} + B_1^{+}(\lambda_*,\xi)\phi_2^{-} + C_1^{+}(\lambda_*,\xi)\psi_1^{-} + D_1^{+}(\lambda_*,\xi)\psi_2^{-}\Big) \\ &+F(\lambda_*,\xi)\Big(A_2^{+}(\lambda_*,\xi)\phi_1^{-} + B_2^{+}(\lambda_*,\xi)\phi_2^{-} + C_2^{+}(\lambda_*,\xi)\psi_1^{-} + D_2^{+}(\lambda_*,\xi)\psi_2^{-}\Big). \end{aligned}$$

Recalling that $\varphi_*(x_1;\lambda,\xi)$ must decay as $x_1 \to +\infty$, and employing the linear independence of ψ_1^{-} and ψ_2^{-}, we conclude

$$E(\lambda_*,\xi)C_1^{+}(\lambda_*,\xi) + F(\lambda_*,\xi)C_2^{+}(\lambda_*,\xi) = 0 = E(\lambda_*,\xi)D_1^{+}(\lambda_*,\xi) + F(\lambda_*,\xi)D_2^{+}(\lambda_*,\xi),$$

from which the identity is immediate. Moreover, in the undercompressive case, since $\phi_2^{+}(x_1)$ is a slow mode, we have

$$\begin{aligned} \phi_*(x_1,\lambda_*,\xi) =&E(\lambda_*,\xi)\phi_1^{+}(x_1;\lambda_*,\xi) \\ =&E(\lambda_*,\xi)\Big(A_1^{+}(\lambda_*,\xi)\phi_1^{-} + B_1^{+}(\lambda_*,\xi)\phi_2^{-} + C_1^{+}(\lambda_*,\xi)\psi_1^{-} + D_1^{+}(\lambda_*,\xi)\psi_2^{-}\Big), \end{aligned}$$

from which we conclude

$$C_1^+(\lambda_*, \xi) = D_1^+(\lambda_*, \xi) = 0.$$

Similarly, we find

$$\frac{B_2^-(\lambda_*, \xi)}{B_1^-(\lambda_*, \xi)} = \frac{C_2^-(\lambda_*, \xi)}{C_1^-(\lambda_*, \xi)} = \frac{D_2^-(\lambda_*, \xi)}{D_1^-(\lambda_*, \xi)} = 0.$$

In addition to these scattering expansions for $\phi_k^\pm$, we record similar expansions for $N_k^\pm$. Computing directly, we find for $y_1 \le 0$,

$$\begin{aligned}
N_1^+(y; \lambda, \xi) =& \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[(A_1^+ C_2^+ - C_1^+ A_2^+) \frac{W(\phi_2^-, \phi_1^-, \psi_1^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (A_1^+ D_2^+ - D_1^+ A_2^+) \frac{W(\phi_2^-, \phi_1^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^+ D_2^+ - D_1^+ C_2^+) \frac{W(\phi_2^-, \psi_1^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_2^+(y; \lambda, \xi) =& - \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[(B_1^+ C_2^+ - C_1^+ B_2^+) \frac{W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (B_1^+ D_2^+ - D_1^+ B_2^+) \frac{W(\phi_1^-, \phi_2^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^+ D_2^+ - D_1^+ C_2^+) \frac{W(\phi_1^-, \psi_1^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_1^-(y; \lambda, \xi) =& - \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[C_2^+ \frac{W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + D_2^+ \frac{W(\phi_1^-, \phi_2^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_2^-(y; \lambda, \xi) =& \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[C_1^+ \frac{W(\phi_1^-, \phi_2^-, \psi_1^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + D_1^+ \frac{W(\phi_1^-, \phi_2^-, \psi_2^-)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big].
\end{aligned} \tag{3.2}$$

Similarly, for $y_1 \ge 0$,

$$\begin{aligned}
N_1^+(y; \lambda, \xi) =& \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[C_2^- \frac{W(\psi_1^+, \phi_1^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + D_2^- \frac{W(\psi_2^+, \phi_1^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_2^+(y; \lambda, \xi) =& - \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[C_1^- \frac{W(\psi_1^+, \phi_1^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + D_1^- \frac{W(\psi_2^+, \phi_1^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_1^-(y; \lambda, \xi) =& - \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[(A_1^- C_2^- - C_1^- A_2^-) \frac{W(\phi_1^+, \psi_1^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (A_1^- D_2^- - D_1^- A_2^-) \frac{W(\phi_1^+, \psi_2^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^- D_2^- - D_1^- C_2^-) \frac{W(\psi_1^+, \psi_2^+, \phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big] \\
N_2^-(y; \lambda, \xi) =& \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}} \Big[(B_1^- C_2^- - C_1^- B_2^-) \frac{W(\phi_2^+, \psi_1^+, \phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (B_1^- D_2^- - D_1^- B_2^-) \frac{W(\phi_2^+, \psi_2^+, \phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^- D_2^- - D_1^- C_2^-) \frac{W(\psi_1^+, \psi_2^+, \phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \Big].
\end{aligned} \tag{3.3}$$

We will proceed by considering the compressive case with $y_1 \le x_1 \le 0$ and the undercompressive case with $0 \le y_1 \le x_1$. The remaining cases are similar. We recall from Lemma 3.1 that for the Lax case the slow ODE solutions satisfy (for $n = 0, 1, 2, 3$)

$$\begin{aligned}
\partial_{x_1}^n \psi_2^-(x_1; \lambda, \xi) =& e^{\mu_2^-(\lambda,\xi)x_1} (\mathbf{O}(\rho^n) + \mathbf{O}(e^{-\alpha|x_1|})) \\
\partial_{x_1}^n \psi_1^+(x_1; \lambda, \xi) =& e^{\mu_3^+(\lambda,\xi)x_1} (\mathbf{O}(\rho^n) + \mathbf{O}(e^{-\alpha|x_1|})),
\end{aligned}$$

while for the undercompressive case the slow ODE solutions satisfy

$$\begin{aligned}\partial_{x_1}^n \psi_2^-(x_1;\lambda,\xi) =& e^{\mu_2^-(\lambda,\xi)x_1}(\mathbf{O}(\rho^n)+\mathbf{O}(e^{-\alpha|x_1|}))\\ \partial_{x_1}^n \phi_2^+(x_1;\lambda,\xi) =& e^{\mu_2^+(\lambda,\xi)x_1}(\mathbf{O}(\rho^n)+\mathbf{O}(e^{-\alpha|x_1|})),\end{aligned}$$

Compressive case, $y_1 \le x_1 \le 0$. In either the compressive or undercompressive case for $y_1 \le x_1 \le 0$, we begin with the representation

$$G_{\lambda,\xi}(x_1,y) = \phi_1^+(x_1;\lambda,\xi)N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi)N_2^-(y;\lambda,\xi),$$

and expand the ϕ_k^+ and N_k^- as in (3.1) and (3.2) to obtain (suppressing λ and ξ dependence for notational brevity)

$$\begin{aligned}G_{\lambda,\xi}(x_1,y) =& \Big(A_1^+\phi_1^-(x_1) + B_1^+\phi_2^-(x_1) + C_1^+\psi_1^-(x_1) + D_1^+\psi_2^-(x_1)\Big)\\ &\times\Big(-\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[C_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)} + D_2^+\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)}\Big]\\ &+\Big(A_2^+\phi_1^-(x_1) + B_2^+\phi_2^-(x_1) + C_2^+\psi_1^-(x_1) + D_2^+\psi_2^-(x_1)\Big)\\ &\times\Big(\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[C_1^+\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)} + D_1^+\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)}\Big]\\ =&\Big(\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[(A_2^+C_1^+ - A_1^+C_2^+)\phi_1^-(x_1) + (B_2^+C_1^+ - B_1^+C_2^+)\phi_2^-(x_1)\\ &\quad + (D_2^+C_1^+ - D_1^+C_2^+)\psi_2^-(x_1)\Big]\frac{W(\phi_1^-,\phi_2^-,\psi_1^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)}\\ &+\Big(\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[(A_2^+D_1^+ - A_1^+D_2^+)\phi_1^-(x_1) + (B_2^+D_1^+ - B_1^+D_2^+)\phi_2^-(x_1)\\ &\quad + (C_2^+D_1^+ - C_1^+D_2^+)\psi_1^-(x_1)\Big]\frac{W(\phi_1^-,\phi_2^-,\psi_2^-)(y_1)}{b^{1111}(y_1)W_{\lambda,\xi}(y_1)}.\end{aligned} \tag{3.4}$$

Using now the estimates of Lemma 2.1 and the observation that according to our scaling $\phi_1^-(x_1;\lambda,\xi) = \bar u_{x_1}(x_1) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})$, with additionally B_1^+, C_1^+, and D_1^+ are all $\mathbf{O}(\rho)$, we find

$$\begin{aligned}G_{\lambda,\xi}(x_1,y) =& \Big(-\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[\mathbf{O}(1)(\bar u_{x_1}(x_1) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\\ &\quad + \mathbf{O}(D(\lambda,\xi))e^{\mu_2^- x_1}\Big]\frac{\mathbf{O}(1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1}\\ &+\Big(-\frac{e^{-i\xi\cdot\tilde y}}{(2\pi)^{\frac{d-1}{2}}}\Big)\Big[\mathbf{O}(1)(\bar u_{x_1}(x_1) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})) + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\\ &\quad + \mathbf{O}(D(\lambda,\xi))e^{\mu_1^- x_1}\Big]\frac{\mathbf{O}(1)}{D(\lambda,\xi)}e^{-\mu_1^- y_1}\\ =& e^{-i\xi\cdot\tilde y}\Big[\mathbf{O}(1)e^{\mu_2^-(x_1-y_1)} + \frac{\mathbf{O}(1)\bar u_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^- y_1}\Big].\end{aligned}$$

For the first derivative in y_1, we compute

$$\partial_{y_1}G_{\lambda,\xi}(x_1,y) = \phi_1^+(x_1;\lambda,\xi)\partial_{y_1}N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi)\partial_{y_1}N_2^-(y;\lambda,\xi),$$

where according to Lemma 2.1(iii)–(iv),

$$\partial_{y_1} N_1^-(y;\lambda,\xi) = \frac{\mathbf{O}(\rho)}{D(\lambda,\xi)} e^{-\mu_2^- y_1}$$
$$\partial_{y_1} N_2^-(y;\lambda,\xi) = \frac{\mathbf{O}(\rho^2)}{D(\lambda,\xi)} e^{-\mu_2^- y_1}.$$

Computing as above, then, we find

$$\partial_{y_1} G_{\lambda,\xi}(x_1,y) = e^{-i\xi\cdot\tilde{y}} \Big[\mathbf{O}(\rho) e^{\mu_2^- x_1 - \mu_2^- y_1} + \frac{\mathbf{O}(\rho)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^- y_1} + \frac{\mathbf{O}(\rho^2)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)} e^{-\mu_2^- y_1}\Big].$$

Undercompressive case, $0 \le y_1 \le x_1$. In either the compressive or undercompressive case for $0 \le y_1 \le x_1$, we begin with the representation

$$G_{\lambda,\xi}(x_1,y) = \phi_1^+(x_1;\lambda,\xi) N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi) N_2^-(y;\lambda,\xi),$$

and expand the N_k^- as in to obtain (suppressing λ and ξ dependence for notational brevity)

$$\begin{aligned}
G_{\lambda,\xi}(x_1,y) = &-\frac{e^{-i\xi\cdot\tilde{y}}\phi_1^+(x_1)}{(2\pi)^{\frac{d-1}{2}}}\Big[(A_1^- C_2^- - C_1^- A_2^-)\frac{W(\phi_1^+,\psi_1^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (A_1^- D_2^- - D_1^- A_2^-)\frac{W(\phi_1^+,\psi_2^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^- D_2^- - D_1^- C_2^-)\frac{W(\psi_1^+,\psi_2^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)}\Big] \\
&+\frac{e^{-i\xi\tilde{y}}\phi_2^+(x_1)}{(2\pi)^{\frac{d-1}{2}} b^{1111}}\Big[(B_1^- C_2^- - C_1^- B_2^-)\frac{W(\phi_2^+,\psi_1^+,\phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} \\
&+ (B_1^- D_2^- - D_1^- B_2^-)\frac{W(\phi_2^+,\psi_2^+,\phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)} + (C_1^- D_2^- - D_1^- C_2^-)\frac{W(\psi_1^+,\psi_2^+,\phi_1^+)(y_1)}{W_{\lambda,\xi}(y_1) b^{1111}(y_1)}\Big] \\
= &\frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}}(\bar{u}_{x_1} + \mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|}))[\frac{\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} + \mathbf{O}(1)e^{-\mu_1^+ y_1}] + \frac{e^{-i\xi\cdot\tilde{y}}}{(2\pi)^{\frac{d-1}{2}}}\mathbf{O}(1)e^{\mu_2^+(x_1-y_1)} \\
= &e^{-i\xi\cdot\tilde{y}}\Big[\mathbf{O}(1)e^{\mu_2^+(x_1-y_1)} + \frac{\bar{u}_{x_1}(x_1)\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)}\Big].
\end{aligned}$$

For the first y_1 derivative, we compute as above

$$\begin{aligned}
\partial_{y_1} G_{\lambda,\xi}(x_1,y) = &\phi_1^+(x_1;\lambda,\xi)\partial_{y_1} N_1^-(y;\lambda,\xi) + \phi_2^+(x_1;\lambda,\xi) N_2^-(y;\lambda,\xi) \\
= &e^{-i\xi\cdot\tilde{y}}\Big[\mathbf{O}(\rho)e^{\mu_2^+(x_1-y_1)} + \frac{\mathbf{O}(e^{-\eta|y_1|})\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} \\
&+ \mathbf{O}(e^{-\eta|x_1-y_1|}) + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})\mathbf{O}(e^{-\eta|y_1|})}{D(\lambda,\xi)}\Big].
\end{aligned}$$

□

The estimates of Lemma 3.1 in the Lax case are to be compared with those of Proposition 2.5 in [HoffZ.1].

