Stability for Transition Front Solutions in Multidimensional Cahn-Hilliard Systems

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March 9, 2015

Abstract

We consider nonlinear stability for planar transition front solutions $\bar{u}(x_1)$ arising in multidimensional (i.e., $x \in \mathbb{R}^n$) Cahn-Hilliard systems. In previous work the author has established conditions under which such waves are spectrally and linearly stable, and in this analysis it is shown that linear stability implies nonlinear stability for such systems.

1 Introduction

We consider Cahn-Hilliard systems on $x \in \mathbb{R}^n$,

$$\frac{\partial u_j}{\partial t} = \nabla \cdot \Big\{ \sum_{k=1}^{m} M_{jk}(u) \nabla \Big((-\Gamma \Delta u)_k + F_{u_k}(u) \Big) \Big\}, \tag{1.1}$$

for $j = 1, 2, \dots, m$. Here, $F : \mathbb{R}^n \to \mathbb{R}$, and Γ and M are $m \times m$ matrices. For notational convenience, we will often use the tensor form

$$u_t = \nabla \cdot \Big\{ M(u) D_x \Big(-\Gamma \Delta u + D_u F \Big) \Big\}, \tag{1.2}$$

where the operator D is a Jacobian operator as described, for example, in [4].

For convenient reference, we collect some assumptions that will be made throughout the analysis.

(H0) (Assumptions on Γ) Γ denotes a constant, symmetric, positive definite $m \times m$ matrix.

(H1) (Assumptions on F) $F \in C^4(\mathbb{R}^m)$, and F has at least two distinct local minimizers at which the Hessian matrix $D_u^2 F(u)$ is positive definite and (by subtracting an appropriate hyperplane from F if necessary) we can take F to be zero. We denote this class of values

$$\mathcal{M} := \{u \in \mathbb{R}^m : F(u) = 0, D_u F(u) = 0, D_u^2 F(u) \text{ is positive definite}\}.$$

(H2) (Transition front existence) There exists a transition front solution to (1.1) $\bar{u}(x_1)$ so that

$$-\Gamma \bar{u}'' + F'(\bar{u}) = 0, \tag{1.3}$$

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$.

(H3) (Assumptions on M) $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the wave; i.e., there exists $\theta > 0$ so that for all $y \in \mathbb{R}^m$ and all $x_1 \in \mathbb{R}$ we have

$$y^T M(\bar{u}(x_1)) y \geq \theta |y|^2;$$

and $M_\pm = M(u_\pm)$ are symmetric.

(H4) (Endstate Assumptions) We set $B_\pm := D_u^2 F(u_\pm)$ (a symmetric, positive definite matrix) and assume one of the following holds: (H4a) the matrices $M_\pm B_\pm$ have distinct eigenvalues, as do the matrices $\Gamma^{-1} B_\pm$; *or* (H4b) one or more of these matrices has a repeated eigenvalue, but the solutions $\mu = \mu(\sigma)$ of

$$\det \Big(-\mu^4 M_\pm \Gamma + \mu^2 (M_\pm B_\pm + 2\sigma\kappa_0 M_\pm \Gamma) - \sigma(\lambda_0 I + \kappa_0 M_\pm B_\pm + \sigma \kappa_0^2 M_\pm \Gamma) \Big) = 0 \tag{1.4}$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\sigma)$ is analytic in σ for $|\sigma|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\sigma)$ can be written as $\mu(\sigma) = \sqrt{\sigma} h(\sigma)$, where h is analytic in σ for $|\sigma|$ sufficiently small. Here $|(\lambda_0, \kappa_0)| = 1$, and $(\sigma\lambda_0, \sigma\kappa_0) \in \mathcal{S}_\epsilon$ for $\epsilon > 0$ sufficiently small. (The set $\mathcal{S}_\epsilon$ is defined in Definition 2.1.)

Regarding (H2), we note that Alikakos and others have established that transition front solutions arise precisely as minimizers of the energy functional

$$E(\bar{u}) = \int_{-\infty}^{+\infty} F(\bar{u}) + \frac{1}{2} \langle \Gamma \bar{u}, \bar{u} \rangle dx_1, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ denotes Euclidean inner product. (See [1, 2, 25]).

The system (1.1) is a standard model of certain phase separation processes, and its physicality is discussed in detail in [16] and the references cited there. Our interest in this analysis is to establish that $\bar{u}(x_1)$ is stable for an appropriate class of initial perturbations.

It is well known that for the case of one space dimension solutions $u(x,t)$ of Cahn–Hilliard systems initialized by $u(x,0)$ near a standing wave solution $\bar{u}(x)$ will not generally approach $\bar{u}(x)$ time-asymptotically, but rather will approach a translate of $\bar{u}(x)$ determined by an integral of the initial perturbation. In [17, 18], a local tracking function $\delta(t)$ was employed to track shifts so that at each time the shapes of $u(x,t)$ and $\bar{u}(x)$ were compared, not the relative positions. In the case $n \geq 2$, $u(x,t)$ does not approach a shifted wave asymptotically, but local shifts along the transition front serve to hinder the analysis (they reduce the rate of decay of the perturbations and consequently nonlinearities become more difficult to control). In the current analysis, we employ a shift function that depends both on t and the transverse variable $\tilde{x} = (x_2, x_3, \ldots, x_n)$, defining our perturbation as in [9, 21] by

$$v(x,t) := u(x,t) - \bar{u}(x_1 - \delta(\tilde{x}, t)), \tag{1.6}$$

where $\delta(\tilde{x}, t)$ denotes a shift function to be chosen during the analysis.

Upon substitution of (1.6) into (1.1) we obtain

$$(\partial_t - L)v = (\partial_t - L)(\delta \bar{u}'(x_1)) + \nabla \cdot \mathcal{Q}, \tag{1.7}$$

where

$$Lv := \nabla \cdot \Big\{ \bar{M}(x_1) D_x \Big(-\Gamma \Delta v + \bar{B}(x_1) v \Big) \Big\}, \tag{1.8}$$

with

$$\begin{aligned} \bar{M}(x_1) &:= M(\bar{u}(x_1)) \\ \bar{B}(x_1) &:= D_u^2 F(\bar{u}(x_1)), \end{aligned} \tag{1.9}$$

and $\mathcal{Q}$ is a (matrix-valued) collection of nonlinear terms that will be specified below.

The eigenvalue problem for L can be expressed as $L\phi = \lambda \phi$, and we take the Fourier transform of this equation in the transverse variable $\tilde{x}$, using the scaling

$$\hat{\phi}(x_1, \xi) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\tilde{x}\cdot\xi} \phi(x_1, \tilde{x}) d\tilde{x}. \tag{1.10}$$

The eigenvalue problem transforms to

$$L_\xi \hat{\phi} = -A_\xi H_\xi \hat{\phi} = \lambda \hat{\phi}, \tag{1.11}$$

where

$$\begin{aligned} A_\xi &:= -\partial_{x_1} \bar{M}(x_1) \partial_{x_1} + |\xi|^2 \bar{M}(x_1) \\ H_\xi &:= -\Gamma \partial^2_{x_1 x_1} + \bar{B}(x_1) + |\xi|^2 \Gamma. \end{aligned} \tag{1.12}$$

We note that under our current assumptions A_ξ and H_ξ are both self-adjoint (though of course L_ξ is not). For convenient reference, we collect here a set of conditions on (1.11) that follow from our assumptions (H0)-(H4).

For convenient reference, we collect here a set of conditions on (1.8) that follow from our assumptions (H0)-(H4).

(C0) Same as (H0).

(C1) $\bar{B} \in C^2(\mathbb{R})$ is symmetric; there exists a constant $\alpha_B > 0$ so that

$$\partial_{x_1}^j (\bar{B}(x_1) - B_\pm) = \mathbf{O}(e^{-\alpha_B |x_1|}), \quad x_1 \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both positive definite matrices.

(C2) $\bar{M} \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial_{x_1}^j (\bar{M}(x_1) - M_\pm) = \mathbf{O}(e^{-\alpha_M |x_1|}), \quad x_1 \to \pm\infty,$$

for $j = 0, 1, 2$; $\bar{M}(x_1)$ is uniformly positive definite on $\mathbb{R}$. We will set $\alpha := \min\{\alpha_B, \alpha_M\}$.

(C3) Same as (H4).

Before recalling the spectral theorem of [13], we clarify our terminology for the spectrum of L_ξ (which follows [15]; see particularly the appendix to Chapter 5).

Definition 1.1. *We define the point spectrum of L_ξ, denoted $\sigma_{pt}(L_\xi)$, as the set*

$$\sigma_{pt}(L_\xi) = \{\lambda \in \mathbb{C} : L_\xi \phi = \lambda \phi \text{ for some } \phi \in H^2(\mathbb{R})\}.$$

We define the essential spectrum of L_ξ, denoted $\sigma_{ess}(L_\xi)$, as the values in $\mathbb{C}$ that are not in the resolvent set of L_ξ and are not isolated eigenvalues of finite multiplicity.

We note that $\sigma(L_\xi) = \sigma_{pt}(L_\xi) \cup \sigma_{ess}(L_\xi)$, but the sets $\sigma_{pt}(L_\xi)$ and $\sigma_{ess}(L_\xi)$ are not necessarily disjoint. We will see that for real values of ξ the spectrum of L_ξ is confined to the real line (though L_ξ is not self-adjoint), and is bounded above. We will refer to the largest (right-most) eigenvalue of L_ξ as its *leading* eigenvalue, and we will denote this eigenvalue $\lambda_*(\xi)$.

The assumptions for the spectral theorem of [13] are all straightforward, except for a condition associated with the stability of $\bar{u}$ with respect to (1.1) in $\mathbb{R}$. That is, since $\bar{u}$ is a function of only one variable, it can be viewed as a stationary solution for a Cahn-Hilliard system on $\mathbb{R}$,

$$u_t = \Big\{ M(u)(-\Gamma u_{xx} + D_u F)_x \Big\}_x. \tag{1.13}$$

In [16], the authors identify a spectral stability criterion for $\bar{u}$ as a solution of (1.13), and verify that it is satisfied for certain example systems. In [17, 18], the authors establish that this spectral condition is sufficient to imply nonlinear stability for $\bar{u}$ as a solution of (1.13).

Although we will postpone our full discussion of this condition until Section 2, we will denote it $(\mathcal{D}_0)$ in the statement of our theorem, and we note here that it is ultimately a transversality condition in the following sense. When (1.3) (in (H2)) is written as a first order autonomous ODE system, our condition ensures that $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

Theorem 1.1 (From [13, 14])**.** *Let Assumptions (H0)-(H4) hold, with $\kappa_0 = 0$ in (H4), and additionally assume M is a constant matrix. Assume Condition $(\mathcal{D}_0)$ holds, and that $\bar{u}$ minimizes the energy (1.5). The spectrum of the operator L_ξ satisfies the following:*

I. *For real values of ξ:*

1. The spectrum $\sigma(L_\xi)$ lies entirely on $\mathbb{R}$.

2. The essential spectrum of L_ξ lies in the union of the two intervals

$$(-\infty, -m_\pm b_\pm |\xi|^2 - m_\pm \gamma |\xi|^4],$$

where $m_\pm$, $b_\pm$, and γ respectively denote the smallest eigenvalues of $M_\pm$, $B_\pm$, and Γ.

3. There exists a constant $\theta_0 > 0$ so that the point spectrum of L_ξ is confined to the interval $(-\infty, -\theta_0|\xi|^4]$.

4. There exists a constant $r > 0$ sufficiently small so that for $|\xi| < r$ the leading eigenvalue of L_ξ, denoted $\lambda_*(\xi)$, satisfies

$$\lambda_*(\xi) = -c_3|\xi|^3(1 + \mathbf{o}(|\xi|)),$$

where

$$c_3 = 4\frac{\int_{-\infty}^{+\infty} F(\bar{u}(x_1))dx_1}{\langle \bar{M}^{-1}[u], [u]\rangle} > 0.$$

Here, $\mathbf{o}(\cdot)$ denotes standard "little-O" notation, and $[\cdot]$ denotes jump, so that $[u] = u_+ - u_-$. Moreover, for any $0 < |\xi_0| < r$, there exists $0 < r_0 < r$ sufficiently small so that $\lambda_*(\xi)$ is analytic on $|\xi - \xi_0| < r_0$.

5. The constant $r > 0$ from Part 4 can be taken sufficiently small so that there exists a constant $\theta_1 > 0$ so that for $|\xi| < r$ the set $\sigma_{pt}(L_\xi)\backslash\{\lambda_*(\xi)\}$ is confined to the interval $(-\infty, -\theta_1|\xi|^2]$.