Lemma 3.2. Let conditions (C0)–(C2) and spectral conditions $(\mathcal{D}_s)$ hold. Suppose additionally that $\rho \ge \bar{M}$, some $\bar{M} > 0$ sufficiently large, and that λ is bounded to the right of the curve defined through

$$\text{Re }\lambda = -\frac{c_1}{L}\Big(|\text{Re }\xi|^4 - C_2|\text{Im }\xi|^4 + |\text{Im }\lambda|\Big), \tag{3.5}$$

where c_1 is as in $(\mathcal{D}_s)$ and L and C_2 may be chosen sufficiently large from $L > 1$ and $C_2 \geq c_2$. Then for some constants $C > 0$ and $\beta > 0$, we have the following estimates on $G_{\lambda,\xi}(x_1, y)$, as constructed in (1.11).

$$\begin{aligned}\Big|e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y)\Big| \leq& C\Big(|\lambda|+|\xi|^4\Big)^{-3/4}e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|}\\ \Big|e^{i\xi\cdot\tilde{y}}\partial_{y_1}G_{\lambda,\xi}(x_1,y)\Big| \leq& C\Big(|\lambda|+|\xi|^4\Big)^{-1/2}e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|}.\end{aligned}$$

Proof. We begin by defining the scaling $r = |\lambda| + |\xi|^4$ and rescale the eigenvalue equation (1.12) through $x_1 \mapsto x_1/r^{1/4}$ to obtain an equation of the form

$$-v_{xxxx} - \frac{\lambda + \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m}{rb^{1111}(x_1/r^{1/4})}v = \sum_{k=0}^{3} F_k(r,x_1,\xi)\frac{\partial^k}{\partial x_1^k}v, \tag{3.6}$$

where $F_0(r,x_1,\xi) = \mathbf{O}(r^{-1/4})$ and $F_k(r,x_1,\xi) = \mathbf{O}(r^{-\frac{4-k}{4}})$, $k = 1,2,3$. Writing (3.6) as a first order system with $W_1 = v$, $W_2 = v_{x_1}$, $W_3 = v_{x_1x_1}$, and $W_4 = v_{x_1x_1x_1}$, we have

$$W' = \tilde{\mathbb{A}}(x_1,\lambda,\xi,r)W + \mathbf{O}(r^{-1/4})W,$$

where

$$\tilde{\mathbb{A}}(x_1,\lambda,\xi,r) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{-\sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m-\lambda}{rb^{1111}(x_1/r^{1/4})} & 0 & 0 & 0\end{pmatrix}.$$

for which the eigenvalues of $\tilde{\mathbb{A}}$ can be written in terms of

$$\tilde{\Lambda} := \frac{\lambda + \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m}{rb^{1111}(x_1/r^{1/4})}$$

as

$$\begin{aligned}\tilde{\mu}_1 =& \sqrt[4]{\tilde{\Lambda}}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\\ \tilde{\mu}_2 =& \sqrt[4]{\tilde{\Lambda}}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\\ \tilde{\mu}_3 =& \sqrt[4]{\tilde{\Lambda}}(+\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\\ \tilde{\mu}_4 =& \sqrt[4]{\tilde{\Lambda}}(+\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}),\end{aligned}$$

with associated eigenvectors $\tilde{W}_k = (1, \tilde{\mu}_k, \tilde{\mu}_k^2, \tilde{\mu}_k^3)^{\mathrm{tr}}$. In our expansion for $G_{\lambda,\xi}(x_1, y)$, we will associate a growth mode or a decay mode with each of the $\tilde{\mu}_k$. We will develop the proof of Lemma 3.2 for the decay rate $\tilde{\mu}_1$. The remaining cases are similar.

We let P be the matrix constructed from the eigenvectors associated with the $\tilde{\mu}_k$ and define $V := P(x)^{-1}W$, for which we have

$$V' = \tilde{D}V + \mathbf{O}(r^{-1/4})V, \tag{3.7}$$

where $\tilde{D}$ is a diagonal matrix with the eigenvalues $\tilde{\mu}_k$ along the diagonal. Clearly, equation (3.7) has two solutions that decay at $+\infty$ and two solutions that decay at $-\infty$, with rates given by the $\tilde{u}_k$. As $r \to \infty$, we obtain solutions of the form

$$e^{\tilde{\mu}_1 x_1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e^{\tilde{\mu}_2 x_1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad e^{\tilde{\mu}_3 x_1}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad e^{\tilde{\mu}_4 x_1}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

We are concerned here with the solution associated with $\tilde{\mu}_1$, so by continuous dependence of our solutions on r, we can take $|V_1| \geq C_1(|V_2| + |V_3| + |V_4|)$, for $C_1 > 1$. We define

$$\begin{aligned}e_1 :=& V_1\bar{V}_1 + V_2\bar{V}_2\\ e_2 :=& V_3\bar{V}_3 + V_4\bar{V}_4,\end{aligned}$$

where by $\bar{\cdot}$ we mean complex conjugate. Computing directly, we find

$$\begin{aligned}e_1' =& V_1'\bar{V}_1 + V_1\bar{V}_1' + V_2'\bar{V}_2 + V_2\bar{V}_2'\\ =& \tilde{\mu}_1 V_1\bar{V}_1 + \bar{\tilde{\mu}}_1 V_1\bar{V}_1 + \tilde{\mu}_2 V_2\bar{V}_2 + \bar{\tilde{\mu}}_2 V_2\bar{V}_2 + \mathbf{O}(r^{-1/4})[\mathbf{O}(e_1) + \mathbf{O}(e_2)]\\ =& 2\mathrm{Re}\tilde{\mu}_1 V_1\bar{V}_1 + 2\mathrm{Re}\tilde{\mu}_2 V_2\bar{V}_2 + \mathbf{O}(r^{-1/4})[\mathbf{O}(e_1) + \mathbf{O}(e_2)].\end{aligned}$$

In order to establish estimates on Re $\tilde{\mu}_1$ and Re $\tilde{\mu}_2$, we write $\tilde{\Lambda}$ in the polar form

$$\tilde{\Lambda} = |\tilde{\Lambda}|e^{i\theta}.$$

We have, then

$$\text{Re } \tilde{\mu}_1 = \text{Re } \sqrt[4]{\tilde{\Lambda}}\Big(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\Big) = -\frac{\sqrt{2}}{2}|\tilde{\Lambda}|^{1/4}\Big(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\Big).$$

We require

$$\cos\frac{\theta}{4} - \sin\frac{\theta}{4} \geq \eta_0 > 0, \tag{3.8}$$

for some constant η_0. In the case Re $\tilde{\Lambda} \geq 0$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so that (3.8) holds trivially. In the case Re $\tilde{\Lambda} < 0$, we must have

$$\frac{|\text{Im } \tilde{\Lambda}|}{-\text{Re } \tilde{\Lambda}} \geq \eta_1 > 0,$$

or

$$\frac{|\lambda + \text{Im } \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m|}{-\text{Re } \lambda - \text{Re } \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m} \geq \eta_1,$$

for some constant η_1. Computing directly, and using (3.5), we have

$$\begin{aligned}-\eta_1\text{Re } \lambda - &\eta_1\text{Re} \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\\ \leq& -\eta_1\text{Re } \lambda - \eta_1\theta|\text{Re } \xi|^4 - \eta_1\theta|\text{Im } \xi|^4 + \eta_1 C_\theta|\text{Re } \xi|^2|\text{Im } \xi|^2\\ \leq& \frac{\eta_1 c_1}{L}\Big(|\text{Re } \xi|^4 - C_2|\text{Im } \xi|^4 + |\text{Im } \lambda|\Big)\\ &- \eta_1\theta|\text{Re } \xi|^4 - \eta_1\theta|\text{Im } \xi|^4 + \eta_1 C_\theta|\text{Re } \xi|^2|\text{Im } \xi|^2.\end{aligned}$$

Since C_θ is fixed, we can choose C_2 sufficiently large so that

$$
\begin{aligned}
-\eta_1 \mathrm{Re}\ \lambda - \eta_1 \mathrm{Re} & \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m \\
&\leq \frac{\eta_1 c_1}{L}\Big(|\mathrm{Im}\ \lambda| - C_3|\mathrm{Re}\ \xi|^4 - C_4|\mathrm{Im}\ \xi|^4\Big) \\
&\leq \Big|\mathrm{Im}\ \lambda + \mathrm{Im} \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big|.
\end{aligned}
$$

We conclude that on this domain of λ and ξ,

$$
\mathrm{Re}\ \tilde{\mu}_1 \leq -\beta_1|\tilde{\Lambda}|,
$$

for some $\beta_1 > 0$, and similarly,

$$
\mathrm{Re}\ \tilde{\mu}_2 \leq -\beta_1|\tilde{\Lambda}|.
$$

We turn now to the estimate of $|\tilde{\Lambda}|$. Observing that the denominator in $\tilde{\Lambda}$ is bounded above and below by a constant multiple of r, we focus on the numerator

$$
N = \lambda + \sum_{jklm\neq 0} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m.
$$

In the case $\mathrm{Re}\ \lambda \geq 0$, we compute

$$
\begin{aligned}
\mathrm{Re}\ \Big(\lambda + & \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big) \\
&= \mathrm{Re}\ \Big(\frac{\lambda}{2} + \frac{\lambda}{2} + \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big) \\
&\geq \frac{1}{2}\mathrm{Re}\ \lambda - \frac{c_1}{2L}\Big(|\mathrm{Re}\ \xi|^4 - C_2|\mathrm{Im}\ \xi|^4 + |\mathrm{Im}\ \lambda|\Big) \\
&\quad + \theta|\mathrm{Re}\ \xi|^4 + \theta|\mathrm{Im}\ \xi|^4 - C|\mathrm{Re}\ \xi|^2|\mathrm{Im}\ \xi|^2 \\
&\geq \frac{1}{2}\mathrm{Re}\ \lambda - \frac{c_1}{2L}|\mathrm{Im}\ \lambda| + C_3|\mathrm{Re}\ \xi|^4 + C_4|\mathrm{Im}\ \xi|^4.
\end{aligned}
$$

In the case that

$$
\frac{1}{2}\mathrm{Re}\ \lambda + C_3|\mathrm{Re}\ \xi|^4 + C_4|\mathrm{Im}\ \xi|^4 \geq \frac{c_1}{L}|\mathrm{Im}\ \lambda|,
$$

we can conclude that

$$
\mathrm{Re}\ \Big(\lambda + \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big) \geq \eta_2 r, \tag{3.9}
$$

for some constant η_2. On the other hand, if

$$
\frac{1}{2}\mathrm{Re}\ \lambda + C_3|\mathrm{Re}\ \xi|^4 + C_4|\mathrm{Im}\ \xi|^4 < \frac{c_1}{L}|\mathrm{Im}\ \lambda|,
$$

(and L is taken sufficiently large), we have

$$
\begin{aligned}
\Big|\mathrm{Im}\ \Big(\lambda + & \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big| \\
&\geq |\mathrm{Im}\ \lambda| - C_5|\mathrm{Re}\ \xi||\mathrm{Im}\ \xi|^3 - C_6|\mathrm{Re}\ \xi|^3|\mathrm{Im}\ \xi| \\
&\geq \eta_2 r.
\end{aligned} \tag{3.10}
$$