II. *Moreover, if we allow complex values of ξ ($\xi = \xi_R + i\xi_I$) so that $|\xi|^2$ becomes*

$$\zeta = |\xi_R|^2 - |\xi_I|^2 + 2i\langle \xi_R, \xi_I\rangle,$$

then:

6. There exist constants c_1 and θ_1 sufficiently small, and a constant C_{θ_1} sufficiently large, so that the essential spectrum for L_ξ is bounded to the left of a wedge contour described by

$$Re\,\lambda + c_1|Im\,\lambda| = -\theta_1\Big(|\xi_R|^2 + |\xi_R|^4\Big) + C_{\theta_1}\Big(|\xi_I|^2 + |\xi_I|^4\Big).$$

The designation C_{θ_1} indicates that θ_1 and C_{θ_1} are chosen together, and one can be varied at the expense of a change in the other.

7. The perturbation expression for $\lambda_*(\xi)$ given in Part 4 continues to hold for complex values of ξ (with $|\xi|^3$ replaced by $\zeta^{3/2}$), and there exist constants c_2 and θ_2 sufficiently small, and a constant C_{θ_2} sufficiently large, so that the remainder of the point spectrum is bounded for $|\zeta|$ sufficiently small to the left of a contour described by

$$Re\,\lambda + c_2|Im\,\lambda| = -\theta_2|\xi_R|^2 + C_{\theta_2}|\xi_I|^2.$$

8. There exist constants c_3 and θ_3 sufficiently small, and a constant C_{θ_3} sufficiently large, so that the point spectrum for L_ξ is bounded to the left of a contour described by

$$Re\,\lambda + c_3|Im\,\lambda| = -\theta_3|\xi_R|^4 + C_{\theta_3}\Big(1 + |\xi_R|^2 + |\xi_I|^2 + |\xi_I|^4\Big).$$

The main observations summarized in Theorem 1.1 are as follows: Part I asserts that for real values of ξ the spectrum of L_ξ lies entirely in the stable (i.e., negative-real) half-plane, and indeed the leading eigenvalue moves into the stable half-plane like $|\xi|^3$. Moreover, the remainder of the spectrum (both point and essential) separates from $\lambda_*(\xi)$ by moving into the stable half-plane at the faster rate $|\xi|^2$ (faster for $|\xi|$ small). Part II asserts that similar, if more complicated, dynamics continue to hold for complex values of ξ. Parts I is proven in [13], while Part II is proven in [14]. Finally, we note that only Parts 4, 7, and 8 require M to be constant.

If we let $G(x,t;y)$ denote a Green's function for L so that

$$(\partial_t - L)G = 0; \quad G(x,0;y) = \delta_y(x)I, \tag{1.14}$$

where in this case $\delta_y(x)$ denotes a Dirac delta function, then we can express solutions of (1.7) as

$$v(x,t) - \bar{u}'(x_1)\delta(\tilde{x},t) = \int_{\mathbb{R}^n} G(x,t;y)v_0(y)dy + \int_0^t \int_{\mathbb{R}} G(x,t-s;y)\nabla\cdot\mathcal{Q}(y,s)dyds. \tag{1.15}$$

As we'll clarify in Theorem 1.2 we can express G as

$$G(x,t;y) = \bar{u}'(x_1)e(\tilde{x},t;y) + \tilde{G}(x,t;y), \tag{1.16}$$

where, roughly speaking, $e(\tilde{x},t;y)$ encodes information associated with the shift $\delta(\tilde{x},t)$, and $\tilde{G}(x,t;y)$ encodes information away from the transition layer. We obtain

$$\begin{aligned} v(x,t) - \bar{u}'(x_1)\delta(\tilde{x},t) &= \bar{u}'(x_1)\int_{\mathbb{R}^n} e(\tilde{x},t;y)v_0(y)dy + \int_{\mathbb{R}^n}\tilde{G}(x,t;y)v_0(y)dy \\ &+ \bar{u}'(x_1)\int_0^t\int_{\mathbb{R}^n} e(\tilde{x},t-s;y)\nabla\cdot\mathcal{Q}(y,s)dyds + \int_0^t\int_{\mathbb{R}^n}\tilde{G}(x,t-s;y)\nabla\cdot\mathcal{Q}(y,s)dyds. \end{aligned} \tag{1.17}$$

We now choose $\delta(\tilde{x},t)$ so that

$$\delta(\tilde{x},t) = -\int_{\mathbb{R}^n} e(\tilde{x},t;y)v_0(y)dy - \int_0^t\int_{\mathbb{R}^n} e(\tilde{x},t-s;y)\nabla\cdot\mathcal{Q}(y,s)dyds. \tag{1.18}$$

Upon combining (1.17) and (1.18), and integrating the nonlinear terms by parts, we obtain the system of $m+1$ integral equations

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^n}\tilde{G}(x,t;y)v_0(y)dy - \int_0^t\int_{\mathbb{R}^n}\sum_{j=1}^n \tilde{G}_{y_j}(x,t-s;y)\mathcal{Q}_j(y,s)dyds \\ \delta(\tilde{x},t) &= -\int_{\mathbb{R}^n} e(\tilde{x},t;y)v_0(y)dy + \int_0^t\int_{\mathbb{R}^n}\sum_{j=1}^n e_{y_j}(\tilde{x},t-s;y)\mathcal{Q}_j(y,s)dyds. \end{aligned} \tag{1.19}$$

In addition, we can augment this system with integral equations for derivatives of v and δ as necessary by differentiating through the integral signs, which will be justified by estimates on $\tilde{G}(x,t;y)$ and $e(\tilde{x},t;y)$. Our primary goal in this analysis is to use nonlinear iteration to establish existence and asymptotic behavior of solutions to this system. In order to accomplish this, we require detailed estimates on $\tilde{G}(x,t;y)$ and $e(\tilde{x},t;y)$, as established in [14].

In order to efficiently describe some logarithmic behavior that arises in a theorem from [14], we make the following definition.

Definition 1.2. *We define a function $h_{p,n}(t)$ for all $1 \leq p \leq \infty$, $n = 2, 3, \ldots,$ and $t > 0$. Precisely, we take $h_{p,2}(t) \equiv 1$ for all $1 \leq p \leq \infty$, and for $n = 3, 4, \ldots$ we set*

$$h_{p,n}(t) = \begin{cases} \ln(e+t) & p = 1 \\ 1 & p > 1. \end{cases}$$

In addition, for $1 \leq p < \infty$ we will denote the L^p norm in the transverse variable as

$$\|u(x,t)\|_{L^p_{\tilde{x}}} := \Big(\int_{\mathbb{R}^n} |u(x_1, \tilde{x}, t)|^p d\tilde{x} \Big)^{\frac{1}{p}},$$

and we define $\|u(x,t)\|_{L^\infty_{\tilde{x}}}$ in an analogous fashion.

Finally, for positive constants K and T that arise in the statement of Theorem 1.2, we will let $\chi^{II}(x,t;y)$ denote the characteristic function for the set

$$\mathcal{S}^{II} := \{(x,t;y) : t \geq T, |x-y| \leq Kt\},$$

and we will let $\chi^{III}(x,t;y)$ denote the characteristic function for the complement of $\mathcal{S}^{II}$ (in $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n$). We can then write

$$\tilde{G}(x,t;y) = \tilde{G}^{II}(x,t;y) + \tilde{G}^{III}(x,t;y),$$

where

$$\begin{aligned} \tilde{G}^{II}(x,t;y) &= \tilde{G}(x,t;y)\chi^{II}(x,t;y) \\ \tilde{G}^{III}(x,t;y) &= \tilde{G}(x,t;y)\chi^{III}(x,t;y). \end{aligned}$$

The following theorem is established in [14]; we note that the spectral condition $(\mathcal{D}_\xi)$ will be stated precisely in Section 2.

Theorem 1.2 (From [14])**.** *Suppose Conditions (C0)-(C3) hold, along with spectral condition $(\mathcal{D}_\xi)$, and suppose the conclusions of Theorem 1.1 hold, possibly under weaker hypotheses. Then given any time $T > 0$ there exist constants $\eta > 0$ (sufficiently small), and $C > 0$, $K > 0$, $M > 0$ (sufficiently large) so that the Green's function described in (1.2) can be bounded as follows: there exists a splitting*

$$G(x,t;y) = \bar{u}'(x_1)e(\tilde{x},t;y) + \tilde{G}(x,t;y),$$

so that:
(I) Transition layer terms.

$$\|e(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}}$$

$$\|e_{y_1}(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{1}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}},$$

$$\|e_t(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}}$$

$$\|e_{ty_1}(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{4}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}},$$

and for any multiindex β in $\tilde{x}$ and $\tilde{y}$, with $|\beta| \le 3$,

$$\|\partial^\beta e(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}}$$

$$\|\partial^\beta e_{y_1}(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{1+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}}$$

$$\|\partial^\beta e_t(\tilde{x},t;y)\|_{L^p_{\tilde{x}}} \le C(1+t)^{-\frac{3+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{y_1^2}{Mt}}.$$

(II) Asymptotic terms. For $|x-y| \le Kt$, $t \ge T$

$$\|\tilde{G}^{II}(x,t;y)\|_{L^p_{\tilde{x}}} \le C\Big(t^{-\frac{1}{2}-\frac{n-1}{2}(1-\frac{1}{p})} + t^{-\frac{2}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t)\Big) e^{-\frac{(x_1-y_1)^2}{Mt}}$$

$$\|\tilde{G}^{II}_{x_1}(x,t;y)\|_{L^p_{\tilde{x}}} \le C\Big(t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) + t^{-\frac{1}{2}-\frac{n-1}{2}(1-\frac{1}{p})} e^{-\eta|x_1|} + t^{-\frac{2}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\eta|x_1|}\Big) e^{-\frac{(x_1-y_1)^2}{Mt}}$$

$$\|\tilde{G}^{II}_{y_1}(x,t;y)\|_{L^p_{\tilde{x}}} \le Ct^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{(x_1-y_1)^2}{Mt}}$$

$$\|\tilde{G}^{II}_{x_1 y_1}(x,t;y)\|_{L^p_{\tilde{x}}} \le C\Big(t^{-\frac{4}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) + t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\eta|x_1|}\Big) e^{-\frac{(x_1-y_1)^2}{Mt}},$$

and for any multiindex β in $\tilde{x}$ and $\tilde{y}$, with $|\beta| \le 3$,

$$\|\partial^\beta \tilde{G}^{II}(x,t;y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{2+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{(x_1-y_1)^2}{Mt}}$$

$$\|\partial^\beta \tilde{G}^{II}_{x_1}(x,t;y)\|_{L^p_{\tilde{x}}} \le C\Big(t^{-\frac{3+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) + t^{-\frac{2+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\eta|x_1|}\Big) e^{-\frac{(x_1-y_1)^2}{Mt}}$$

$$\|\partial^\beta \tilde{G}^{II}_{y_1}(x,t;y)\|_{x_2} \le Ct^{-\frac{3+|\beta|}{3}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{(x_1-y_1)^2}{Mt}}.$$

(III) Local terms. For $|x-y| \ge Kt$ or $0 < t < T$, and for any multiindex α in x and y with $|\alpha| \le 3$

$$\|\partial^\alpha \tilde{G}^{III}(x,t;y)\|_{L^p_{\tilde{x}}} \le Ct^{-\frac{1+|\alpha|}{4}-\frac{1}{4}(1-\frac{1}{p})} e^{-\frac{(x_1-y_1)^{4/3}}{Mt^{1/3}}}.$$

Moreover, precisely the same estimates hold if the L^2 norm in $\tilde{x}$ is replaced by the transverse L^2 norm in $\tilde{y}$.

Both for the statement of our main theorem and for the analysis to follow, it will be convenient to set notation for some unwieldy expressions that will commonly occur. We define:

$$
\begin{aligned}
\Theta(x_1,t) &:= (1+t)^{-1/2} e^{-\frac{x_1^2}{Lt}} + (1+|x_1|+\sqrt{t})^{-3/2};\\
A_0(x_1,t;p) &:= \Big((1+t)^{-\frac{n-1}{2}(1-\frac{1}{p})} + (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{6}} h_{p,n}(t)\Big)\Theta(x_1,t);\\
A_1(x_1,t;p) &:= \Big(t^{-1/4}(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{4}} + t^{-1/4}(1+t)^{-\frac{n-1}{2}(1-\frac{1}{p})+\frac{1}{4}} e^{-\eta|x_1|}\\
&\quad + t^{-1/4}(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})+\frac{1}{12}} h_{p,n}(t) e^{-\eta|x_1|}\Big)\Theta(x_1,t),
\end{aligned}
\tag{1.20}
$$

and

$$
\begin{aligned}
A_k(x_1,t;p) &:= t^{-1/4}(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})+\frac{1}{12}-\frac{1}{3}} h_{p,n}(t)\Theta(x_1,t); \quad k=2,3,\dots,n;\\
B_\beta(t;p) &:= (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{|\beta|}{3}} h_{p,n}(t); \quad |\beta|\le 3\\
\dot{B}(t;p) &:= (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-1} h_{p,n}(t).
\end{aligned}
\tag{1.21}
$$

Here, L is some sufficiently large constant.

The primary goal of the current analysis is to show that the estimates stated in Theorem 1.2 lead to the following theorem.

Theorem 1.3. *Let $\bar{u}(x_1)$ be a planar transition front solution to (1.1). Suppose that conditions (H0)-(H4) hold, along with spectral condition $(\mathcal{D}_\xi)$, and that the conclusions of Theorem 1.1 hold, possibly under weaker hypotheses. Then for Hölder continuous initial data $u_0 \in C^\gamma(\mathbb{R}^n)$, $0<\gamma<1$, with*

$$\|u_0(x)-\bar{u}(x_1)\|_{L^1_{\tilde{x}}} + \|u_0(x)-\bar{u}(x_1)\|_{L^\infty_{\tilde{x}}} \le \frac{\epsilon}{(1+|x_1|)^{\frac{3}{2}}},$$

for $\epsilon>0$ sufficiently small, there exists a unique solution to (1.1)

$$u \in C^{4+\gamma,1+\frac{\gamma}{4}}(\mathbb{R}^n\times(0,\infty)) \cap C^{\gamma,\frac{\gamma}{4}}(\mathbb{R}^n\times[0,\infty))$$

and a shift

$$\delta \in C^{3,1}(\mathbb{R}^{n-1}\times[0,\infty))$$

so that

$$
\begin{aligned}
\|v(x,t)\|_{L^p_{\tilde{x}}} &\le CA_0(x_1,t;p);\\
\|\partial_{x_i} v(x,t)\|_{L^p_{\tilde{x}}} &\le CA_i(x_1,t;p); \quad i=1,2,\dots,n,
\end{aligned}
$$

and

$$
\begin{aligned}
\|D^\beta_{\tilde{x}}\delta(\tilde{x},t)\|_{L^p_{\tilde{x}}} &\le CB_\beta(t;p); \quad |\beta|\le 3;\\
\|\partial_t\delta(\tilde{x},t)\|_{L^p_{\tilde{x}}} &\le C\dot{B}(t;p).
\end{aligned}
$$

The remainder of the paper is organized as follows. In Section 2 we describe spectral condition $(\mathcal{D}_\xi)$. In Section 3, we analyze and describe the nonlinearity $\mathcal{Q}$, and in Section 4 we establish estimates on the linear and nonlinear integrtals in (1.19). In Section 5 we carry out the nonlinear iteration that proves Theorem 1.3.

2 Spectral Condition $\mathcal{D}_\xi$

The purpose of this section is to review enough material from [13] and [14] so that we can state spectral condition $(\mathcal{D}_\xi)$. We express the Evans function for (1.11) in terms of asymptotically growing and decaying solutions for this equation. As $x_1 \to \pm\infty$ this equation is asymptotically close to the constant coefficient equations

$$-M_\pm\Gamma\phi'''' + (M_\pm B_\pm + 2|\xi|^2 M_\pm\Gamma)\phi'' - (\lambda I + |\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm\Gamma)\phi = 0. \tag{2.1}$$

If we search for solutions of the form $\phi(x_1) = e^{\mu x_1} r$, where μ is a scalar constant and $r \in \mathbb{C}^m$ is a constant vector (constant in x_1) we obtain the associated eigenvalue problem

$$\Big\{ - \mu^4 M_\pm\Gamma + \mu^2(M_\pm B_\pm + 2|\xi|^2 M_\pm\Gamma) - (\lambda I + |\xi|^2 M_\pm B_\pm + |\xi|^4 M_\pm\Gamma)\Big\} r = 0. \tag{2.2}$$

In this last expression, it will be convenient to set $\kappa := |\xi|^2$ to get

$$\Big\{ - \mu^4 M_\pm\Gamma + \mu^2(M_\pm B_\pm + 2\kappa M_\pm\Gamma) - (\lambda I + \kappa M_\pm B_\pm + \kappa^2 M_\pm\Gamma)\Big\} r = 0. \tag{2.3}$$

At this stage, we introduce a radial variable σ defined so that

$$(\lambda, \kappa) = \sigma(\lambda_0, \kappa_0), \tag{2.4}$$

where $|(\lambda_0, \kappa_0)| = 1$, which allows us to express our asymptotic eigenvalue problem as

$$\Big\{ - \mu^4 M_\pm\Gamma + \mu^2(M_\pm B_\pm + 2\sigma\kappa_0 M_\pm\Gamma) - \sigma(\lambda_0 I + \kappa_0 M_\pm B_\pm + \sigma\kappa_0^2 M_\pm\Gamma)\Big\} r = 0. \tag{2.5}$$

(Our use of this radial variable follows particularly [27, 29].) Following [16] we set the notation

$$\begin{aligned} \sigma(M_\pm B_\pm) &= \{\beta_j^\pm\}_{j=1}^m \\ \sigma(M_\pm\Gamma) &= \{\gamma_j^\pm\}_{j=1}^m, \\ \sigma(\Gamma^{-1}B_\pm) &= \{\nu_j^\pm\}_{j=1}^m, \end{aligned} \tag{2.6}$$

where $\sigma(\cdot)$ denotes the collection of eigenvalues and we choose our ordering so that $j < k$ implies $\beta_j^\pm \le \beta_k^\pm$, $\gamma_j^\pm \le \gamma_k^\pm$, and $\nu_j^\pm \le \nu_k^\pm$. The fact that the eigenvalues for these matrices are all real and positive follows from symmetry and positivity of Γ, $B_\pm$, and $M_\pm$, as discussed in more detail in [16].

As shown in [14] we can express the $\{\mu_j^\pm\}_{j=1}^{4m}$ analytically as functions of the variable

$$s := \sqrt{\sigma} = (|\lambda|^2 + |\xi|^4)^{1/4}, \tag{2.7}$$

and we summarize relations derived in [14] as follows, for $j = 1, 2, \dots, m$:

$$\begin{aligned}
\mu_j^\pm(s) &= -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(s^2) \\
\mu_{m+j}^\pm(s) &= -\sqrt{\frac{\lambda}{\beta_j^\pm} + \kappa + \mathbf{O}(|s|^3)} \\
\mu_{2m+j}^\pm &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm} + \kappa + \mathbf{O}(|s|^3)} \\
\mu_{3m+j}^\pm(s) &= +\sqrt{\nu_j^\pm} + \mathbf{O}(s^2).
\end{aligned} \tag{2.8}$$

(As discussed in [14], these relations are true under additional assumptions on λ and ξ that won't play a direct role here.)

In the following lemma, we collect estimates on solutions to (1.11). First, we define a domain of applicability, which is simply specified to avoid branches that arise in the specifications of $\{\mu_j^\pm\}_{j=1}^{4m}$. For $\kappa \in \mathbb{C}$ (i.e., allowing for complexification of ξ, as discussed in [14]), we must remain away from branches $\lambda/\beta_j^\pm + \kappa \in (-\infty, 0]$, which we denote $b_j^{\kappa,\pm}$. Given any $\kappa \in \mathbb{C}$, we denote the collection of all such branches

$$\mathcal{B}_\kappa = \cup_{j,\pm} b_j^{\kappa,\pm}.$$

Definition 2.1. *For $\epsilon > 0$, we will denote by $\mathcal{S}_\epsilon$ the following set:*

$$\mathcal{S}_\epsilon := \Big\{(\lambda,\kappa) : |s| < \epsilon, \quad \lambda \notin \mathcal{B}_\kappa\Big\}.$$

Lemma 2.1 (From [13])**.** *Under Conditions (C0)-(C3), there exist constants $\epsilon, \eta > 0$ so that the following estimates hold uniformly in $(\lambda,\kappa) \in \mathcal{S}_\epsilon$ on a choice of linearly independent solutions of the eigenvalue problem (1.11):*

(I) For $x_1 \le 0$, $k = 0, 1, 2, 3$, and $j = 1, 2, \dots, m$,

$$\partial_{x_1}^k \phi_j^-(x_1; s) = e^{\mu_{2m+j}^-(s)x_1}\Big((\mu_{2m+j}^-)^k r_{2m+1-j}^- + \mathbf{O}(e^{-\eta|x_1|})\Big); \qquad (slow)$$

$$\partial_{x_1}^k \phi_{m+j}^-(x_1; s) = e^{\mu_{3m+j}^-(s)x_1}\Big((\mu_{3m+j}^-)^k r_{m+1-j}^- + \mathbf{O}(e^{-\eta|x_1|})\Big); \qquad (fast)$$

and

$$\partial_{x_1}^k \psi_j^-(x_1; s) = e^{\mu_j^-(s)x_1}\Big((\mu_j^-)^k r_j^- + \mathbf{O}(e^{-\eta|x_1|})\Big); \qquad (fast)$$

$$\begin{aligned}
\partial_{x_1}^k \psi_{m+j}^-(x_1; s) = {}& \frac{1}{\mu_{m+j}^-}\Big((\mu_{m+j}^-)^k e^{\mu_{m+j}^-(s)x_1} - (-\mu_{m+j}^-)^k e^{-\mu_{m+j}^-(s)x_1}\Big) r_{m+j}^- \\
&+ \mathbf{O}(e^{-\eta|x_1|}). \qquad (slow)
\end{aligned}$$

(II) For $x_1 \geq 0$, $k = 0, 1, 2, 3$, and $j = 1, 2, \ldots, m$,

$$\partial_{x_1}^k \phi_j^+(x_1; s) = e^{\mu_j^+(s)x_1}\Big((\mu_j^+)^k r_j^+ + \mathbf{O}(e^{-\eta|x_1|})\Big); \qquad \textit{(fast)}$$

$$\partial_{x_1}^k \phi_{m+j}^+(x_1; s) = e^{\mu_{m+j}^+(s)x_1}\Big((\mu_{m+j}^+)^k r_{m+j}^+ + \mathbf{O}(e^{-\eta|x_1|})\Big); \qquad \textit{(slow)}$$

and

$$\begin{aligned}\partial_{x_1}^k \psi_j^+(x_1; s) = \frac{1}{\mu_{2m+j}^+}&\Big((\mu_{2m+j}^+)^k e^{\mu_{2m+j}^+(s)x_1} - (-\mu_{2m+j}^+)^k e^{-\mu_{2m+j}^+(s)x_1}\Big) r_{2m+1-j}^+ \\ &+ \mathbf{O}(e^{-\eta|x_1|}); \qquad \textit{(slow)}\\ \partial_{x_1}^k \psi_{m+j}^+(x_1; s) = e^{\mu_{3m+j}^+(s)x_1}&\Big((\mu_{3m+j}^+)^k r_{m+1-j}^+ + \mathbf{O}(e^{-\eta|x_1|})\Big). \qquad \textit{(fast)}\end{aligned}$$

Throughout the statement, we have suppressed dependence on λ_0 and κ_0.

Remark 2.1. *Since $\bar{u}'(x_1)$ decays at exponential rate as $x_1 \to \pm\infty$, it must be the case that $\bar{u}'(x_1)$ is a linear combination of the fast-decaying solutions $\{\phi_{m+j}^-(x_1; 0)\}_{j=1}^m$ and of the fast-decaying solutions $\{\phi_j^+(x_1; 0)\}_{j=1}^m$. Focusing for specificity on the latter, we note that the linear combination will not contain any solutions that decay at a slower rate than $\bar{u}'(x_1)$, We are justified then in letting J^+ denote the index of the slowest decaying solution that appears in the linear combination, or if multiple solutions have the same rate one of these indices. Noting that the faster decaying solutions can be subsumed into the exponential errors in Lemma 2.1, we can write*

$$\bar{u}'(x_1) = \phi_{J^+}^+(x_1; 0),$$

where in the case of multiple solutions with the same decay rate we may have to revise our original (arbitrary) selection of the eigenvector $r_{J^+}^+$. Proceeding similarly for $x_1 < 0$ and appealing to analyticity in σ, we conclude

$$\begin{aligned}\phi_{J^-}^-(x_1; s) &= \bar{u}'(x_1) + \mathbf{O}(s^2 e^{-\eta|x_1|})\\ \phi_{J^+}^+(x_1; s) &= \bar{u}'(x_1) + \mathbf{O}(s^2 e^{-\eta|x_1|}).\end{aligned} \tag{2.9}$$

We now set some convenient notation.

Definition 2.2. *Suppose $\{\phi_j\}_{j=1}^N$ denote N vectors, each of length $M \leq N$, and dependent on a single independent variable, and suppose $N/M = l$, where l is an integer. Then we set the Wronskian notation*

$$W(\phi_1, \phi_2, \ldots, \phi_N) := \det \begin{pmatrix} \phi_1 & \phi_2 & \ldots & \phi_N \\ {\phi_1}' & {\phi_2}' & \ldots & {\phi_N}' \\ \vdots & \vdots & \vdots & \vdots \\ {\phi_1}^{(l-1)} & {\phi_2}^{(l-1)} & \ldots & {\phi_N}^{(l-1)} \end{pmatrix}, \tag{2.10}$$

where $'$ and $^{(l-1)}$ denote usual differentiation with respect to the independent variable.