In the case Re $\lambda \le 0$, we compute (for L and C_2 chosen sufficiently large)

$$\begin{aligned}\text{Re }\Big(\lambda + &\sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m\Big)\\ &\ge -\frac{c_1}{L}\Big(|\text{Re }\xi|^4 - C_2|\text{Im }\xi|^4 + |\text{Im }\lambda|\Big)\\ &\quad + \theta|\text{Re }\xi|^4 + \theta|\text{Im }\xi|^4 - C|\text{Re }\xi|^2|\text{Im }\xi|^2\\ &\ge -\frac{c_1}{L}|\text{Im }\lambda| + C_3|\text{Re }\xi|^4 + C_4|\text{Im }\xi|^4.\end{aligned}$$

We have, then, either

$$C_3|\text{Re }\xi|^4 + C_4|\text{Im }\xi|^4 \ge 2\frac{c_1}{L}|\text{Im }\lambda|,$$

for which (3.9) is clear, or

$$C_3|\text{Re }\xi|^4 + C_4|\text{Im }\xi|^4 < 2\frac{c_1}{L}|\text{Im }\lambda|,$$

for which we have (3.10). We conclude that

$$\beta_1|\tilde\Lambda| = \beta_1\Big|\frac{\lambda + \sum_{jklm\neq 1} b^{jklm}(x_1/r^{1/4})\xi_j\xi_k\xi_l\xi_m}{rb^{1111}(x_1/r^{1/4})}\Big| \ge \tilde\beta > 0.$$

We have, then,

$$e_1' \le -\tilde\beta e_1 + \mathbf{O}(r^{-1/4})[\mathbf{O}(e_1) + \mathbf{O}(e_2)],$$

and similarly

$$e_2' \le \tilde\beta e_2 + \mathbf{O}(r^{-1/4})[\mathbf{O}(e_1) + \mathbf{O}(e_2)].$$

Following [ZH] and recalling that V_1 is the dominant component, we consider the ratio $z = \frac{e_2}{e_1}$, for which

$$\begin{aligned}z' =& \frac{e_1e_2' - e_2e_1'}{e_1^2} = \frac{e_2'}{e_1} - z\frac{e_1'}{e_1}\\ \ge& 2\tilde\beta z + \mathbf{O}(r^{-1/4})(1 + \mathbf{O}(z) + \mathbf{O}(z^2)).\end{aligned}$$

Integrating on $[x_1, +\infty)$, and keeping in mind that $z(x_1)$ is bounded by construction, we derive the integral equation

$$z(x_1) = -\int_{x_1}^{+\infty} e^{-2\tilde\beta(\eta - x_1)}\Big[\mathbf{O}(r^{-1/4})(1 + \mathbf{O}(z) + \mathbf{O}(z^2))\Big]d\eta,$$

from which we conclude

$$z(x_1) = \mathbf{O}(r^{-1/4}).$$

(See [HowardZ.1].) We have, then, the relation $e_2(x_1) = \mathbf{O}(r^{-1/4})e_1(x_1)$, from which

$$e_1' \le -\frac{\tilde\beta}{2}e_1,$$

for r sufficiently large. Integrating on $[x_1, y_1]$, we obtain

$$\frac{e_1(x_1)}{e_1(y_1)} \le Ce^{-\frac{\tilde\beta}{2}|x_1 - y_1|}.$$

We can conclude from the dominance of V_1 and the relation

$$W_1 = V_1 + V_2 + V_3 + V_4$$

that

$$\Big|\frac{W_1(x_1)}{W_1(y_1)}\Big| \le C' e^{-\frac{\tilde{\beta}}{4}|x_1-y_1|}.$$

Finally, returning to our original scaling, we have that the decay mode associated with μ_1, say ϕ_1^+, satisfies

$$\Big|\frac{\phi_1^+(x_1)}{\phi_1^+(y_1)}\Big| \le C' e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|},$$

for some $\beta > 0$.

The remaining cases are similar.

We now estimate $G_{\lambda,\xi}(x_1,y)$ from expansion (1.11). We focus on the term $e^{i\xi\cdot\tilde{y}}\phi_1^+(x_1)N_1^-(y)$, for which we have (suppressing λ and ξ dependence for notational brevity)

$$\begin{aligned}
\Big|e^{i\xi\cdot\tilde{y}}\phi_1^+(x_1)N_1^-(y)\Big| =& \Big|(2\pi)^{\frac{1-d}{2}}\frac{\phi_1^+(x_1)W(\phi_1^-,\phi_1^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1)}\Big| \\
\le& (2\pi)^{\frac{1-d}{2}}\Big|\frac{\phi_1^+(x_1)}{\phi_1^+(y_1)}\Big|\Big|\frac{\phi_1^+(y_1)W(\phi_1^-,\phi_1^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1)}\Big| \\
\le& C\Big|\frac{\phi_1^+(y_1)W(\phi_1^-,\phi_1^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1)}\Big| e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|}.
\end{aligned}$$

Here, we observe that the expression

$$\frac{\phi_1^+(y_1)W(\phi_1^-,\phi_1^+,\phi_2^+)(y_1)}{W_{\lambda,\xi}(y_1)}$$

is a summand of $(2\pi)^{\frac{d-1}{2}}e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y)$, evaluated at $x_1 = y_1$, and consequently is bounded for λ to the right of essential spectrum. We mention for clarity that the term $e^{i\xi\cdot\tilde{y}}$ has not been lost due to the norm (in fact, ξ will be complexified, so its norm is not generally 1), but rather has simply been factored out. According to our representation $W = PV$ from the proof of Lemma 2.2, the ρ behavior of the $\phi_k^\pm$ is characterized by

$$\partial_{y_1}^n \phi_k^\pm(y_1) = \mathbf{O}(\rho^{n/4})$$

for all values of ρ such that the derivatives exist. Computing the determinants $W(\phi_1^-,\phi_1^+,\phi_2^+)(y_1)$ and $W_{\lambda,\xi}(y_1)$ directly, we find

$$\Big|\frac{\phi_1^+(y_1)W(\phi_1^-,\phi_1^+,\phi_2^+)}{W_{\lambda,\xi}(y_1)}\Big| = \mathbf{O}(\rho^{-3/4}).$$

We compute derivative estimates similarly, proceeding again by direct calculation as in the proof of Lemma 2.1. □

Lemma 3.3. Under conditions (C0)–(C2), away from essential spectrum, and for $\delta \le \rho \le \bar{M}$, $\delta > 0$ as in Lemma 3.1 and $\bar{M} > 0$ as in Lemma 3.2, we have the following estimates on $G_{\lambda,\xi}(x_1,y)$, as constructed in (1.11).

$$\Big|\partial_{y_1}^n G_{\lambda,\xi}(x_1,y)\Big| = \mathbf{O}(1).$$

Proof. Lemma 3.3 is clear from our construction of $G_{\lambda,\xi}(x_1,y)$ through representation (1.11). □

4 Estimates on $G(t,x;y)$

We now employ the estimates of the Lemmas of Section 3 to derive estimates on the Green's function $G(t,x;y)$ through Fourier–Laplace inversion

$$G(t,x;y) = \frac{1}{(2\pi)^d i}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot\tilde{x}}\int_{\Gamma} e^{\lambda t} G_{\lambda,\xi}(x_1,y)d\lambda d\xi, \tag{4.1}$$

where for each $\xi\in\mathbb{R}^{d-1}$, the contour Γ must encircle the poles of $G_{\lambda,\xi}(x_1,y)$ (which correspond with point spectrum of the operator L_ξ).

Before beginning the detailed proof of Theorem 1.1, we give a brief overview of the approach taken and set some notation. In each case of Lemma 3.1, and in the estimates of Lemma 3.2, the estimate on $G_{\lambda,\xi}(x_1,y)$ is divided into a number of terms that can each be integrated separately against $e^{i\xi\cdot\tilde{x}+\lambda t}$. For each term, the contour of integration Γ will both depend on t, x, y, and ξ, and we rely on the Cauchy theorem for invariance of the result. (In certain cases we will also complexify $\xi=\xi_R+i\xi_I$, for which the complex part ξ_I will depend on t, x, and y.) In particular, though our contours of integration depend on t, x, and y, we can differentiate (4.1) without considering this dependence. In the event that $|x_1-y_1|\gg t$, we will find it advantageous to select a contour that crosses the real axis far to the right of the imaginary axis and proceeds toward and into the negative real half-plane as roughly $\mathrm{Re}\ \lambda=\lambda_R-(\mathrm{Im}\ \lambda)^4$, for some appropriately chosen real λ_R. In the event that $|x_1-y_1|\ll t$, we will find it advantageous to follow a similar contour that passes through $\lambda_R<0$. In either case, we only follow our contour of choice until it stikes the contour Γ_{bound}, which aside from the curve $\lambda_*(\xi)$, lies entirely to the right of the point spectrum of L_ξ. Throughout the analysis, for chosen contour Γ, we will use the notation $\bar{\Gamma}$ to indicate the truncated portion of contour we follow prior to striking Γ_d.

4.1 Small t estimates ($|x_1-y_1|\geq Kt$)

In the case $|x_1-y_1|\geq Kt$, K sufficiently large, we proceed from the estimates of Lemma 3.2 along the contour described through

$$\mathrm{Re}\ \lambda = R-\frac{c_1}{L}(|\mathrm{Re}\ \xi|^4+|\mathrm{Im}\ \lambda|),$$

where L is as in Lemma 3.2,

$$R := \frac{|x-y|^{4/3}}{L_1 t^{4/3}},$$

and ξ will be complexified as $\xi=\xi_R+i\tilde{w}$, with

$$\tilde{w} := \frac{|x-y|^{1/3}}{L_2 t^{1/3}}\cdot\frac{(\tilde{x}-\tilde{y})}{|\tilde{x}-\tilde{y}|}.$$

Comparing R with $\tilde{w}$, we observe that $|\mathrm{Im}\ \xi|^4=\frac{L_1}{L_2^4}R$, for which

$$\begin{aligned}\mathrm{Re}\ \lambda =& R-\frac{c_1}{L}\Big(|\mathrm{Re}\ \xi|^4+|\mathrm{Im}\ \lambda|\Big)\\ =& -\frac{c_1}{L}\Big(|\mathrm{Re}\ \xi|^4-\frac{L}{c_1}\frac{L_2^4}{L_1}|\mathrm{Im}\ \xi|^4+|\mathrm{Im}\ \lambda|\Big).\end{aligned}$$

We can choose L_1 and L_2 so that the spectral assumption of Lemma 3.2 holds, and we have

$$\Big|e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y)\Big|\leq C\Big(|\lambda|+|\xi|^4\Big)^{-3/4}e^{-\beta(|\lambda|+|\xi|^4)^{1/4}|x_1-y_1|}.$$

Indexed by $k = \text{Im } \lambda$, our contour becomes

$$\lambda = R - \frac{c_1}{L}\Big(|\text{Re } \xi|^4 + |k|\Big) + ik,$$

for which we have

$$|\lambda| + |\xi|^4 \geq \tilde{C}(R + |k|),$$

for some constant $\tilde{C}$. The integral of interest becomes

$$\begin{aligned}
|G(t,x;y)| \leq& C \int_{\mathbb{R}^{d-1}} \int_{\Gamma_{\lambda(\xi)}} \Big|e^{\lambda t + i\xi\cdot(\tilde{x}-\tilde{y})}\Big|\Big|e^{i\xi\cdot\tilde{y}} G_{\lambda,\xi}(x_1,y)\Big| d\lambda d\xi_R \\
\leq& C \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} e^{(R - \frac{c_1}{L}|\xi_R|^4 - \frac{c_1}{L}|k|)t - \frac{|x-y|^{1/3}}{L_2 t^{1/3}}|\tilde{x}-\tilde{y}|} \Big(R + |k|\Big)^{-3/4} e^{-\beta_1 (R+|k|)^{1/4}|x_1-y_1|} dk d\xi_A \\
\leq& C e^{\frac{|x-y|^{4/3}}{L_1 t^{1/3}} - \frac{|x-y|^{1/3}}{L_2 t^{1/3}}|\tilde{x}-\tilde{y}| - \beta_1 \frac{|x-y|^{1/3}}{L_1^{1/4} t^{1/3}}|x_1-y_1|} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \Big(R + |k|\Big)^{-3/4} e^{-\frac{c_1}{L}|\xi_R|^4 t - \frac{c_1}{L}|k| t} dk d\xi_A \\
\leq& C t^{-\frac{d}{4}} e^{-\frac{|x-y|^{4/3}}{M t^{1/3}}}.
\end{aligned}$$

Derivative estimates are almost identical. Since $|x - y| \geq Kt$, these estimates can be subsumed into those of Theorem 1.1.

4.2 Large t estimates ($|x_1 - y_1| \leq Kt$)

In the case $|x_1 - y_1| \leq Kt$, we proceed from the estimates of Lemma 3.1. In the course of our proof, we will find the following technical lemma convenient.