We will define a Wronskian that might appropriately be regarded as an Evans type function for this problem. This function will depend on s, with λ_0 and κ_0 regarded as parameters. First, we set

$$\mathcal{D}(\lambda,\kappa) := W(\underbrace{\phi_1^+,\dots,\phi_m^+}_{\text{fast}},\overbrace{\phi_{m+1}^+,\dots,\phi_{2m}^+}^{\text{slow}},\overbrace{\phi_1^-,\dots,\phi_m^-}^{\text{slow}},\underbrace{\phi_{m+1}^-,\dots,\phi_{2m}^-}_{\text{fast}}),$$

and then with $s=\sqrt{\sigma}$ set

$$D(s) := \mathcal{D}(\lambda_0\sigma,\kappa_0\sigma), \tag{2.11}$$

where the dependence on λ_0 and κ_0 has been suppressed on the left hand side.

If we take $\kappa = 0$ in $\mathcal{D}$, we obtain precisely the Evans function associated with $\bar{u}(x_1)$ viewed as a solution to the scalar system (1.13). In [16], the authors analyze this function, and following the notation used there we specify it as $D_a(\zeta) = \mathcal{D}(\lambda, 0)$, where $\zeta = \sqrt{\lambda}$. In particular, it is shown in [16] That under the assumptions (H0)-(H4), with $\kappa_0 = 0$ in (H4), we have $D_a^{(k)}(0) = 0$ for $k = 0, 1, \dots, m$, and transversality (as described in the paragraph immediately preceding Theorem 1.1) is determined by the following condition.

Condition ($\mathcal{D}_0$).

$$\frac{d^{m+1}D_a}{d\zeta^{m+1}}(0) \neq 0.$$

Remark 2.2. *As discussed in Remark 3.1 of [14], the lowest (possible) order non-zero derivative of $D(s)$ will be the $(m+1)$st derivative, with two derivatives on exactly one of $\phi_{J^-}^-$ and $\phi_{J^+}^+$ and one derivative on each of $m-1$ slow-decaying solutions. Similarly as in [16] we denote these terms*

$$\frac{2}{(m+1)!}D^{(m+1)}(0) = \sum_{j_1,j_2,\dots,j_{m-1}=1}^{(2m)} \tilde{\mathcal{W}}_{j_1,j_2,\dots,j_{m-1}}, \tag{2.12}$$

where the notation $\sum_{j_1,j_2,\dots,j_{m-1}=1}^{(2m)}$ denotes summation for which j_1 goes from 1 to $m+2$, j_2 goes from j_1+1 to $m+3$, and so on until j_{m-1} goes from $j_{m-2}+1$ to $2m$.

We note that there are precisely $2m$ slow decay modes, $\{\phi_j^-\}_{j=1}^m$ and $\{\phi_j^+\}_{j=m+1}^{2m}$, and so we can refer to them unambiguously with a set of indices running from 1 to $2m$. In this way, the summand $\tilde{\mathcal{W}}_{j_1,j_2,\dots,j_{m-1}}$ refers to the term in $D^{(m+1)}(0)$ for which derivatives appear on the slow modes with indices $j_1, j_2, \dots, j_{m-1}$. For example, for $m = 2$ we only have one index and it ranges from 1 to 4. For $m = 3$ we have two indices, and j_1 ranges from 1 to 5 while j_2 ranges from 2 to 6.

We conclude this remark by noting that we use tildes on $\mathcal{W}$ here to distinguish it from the coefficients in [16], which will also play a role in our analysis, and will be designated as in [16] without tildes.

2.1 The Case $m = 2$

Before reviewing the general case, we focus on the case $m = 2$. For specificity we'll assume $J^- = 3$ and $J^+ = 2$, which is expected in the sense that $\bar{u}'$ will generally be a linear combination of the solutions that decay at exponential rate when $s = 0$, and generically these linear combinations will contain the slowest decaying solutions. First, for the case $n = 1$ it's shown in [16] that

$$\frac{1}{3} D_a'''(0) = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4,$$

and the transversality condition in one space dimension is precisely that this sum be nonzero. Correspondingly, in multiple space dimensions we have

$$\frac{1}{3} D'''(0) = \lambda_0 \Big(\mathcal{W}_1 \sqrt{\lambda_0 + \beta_2^- \kappa_0} + \mathcal{W}_2 \sqrt{\lambda_0 + \beta_1^- \kappa_0} + \mathcal{W}_3 \sqrt{\lambda_0 + \beta_1^+ \kappa_0} + \mathcal{W}_4 \sqrt{\lambda_0 + \beta_2^+ \kappa_0} \Big),$$

which corresponds with

$$\frac{1}{3} D'''(0) s^3 = \lambda \Big(\mathcal{W}_1 \sqrt{\lambda + \beta_2^- |\xi|^2} + \mathcal{W}_2 \sqrt{\lambda + \beta_1^- |\xi|^2} + \mathcal{W}_3 \sqrt{\lambda + \beta_1^+ |\xi|^2} + \mathcal{W}_4 \sqrt{\lambda + \beta_2^+ |\xi|^2} \Big).$$

In this case, the condition $D_a'''(0) \neq 0$ does not provide enough information about $D'''(0)$, and we make the stronger assumption that the $\{\mathcal{W}_j\}_{j=1}^4$ are all non-zero, and all have the same sign. This has been verified for an example case in [16], and we also note that in the framework of [16] each of the $\{\mathcal{W}_j\}_{j=1}^4$ has to be computed individually, so there is no extra work associated with checking this stronger condition. This condition will be stated more precisely in the next subsection.

2.2 The General Case $m \geq 3$

For the general case, it is shown in [13] that

$$\frac{2}{(m+1)!} D^{(m+1)}(0) = \lambda_0 \sum_{j_1, j_2, \dots, j_{m-1}=1}^{(2m)} \mathcal{W}_{j_1, j_2, \dots, j_{m-1}} \sqrt{\lambda_0 + \beta(j_1)\kappa_0} \cdots \sqrt{\lambda_0 + \beta(j_{m-1})\kappa_0},$$

and correspondingly

$$\frac{2}{(m+1)!} D^{(m+1)}(0) s^{m+1} = \lambda \sum_{j_1, j_2, \dots, j_{m-1}=1}^{(2m)} \mathcal{W}_{j_1, j_2, \dots, j_{m-1}} \sqrt{\lambda + \beta(j_1)|\xi|^2} \cdots \sqrt{\lambda + \beta(j_{m-1})|\xi|^2}, \tag{2.13}$$

where $\beta(j_i)$ denotes the value $\beta_j^\pm$ corresponding with $\phi_{j_i}^\pm$. Here, the coefficients $\mathcal{W}_{j_1, j_2, \dots, j_{m-1}}$ are precisely the values from [16] from the relation

$$\frac{2}{(m+1)!} D_a^{(m+1)}(0) = \sum_{j_1, j_2, \dots, j_{m-1}=1}^{(2m)} \mathcal{W}_{j_1, j_2, \dots, j_{m-1}}.$$

Similarly as in the case $m = 2$, we make the following assumption.

Condition $(\mathcal{D}_\xi)$**.** We assume that at least one of the coefficients $\mathcal{W}_{j_1,j_2,\ldots,j_{m-1}}$ in (2.13) is non-zero, denoted $\mathcal{W}_J$, and that the remaining coefficients are either 0 or of the same sign as $\mathcal{W}_J$.

3 The Nonlinearity

In this section, we derive the precise form of our nonlinearity $\mathcal{Q}$. As a start, we rearrange (1.6) and differentiate with respect to t to obtain

$$u_t = \bar{u}'(x_1 - \delta(\tilde{x}, t))(-\delta_t) + v_t, \tag{3.1}$$

and additionally we note

$$\bar{u}'(x_1 - \delta(\tilde{x}, t)) = \bar{u}'(x_1) + \mathbf{O}(e^{-\eta|x_1|}\delta). \tag{3.2}$$

Notice in particular that we can express the $\mathbf{O}(\cdot)$ term as a derivative, allowing us to conclude

$$u_t = v_t - \bar{u}'(x_1)\delta_t + \frac{\partial}{\partial x_1} Q_0(x_1, \delta), \tag{3.3}$$

where

$$|Q_0| \le Ce^{-\eta|x_1|}|\delta\delta_t|. \tag{3.4}$$

I.e.,

$$Q_0 = (\bar{u}(x_1) - \bar{u}(x_1 - \delta(\tilde{x}, t)))\delta_t(\tilde{x}, t),$$

which satisfies (3.4).

Equation (3.3) will become the left-hand side of (1.2), and for the right-hand side we begin by observing that

$$\begin{aligned} \Delta\bar{u}(x_1 - \delta) &= \bar{u}''(x_1 - \delta) + \sum_{k=2}^{n} \Big(\bar{u}''(x_1 - \delta)(\delta_{x_k})^2 - \bar{u}'(x_1 - \delta)\delta_{x_k x_k}\Big) \\ &= \bar{u}''(x_1 - \delta) - \bar{u}'(x_1)\tilde{\Delta}\delta + Q_\Delta, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \tilde{\Delta} &= \sum_{k=2}^{n} \frac{\partial^2}{\partial x_k^2} \\ |Q_\Delta| &\le Ce^{-\eta|x_1|} \sum_{k=2}^{n} (\delta_{x_k}^2 + |\delta\delta_{x_k x_k}|) \\ |\frac{\partial Q_\Delta}{\partial x_1}| &\le Ce^{-\eta|x_1|} \sum_{k=2}^{n} (\delta_{x_k}^2 + |\delta\delta_{x_k x_k}|) \\ |\frac{\partial Q_\Delta}{\partial x_j}| &\le Ce^{-\eta|x_1|} \sum_{k=2}^{n} (\delta_{x_k}\delta_{x_k x_j} + |\delta_{x_j}\delta_{x_k x_k}| + |\delta\delta_{x_k x_k x_j}|), \end{aligned} \tag{3.6}$$

for $j = 2, 3, \ldots, n$. Likewise, we can write

$$D_u F(\bar{u}(x_1 - \delta) + v) = D_u F(\bar{u}(x_1 - \delta)) + D_u^2 F(\bar{u}(x_1))v + Q_F, \tag{3.7}$$

where

$$\begin{aligned} |Q_F| &\le C(|v|^2 + e^{-\eta|x_1|}|\delta v|) \\ |\frac{\partial Q_F}{\partial x_1}| &\le C(|v||v_{x_1}| + e^{-\eta|x_1|}(|v|^2 + |\delta v|)) \\ |\frac{\partial Q_F}{\partial x_j}| &\le C(|v||v_{x_j}| + e^{-\eta|x_1|}(|\delta_{x_j} v| + |\delta v_{x_j}|)), \quad j = 2, 3, \ldots, n. \end{aligned} \tag{3.8}$$

Finally,

$$M(\bar{u}(x_1 - \delta) + v) = M(\bar{u}(x_1)) + Q_M, \tag{3.9}$$

where

$$\begin{aligned} |Q_M| &\le C(|v| + e^{-\eta|x_1|}|\delta|) \\ |\frac{\partial Q_M}{\partial x_1}| &\le C\Big(|v_{x_1}| + e^{-\eta|x_1|}(|v| + |\delta|)\Big) \\ |\frac{\partial Q_M}{\partial x_j}| &\le C\Big(|v_{x_j}| + e^{-\eta|x_1|}(|\delta_{x_j}|)\Big), \quad j = 2, 3, \ldots, n. \end{aligned} \tag{3.10}$$

Upon direct substitution of (3.3), (3.5), (3.7), and (3.9) into (1.2) we find

$$\begin{aligned} &v_t - \bar{u}'(x_1)\delta_t + \frac{\partial}{\partial x_1} Q_0(x_1, \delta) \\ &= \nabla \cdot \Big\{ \Big(\bar{M}(x_1) + Q_M \Big) D_x \Big(- \Gamma \bar{u}''(x_1 - \delta) + \Gamma \bar{u}'(x_1)\tilde{\Delta}\delta \\ &\quad - \Gamma \Delta v - \Gamma Q_\Delta + D_u F(\bar{u}(x_1 - \delta)) + \bar{B}(x_1) v + Q_F \Big) \Big\}. \end{aligned} \tag{3.11}$$

We observe from (1.3) that

$$-\Gamma \bar{u}''(x_1 - \delta) + D_u F(\bar{u}(x_1 - \delta)) = 0,$$

so that

$$\begin{aligned} &v_t - \bar{u}'(x_1)\delta_t + \frac{\partial}{\partial x_1} Q_0(x_1, \delta) \\ &= \nabla \cdot \Big\{ \Big(\bar{M}(x_1) + Q_M \Big) D_x \Big(\Gamma \bar{u}'(x_1)\tilde{\Delta}\delta - \Gamma \Delta v - \Gamma Q_\Delta + \bar{B}(x_1) v + Q_F \Big) \Big\}. \end{aligned} \tag{3.12}$$

In this way, we can express our equation for v as

$$(\partial_t - L)(v - \bar{u}'(x_1)\delta) = \nabla \cdot Q - \frac{\partial Q_0}{\partial x_1}, \tag{3.13}$$

where

$$Q = \bar{M}(x_1)D_x(-\Gamma Q_\Delta + Q_F) + Q_M D_x\Big(\Gamma\bar{u}'(x_1)\tilde{\Delta}\delta - \Gamma\Delta v + \bar{B}(x_1)v - \Gamma Q_\Delta + Q_F\Big). \quad (3.14)$$

We will denote the i^{th} column of Q by Q_i, and likewise we will denote the entry in the j^{th} row of the i^{th} column by Q_{ji}. In our definition of Q, we see that Q_i will consist of terms with an x_i-derivative from the Jacobian operator, and since x_1 has a distinguished role it follows that Q_1 will have a different form from Q_j, $j = 2, 3, \ldots, n$. Finally, in order to incorporate Q_0, we'll let $\mathcal{Q}$ denote the matrix obtained by taking $Q_1 - Q_0$ as the first column, and Q_i, $i = 2, 3, \ldots, n$ for the remaining columns. With this notation, we can express our nolinearity as $\nabla \cdot \mathcal{Q}$, where

$$\nabla \cdot \mathcal{Q} = \sum_{i=1}^{n} \frac{\partial \mathcal{Q}_i}{\partial x_i},$$

and $\{\mathcal{Q}_i\}_{i=1}^n$ denote the columns of $\mathcal{Q}$.

Combining (3.4), (3.6), (3.8), and (3.10) we find

$$\begin{aligned}
|\mathcal{Q}_1| &\le C_1\Big(|v||v_{x_1}| + |v||\Delta v_{x_1}|\Big) \\
&\quad + C_2 e^{-\eta|x_1|}\Big(|\delta\delta_t| + \sum_{j=2}^{n}(\delta_{x_j}^2 + \delta\delta_{x_jx_j}) + |v|^2 + |\delta v| + |\delta\Delta v_{x_1}| + |v\tilde{\Delta}\delta|\Big) \\
|\mathcal{Q}_i| &\le C_1\Big(|v||v_{x_i}| + |v||\Delta v_{x_i}|\Big) \qquad (3.15) \\
&\quad + C_2 e^{-\eta|x_1|}\Big(\sum_{j=2}^{n}(\delta_{x_j}\delta x_j x_i + \delta_{x_i}\delta_{x_jx_j} + \delta\delta_{x_jx_jx_i}) \\
&\quad + |\delta_{x_i}v| + |\delta v_{x_i}| + |\delta\Delta v_{x_i}| + |\delta\tilde{\Delta}\delta_{x_i}| + |v\tilde{\Delta}\delta_{x_i}|\Big),
\end{aligned}$$

for $i = 2, 3, \ldots, n$.

4 Integral Estimates

In this section, we obtain estimates on integrals arising in the linear and nonlinear terms in the equations of (1.19).

4.1 Linear Estimates

Associated with the linear integrals in (1.19), we define

$$v_l(x,t) := \int_{\mathbb{R}^2} \tilde{G}(x,t;y)v_0(y)dy, \quad (4.1)$$

and

$$\delta_l(\tilde{x},t) := -\int_{\mathbb{R}^n} e(\tilde{x},t;y)v_0(y)dy. \tag{4.2}$$

Here, the subscript l is simply notation designating *linear*.

Lemma 4.1. *Let $\tilde{G}(x,t;y)$ and $e(\tilde{x},t;y)$ denote any functions satisfying the estimates stated in Theorem 1.2, let $v_l(x,t)$ and $\delta_l(\tilde{x},t)$ be as defined in (4.1) and (4.2), and suppose*

$$\|v_0(y)\|_{L^1_{\tilde{y}}} \le (1+|y_1|)^{-3/2}.$$

Then there exists a constant $C>0$ sufficiently large so that the following estimates hold:

$$\|v_l(x,t)\|_{L^p_{\tilde{x}}} \le CA_0(x_1,t;p);$$
$$\|\partial_{x_i} v_l(x,t)\|_{L^p_{\tilde{x}}} \le CA_i(x_1,t;p); \quad i=1,2,\dots,n,$$

and for $0\le|\beta|\le 3$,

$$\|\partial^\beta_{\tilde{x}}\delta_l(\tilde{x},t)\|_{L^p_{\tilde{x}}} \le CB_\beta(t;p)$$
$$\|\partial_t\delta_l(\tilde{x},t)\|_{L^p_{\tilde{x}}} \le C\dot{B}(t;p),$$

with $\{A_i\}_{i=0}^n$, $\{B_\beta\}_{|\beta|\le 3}$, and $\dot{B}$ as defined in (1.20) and (1.21).

Remarks on the proof. For the first estimate on $v_l(x,t)$, we begin with the inequality

$$\|v_l(x,t)\|_{L^p_{\tilde{x}}} \le \int_{-\infty}^{+\infty} \sup_{\tilde{y}\in\mathbb{R}^{n-1}} \|\tilde{G}(x,t;y)\|_{L^p_{\tilde{x}}}\|v_0\|_{L^1_{\tilde{y}}} dy_1.$$

At this point, we take advantage of the observation that our estimates on $\|\tilde{G}\|_p$ from Theorem 1.2 differ only by powers of t from the estimates of Theorem 1.1 from [17]. In this way, the proof of Lemma 4.1 in the current analysis follows immediately from the proof of Lemma 5.1 in [17]. □

4.2 Nonlinear Estimates

We now turn to estimates on the nonlinear integrals arising in (1.19),

$$v_n(x,t) = -\int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^n \tilde{G}_{y_j}(x,t-s;y)\mathcal{Q}^l_j(y,s)dyds, \tag{4.3}$$

and

$$\delta_n(x_2,t) = \int_0^t \int_{\mathbb{R}^n} \sum_{j=1}^n e_{y_j}(\tilde{x},t-s;y)\mathcal{Q}^l_j(y,s)dyds, \tag{4.4}$$

where the nonlinearities $\mathcal{Q}^l_j$ are obtained by substituting v_l and δ_l in for v and δ in the expressions for $\mathcal{Q}_j$, $j=1,2,\dots,n$.

We observe at the outset that for $s \in [0, t/2]$ the time decay will be determined by $\tilde{G}$ and e, while for $s \in [t/2, t]$ it will be determined by $\mathcal{Q}^l_j(y,s)$. In light of this, for $s \in [0, t/2]$, we will use the inequality

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^n} \tilde{G}_{y_j}(x, t-s; y) \mathcal{Q}^l_j(y,s) dy \Big\|_{L^p_{\tilde{x}}} \\ &\quad \le \int_{-\infty}^{+\infty} \sup_{\tilde{y} \in \mathbb{R}^{n-1}} \|\tilde{G}_{y_j}(x, t-s; y)\|_{L^p_{\tilde{x}}} \|\mathcal{Q}^l_j(y,s)\|_{L^1_{\tilde{y}}} dy_1, \end{aligned} \tag{4.5}$$

while for $s \in [t/2, t]$ we will use the inequality

$$\begin{aligned} &\Big\| \int_{\mathbb{R}^n} \tilde{G}_{y_j}(x, t-s; y) \mathcal{Q}^l_j(y,s) dy \Big\|_{L^p_{\tilde{x}}} \\ &\le \int_{-\infty}^{+\infty} \sup_{\tilde{y} \in \mathbb{R}^{n-1}} \|\tilde{G}_{y_j}(x, t-s; y)\|^{\frac{1}{p}}_{L^1_{\tilde{x}}} \sup_{\tilde{x} \in \mathbb{R}^{n-1}} \|\tilde{G}_{y_j}(x, t-s; y)\|^{\frac{1}{q}}_{L^1_{\tilde{y}}} \|\mathcal{Q}^l_j(y,s)\|_{L^p_{\tilde{y}}} dy_1. \end{aligned} \tag{4.6}$$

Here, and in the remainder of the analysis, we will often simplify calculations by using sufficiently large constants C, even when more precise constants could be identified (with more work). We will often arrange a series of inequalities for which a new constant will be appropriate at each step, and we'll designate these constants C_1, C_2, etc. Finally, we will recycle this notation, so that the next calculation will begin again with C_1, unrelated to C_1 from the previous calculation.

It's clear from our linear estimates of Lemma 4.1 that for large t, v_l decays much faster than δ_l in t, and likewise derivatives of v_l and δ_l decay at least as fast, respectively, as v_l and δ_l in t. This observation allows us to focus on the terms in $\mathcal{Q}^l$ that will determine the estimates. For $\mathcal{Q}^l_1$ these are

$$|v_l||\partial_{x_1} v_l| + e^{-\eta|x_1|}((\partial_{x_j}\delta_l)^2 + \delta_l \partial^2_{x_j x_j}\delta_l),$$

while for $\mathcal{Q}^l_j$ these are

$$|v_l||\partial_{x_j} v_l| + e^{-\eta|x_1|}((\partial_{x_j}\delta_l)\partial^2_{x_j x_j}\delta_l + \delta_l \partial^3_{x_j x_j x_j}\delta_l).$$

I.e., if we can control these individual terms, we will have control over the full nonlinearity $\mathcal{Q}$.

For $|v_l||\partial_{x_1} v_l|$, we find

$$\begin{aligned} \||v_l||\partial_{x_1} v_l|\|_{L^p_{\tilde{x}}} &\le \|v_l\|_{L^\infty_{\tilde{x}}} \|\partial_{x_1} v_l\|_{L^p_{\tilde{x}}} \le C_1 A_0(x_1, t; \infty) A_1(x_1, t; p) \\ &\le C_2 \Big\{ t^{-\frac{1}{4}} (1+t)^{-\frac{n-1}{3}(2-\frac{1}{p}) - \frac{5}{12}} h_{p,n}(t) + t^{-\frac{1}{4}}(1+t)^{-\frac{n-1}{3} - \frac{n-1}{2}(1-\frac{1}{p}) + \frac{1}{12}} e^{-\eta|x_1|} \\ &+ t^{-\frac{1}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p}) - \frac{1}{12}} h_{p,n}(t) e^{-\eta|x_1|} \Big\} \Theta(x_1, t)^2. \end{aligned}$$

Likewise, for $e^{-\eta|x_1|}(\partial_{x_j}\delta_l)^2$ we find

$$\begin{aligned}\|e^{-\eta|x_1|}(\partial_{x_j}\delta_l)^2\|_{L^p_{\tilde{x}}} &\le e^{-\eta|x_1|}\|\partial_{x_j}\delta_l\|_{L^\infty_{\tilde{x}}}\|\partial_{x_j}\delta_l\|_{L^p_{\tilde{x}}} \le C_1 e^{-\eta|x_1|}B_{\beta_j}(t;\infty)B_{\beta_j}(t;p)\\ &\le C_2(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{2}{3}}h_{p,n}(t)e^{-\eta|x_1|},\end{aligned}$$

where β_j denotes the multiindex with 1 in the j-th position and zeros otherwise. Clearly, $e^{-\eta|x_1|}\delta_l\partial^2_{x_jx_j}\delta_l$ leads to the same estimate, and we see that our preliminary estimate on the nonlinearity $\mathcal{Q}^l_1$ is

$$\|\mathcal{Q}^l_1\|_{L^p_{\tilde{x}}} \le C\Big(t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})+\frac{1}{12}}h_{p,n}(t)\Theta(x_1,t)^2 + t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})+\frac{1}{12}}h_{p,n}(t)e^{-\eta|x_1|}\Big). \tag{4.7}$$

Turning to $\mathcal{Q}^l_j$, $j=2,3,\dots,n$, we observe a slight advantage in the terms involving our shift δ_l. Proceeding otherwise as for $\mathcal{Q}^l_1$ we find

$$\|\mathcal{Q}^l_j\|_{L^p_{\tilde{x}}} \le C\Big(t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{2}(2-\frac{1}{p})+\frac{1}{12}}h_{p,n}(t)\Theta(x_1,t)^2 + t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{1}{4}}h_{p,n}(t)e^{-\eta|x_1|}\Big). \tag{4.8}$$

Lemma 4.2. *Let $\tilde{G}(x,t;y)$ and $e(\tilde{x},t;y)$ denote any functions satisfying the estimates stated in Theorem 1.2, and let $\mathcal{Q}^l_j(x,t)$, $j=1,2,\dots,n$ satisfy the estimates (4.7) and (4.8). Then there exists a constant $C>0$ sufficiently large so that for $1\le p\le\infty$, the following estimates hold:*

$$\begin{aligned}\|v_n(x,t)\|_{L^p_{\tilde{x}}} &\le CA_0(x_1,t;p)\\ \|\partial_{x_j}v_n(x,t)\|_{L^p_{\tilde{x}}} &\le CA_j(x_1,t;p); \quad j=1,2,\dots,n,\end{aligned}$$

and

$$\begin{aligned}\|\partial^\beta_{\tilde{x}}\delta_n(\tilde{x},t)\|_{L^p_{\tilde{x}}} &\le CB_\beta(t;p); \quad |\beta|\le 3\\ \|\partial_t\delta_n(\tilde{x},t)\|_{L^p_{x_2}} &\le C\dot{B}(t;p).\end{aligned}$$

We observe that the key observation of Lemma 4.2 is that the nonlinear integrals $v_n(x,t)$ satisfy precisely the same estimates as the linear integrals.