Lemma 4.1. For the $B_k^{\pm}(\xi)$, as in (2.2), we have the following relations and estimates.

(i) For $\zeta := (-iA, \xi_2, \xi_3, ..., \xi_{d-1})$, we have

$$\sum_{jklm} b_{\pm}^{jklm} \zeta_j \zeta_k \zeta_l \zeta_m = B_0^{\pm}(\xi) - iAB_1^{\pm}(\xi) - A^2 B_2^{\pm}(\xi) + iA^3 B_3^{\pm}(\xi) + A^4 b_{\pm}^{1111}.$$

(ii) Under hypothesis (H2), for any $\xi \in \mathbb{C}^d$,

$$\text{Re } \sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_j\xi_k\xi_l\xi_m \geq \theta |\text{Re } \xi|^4 - C_\theta |\text{Im } \xi|^4.$$

(iii) Under hypothesis (H2), for any $\xi \in \mathbb{C}^d$,

$$\text{Re } \sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_j\xi_k\xi_l\xi_m \geq \theta \text{Re } \Big(b^{1111}(\bar{u}(x_1))\xi_1^4 + \sum_{jklm\neq 1} b^{jklm}(\bar{u}(x_1))\xi_j\xi_k\xi_l\xi_m\Big) - C_\theta |\text{Im } \xi|^4.$$

Proof. Equality (i) can be verified by direct substitution. For (ii), we complexify each ξ_k as $\xi_k = \xi_{k_1} + i\xi_{k_2}$ and compute

$$\begin{aligned}
&\text{Re } \sum_{jklm} b^{jklm}(\bar{u}(x_1))(\xi_{j_1} + i\xi_{j_2})(\xi_{k_1} + i\xi_{k_2})(\xi_{l_1} + i\xi_{l_2})(\xi_{m_1} + i\xi_{m_2}) \\
&\geq \sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_{j_1}\xi_{k_1}\xi_{l_1}\xi_{m_1} + \sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_{j_2}\xi_{k_2}\xi_{l_2}\xi_{m_2} - C_1 |\text{Re } \xi|^2 |\text{Im } \xi|^2 \\
&\geq \theta |\text{Re } \xi|^4 + \theta |\text{Im } \xi|^4 - C_1 |\text{Re } \xi|^2 |\text{Im } \xi|^2.
\end{aligned}$$

Applying a weighted Young's inequality to the final term, we have

$$\sqrt{\epsilon}|\text{Re }\xi|^2\frac{|\text{Im }\xi|^2}{\sqrt{\epsilon}} \le \epsilon\frac{|\text{Re }\xi|^4}{2} + \frac{1}{\epsilon}\frac{|\text{Im }\xi|^4}{2}.$$

By choosing ϵ sufficiently small we conclude (ii) for some constant C_θ.

For (iii), we first observe that by the continuity of the $b^{jklm}(u)$, we have the inequality

$$\text{Re }\Big(b^{1111}(\bar{u}(x_1))\xi_1^4 + \sum_{jklm\neq 1} b^{jklm}(\bar{u}(x_1))\xi_k\xi_k\xi_l\xi_m\Big) \le C_1|\text{Re }\xi|^4 + C_2|\text{Im }\xi|^4,$$

for some constants C_1 and C_2. According to (ii), we have then

$$\begin{aligned}\text{Re }&\sum_{jklm} b^{jklm}(\bar{u}(x_1))\xi_j\xi_k\xi_l\xi_m \ge \theta|\text{Re }\xi|^4 - C_\theta|\text{Im }\xi|^4\\ \ge&\frac{\theta}{C_1}\text{Re }\Big(b^{1111}(\bar{u}(x_1))\xi_1^4 + \sum_{jklm\neq 1} b^{jklm}(\bar{u}(x_1))\xi_k\xi_k\xi_l\xi_m\Big) - \frac{C_2\theta}{C_1}|\text{Im }\xi|^4 - C_\theta|\text{Im }\xi|^2,\end{aligned}$$

from which we have (iii). This concludes the proof of Lemma 4.1. □

Lax case. For the Lax case and $y_1 \le x_1 \le 0$, we have, from Lemma 3.1,

$$e^{i\xi\cdot\tilde{y}}G_{\lambda,\xi}(x_1,y) = \mathbf{O}(1)e^{\mu_2^-(\lambda,\xi)(x_1-y_1)} + \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)}e^{-\mu_2^-(\lambda,\xi)y_1} + \frac{\mathbf{O}(\rho)\mathbf{O}(e^{-\eta|x_1|})}{D(\lambda,\xi)}e^{-\mu_2^-(\lambda,\xi)y_1}.$$

We begin by considering the scattering term, for which the eigenvalue $\lambda_*(\xi)$ does not play a role. In this case, we have

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_\Gamma e^{\lambda t+\mu_2^-(\lambda,\xi)(x_1-y_1)}d\lambda d\xi.$$

Following [ZH], our general approach will be to employ the saddle-point method to choose an optimal contour so long as we remain to the right of Γ_{bound}, and to follow Γ_{bound} out to the point at ∞ (see Figure 2). We define Γ_{bound} as the contour defined outside $B(0,r)$ through

$$\lambda(k) = -c_1(|\text{Re }\xi|^4 - C_2|\text{Im }\xi|^4 + |k|) + ik,$$

and inside $B(0,r)$ by a vertical line connecting points for which it exits $B(0,r)$ (see Figure 2). In the event that $|\xi|$ is large enough so that Γ_{bound} lies entirely to the left of $B(0,r)$, we may proceed simply through integration along Γ_{bound}.

For $\rho < r$

$$\mu_2^-(\lambda,\xi) = -\Big(\frac{1}{a_1^-} + i\frac{B_1^-(\xi)}{(a_1^-)^2}\Big)\Lambda + \frac{B_2^-(\xi)}{(a_1^-)^3}\Lambda^2 + i\frac{B_3^-(\xi)}{(a_1^-)^4}\Lambda^3 - \frac{b_-^{1111}}{(a_1^-)^5}\Lambda^4 + \mathbf{O}(\rho^5),$$

with

$$\Lambda(\lambda,\xi) = \lambda + i\tilde{a}^-\cdot\xi + B_0^-(\xi).$$

For the full exponent, we have, then

$$\begin{aligned}
\lambda t+&i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^-(\lambda,\xi)(x_1-y_1)\\
=&\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})-\Big(\frac{1}{a_1^-}+i\frac{B_1^-(\xi)}{(a_1^-)^2}\Big)\Big(\lambda+i\tilde{a}^-\cdot\xi+B_0^-(\xi)\Big)(x_1-y_1)\\
+&\frac{B_2^-(\xi)}{(a_1^-)^3}\Big(\lambda+i\tilde{a}^-\cdot\xi+B_0^-(\xi)\Big)^2(x_1-y_1)+i\frac{B_3^-(\xi)}{(a_1^-)^4}\Big(\lambda+i\tilde{a}^-\cdot\xi+B_0^-(\xi)\Big)^3(x_1-y_1)\\
-&\frac{b_-^{1111}}{(a_1^-)^5}\Big(\lambda+i\tilde{a}^-\cdot\xi+B_0^-(\xi)\Big)^4(x_1-y_1)+\mathbf{O}(\rho^5)(x_1-y_1)\\
=&\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\Big[-\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi)+\frac{B_2^-(\xi)}{(a_1^-)^3}(\lambda+i\tilde{a}^-\cdot\xi)^2\\
&+i\frac{B_3^-(\xi)}{(a_1^-)^4}(\lambda+i\tilde{a}^-\cdot\xi)^3-\frac{b_-^{1111}}{(a_1^-)^5}(\lambda+i\tilde{a}^-\cdot\xi)^4-i\frac{B_1^-(\xi)}{(a_1^-)^2}(\lambda+i\tilde{a}\cdot\xi)-\frac{1}{a_1^-}B_0^-(\xi)\Big](x_1-y_1)\\
&+\mathbf{O}(\rho^5)(x_1-y_1).
\end{aligned}$$

Setting

$$\zeta:=(-\frac{i}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi),\xi_2,\xi_3,...,\xi_d),$$

and using Lemma 4.1(i), we can re-write this exponent as

$$\begin{aligned}
\lambda t+&i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^-(\lambda,\xi)(x_1-y_1)\\
=&\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\Big(-\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi)-\frac{1}{a_1^-}\sum_{jklm}b_-^{jklm}\zeta_j\zeta_k\zeta_l\zeta_m\Big)(x_1-y_1)\\
+&\mathbf{O}(\rho^5)(x_1-y_1).
\end{aligned}$$

According to Lemma 4.1(iii), we have

$$\begin{aligned}
\mathrm{Re}\ &\Big(\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^-(\lambda,\xi)(x_1-y_1)\Big)\\
\le&\mathrm{Re}\ \Big(\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})-\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi)(x_1-y_1)\Big)\\
&-\frac{\theta}{a_1^-}\mathrm{Re}\ \Big(b_-^{1111}\frac{(\lambda+i\tilde{a}\cdot\xi)^4}{(a_1^-)^4}+B_0^-(\xi)\Big)(x_1-y_1)\\
&+\frac{C_\theta}{a_1^-}|\mathrm{Im}\ \zeta|^4(x_1-y_1)+\mathbf{O}(\rho^5)\\
=&\mathrm{Re}\ \Big(\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})\Big)\\
&-\mathrm{Re}\ \Big(\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi+\theta B_0^-(\xi))+\frac{\theta}{(a_1^-)^5}b_-^{1111}(\lambda+i\tilde{a}^-\cdot\xi+\theta B_0^-(\xi))^4\Big)(x_1-y_1)\\
&+\frac{C_\theta}{a_1^-}|\mathrm{Im}\ \zeta|^4(x_1-y_1)+\mathbf{O}(\rho^5)(x_1-y_1).
\end{aligned}$$

We choose a contour along which

$$
\begin{aligned}
-&\frac{1}{a_1^-}\Big(\lambda + i\tilde{a}^- \cdot \xi + \theta B_0^-(\xi)\Big) - \frac{\theta}{(a_1^-)^5} b_-^{1111}\Big(\lambda + i\tilde{a}^- \cdot \xi + \theta B_0^-(\xi)\Big)^4 \\
&= -\frac{1}{a_1^-}\lambda_R - \frac{\theta b_-^{1111}}{(a_1^-)^5}\lambda_R^4 + ik.
\end{aligned}
$$

We determine the form of $\lambda(k)$ along this contour by considering the expansion

$$
\begin{aligned}
\lambda(k) + &i\tilde{a}^- \cdot \xi + \theta B_0^-(\xi) \\
&= \lambda_R + A_1 k + A_2 k^2 + A_3 k^3 + A_4 k^4 + \mathbf{O}(k^5),
\end{aligned}
$$

for which we determine

$$
\begin{aligned}
\lambda(k) = &\lambda_R - i\tilde{a}^- \cdot \xi - i a_1^- k - \theta B_0^-(\xi) \\
&+ \frac{6\theta b_-^{1111}}{(a_1^-)^2}\lambda_R^2 k^2 - 4i\lambda_R \frac{\theta b_-^{1111}}{a_1^-} k^3 - \theta b_1^{1111} k^4 + \mathbf{O}((\lambda_R + |k|)^5).
\end{aligned}
$$

Along this contour, then, we have

$$
\begin{aligned}
&\mathrm{Re}\ \Big(\lambda t + i\xi \cdot (\tilde{x} - \tilde{y}) + \mu_2^-(\lambda, \xi)(x_1 - y_1)\Big) \\
&= \mathrm{Re}\ (\lambda t + i\xi \cdot (\tilde{x} - \tilde{y})) \\
&- \mathrm{Re}\ \Big[\frac{1}{a_1^-}(\lambda + i\tilde{a}^- \cdot \xi + \theta B_0^-(\xi)) + \frac{\theta}{(a_1^-)^5}(\lambda + i\tilde{a}^- \cdot \xi + \theta B_0^-(\xi))^4\Big](x_1 - y_1) \\
&\quad + \frac{C_\theta}{a_1^-}|\mathrm{Im}\ \zeta|^4 (x_1 - y_1) + \mathbf{O}(\rho^5)(x_1 - y_1) \\
&= \lambda_R t + \frac{6\theta b_-^{1111}}{(a_1^-)^2}\lambda_R^2 k^2 t - \theta b_-^{1111} k^4 t - \theta B_0^-(\xi) t - \xi_I \cdot (\tilde{x} - \tilde{y} - \tilde{a}^- t) \\
&\quad - (\frac{1}{a_1^-}\lambda_R + \frac{\theta b_-^{1111}}{(a_1^-)^5}\lambda_R^4)(x_1 - y_1) + \frac{C_\theta}{a_1^-}|\mathrm{Im}\ \zeta|^4 (x_1 - y_1) + \mathbf{O}(\rho^5)(x_1 - y_1). \\
&= -\frac{1}{a_1^-}\lambda_R(x_1 - y_1 - a_1^- t) - \xi_I \cdot (\tilde{x} - \tilde{y} - \tilde{a}^- t) - \theta b_-^{1111} k^4 t - \theta B_0^-(\xi) t + \frac{6\theta b_-^{1111}}{(a_1^-)^2}\lambda_R^2 k^2 t \\
&\quad - \frac{\theta b_-^{1111}}{(a_1^-)^5}\lambda_R^4 (x_1 - y_1) + \frac{C_\theta}{a_1^-}|\mathrm{Im}\ \zeta|^4 (x_1 - y_1) + \mathbf{O}(\rho^5)(x_1 - y_1).
\end{aligned}
$$