Proof. In order to establish the estimates of Lemma 4.2, it's useful to observe that

$$\|\mathcal{Q}^l_j\|_{L^p_{x_2}} \le C(\Psi_1(x_1,t;p)+\Psi_2(x_1,t;p)+\Psi_3(x_1,t;p)), \quad j=1,2,\dots,n,$$

where

$$\begin{aligned}\Psi_1(x_1,t;p) &= t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{11}{12}}h_{p,n}(t)e^{-\frac{2x_1^2}{Lt}}\\ \Psi_2(x_1,t;p) &= t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})+\frac{1}{12}}h_{p,n}(t)(1+|x_1|+\sqrt{t})^{-3}\\ \Psi_3(x_1,t;p) &= t^{-\frac{3}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})+\frac{1}{2}}h_{p,n}(t)e^{-\eta|x_1|}.\end{aligned}$$

In fact, it follows from our estimates so far that for $j=2,\dots,n$ the estimates are slightly better,

$$\|\mathcal{Q}^j_1\|_{L^p_{x_2}} \le C(\Psi_1(x_1,t;p)+\Psi_2(x_1,t;p)+(1+t)^{-\frac{1}{4}}\Psi_3(x_1,t;p)); \quad j=2,3,\dots,n,$$

but we won't need to take advantage of this, and the improvement is lost when we incorporate our small-time theory for estimates on higher order derivatives of v.

Beginning with the estimates on δ_n, we observe that similarly as with (4.5) and (4.6) we will use the inequality

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^n} e_{y_j}(\tilde{x}, t-s; y) \mathcal{Q}_j^l(y,s) dy \Big\|_{L^p_{\tilde{x}}} \\ &\quad \le \int_{-\infty}^{+\infty} \sup_{\tilde{y}\in\mathbb{R}^{n-1}} \|e_{y_j}(\tilde{x}, t-s; y)\|_{L^p_{\tilde{x}}} \|\mathcal{Q}_j^l(y,s)\|_{L^1_{\tilde{y}}} dy_1\end{aligned} \tag{4.9}$$

for $s \in [0, t/2]$, while for $s \in [t/2, t]$ we will use the inequality

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^n} e_{y_j}(\tilde{x}, t-s; y) \mathcal{Q}_j^l(y,s) dy \Big\|_{L^p_{\tilde{x}}} \\ &\quad \le \int_{-\infty}^{+\infty} \sup_{\tilde{y}\in\mathbb{R}^{n-1}} \|e_{y_j}(\tilde{x}, t-s; y)\|^{\frac{1}{p}}_{L^1_{\tilde{x}}} \sup_{\tilde{x}\in\mathbb{R}^{n-1}} \|e_{y_j}(\tilde{x}, t-s; y)\|^{\frac{1}{q}}_{L^1_{\tilde{y}}} \|\mathcal{Q}_j^l(y,s)\|_{L^p_{\tilde{y}}} dy_1.\end{aligned} \tag{4.10}$$

For $j = 1$ we have three terms to consider for each of (4.9) and (4.10) (corresponding with Ψ_1, Ψ_2, and Ψ_3). Starting with (4.9)-Ψ_1, we obtain integrals of the form

$$\begin{aligned}\mathcal{I}_1 := \int_0^{\frac{t}{2}} \int_{-\infty}^{+\infty} &(1+(t-s))^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{3}} h_{p,n}(t-s) e^{-\frac{y_1^2}{M(t-s)}} \\ &\times s^{-\frac{1}{4}}(1+s)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(s) e^{-\frac{2y_1^2}{Ls}} dy_1 ds.\end{aligned}$$

Integrating the exponential $e^{-\frac{2y_1^2}{Ls}}$, we see that

$$\begin{aligned}\mathcal{I}_1 &\le C_1 \int_0^{\frac{t}{2}} (1+(t-s))^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{3}} h_{p,n}(t-s) s^{+\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{p,n}(s) ds \\ &\le C_2 (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{3}} h_{p,n}(t) \int_0^{\frac{t}{2}} s^{+\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{p,n}(s) ds \\ &\le C_3 (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{1}{3}} h_{p,n}(t),\end{aligned}$$

where we have observed that the integrand after the first inequality is integrable. In fact, we see that aside from the logarithmic multiplier $h_{p,n}(t)$ we obtain a decay rate better than required by a factor $t^{-1/3}$.

Similarly, using (4.10) we obtain integrals of the form

$$\mathcal{I}_2 := \int_{\frac{t}{2}}^{t} \int_{-\infty}^{+\infty} (1+(t-s))^{-\frac{1}{3}} h_{p,n}(t-s) e^{-\frac{y_1^2}{M(t-s)}} s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(s) e^{-\frac{2y_1^2}{Ls}} dy_1 ds,$$

for which we obtain (now integrating $e^{-\frac{y_1^2}{M(t-s)}}$)

$$\mathcal{I}_2 \leq C_1 \int_{\frac{t}{2}}^{t} (t-s)^{+\frac{1}{2}}(1+(t-s))^{-\frac{1}{3}} h_{p,n}(t-s) s^{-\frac{1}{4}}(1+s)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(s) ds$$
$$\leq C_2 t^{-\frac{1}{4}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(t) \int_{\frac{t}{2}}^{t} (t-s)^{+\frac{1}{2}}(1+(t-s))^{-\frac{1}{3}} h_{p,n}(t-s) ds$$
$$\leq C_3 t^{-\frac{1}{12}}(1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} (h_{p,n}(t))^2,$$

which is much better than required.

The remaining estimates on $\delta_n(\tilde{x},t)$ are established in a nearly identical fashion, and we omit the details.

Turning now to the estimates on $v_n(x,t)$, we note at the outset that we have different estimates on $\tilde{G}(x,t-s;y)$ for different values of x, y, t, and s. For $t-s \leq T$ we have the estimate stated in Part III of Theorem 1.2 (with t replaced by $t-s$) for all x and y, while for $t-s > T$ we have different estimates for the cases $|x-y| > K(t-s)$ and $|x-y| \leq K(t-s)$.

We'll start with the case $t \leq T$, and it will be convenient to fix $T=2$. In this case, we certainly have $t-s \leq T$, and so our estimate is

$$\|\tilde{G}^{III}_{y_j}(x,t-s;y)\|_{L^p_{\tilde{x}}} \leq C(t-s)^{-\frac{1}{2}-\frac{1}{4}(1-\frac{1}{p})} e^{-\frac{(x_1-y_1)^2}{M(t-s)^{1/3}}}.$$

In principle, we need to integrate this estimate against each of $\Psi_1(y_1,s;p)$, $\Psi_2(y_1,s;p)$, and $\Psi_3(y_1,s;p)$, but the calculations are similar for each case, so we carry out the details only for $\Psi_1(y_1,s;p)$. In this case, we can use (4.5) to obtain estimates

$$\mathcal{I}_1 = \int_0^t \int_{-\infty}^{+\infty} (t-s)^{-\frac{1}{2}-\frac{1}{4}(1-\frac{1}{p})} e^{-\frac{(x_1-y_1)^{4/3}}{M(t-s)^{1/3}}} s^{-\frac{1}{4}}(1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{p,n}(s) e^{-\frac{2y_1^2}{Ls}} dy_1 ds.$$

The key observation to make here is that for $t \leq T$, the terms

$$e^{-\frac{(x_1-y_1)^{4/3}}{M(t-s)^{1/3}}}; \quad \text{and} \quad e^{-\frac{2y_1^2}{Ls}}$$

both decay exponentially in some scaling of the space coordinates. If $|y_1| \leq |x_1|/2$ then

$$e^{-\frac{(x_1-y_1)^{4/3}}{M(t-s)^{1/3}}} \leq e^{-\frac{x_1^{4/3}}{2^{4/3}MT^{1/3}}},$$

while for $|y_1| > |x_1|/2$

$$e^{-\frac{2y_1^2}{Ls}} \leq e^{-\frac{x_1^2}{2LT}}.$$

In either case, we obtain exponential decay in $|x_1|$, and for bounded times this is more than sufficient. (I.e., it can be effectively viewed as exponential decay in both x_1 and t, which gives estimates smaller than $A_0(x_1,t;p)$.)

For $t > 2$, we divide our integrals first as

$$\int_0^t \int_{-\infty}^{+\infty} = \int_0^{t-1} \int_{-\infty}^{+\infty} + \int_{t-1}^{t} \int_{-\infty}^{+\infty} =: I_1 + I_2.$$

For I_2 we have $t - s \leq 1$, and so we can again proceed with the estimates of Part III of Theorem 1.2. In this case, we do not get exponential decay in time, but we get sufficient t-decay from the nonlinearities since $s \geq t - 1$.

For I_1, we use the inequality

$$\|\tilde{G}_{y_j}(x, t-s; y)\|_{L^p_{\tilde{x}}} \leq \|\tilde{G}^{II}_{y_j}(x, t-s; y)\|_{L^p_{\tilde{x}}} + \|\tilde{G}^{III}_{y_j}(x, t-s; y)\|_{L^p_{\tilde{x}}}.$$

For the estimates involving $\tilde{G}^{III}_{y_j}$, we again proceed with the estimates from Part III of Theorem 1.2, while for the estimates involving $\tilde{G}^{II}_{y_j}$, we proceed with the estimates from Part II of Theorem 1.2. Focusing on the latter, we can view the analysis as divided into 36 different cases. It's perhaps convenient to organize these cases at four levels:

a) v_n; $\partial_{x_1} v_n$; $\partial_{x_k} v_n$, $k = 2, 3, \ldots, n$;

b) $j = 1$; $j = 2, 3, \ldots, n$;

c) $s \in [0, t/2]$; $s \in [t/2, t-1]$;

d) Ψ_1; Ψ_2; Ψ_3.

We can now refer to cases by an ordered sequence of four numbers. For example Case 1.1.2.2 refers to v_n, $j = 1$, $s \in [t/2, t-1]$, and Ψ_2. The various arguments we use will all be apparent from three cases, 1.1.1.1-2 and 1.1.2.1, so these are the only cases we consider in detail.

Case 1.1.1.1. We begin with Case 1.1.1.1 (i.e., v_n, $j = 1$, $s \in [0, t/2]$, and Ψ_1), for which we use (4.5) to obtain integrals of the form

$$\mathcal{J}_1 = \int_0^{\frac{t}{2}} \int_{-\infty}^{+\infty} (t-s)^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t-s) e^{-\frac{(x_1-y_1)^2}{Mt}} s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{1,n}(s) e^{-\frac{2y_1^2}{Ls}} dy_1 ds. \tag{4.11}$$

In evaluating integrals of this form, we will make use of the following equality from [17]:

$$\int_{-\infty}^{+\infty} e^{-\frac{(x_1-y_1)^2}{M(t-s)}} e^{-\frac{2y_1^2}{Ls}} dy_1 = \sqrt{\frac{(L/2)Ms(t-s)}{(L/2)s + M(t-s)}} e^{-\frac{x_1^2}{(L/2)s+M(t-s)}}, \tag{4.12}$$

which implies the inequality

$$\int_{-\infty}^{+\infty} e^{-\frac{(x_1-y_1)^2}{M(t-s)}} e^{-\frac{2y_1^2}{Ls}} dy_1 \leq C t^{-\frac{1}{2}} (t-s)^{\frac{1}{2}} s^{\frac{1}{2}} e^{-\frac{x_1^2}{Mt}}, \tag{4.13}$$

where we've taken $\frac{L}{2} \leq M$.

Using (4.13) we see that

$$\begin{aligned}
\mathcal{J}_1 &\leq C_1 t^{-\frac{1}{2}} e^{-\frac{x_1^2}{Mt}} \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t-s) s^{+\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{1,n}(s) ds \\
&\leq C_2 t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{x_1^2}{Mt}} \int_0^{\frac{t}{2}} s^{+\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{17}{12}} h_{1,n}(s) ds \\
&\leq C_3 t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\frac{x_1^2}{Mt}},
\end{aligned}$$

using integrability in s.