According to Lemma 4.1(ii), we can conclude the estimate

$$
\begin{aligned}
&\mathrm{Re}\ \Big(\lambda t + i\xi \cdot (\tilde{x} - \tilde{y}) + \mu_2^-(\lambda, \xi)(x_1 - y_1)\Big) \\
&\leq -\frac{1}{a_1^-}\lambda_R(x_1 - y_1 - a_1^- t) - \xi_I \cdot (\tilde{x} - \tilde{y} - \tilde{a}^- t) - \theta b_-^{1111} k^4 t - \theta^2 |\xi_R|^4 t \\
&+ \frac{6\theta b_-^{1111}}{(a_1^-)^2}\lambda_R^2 k^2 t + \tilde{C}_\theta(\lambda_R^4 + |\xi_I|^4)(x_1 - y_1) + \mathbf{O}(\rho^5)(x_1 - y_1),
\end{aligned}
\tag{4.2}
$$

for some constant $\tilde{C}_\theta$, and where we have used the observation

$$\begin{aligned}\text{Im } \zeta_1 =& \text{Im } (-\frac{i}{a_1^-}\lambda + \frac{1}{a_1^-}\tilde{a}^- \cdot \xi) = -\frac{1}{a_1^-}\text{Re } \lambda + \frac{1}{a_1^-}\tilde{a}^- \cdot \xi_I \\ =& -\frac{1}{a_1^-}\Big(\lambda_R - \tilde{a}\cdot\xi_I - \theta \text{Re } B_0^-(\xi) + \frac{6\theta b_-^{1111}}{(a_1^-)^2}\lambda_R^2 k^2 - \theta b_-^{1111}k^4\Big) \\ &+ \mathbf{O}((\lambda_R + |k|)^5) + \frac{1}{a_1^-}\tilde{a}^- \cdot \xi_I,\end{aligned}$$

so that

$$|\text{Im } \zeta|^4 \le M(\lambda_R^4 + |\xi_I|^4) + \mathbf{O}(\rho^5).$$

We proceed now by taking an appropriate choice of λ_R and ξ_I. For the scattering term, we take

$$\begin{aligned}\lambda_R =& \Big(\frac{x_1 - y_1 - a_1^- t}{L_1 t}\Big)^{1/3} \\ \xi_I =& \Big(\frac{\tilde{x} - \tilde{y} - \tilde{a}^- t}{L_2 t}\Big)^{1/3}.\end{aligned}$$

For this choice, we have

$$\begin{aligned}&\text{Re } \Big(\lambda t + i\xi\cdot(\tilde{x} - \tilde{y}) + \mu_2^-(\lambda,\xi)(x_1 - y_1)\Big) \\ =& -\frac{1}{a_1^-}\lambda_R^4 L_1 t - \xi_2^4 L_2 t - \theta b_-^{1111}k^4 t - \theta^2|\xi_R|^4 t + \lambda_R^2 L_1^{1/3}\frac{6\theta b_-^{1111}}{(a_1^-)^2}k^2 t \\ &+ C(\lambda_R^4 + \xi_I^4)|x_1 - y_1| + \mathbf{O}(\rho^5)(x_1 - y_1).\end{aligned}$$

For L_1 and L_2 sufficiently large, and by Young's inequality, we conclude there exist constants M_1, M_2, η_1, and η_2 so that

$$\begin{aligned}&\text{Re } \Big(\lambda t + i\xi\cdot(\tilde{x} - \tilde{y}) + \mu_2^-(\lambda,\xi)(x_1 - y_1)\Big) \\ \le& -\frac{(x_1 - y_1 - a_1^- t)^{4/3}}{M_1 t^{1/3}} - \frac{|\tilde{x} - \tilde{y} - \tilde{a}^- t|}{M_2 t^{1/3}} - \eta_1 k^4 t - \eta_2 |\xi_R|^4 t.\end{aligned}$$

Our integral becomes

$$\begin{aligned}&\Big|\int_{\bar{\mathbb{R}}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_{\bar{\Gamma}} e^{\lambda t + \mu_2^-(\lambda,\xi)(x_1-y_1)}d\lambda d\xi\Big| \\ &\le C e^{-\frac{|x-y-a_- t|^{4/3}}{M t^{1/3}}}\int_{\mathbb{R}^{d-1}} e^{-\eta_2|\xi_R|^4 t}\int_{\mathbb{R}} e^{-\eta_1 k^4 t}dk \\ &\le C t^{-d/4} e^{-\frac{|x-y-a_- t|^{4/3}}{M t^{1/3}}},\end{aligned}$$

where $\bar{\mathbb{R}}^{d-1}$ and $\bar{\Gamma}$ represent truncated contours in the ball $B(0,r)$ (see Figure 2). We follow the contour Γ until we strike Γ_{bound}, which we follow to the point at ∞. Critically, according to $(\mathcal{D}_s)$ there are no eigenvalues inside the ball $B(0,r)$ and no eigenvalues on or to the right of Γ_{bound}.

Excited term. We next consider the excited term, for which

$$e^{i\xi\cdot\tilde{y}} S_{\lambda,\xi}(x_1, y) = \frac{\mathbf{O}(1)\bar{u}_{x_1}(x_1)}{D(\lambda,\xi)} e^{-\mu_2^-(\lambda,\xi)y_1}.$$

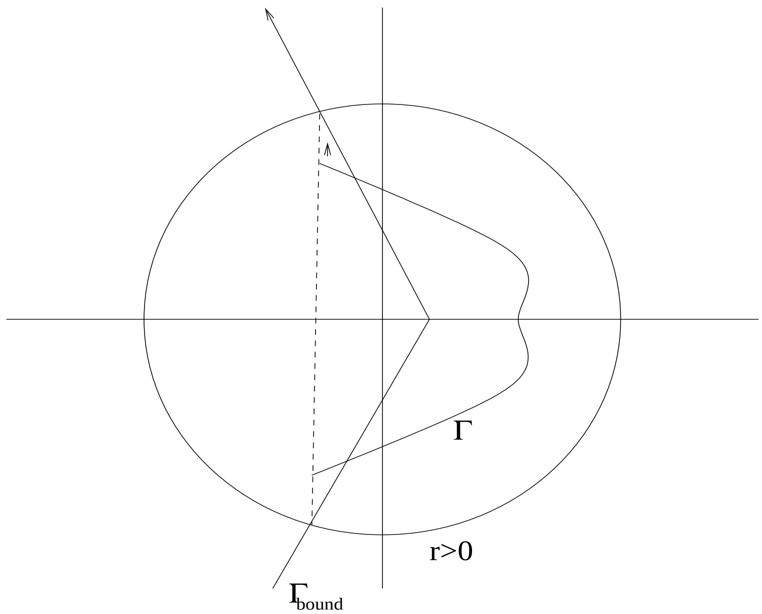

Figure 2: Principal contours.

The critical new difficulty here is that we must keep track of the zero of the Evans function, $\lambda_*(\xi)$. The fundamental integral takes the form

$$e(t,\tilde{x};y) = \int_{\mathbb{R}^{d-1}} \mathbf{O}(1) e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_\Gamma \frac{e^{\lambda t - \mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)} d\lambda d\xi.$$

In principle, we proceed as with the scattering term, though the choices of λ_R and ξ_I become more delicate. We have, in general, two things to consider: 1. the size of $D(\lambda,\xi)^{-1}$ for λ near $\lambda_*(\xi)$ (at which we have a pole), and 2. the residue picked up when our contour fails to encircle $\lambda_*(\xi)$. According to condition $(\mathcal{D}_s)$, there exists a neighborhood V of zero in complex ξ-space so that $\lambda_*(\xi)$ is the unique zero of $D(\lambda,\xi)$. By analyticity of $D(\lambda,\xi)$ in $\rho < r$, we can write

$$\frac{1}{D(\lambda,\xi)} = \frac{g(\lambda,\xi)}{\lambda - \lambda_*},$$

for some function $g(\lambda,\xi)$ analytic in $\rho < r$. By analyticity, $|g(\lambda,\xi)|$ is bounded over any truncated domain in λ-ξ space, and consequently we are justified for $\rho < r$ in considering the integral

$$\int_{\bar{\mathbb{R}}^{d-1}} \mathbf{O}(1) e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_{\bar{\Gamma}} \frac{e^{\lambda t - \mu_2^-(\lambda,\xi)y_1}}{\lambda - \lambda_*} d\lambda d\xi,$$

where away from the truncated contours $\bar{\mathbb{R}}^{d-1}$ and $\bar{\Gamma}$ we proceed along the contour Γ_{bound}. Our general contour from the scattering analysis remains unchanged, though we must carefully choose λ_R and ξ_I in a number of cases. As in the case of the scattering estimate, we take an optimal contour chosen by the saddle-point method until we strike the contour Γ_{bound}, which we follow out to the point at ∞. For notational convenience, we define

$$w := (w_1, \tilde{w}) = \left(\frac{-y_1 - a_1^- t}{t}, \frac{\tilde{x} - \tilde{y} - \tilde{a}^- t}{t}\right).$$

Following the basic approach of the one-dimensional systems analysis of [ZH], we will find it convenient to divide the analysis into several cases. The primary consideration in selecting these cases is the location of the leading eigenvalue $\lambda_*(\xi)$. We have

$$(\lambda_R, \xi_I) = \begin{cases} \Big(\pm (\epsilon/L)^{1/3}, 0\Big) & \epsilon \le |w_1| \le K, 0 \le |\tilde{w}| \le K \quad \text{(a)}^{\pm} \\ \Big(\pm (\epsilon/L_1)^{1/3}, (\epsilon/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}\Big) & 0 \le |w_1| \le K, \epsilon \le |\tilde{w}| \le K \quad \text{(b)}^{\pm} \\ \Big((w_1/L)^{1/3}, 0\Big) & t^{-3/4} \le |w_1|, |\tilde{w}| \le \epsilon, |w_1| \ge N|\tilde{w}| \quad \text{(c)}^{\pm} \\ \Big((|\tilde{w}|/L_1)^{1/3}, (|\tilde{w}|/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}\Big) & t^{-3/4} \le |w_1|, |\tilde{w}| \le \epsilon, |w_1| \le N|\tilde{w}| \quad \text{(d)}^{\pm} \\ \Big((w_1/L)^{1/3}, 0\Big) & 0 \le |\tilde{w}| \le t^{-3/4} \le |w_1| \le \epsilon \quad \text{(e)}^{\pm} \\ \Big((|\tilde{w}|/L_1)^{1/3}, (|\tilde{w}|/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}\Big) & 0 \le |w_1| \le t^{-3/4} \le |\tilde{w}| \le \epsilon \quad \text{(f)}^{\pm} \\ \Big(t^{-1/4}/L_1, (t^{-1/4}/L_2)\frac{\tilde{w}}{|\tilde{w}|}\Big) & 0 \le |w_1|, |\tilde{w}| \le t^{-3/4} \quad \text{(g)}^{\pm}, \end{cases}$$

where cases $(\cdot)^{\pm}$ correspond with $w_1 \gtrless 0$ and consequently $\lambda_R \gtrless 0$.

Case (a)$^+$. ($\epsilon \le w_1 \le K$, $0 \le \tilde{w} \le K$). In this case, we choose $\lambda_R = (\epsilon/L)^{1/3}$ and $\xi_I = 0$, for which for $\rho < r$, we have $|\lambda - \lambda_*(\xi)|^{-1} = \mathbf{O}(1)$. For the exponent, we find

$$\begin{aligned} &\text{Re } \Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big) \\ &= -\frac{1}{a_1^-}(\epsilon/L)^{1/3}(-y_1 - a_1^- t) - \theta b_-^{1111}k^4 t + \frac{6\theta b_-^{1111}}{(a_1^-)^2 L^{2/3}}\epsilon^{2/3}k^2 t - \theta^2|\xi_R|^4 t - \frac{\theta b_-^{1111}}{(a_1^-)^5 L^{4/3}}\epsilon^{4/3}(-y_1) \\ &\quad + \tilde{C}_\theta \frac{\epsilon^{4/3}}{L^{4/3}}|y_1| + \mathbf{O}(\rho^5)|y_1| \\ &\le -\frac{1}{a_1^-}(\epsilon^{4/3}/L^{1/3})t - \theta b_-^{1111}k^4 t - \theta^2|\xi_R|^4 + \frac{6\theta b_-^{1111}}{(a_1^-)^2 L^{2/3}}\epsilon^{2/3}k^2 t + \tilde{C}_\theta\frac{\epsilon^{4/3}}{L^{4/3}}|y_1| + \mathbf{O}(\rho^5)|y_1|. \end{aligned}$$

Here, $|y_1| \le Kt$, so by choosing ϵ sufficiently small, we can conclude

$$\begin{aligned} \Big|\int_{\mathbb{R}_*^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_{\Gamma_*}\frac{e^{\lambda t - \mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda d\xi\Big| \\ \le C t^{-d/4}e^{-t/M}. \end{aligned}$$

We follow this contour until we strike the horizontal contour defined through $\lambda(k) = -d + ik$, which connects the branches of Γ_{bound} (see Figure 2.) Along this contour, we have exponential decay in time, which in the case $|x-y| \le Kt$ (currently under consideration), provides an estimate that can be subsumed.

Away from our ball around the origin, we must continue to integrate along the contour described through

$$\lambda(k) = -\frac{c_1}{L}\Big(|\xi_R|^4 - C_2|\xi_I|^4 + |k|\Big) + ik,$$

or in this case, with $\xi_I = 0$,

$$\lambda(k) = -\frac{c_1}{L}\Big(|\xi_R|^4 + |k|\Big) + ik.$$

Along this contour $|\lambda - \lambda_*|^{-1} = \mathbf{O}(1)$ by condition $(\mathcal{D}_s)$, and choosing L sufficiently large we can avoid essential spectrum and insure the exponent satisfies

$$\text{Re } \Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big) \le -\eta t - \frac{c_1}{L}(|\xi_R|^4 + |k|)t.$$

Case (b)$^+$. $(0 \le w_1 \le K, \epsilon \le |\tilde{w}| \le K)$ In this case, we choose $\lambda_R = (\epsilon/L_1)^{1/3}$ and $\xi_I = (\epsilon/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}$, for which for $\rho < r$, we have $|\lambda - \lambda_*(\xi)|^{-1} = \mathbf{O}(1)$. In this case our bounding contour takes the form

$$\lambda(k) = -c_1\Big(|\xi_R|^4 - \frac{C_2}{L_2^{4/3}}\epsilon^{4/3} + |k|\Big) + ik.$$

According to (4.2), we have

$$\begin{aligned}
&\mathrm{Re}\,\Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big)\\
&\le -\frac{1}{a_1^-}(\epsilon/L_1)^{1/3}(-y_1 - a_1^- t) - (\epsilon/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}\cdot(\tilde{x}-\tilde{y}-\tilde{a}^- t)\\
&\quad - \theta b_-^{1111}k^4 t - \theta^2|\xi_R|^4 + \frac{6\theta b_-^{1111}}{(a_1^-)^2}(\epsilon/L_1)^{2/3}k^2 t + \tilde{C}_\theta\epsilon^4\Big(\frac{1}{L_1^{4/3}} + \frac{1}{L_2^{4/3}}\Big)|x_1-y_1| + \mathbf{O}(\rho^5)|x_1-y_1|.
\end{aligned}$$

In this case, $|\tilde{w}| \ge \epsilon$, and we have

$$(\epsilon/L_2)^{1/3}\frac{\tilde{w}}{|\tilde{w}|}\cdot(\tilde{x}-\tilde{y}-\tilde{a}^- t) \ge \frac{\epsilon^{2/3}}{L_2^{1/3}}t,$$

from which for L_2 sufficiently large we obtain exponential decay in t, which can be subsumed.

Case (c)$^+$. $(t^{-3/4} \le w_1, |\tilde{w}| \le \epsilon, w_1 \ge N|\tilde{w}|)$ Choosing $\lambda_R = (w_1/L)^{1/3}$ and $\xi_I = 0$, we observe that for $w_1 \ge t^{-3/4}$, $|\lambda - \lambda_*|^{-1} \le Ct^{1/4}$. For the exponent, we have from (4.2)

$$\begin{aligned}
&\mathrm{Re}\,\Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big)\\
&\le -\frac{1}{a_1^- L^{1/3}}w_1^{4/3}t - \theta b_-^{1111}k^4 t - \theta^2|\xi_R|^4 t\\
&\quad + \frac{6\theta b_-^{1111}}{(a_1^-)^2}\frac{w_1^{2/3}}{L^{2/3}}k^2 t + \tilde{C}_\theta\frac{w_1^{4/3}}{L^{4/3}}(x_1-y_1) + \mathbf{O}(\rho^5)(x_1-y_1),
\end{aligned}$$

where by choosing L sufficiently large and applying Young's inequality, we conclude

$$\begin{aligned}
&\mathrm{Re}\,\Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) + \mu_2^-(\lambda,\xi)(x_1-y_1)\Big)\\
&\le -\eta w_1^{4/3}t - \tilde{\theta}b_-^{1111}k^4 t - \theta^2|\xi_R|^4 t.
\end{aligned}$$

In this case $\tilde{w}$ decay is a consequence of the inequality $|w_1| \ge N|\tilde{w}|$. Combining these observations we conclude an esimate by

$$\mathbf{O}(t^{-\frac{d-1}{4}})e^{-\frac{|\tilde{x}-\tilde{y}-\tilde{a}^- t|^{4/3}}{Mt^{1/3}}}e^{-\frac{(y_1+a_1^- t)^2}{Mt}}.$$

Case (d)$^+$ follows from the analysis of Case (c)$^+$ and the observation that for $|w_1| \le N|\tilde{w}|$, the w_1 decay follows from $|\tilde{w}|$ decay. Similarly, the analysis of Case (e)$^+$ is similar to the analysis of Case (c)$^+$, while the analysis of Case (f)$^+$ is similar to the analysis of Case (d)$^+$.

In the case (g)$^+$, we choose $\lambda_R = t^{-1/4}/L_1$ and $\xi_I = \frac{\tilde{w}}{|\tilde{w}|}t^{-1/4}/L_2$. For $L_2 \gg L_1$, we have $|\lambda - \lambda_*|^{-1} \le Ct^{1/4}$, with exponential estimate

$$\begin{aligned}
&\mathrm{Re}\,\Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big)\\
&\le -\frac{1}{a_1^-}\frac{t^{-1/4}}{L_1}(x_1 - y_1 - a_1^- t) - \frac{t^{-1/4}}{L_2}\frac{\tilde{w}}{|\tilde{w}|}\cdot(\tilde{x}-\tilde{y}-\tilde{a}^- t) - \theta b_-^{1111}k^4 t - \theta^2|\xi_R|^4 t\\
&\quad + \frac{6\theta b_-^{1111}}{(a_1^-)^2}\frac{t^{-1/2}}{L_1^2}k^2 t + \tilde{C}_\theta\Big(\frac{t^{-1}}{L_1^4} + \frac{t^{-1}}{L_2^4}\Big)(x_1-y_1) + \mathbf{O}(\rho^5)(x_1-y_1).
\end{aligned}$$

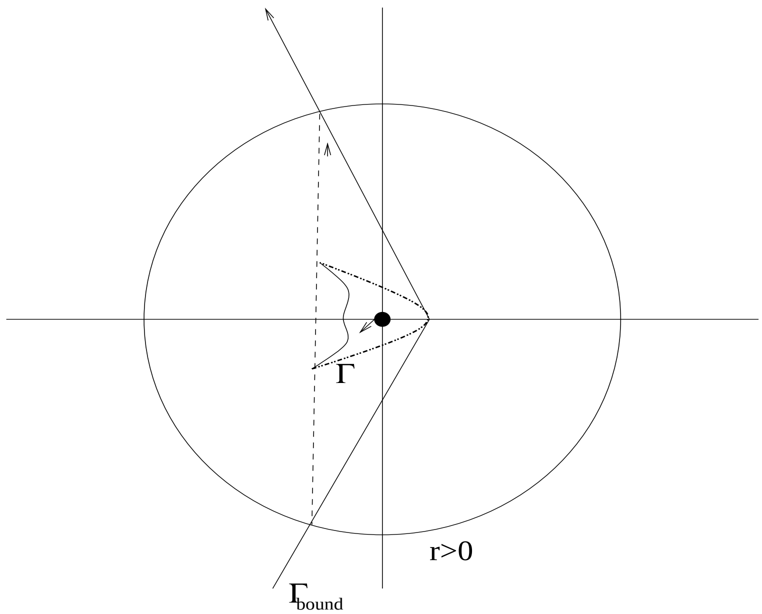

Figure 3: Contours for $|y_1| \le |a_1^-|t$.

We observe that for $t^{-3/4} \ge w_1$, we have $t^{-1/4} \ge w_1^{1/3}$, and similarly for $|\tilde{w}|$, so that we can conclude the estimate

$$\begin{aligned}\operatorname{Re}\Big(\lambda t + i\xi\cdot(\tilde{x}-\tilde{y}) - \mu_2^-(\lambda,\xi)y_1\Big)\\ \le -\eta\frac{w_1^{4/3}}{L_1}t - \frac{|\tilde{w}|^{4/3}}{L_2}t - \tilde{\theta} b_-^{1111}k^4 t - \theta^2|\xi_R|^4 t + C.\end{aligned}$$

The cases $(\cdot)^-$ follow similarly, except that λ_R is now chosen negative so that our contour Γ does not contain the leading eigenvalue $\lambda_*(\xi)$. In this case, we take a contour entirely to the left of the imaginary axis, augmented by a contour that picks up the pole at λ_* (see Figure 3). The critical new issue arising with these contours is the analysis of the residue term.

Cases (a)$^-$–(g)$^-$. In each of the cases (a)$^-$ through (g)$^-$, we compute

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_{\Gamma_{\text{res}}}\frac{e^{\lambda t-\mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda d\xi = \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\Big[\int_{\Gamma_R}\frac{e^{\lambda t-\mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda + \int_{\Gamma_{\text{loop}}}\frac{e^{\lambda t-\mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda\Big]d\xi.$$

For the integral

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_{\Gamma_R}\frac{e^{\lambda t-\mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda d\xi,$$

we compute almost exactly as in the cases (a)$^+$ through (g)$^+$. For the integration over Γ_{loop} we proceed through Cauchy's integral formula to get (for some constant c)

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})}\int_{\Gamma_{\text{loop}}}\frac{e^{\lambda t-\mu_2^-(\lambda,\xi)y_1}}{D(\lambda,\xi)}d\lambda d\xi = c\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})+\lambda_* t-\mu_2^-(\lambda_*,\xi)y_1}d\xi,$$

where according to $(\mathcal{D}_s)$

$$\lambda_*(\xi) = -i\tilde{a}_{\rm ave}\cdot\xi - \lambda_2^{jk}\xi_j\xi_k + i\lambda_3^{jkl}\xi_j\xi_k\xi_l - \lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m + \mathbf{O}(|\xi|^5).$$

Expanding $\mu_2^-(\lambda_*,\xi)$, we have

$$\begin{aligned}
&i\xi\cdot(\tilde{x}-\tilde{y}) + \lambda_*(\xi)t - \mu_2^-(\lambda_*(\xi),\xi)y_1\\
=&i\xi\cdot(\tilde{x}-\tilde{y}) + \lambda_*(\xi)t + \frac{1}{a_1^-}(\lambda_* + i\tilde{a}^-\cdot\xi + B_0^-(\xi))y_1\\
+&i\frac{B_1^-(\xi)}{(a_1^-)^2}(\lambda_* + i\tilde{a}^-\cdot\xi + B_0^-(\xi))y_1 - \frac{B_2^-(\xi)}{(a_1^-)^3}(\lambda_* + i\tilde{a}^-\cdot\xi + B_0^-(\xi))^2y_1\\
-&i\frac{B_3^-(\xi)}{(a_1^-)^4}(\lambda_* + i\tilde{a}^-\cdot\xi + B_0^-(\xi))^3y_1 + \frac{b_-^{1111}}{(a_1^-)^5}(\lambda_* + i\tilde{a}^-\cdot\xi + B_0^-(\xi))^4y_1 + \mathbf{O}(\rho^5)y_1.
\end{aligned}$$