Case 1.1.1.2. Likewise, for Case 1.1.1.2 we have integrals of the form

$$\begin{aligned}
\mathcal{J}_2 = \int_0^{\frac{t}{2}} \int_{-\infty}^{+\infty} &(t-s)^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t-s) e^{-\frac{(x_1-y_1)^2}{M(t-s)}} \\
&\times s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{5}{12}} h_{1,n}(s) (1+|y_1|+\sqrt{s})^{-3} dy_1 ds.
\end{aligned}$$

In this case, it's convenient to divide the integration over y_1 into two cases, $|x_1 - y_1| \leq \gamma |x_1|$ and $|x_1 - y_1| > \gamma |x_1|$, for some $0 < \gamma < 1$, which will be chosen close to 1. In the case $|x_1 - y_1| > \gamma |x_1|$ we have

$$e^{-\frac{(x_1-y_1)^2}{Mt}} \leq e^{-\gamma^2 \frac{x_1^2}{Mt}} \leq e^{-\frac{x_1^2}{Lt}},$$

where we've taken $L \geq M/\gamma^2$ (so that $M/\gamma^2 \leq L \leq 2M$, which is possible for γ close to 1). On the other hand, if $|x_1 - y_1| \leq \gamma |x_1|$, then we must have $|y_1| \geq (1-\gamma)|x_1|$, in which case

$$(1+|y_1|+\sqrt{s})^{-3} \leq (1+(1-\gamma)|x_1|+\sqrt{s})^{-3}.$$

First, if we denote by $\mathcal{K}_1$ the part of $\mathcal{J}_2$ associated with $|x_1 - y_1| \leq \gamma |x_1|$, we integrate the kernel to find

$$\begin{aligned}
\mathcal{K}_1 &\leq C_1 \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t-s) s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{5}{12}} h_{1,n}(s) (1+|x_1|+\sqrt{s})^{-3} ds \\
&\leq C_2 t^{-\frac{1}{2}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) \int_0^{\frac{t}{2}} s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{5}{12}} h_{1,n}(s) (1+|x_1|+\sqrt{s})^{-3} ds \\
&\leq C_3 t^{-\frac{1}{2}-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) (1+|x_1|)^{-3/2},
\end{aligned}$$

in which we've been able to use $(1+\sqrt{s})^{-3/2}$ in order to get integrability in s (much more than we needed). For $|x_1| \geq \sqrt{t}$ this decay in $|x_1|$ provides decay in $\sqrt{t}$ as well, while for $|x_1| < \sqrt{t}$ we can subsume this into the kernel decay.

Likewise, we denote by $\mathcal{K}_2$ the part of $\mathcal{J}_2$ associated with $|x_1 - y_1| > \gamma|x_1|$, and integrate the algebraic decay to find

$$\begin{aligned}
\mathcal{K}_2 &\le C_1 \int_0^{\frac{t}{2}} (t-s)^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t-s) e^{-\gamma^2 \frac{x_1^2}{Mt}} s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{5}{12}} h_{1,n}(s)(1+\sqrt{s})^{-2} ds \\
&\le C_2 t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\gamma^2 \frac{x_1^2}{Mt}} \int_0^{\frac{t}{2}} s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}-\frac{5}{12}} h_{1,n}(s)(1+\sqrt{s})^{-2} ds \\
&\le C_3 t^{-1-\frac{n-1}{3}(1-\frac{1}{p})} h_{p,n}(t) e^{-\gamma^2 \frac{x_1^2}{Mt}},
\end{aligned}$$

which can be subsumed into the kernel estimate.

Case 1.1.2.1. Since the nonlinearities are generally smaller than the kernels, the claimed estimates are much easier to obtain for $s \in [t/2, t-1]$, and we only consider one case.

For Case 1.1.2.1 we use (4.6) to obtain integrals of the form

$$\begin{aligned}
\mathcal{J}_3 = \int_{\frac{t}{2}}^{t-1} \int_{-\infty}^{+\infty} & (t-s)^{-1} h_{p,n}(t-s) e^{-\frac{(x_1-y_1)^2}{M(t-s)}} \\
& \times s^{-\frac{1}{4}} (1+s)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(s) e^{-\frac{2y_1^2}{Ls}} dy_1 ds,
\end{aligned}$$

and using (4.13) we can estimate these as

$$\begin{aligned}
\mathcal{J}_3 &\le C_1 t^{-\frac{1}{2}} e^{-\frac{x_1^2}{Mt}} \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac{1}{2}} h_{1,n}(t-s) s^{+\frac{1}{4}} (1+s)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(s) ds \\
&\le C_2 t^{-\frac{1}{4}} (1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} h_{p,n}(t) e^{-\frac{x_1^2}{Mt}} \int_{\frac{t}{2}}^{t-1} (t-s)^{-\frac{1}{2}} h_{1,n}(t-s) ds \\
&\le C_3 t^{+\frac{1}{4}} (1+t)^{-\frac{n-1}{3}(2-\frac{1}{p})-\frac{17}{12}} (h_{p,n}(t))^2 e^{-\frac{x_1^2}{Mt}},
\end{aligned}$$

which is much better than required.

5 Nonlinear Iteration

In analyzing the nonlinearities $\mathcal{Q}$, we must keep track of the following quantities: v, $\{\partial_{x_k} v\}_{k=1}^n$, $\{\partial_x^\alpha v\}_{|\alpha|=3}$, δ, δ_t, and $\{D_{\tilde{x}}^\beta \delta\}_{|\beta|\le 3}$. The third order estimates on v will be accommodated by a short time analysis, but the remaining terms will be carried through the nonlinear iteration.

The goal is simply to show that we (at least) recover the linear estimates, and one relatively straightfoward way to think about this process is in terms of ratios such as

$$\frac{\|v(x,t)\|_{L^p_{\tilde{x}}}}{A_0(x_1,t;p)}.$$

We will show that such ratios are bounded, and Theorem 1.3 will be an immediately consequence.

We define

$$\zeta(t) := \sup_{\substack{1\le p\le\infty \\ (y_1,s)\in\mathbb{R}\times[0,t]}} \Big\{ \frac{\|v(y,s)\|_{L^p_{\tilde y}}}{A_0(y_1,s)} + \sum_{k=1}^n \frac{\|\partial_{y_k} v(y,s)\|_{L^p_{\tilde y}}}{A_k(y_1,s)} + \sum_{|\beta|\le 3} \frac{\|\partial^\beta_{\tilde y}\delta(\tilde y,s)\|_{L^p_{\tilde y}}}{B_\beta(s)} + \frac{\|\partial_s \delta(\tilde y,s)\|_{L^p_{\tilde y}}}{\dot B(s)} \Big\}. \tag{5.1}$$

5.1 Short-time Theory for the Solution

For our short-time theory, we verify that equation (1.1) satisfies the uniform parabolicity described in [13], and then apply the results of that reference. To begin, we observe that (1.1) can be expressed as

$$\frac{\partial u_j}{\partial t} = \sum_{l=1}^n \frac{\partial}{\partial x_l}\Big\{ \sum_{k=1}^m M_{jk}(u)\Big(-\sum_{i=1}^m \Gamma_{ki}\Delta\partial_{x_l} u_i + D_u F_{u_k}(u)\partial_{x_l} u\Big)\Big\},$$

from which we see that the highest order term on the right-hand side (which determines parabolicity) can be expressed as

$$\begin{aligned} &\sum_{l=1}^n \frac{\partial}{\partial x_l}\Big\{ \sum_{i=1}^m (-M(u)\Gamma)_{ji} \sum_{q=1}^n \partial^3_{x_q x_q x_l} u_i \Big\} \\ &\quad = \sum_{l=1}^n \frac{\partial}{\partial x_l}\Big\{ \sum_{i=1}^m \sum_{|\alpha|=3} -A^{ji}_{\alpha,l} D^\alpha u_i \Big\}, \end{aligned}$$

where $A_{\alpha l}$ is either the matrix $M(u)\Gamma$ or 0 depending on the values of α and l. Precisely, if $l=1$, it is the matrix $M(u)\Gamma$ for $\alpha=(3,0,\dots,0)$, $(1,2,0,\dots,0)$, $\dots$, $(1,0,\dots,0,2)$, while for $l=2$ it is the matrix $M(u)\Gamma$ for $\alpha=(2,1,0,\dots,0)$, $(0,3,0,\dots,0)$, $\dots$, $(0,1,\dots,0,2)$, and similarly for $l=3,\dots,n$. For uniform parabolicity as defined in [12] (following [5], p. 239), we compute

$$\sum_{l=1}^n \sum_{|\alpha|=3} -A_{\alpha,l}(i\xi)^\alpha i\xi_l = -M(u)\Gamma \sum_{l=1}^n \sum_{q=1}^n \xi_q^2\xi_l^2 = -|\xi|^4 M(u)\Gamma.$$

Uniform parabolicity is determined by evaluation at $|\xi|=1$; in particular, our equation is uniformly parabolic if all eigenvalues of the resulting matrix have negative real part. Since Γ is positive definite, and $M(u)$ is uniformly positive definite, that is the case here.

We conclude from Theorem 5.1 in [12] that for any $\tau\ge 0$, if $u(\cdot,\tau)\in C^\gamma(\mathbb{R}^n)$ for some $0<\gamma<1$ (i.e., Hölder continuity) then on some sufficiently small interval $[\tau,\tilde T]$ we have

$$u \in C^{\gamma,\frac{\gamma}{4}}(\mathbb{R}^n\times[\tau,\tilde T]) \cap C^{4+\gamma,1+\frac{\gamma}{4}}(\mathbb{R}^n\times[\sigma,\tilde T]),$$

for any $\sigma \in (\tau, T)$. Moreover, there exists a Green's function $\mathcal{G}(x,t;\xi,\tau)$ so that

$$u(x,t) = \int_{\mathbb{R}^n} \mathcal{G}(x,t;\xi,\tau)u(\xi,\tau)d\xi,$$

where for any multiindex $|\alpha| \leq 3$ there exist constants c and C so that

$$|D_x^\alpha \mathcal{G}(x,t;\xi,\tau)| \leq C(t-\tau)^{-\frac{n+|\alpha|}{4}} e^{-c\frac{|x-\xi|^{4/3}}{(t-\tau)^{1/3}}}. \tag{5.2}$$

for $t \in [\tau, \tilde{T}]$.

In the following calculations, we'll use two straightforward lemmas that are stated here without proof.

Lemma 5.1. *Let $\gamma > -1$, $\alpha, m > 0$. Then*

$$\int_0^{+\infty} \tau^\gamma e^{-\alpha\tau^m} d\tau = \frac{1}{m}\alpha^{-\frac{1+\gamma}{m}}\Gamma(\frac{\gamma+1}{m}).$$

Lemma 5.2. *Let $\eta, \alpha, m > 0$. Then for any $\epsilon > 0$*

$$\tau^\eta e^{-\alpha\tau^m} \leq C\alpha^{-\eta/m} e^{-(\alpha-\epsilon)\tau^m},$$

where C depends on η, m, and ϵ, but not on α.

Fixing $\tau \geq 0$ and $t \in [\tau, \tilde{T}]$, we observe that

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \mathcal{G}(x,t;\xi,\tau)u(\xi,\tau)d\xi \\ &= \int_{\mathbb{R}^n} \mathcal{G}(x,t;\xi,\tau)u(x,\tau)d\xi + \int_{\mathbb{R}^n} \mathcal{G}(x,t;\xi,\tau)(u(\xi,\tau)-u(x,\tau))d\xi \\ &=: I_1 + I_2. \end{aligned}$$

As noted in [12], we have the identity

$$\int_{\mathbb{R}^n} \mathcal{G}(x,t;\xi,\tau)d\xi = I, \tag{5.3}$$

and it follows that $I_1 = u(x,\tau)$. For I_2, we have

$$\begin{aligned} |I_2| &\leq C_1 \int_{\mathbb{R}^n} |\xi - x|^\gamma (t-\tau)^{-\frac{n}{4}} e^{-c\frac{|x-\xi|^{4/3}}{(t-\tau)^{1/3}}} d\xi \\ &\leq C_2 (t-\tau)^{\frac{\gamma-n}{4}} \int_{\mathbb{R}^n} e^{-\tilde{c}\frac{|x-\xi|^{4/3}}{(t-\tau)^{1/3}}} d\xi \leq C_3 (t-\tau)^{\frac{\gamma}{4}}. \end{aligned}$$

We see that

$$u(x,t) = u(x,\tau) + \mathbf{O}((t-\tau)^{\frac{\gamma}{4}}).$$

Likewise,

$$\begin{aligned} D_x^\alpha u(x,t) &= \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}(x,t;\xi,\tau)u(\xi,\tau)d\xi \\ &= \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}(x,t;\xi,\tau)u(x,\tau)d\xi + \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}(x,t;\xi,\tau)(u(\xi,\tau) - u(x,\tau))d\xi \\ &=: J_1 + J_2. \end{aligned}$$

We observe from (5.3) that $J_1 = 0$, and proceeding similarly as in the previous calculation, we find

$$|D_x^\alpha u(x,t)| \le C(t-\tau)^{\frac{\gamma - |\alpha|}{4}}; \quad 1 \le |\alpha| \le 3.$$

5.2 Short-time Theory for the Shift

We now have a solid understanding of the short-time behavior of $u(x,t)$. Recalling that

$$u(x,t) = \bar{u}(x_1 - \delta(\tilde{x},t)) + v(x,t), \tag{5.4}$$

we see that if we additionally obtain information about $\delta(\tilde{x},t)$ we can make conclusions about $v(x,t)$ as well. Following [17] we proceed as follows: we carry out an iteration argument in δ in an appropriate function space, and for estimates involving v we use our estimates on u, the function space for δ, and (5.4).