Expanding $\lambda_*(\xi)$ in ξ, we find

$$\begin{aligned}
&i\xi\cdot(\tilde{x}-\tilde{y}) + \lambda_*(\xi)t - \mu_2^-(\lambda_*(\xi),\xi)y_1\\
&=i\xi\cdot\Big(\tilde{x}-\tilde{y}-\tilde{a}_{\rm eff}^-t\Big) - (t+\frac{y_1}{a_1^-})\lambda_2^{jk}\xi_j\xi_k\\
&\quad + i(t+\frac{y_1}{a_1^-})\lambda_3^{jkl}\xi_j\xi_k\xi_l - B_{\rm eff}^{jklm}(t,y_1)\xi_j\xi_k\xi_l\xi_m t + \mathbf{O}(|\xi|^5)y_1,
\end{aligned}$$

where

$$\tilde{a}_{\rm eff}(t,y_1) := \tilde{a}_{\rm ave} + \frac{y_1}{a_1^-t}(\tilde{a}_{\rm ave} - \tilde{a}^-)$$

and

$$B_{\rm eff}^{jklm}(t,y_1)\xi_j\xi_k\xi_l\xi_m = (1+\frac{y_1}{a_1^-t})\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m - \frac{y_1}{a_1^-t}b_-^{jklm}\zeta_j\zeta_k\zeta_l\zeta_m,$$

with

$$\zeta := (\frac{\tilde{a}_{\rm ave}^- - \tilde{a}^-}{a_1^-}\cdot\xi, \xi_2, ..., \xi_d).$$

According to Lemma 4.1, we have (for $y_1<0$)

$$\begin{aligned}
&\mathrm{Re}\,\Big(B_{\rm eff}^{jklm}(t,y_1)\xi_j\xi_k\xi_l\xi_m\Big) = \mathrm{Re}\,\Big((1+\frac{y_1}{a_1^-t})\lambda_4^{jklm}\xi_j\xi_k\xi_l\xi_m + \frac{|y_1|}{a_1^-t}b_-^{jklm}\zeta_j\zeta_k\zeta_l\zeta_m\Big)\\
&\ge(1+\frac{y_1}{a_1^-t})\lambda_4^0\Big(|\xi_R|^4 - C|\xi_I|^4\Big) + \frac{\theta|y_1|}{a_1^-t}\mathrm{Re}\,\Big(b_-^{1111}(\frac{i(\tilde{a}^- - \tilde{a}_{\rm ave})\cdot\xi}{a_1^-})^4 + b_-^{jklm}\xi_j\xi_k\xi_l\xi_m\Big) - C|\xi_I|^4\\
&\ge\Big((1+\frac{y_1}{a_1^-t})\lambda_4^0 + \frac{\theta^2|y_1|}{a_1^-t}\Big)|\xi_R|^4 - C|\xi_I|^4 \ge c_1|\xi_R|^4 - C_2|\xi_I|^4.
\end{aligned}$$

In evaluating our residue integral, we observe that the optimal choice for ξ_I is

$$\xi_I^{\rm eff} = \frac{1}{L^{1/3}}|\tilde{w}_{\rm eff}|^{1/3}\frac{\tilde{w}_{\rm eff}}{|\tilde{w}_{\rm eff}|},$$

where

$$\tilde{w}_{\rm eff} = \frac{\tilde{x}-\tilde{y}-\tilde{a}_{\rm eff}^-t}{t},$$

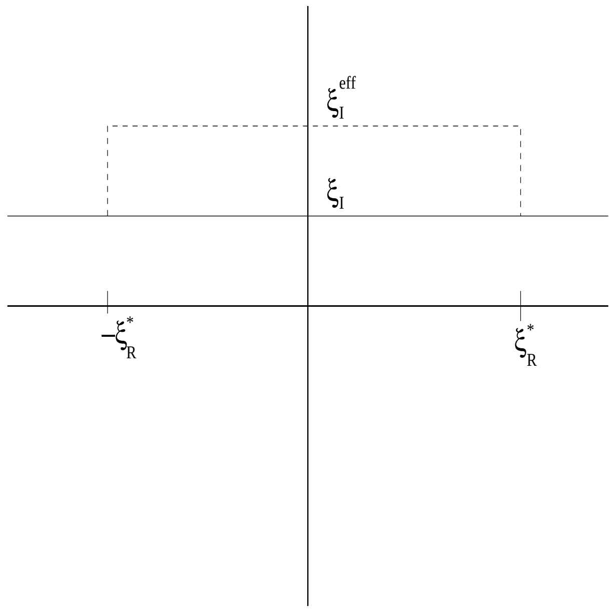

Figure 4: Lifting of ξ for residue analysis.

for which we obtain

$$\begin{aligned}
&\mathrm{Re}\Big(i\xi\cdot(\tilde{x}-\tilde{y})+\lambda_*(\xi)t-\mu_2^-(\lambda_*(\xi),\xi)y_1\Big)\\
&\quad\le-\frac{|\tilde{w}_{\mathrm{eff}}|^{4/3}}{L^{1/3}}t+(t+\frac{y_1}{a_1^-})\Big[-\lambda_2^0|\xi_R|^2+C_2\frac{|\tilde{w}_{\mathrm{eff}}|^{2/3}}{L^{2/3}}+C_3|\xi_R|^2\frac{|\tilde{w}_{\mathrm{eff}}|^{1/3}}{L^{1/3}}+C_3\frac{|\tilde{w}_{\mathrm{eff}}|}{L}\Big]\\
&\quad-c_1|\xi_R|^4t+C_4\frac{|\tilde{w}_{\mathrm{eff}}|^{4/3}}{L^{4/3}}t+\mathbf{O}(|\xi|^5)t.
\end{aligned}$$

Following [HoffZ.1], we must alter our chosen contour from $\xi=\xi_R+i\xi_I$ to $\xi=\xi_R+i\xi_I^{\mathrm{eff}}$, which creates new vertical strips of contour (see Figure 4). Along these strips, $|\xi_R|$ is bounded away from 0 and we have exponential decay in t. Since all cases under consideration have $|\tilde{x}-\tilde{y}|\le Kt$ for some K sufficiently large, we conclude an estimate along these strips by

$$Ce^{-(|\tilde{x}-\tilde{y}|+t)}.$$

Between $-\xi_R^*$ and ξ_R^*, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}_*^{d-1}}e^{i\xi\cdot(\tilde{x}-\tilde{y})+\lambda_*t-\mu_2^-(\lambda_*,\xi)y_1}d\xi\Big|\le\int_{\mathbb{R}_*^{d-1}}e^{\mathrm{Re}(i\xi\cdot(\tilde{x}-\tilde{y})+\lambda_*t-\mu_2^-(\lambda_*,\xi)y_1)}d\xi_R\\
&\quad\le C\int_{\mathbb{R}_*^{d-1}}e^{-\frac{1}{L^{1/3}}|\tilde{w}_{\mathrm{eff}}|^{4/3}t+(t+\frac{y_1}{a_1^-})\Big[-\lambda_2^0|\xi_R|^2+C_2\frac{|\tilde{w}_{\mathrm{eff}}|^{2/3}}{L^{2/3}}+C_3|\xi_R|^2\frac{|\tilde{w}_{\mathrm{eff}}|^{1/3}}{L^{1/3}}+C_3\frac{|\tilde{w}_{\mathrm{eff}}|}{L}\Big]-c_1|\xi_R|^4t+C_2\frac{|\tilde{w}_{\mathrm{eff}}|^{4/3}}{L^{4/3}}t}d\xi_R.
\end{aligned}$$

We observe that in the case of Condition (2a) from spectral condition $(\mathcal{D}_s)$, all expressions inside the square brackets are zero. In this case, upon choosing L sufficiently large, we conclude an estimate by

$$Ct^{-\frac{d-1}{4}}e^{-\eta|\tilde{w}_{\mathrm{eff}}|^{4/3}t}=Ct^{-\frac{d-1}{4}}e^{-\frac{(\tilde{x}-\tilde{y}-\tilde{a}_{\mathrm{eff}}^-(t,y_1)t)^{4/3}}{Mt^{1/3}}}.$$

Alternatively, in the case of Condition (2b) from spectral condition ($\mathcal{D}_s$), the expression inside the square brackets are all present, and the expression $c_1|\xi_R|^4 t$ must be replaced with $c_1|\xi_R|^4|y_1| - C_4|\xi_R|^4|y_1 + a_1^- t|$. We observe that for ξ_R sufficiently small (before we strike Γ_{bound}), the growth $C_4|\xi_R|^4|y_1 + a_1^- t|$ is dominated by second order decay. In this case, the second order growth term,

$$(t + \frac{y_1}{a_1^-})C_2\frac{|\tilde{w}_{\text{eff}}|^{2/3}}{L^{2/3}},$$

is only dominated by fourth order decay $(-L^{-1/3}|\tilde{w}_{\text{eff}}|^{4/3}t)$ in the case

$$|\tilde{w}_{\text{eff}}| \geq C(1 + \frac{y_1}{a_1^- t})^{\frac{3}{2}}.$$

That is, for small time, with time measured from the moment the signal strikes the shock layer ($y_1 = -a_1^- t$), the higher order regularity dominates behavior. In the case $|\tilde{w}_{\text{eff}}| \leq C(1 + \frac{y_1}{a_1^- t})^{\frac{3}{2}}$, we must choose in lieu of our fourth order scaling, the second order scaling

$$\xi_I^{\text{eff}} := \frac{\tilde{w}_{\text{eff}}}{L(1 + \frac{y_1}{a_1^- t})},$$

for which we have,

$$\begin{aligned}
&\text{Re}\Big(i\xi\cdot(\tilde{x} - \tilde{y}) + \lambda_*(\xi)t - \mu_2^-(\lambda_*(\xi), \xi)y_1\Big) \\
&\quad \leq -\frac{|\tilde{w}_{\text{eff}}|^2}{L|1 + \frac{y_1}{a_1^-}|}t + (t + \frac{y_1}{a_1^-})\Big[-\lambda_2^0|\xi_R|^2 + C_2\frac{|\tilde{w}_{\text{eff}}|^2}{L^2(1 + \frac{y_1}{a_1^- t})^2} + C_3|\xi_R|^2\frac{|\tilde{w}_{\text{eff}}|}{L(1 + \frac{y_1}{a_1^- t})} + C_3\frac{|\tilde{w}_{\text{eff}}|^3}{L^3(1 + \frac{y_1}{a_1^- t})^3}\Big] \\
&\quad\quad - c_1|\xi_R|^4 y_1 + C_4|\xi_R|^4|y_1 + a_1^- t| + C_4\frac{|\tilde{w}_{\text{eff}}|^4}{L^4(1 + \frac{y_1}{a_1^-})^4}t + \mathbf{O}(|\xi|^5)t. \qquad (4.3)
\end{aligned}$$

We observe here that in the case $|\tilde{w}_{\text{eff}}| \leq C(1 + \frac{y_1}{a_1^- t})^{\frac{3}{2}}$, the first time on the right-hand side of (4.3) dominates the remaining growth terms (for ξ_R small), and we determine an estimate by

$$C\Big(y_1^{-\frac{d}{4}} \wedge |y_1 + a_1^- t|^{-\frac{d}{2}}\Big)e^{-\frac{(\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^- t)^2}{M|y_1 + a_1^- t|}}.$$

With regard to large t behavior, we observe that

$$y_1^{-\frac{d}{4}} \wedge |y_1 + a_1^- t|^{-\frac{d}{2}} \leq Ct^{-\frac{d}{4}},$$

for which we have an estimate by

$$C\Big(t^{-\frac{d}{4}} \wedge |y_1 + a_1^- t|^{-\frac{d}{2}}\Big)e^{-\frac{(\tilde{x} - \tilde{y} - \tilde{a}_{\text{eff}}^- t)^2}{M|y_1 + a_1^- t|}}.$$

On the other hand, small t behavior is controlled by integration along Γ_{bound}, for which we again have fourth order behavior.

Transmission estimate (Undercompressive case, $y_1 < 0 < x_1$). The most fundamentally new estimate in this analysis is the scattering term for an undercompressive shock, which correponds with mass passing through the shock layer. Undercompressive shocks do not arise in the second-order regularization of [HoffZ.1], and though undercompressive shocks are considered in the general systems analysis of [Z.1], the estimates obtained are not as detailed as those we require. We consider the integral

$$\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(\tilde{x}-\tilde{y})} \int_\Gamma e^{\lambda t+\mu_2^+(\lambda,\xi)x_1-\mu_2^-(\lambda,\xi)y_1} d\lambda d\xi.$$

In this case,

$$\begin{aligned}\lambda t+&i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^+(\lambda,\xi)x_1-\mu_2^-(\lambda,\xi)y_1\\ =&\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\Big(-\frac{1}{a_1^+}(\lambda+i\tilde{a}^+\cdot\xi)-\frac{1}{a_1^+}b_+^{jklm}\zeta_j^+\zeta_k^+\zeta_l^+\zeta_m^+\Big)x_1\\ &+\Big(\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi)-\frac{1}{a_1^-}b_-^{jklm}\zeta_j^-\zeta_k^-\zeta_l^-\zeta_m^-\Big)y_1,\end{aligned}$$

where

$$\zeta^\pm=\Big(-\frac{i}{a_1^\pm}(\lambda+i\tilde{a}^\pm\cdot\xi),\xi_2,\xi_3,...,\xi_d\Big).$$

We apply Lemma 4.1(iii) to the ζ terms, which gives

$$\mathrm{Re}\,\Big(\frac{1}{a_1^\pm}b_\pm^{jklm}\zeta_j^\pm\zeta_k^\pm\zeta_l^\pm\zeta_m^\pm\Big)\geq\frac{\theta}{a_1^\pm}\mathrm{Re}\,\Big(b_\pm^{1111}(\zeta_1^\pm)^4+b_\pm^{jklm}\xi_j\xi_k\xi_l\xi_m\Big)-C|\mathrm{Im}\;\zeta^\pm|^4.$$

We have, then

$$\begin{aligned}&\mathrm{Re}\,\Big(\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^+(\lambda,\xi)x_1-\mu_2^-(\lambda,\xi)y_1\Big)\\ &\leq\mathrm{Re}\,\Big[\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})-\frac{1}{a_1^+}(\lambda+i\tilde{a}^+\cdot\xi)x_1+\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi)y_1\\ &-\frac{\theta}{a_1^+}\Big(b_+^{1111}\frac{(\lambda+i\tilde{a}^+\cdot\xi)^4}{(a_1^+)^4}+B_0^+(\xi)\Big)x_1+\frac{\theta}{a_1^-}\Big(b_-^{1111}\frac{(\lambda+i\tilde{a}^-\cdot\xi)^4}{(a_1^-)^4}+B_0^-(\xi)\Big)y_1\Big]\\ &+C_+|\mathrm{Im}\;\zeta^+|^4|x_1|+C_-|\mathrm{Im}\;\zeta^-|^4|y_1|+\mathbf{O}(\rho^5)|x_1|+\mathbf{O}(\rho^5)|y_1|.\end{aligned}$$

Rearranging terms, we can re-write this as

$$\begin{aligned}\mathrm{Re}\,&\Big(\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})+\mu_2^+(\lambda,\xi)x_1-\mu_2^-(\lambda,\xi)y_1\Big)\leq\mathrm{Re}\,\Big[\lambda t+i\xi\cdot(\tilde{x}-\tilde{y})\\ &-\Big(\frac{1}{a_1^+}(\lambda+i\tilde{a}^+\cdot\xi+\theta B_0^+(\xi))+\frac{\theta b_+^{1111}}{(a_1^+)^5}(\lambda+i\tilde{a}^+\cdot\xi+\theta B_0^+(\xi))^4\Big)x_1\\ &+\Big(\frac{1}{a_1^-}(\lambda+i\tilde{a}^-\cdot\xi+\theta B_0^-(\xi))+\frac{\theta b_-^{1111}}{(a_1^-)^5}(\lambda+i\tilde{a}^-\cdot\xi+\theta B_0^-(\xi))^4\Big)y_1\Big]\\ &+C_+|\mathrm{Im}\;\zeta^+|^4|x_1|+C_-|\mathrm{Im}\;\zeta^-|^4|y_1|+\mathbf{O}(\rho^5)|x_1|+\mathbf{O}(\rho^5)|y_1|.\end{aligned}$$

Selecting an optimal contour is complicated in this case by the transitional behavior of the signal. In light of this, we define two possible contours through the representation

$$\begin{aligned}&\frac{1}{a_1^\pm}(\lambda+i\tilde{a}^\pm\cdot\xi+\theta B_0^\pm(\xi))+\frac{\theta b_\pm^{1111}}{(a_1^\pm)^5}(\lambda+i\tilde{a}^\pm\cdot\xi+\theta B_0^\pm(\xi))^4\\ &=\frac{1}{a_1^\pm}(\lambda_R-\tilde{a}^\pm\cdot\xi_I)+\frac{\theta b_\pm^{1111}}{(a_1^\pm)^5}(\lambda_R-\tilde{a}^\pm\cdot\xi_I)^4+ik.\end{aligned}$$

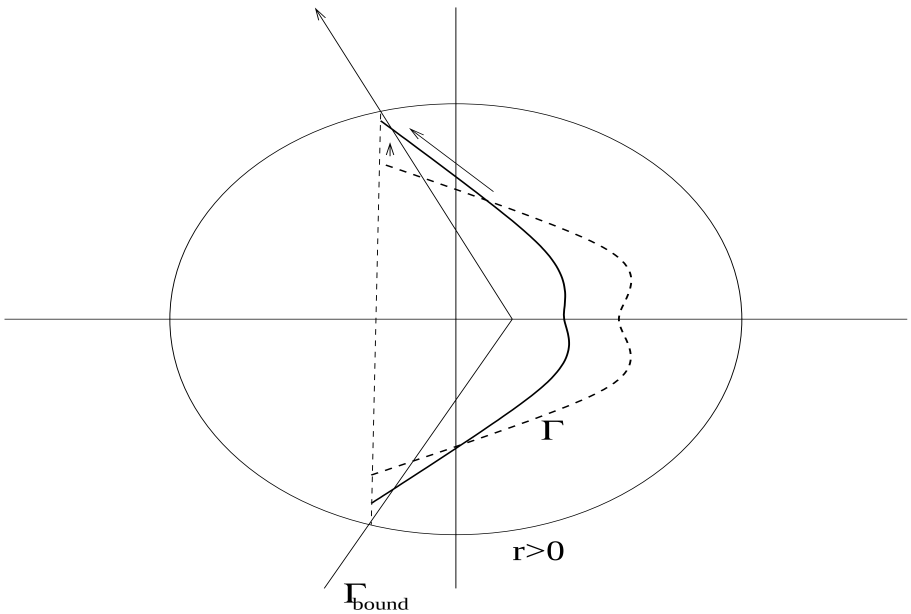

Figure 5: Contours in the case $y_1 \le 0 \le x_1$, undercompressive case.

Proceeding as in our analysis of the Lax case, we find that $\lambda(k)$ satisfies one of the contours

$$\begin{aligned}\lambda_\pm(k) =& (\lambda_R - \tilde{a}^\pm \cdot \xi_I) - i\tilde{a}^\pm \cdot \xi - ia_1^\pm k - \theta B_0^\pm(\xi) + \frac{6\theta b_\pm^{1111}}{(a_1^\pm)^2}(\lambda_R - \tilde{a}^\pm \cdot \xi_I)^2 k^2 \\ &- 4i(\lambda_R - \tilde{a}^\pm \cdot \xi_I)\frac{\theta b_\pm}{a_1^\pm}k^3 - \theta b_\pm^{1111}k^4 + \mathbf{O}((|\lambda_R - \tilde{a}^\pm \cdot \xi_I| + |k|)^5).\end{aligned}$$

Of these two possible contours, we always take the right most, switching from one contour to the next at intersections (see Figure 5). We observe, by continuity, that with this choice the real part of the contour we follow is exact, while the real part of the remaining contour is an upper bound.

We have

$$\begin{aligned}\mathrm{Re}\,\Big(\lambda t + i\xi \cdot (\tilde{x} - \tilde{y}) + \mu_2^+(\lambda,\xi)x_1 - \mu_2^-(\lambda,\xi)y_1\Big) &\le \mathrm{Re}\,(\lambda t + i\xi(\tilde{x} - \tilde{y})) \\ &- \Big(\frac{1}{a_1^+}(\lambda_R - \tilde{a}^+ \cdot \xi_I) + \frac{\theta b_+^{1111}}{(a_1^+)^5}(\lambda_R - \tilde{a}^+ \cdot \xi_I)^4\Big)x_1 \\ &+ \Big(\frac{1}{a_1^-}(\lambda_R - \tilde{a}^- \cdot \xi_I) + \frac{\theta b_-^{1111}}{(a_1^-)^5}(\lambda_R - \tilde{a}^- \cdot \xi_I)^4\Big)y_1 \\ &+ C_+|\mathrm{Im}\,\zeta^+|^4|x_1| + C_-|\mathrm{Im}\,\zeta^-|^4|y_1| + \mathbf{O}(\rho^5)|x_1| + \mathbf{O}(\rho^5)|y_1|.\end{aligned}$$

Expanding $\lambda(k)$ and rearranging terms, we have

$$\begin{aligned}
&\text{Re }\Big(\lambda t + i\xi\cdot(\tilde x-\tilde y)+\mu_2^+(\lambda,\xi)x_1-\mu_2^-(\lambda,\xi)y_1\Big)\\
\le&\lambda_R(t-\frac{x_1}{a_1^+}+\frac{y_1}{a_1^-})-\xi_I\cdot(\tilde x-\tilde y-\frac{\tilde a^+}{a_1^+}x_1+\frac{\tilde a^-}{a_1^-}y_1)-\theta B_0^\pm(\xi)t-\theta b_\pm^{1111}k^4t\\
&+\frac{6\theta b_\pm^{1111}}{(a_1^-)^2}(\lambda_R-\tilde a^\pm\cdot\xi_I)^2k^2t-\frac{\theta b_+^{1111}}{(a_1^+)^5}(\lambda_R-\tilde a^+\cdot\xi_I)^4x_1+\frac{\theta b_-^{1111}}{(a_1^-)^5}(\lambda_R-\tilde a^-\cdot\xi_I)^4y_1\\
&+C_+|\text{Im }\zeta^+|^4|x_1|+C_-|\text{Im }\zeta^-|^4|y_1|+\mathbf{O}(\rho^5)|x_1|+\mathbf{O}(\rho^5)|y_1|\\
\le&(\lambda_R-\tilde a^-\cdot\xi_I)(t-\frac{x_1}{a_1^+}+\frac{y_1}{a_1^-})-\xi_I\cdot(\tilde x-\tilde y-(\frac{\tilde a^+-\tilde a^-}{a_1^+t}x_1+\tilde a^-)t)-\theta B_0^\pm(\xi)t-\theta b_\pm^{1111}k^4t\\
&C_1(\lambda_R^4+|\xi_I|^4)t+C_2\rho^5(|x_1|+|y_1|).
\end{aligned}$$

Observing the relation

$$\frac{\tilde a^+-\tilde a^-}{a_1^+t}x_1+\tilde a^-=\frac{\tilde a^+-\tilde a^-}{a_1^-t}y_1+\tilde a^+,$$

we redefine w from the previous analyses as

$$w=(w_1,\tilde w)=\Big(\frac{\frac{x_1}{a_1^+}-\frac{y_1}{a_1^-}-t}{t},\frac{\tilde x-\tilde y-(\frac{\tilde a^+-\tilde a^-}{a_1^-t}y_1+\tilde a^+)t}{t}\Big).$$

We now choose λ_R and ξ_I according to the following scheme.

$$(\lambda_R-\tilde a^-\cdot\xi_I,\xi_I)=\begin{cases}
\Big(\pm(\epsilon/L)^{1/3},0\Big) & \epsilon\le|w_1|\le K, 0\le|\tilde w|\le K \quad \text{(a)}^\pm\\
\Big(\pm(\epsilon/L_1)^{1/3},(\epsilon/L_2)^{1/3}\frac{\tilde w}{|\tilde w|}\Big) & 0\le|w_1|\le K,\epsilon\le|\tilde w|\le K \quad \text{(b)}^\pm\\
\Big((w_1/L)^{1/3},0\Big) & t^{-3/4}\le|w_1|,|\tilde w|\le\epsilon,|w_1|\ge N|\tilde w| \quad \text{(c)}^\pm\\
\Big((|\tilde w|/L_1)^{1/3},(|\tilde w|/L_2)^{1/3}\frac{\tilde w}{|\tilde w|}\Big) & t^{-3/4}\le|w_1|,|\tilde w|\le\epsilon,|w_1|\le N|\tilde w| \quad \text{(d)}^\pm\\
\Big((w_1/L)^{1/3},0\Big) & 0\le|\tilde w|\le t^{-3/4}\le|w_1|\le\epsilon \quad \text{(e)}^\pm\\
\Big((|\tilde w|/L_1)^{1/3},(|\tilde w|/L_2)^{1/3}\frac{\tilde w}{|\tilde w|}\Big) & 0\le|w_1|\le t^{-3/4}\le|\tilde w|\le\epsilon \quad \text{(f)}^\pm\\
\Big(t^{-1/4}/L_1,(t^{-1/4}/L_2)\frac{\tilde w}{|\tilde w|}\Big) & 0\le|w_1|,|\tilde w|\le t^{-3/4} \quad \text{(g)}^\pm,
\end{cases}$$

where again the $\pm$ refer to $w_1\gtrless 0$. With this choice, the analysis follows exactly as in the Lax case. $\square$

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