As a start, we observe that for any $0 \le \tau \le t$ we can write

$$\begin{aligned} \delta(\tilde{x},t) = \delta(\tilde{x},\tau) &- \int_{\mathbb{R}^n} \Big(e(\tilde{x},t;y) - e(\tilde{x},\tau;y)\Big) v_0(y)dy \\ &+ \int_0^\tau \int_{\mathbb{R}^n} \sum_{j=1}^n \Big(e_{y_j}(\tilde{x},t-s;y) - e_{y_j}(\tilde{x},\tau-s;y)\Big) \mathcal{Q}_j(y,s)dyds \\ &+ \int_\tau^t \int_{\mathbb{R}^n} \sum_{j=1}^n e_{y_j}(\tilde{x},t-s;y)\mathcal{Q}_j(y,s)dyds. \end{aligned}$$

Fix any $\tau \ge 0$, and suppose $\delta(\tilde{x},s)$ and $v(x,s)$ exist up to time $s = \tau$ with $\delta \in C^{3,1}(\mathbb{R}^{n-1} \times [0,\tau])$ and v satisfying the estimates

$$|D_x^\alpha v(x,t)| \le C_1(t-\tau)^{\frac{\gamma-|\alpha|}{4}} + C_2, \tag{5.5}$$

for $0 \le |\alpha| \le 3$. For constants C and T, and define the space of functions

$$\mathcal{S} = \Big\{\delta \in C^{3,1}(\mathbb{R}^{n-1} \times [\tau,\tau+T]) : \delta(\tilde{x},\tau) = \delta^\tau(\tilde{x}), \|\delta\|_{3,1} \le C\Big\},$$

where $\|\delta\|_{3,1}$ denotes the usual $C^{3,1}$ norm.

For the purpose of an iteration, we define the map

$$\begin{aligned}\mathcal{T}\delta := \delta(\tilde{x},\tau) &- \int_{\mathbb{R}^n}\Big(e(\tilde{x},t;y)-e(\tilde{x},\tau;y)\Big)v_0(y)dy\\&+\int_0^\tau\int_{\mathbb{R}^n}\sum_{j=1}^n\Big(e_{y_j}(\tilde{x},t-s;y)-e_{y_j}(\tilde{x},\tau-s;y)\Big)\mathcal{Q}_j(y,s)dyds\\&+\int_\tau^t\int_{\mathbb{R}^n}\sum_{j=1}^n e_{y_j}(\tilde{x},t-s;y)\mathcal{Q}_j(y,s)dyds.\end{aligned}$$

We will show that $\mathcal{T}$ is a contraction on $\mathcal{S}$.

It's important to be clear that when $\delta\in\mathcal{S}$ the nonlinear terms $\mathcal{Q}_j$ can be characterized by our short-time theory for u and relation (5.4). That is, we can write

$$\begin{aligned}v(x,t)&=u(x,t)-\bar{u}(x_1-\delta(\tilde{x},t))\\v_{x_1}(x,t)&=u_{x_1}(x,t)-\bar{u}'(x_1-\delta(\tilde{x},t))\\v_{x_j}(x,t)&=u_{x_j}(x,t)-\bar{u}'(x_1-\delta(\tilde{x},t))(-\delta_{x_j});\quad j=2,3,\dots,n,\end{aligned}$$

and similarly for higher order derivatives. We see that v inherits the continuity from u and δ, and that

$$|D_x^\alpha v(x,t)|\le C_1(t-\tau)^{\frac{\gamma-|\alpha|}{4}}+C_2, \tag{5.6}$$

for $0\le|\alpha|\le 3$. Since the terms in Q_j that blow up fastest as s goes to 0 are those associated with third order derivatives on v, we conclude that

$$|Q_j(y,s)|\le K_1 s^{\frac{\gamma-3}{4}}+K_2.$$

In order to check that $\mathcal{T}$ is invariant on $\mathcal{S}$, we note that clearly $\mathcal{T}\delta(\tilde{x},\tau)=\delta^\tau(\tilde{x})$, leaving for verification the condition $\|\mathcal{T}\delta\|_{3,1}\le C$. In order to indicate how we check this condition, we note that we have the inequality

$$\begin{aligned}|\mathcal{T}\delta|&\le|\delta(\tilde{x},\tau)|+\|e(\tilde{x},t;y)-e(\tilde{x},\tau;y)\|_{L^1_y}\|v_0(y)\|_{L^\infty_y}\\&+\int_0^\tau\sum_{j=1}^n\|e_{y_j}(\tilde{x},t-s;y)-e_{y_j}(\tilde{x},\tau-s;y)\|_{L^1_y}\|\mathcal{Q}_j(y,s)\|_{L^\infty_y}ds\\&+\int_\tau^t\sum_{j=1}^n\|e_{y_j}(\tilde{x},t-s;y)\|_{L^1_y}\|\mathcal{Q}_j(y,s)\|_{L^\infty_y}ds\\&=:|\delta(\tilde{x},\tau)|+I_1+I_2+I_3.\end{aligned}$$

For I_1, we observe that

$$e(\tilde{x},t;y)=e(\tilde{x},\tau;y)+e_t(\tilde{x},\tau^*;y)(t-\tau),$$

for some $\tau^* \in (\tau, t)$. In this way, we see that

$$\|e(\tilde{x}, t; y) - e(\tilde{x}, \tau; y)\|_{L^1_y} \leq C_1(1+\tau)^{-1-\frac{1}{3}} \int_{-\infty}^{+\infty} e^{-\frac{y_1^2}{Mt}} dy_1 \leq C_2(1+\tau)^{-1-\frac{1}{3}}\sqrt{t}.$$

We deduce,

$$|I_1| \leq C_2(1+\tau)^{-1-\frac{1}{3}}\sqrt{t}(t-\tau)\|v_0\|_{L^\infty_y}.$$

Likewise, for I_2,

$$\begin{aligned}
&\int_0^\tau \|e_{y_j}(\tilde{x}, t-s; y) - e_{y_j}(\tilde{x}, \tau - s; y)\|_{L^1_y} \|Q_j(y,s)\|_{L^\infty_y} ds \\
&\quad \leq C_1(t-\tau)\int_0^\tau (1+(\tau-s))^{-\frac{4}{3}-\frac{1}{3}} \int_{-\infty}^{+\infty} e^{-\frac{y_1^2}{M(t-s)}} dy_1 (K_1 s^{\frac{\gamma-3}{4}} + K_2) ds \\
&\quad \leq C_2(t-\tau),
\end{aligned}$$

with the main point being integrability in s despite the blow-up as $s \to 0$. The remaining integral I_3 can be analyzed in almost precisely the same way as I_2, and by choosing $t-\tau$ small, we can ensure $|\mathcal{T}\delta(\tilde{x}, t)|$ is as small as we like.

Proceeding similarly, we verify that for $t - \tau$ sufficently small, $\mathcal{T}\delta \in \mathcal{S}$.

Next, we check that $\mathcal{T}$ is a contraction on $\mathcal{S}$. For this calculation, we'll take $\delta_1, \delta_2 \in \mathcal{S}$, and we'll let $\mathcal{Q}_j^{\delta_i}$ denote the nonlinearity associated with δ_i. In particular, we take $u(x,t)$ fixed from Section 5.1, so that $v(x,t)$ is determined from δ via (5.4). In this way, δ_i determines perturbation $\mathcal{Q}_j^{\delta_i}$. Also, $\mathcal{Q}_j^{\delta_1}$ and $\mathcal{Q}_j^{\delta_2}$ will coincide for $s \leq \tau$, so we have

$$\mathcal{T}\delta_1 - \mathcal{T}\delta_2 = \int_\tau^t \int_{\mathbb{R}^n} \sum_{j=1}^n e_{y_j}(\tilde{x}, t-s, y)(\mathcal{Q}_j^{\delta_1} - \mathcal{Q}_j^{\delta_2}) dy ds. \tag{5.7}$$

If we let v^{δ_i} denote the perturbation associated with δ_i, then the perturbation $\mathcal{Q}_j^{\delta_i}$ can be expressed as a polynomial in δ_i, v^{δ_i} and derivatives of these quantities, with coeffecients depending on x_1. In this way, we can express the difference $\mathcal{Q}_j^{\delta_1} - \mathcal{Q}_j^{\delta_2}$ in terms of $\delta_1 - \delta_2$, $v^{\delta_1} - v^{\delta_2}$, and derivatives of these quantities up to third order in space, and including δ_t. In order to express the differences $v^{\delta_1} - v^{\delta_2}$ in terms of $\delta_1 - \delta_2$, we use (5.4) to write

$$v^{\delta_1}(x,t) - v^{\delta_2}(x,t) = \bar{u}(x_1 - \delta_2(\tilde{x}, t)) - \bar{u}(x_1 - \delta_1(\tilde{x}, t)) = \bar{u}'(\zeta)(\delta_1 - \delta_2),$$

for some ζ between $x_1 - \delta_2(\tilde{x}, t)$ and $x_1 - \delta_1(\tilde{x}, t)$. Since $\bar{u}'$ is bounded,

$$|v^{\delta_1} - v^{\delta_2}| \leq C|\delta_1 - \delta_2|,$$

for

$$C = \sup_{x_1 \in \mathbb{R}} |\bar{u}'(x_1)|.$$

Proceeding similarly for derivatives of v, we find that for $s \in [\tau, \tau + T]$

$$\|\mathcal{Q}_j^{\delta_1} - \mathcal{Q}_j^{\delta_2}\|_{L_y^\infty(\mathbb{R}^n)} \leq C\|\delta_1 - \delta_2\|_{3,1}(\tilde{K}_1(s-\tau)^{\frac{\gamma-3}{4}} + \tilde{K}_2). \tag{5.8}$$

Recalling that

$$\begin{aligned}\|e_{y_j}(\tilde{x}, t-s; y)\|_{L_y^1} &\leq C_1 \int_{-\infty}^{+\infty} (1 + (t-s))^{-\frac{1}{3}} h_{1,n}(t-s) e^{-\frac{y_1^2}{M(t-s)}} dy_1 \\ &\leq C_2(1 + (t-s))^{-\frac{1}{3}} h_{1,n}(t-s)(t-s)^{\frac{1}{2}},\end{aligned}$$

we set

$$\mathcal{I}_j := \int_\tau^t \int_{\mathbb{R}^n} e_{y_j}(\tilde{x}, t-s, y)(\mathcal{Q}_j^{\delta_1} - \mathcal{Q}_j^{\delta_2}) dy ds$$

and compute

$$\begin{aligned}|\mathcal{I}_j| &\leq \int_\tau^t \|e_{y_j}(\tilde{x}, t-s, y)\|_{L_y^1} \|\mathcal{Q}_j^{\delta_1} - \mathcal{Q}_j^{\delta_2}\|_{L_y^\infty} ds \\ &\leq C_1 \|\delta_1 - \delta_2\|_{3,1} \int_\tau^t (1 + (t-s))^{-\frac{1}{3}} h_{1,n}(t-s)(t-s)^{\frac{1}{2}} (\tilde{K}_1(s-\tau)^{\frac{\gamma-3}{4}} + \tilde{K}_2) ds \\ &\leq C_2(t-\tau)^{3/4} \|\delta_1 - \delta_2\|_{3,1}.\end{aligned}$$

We see that the multiplier $C_2(t-\tau)^{3/4}$ can be made arbitrarily small by choosing $t-\tau$ small. Proceeding similarly for derivatives, we find

$$\|\mathcal{T}\delta_1 - \mathcal{T}\delta_2\|_{3,1} \leq C(t-\tau)^{3/4}\|\delta_1 - \delta_2\|_{3,1},$$

for some constant C, and so for $t - \tau$ sufficiently small $\mathcal{T}$ is a contraction.

We conclude that if $\delta(\tilde{x}, s)$ exists up to time $s = \tau$ with $\delta \in C^{3,1}(\mathbb{R}^{n-1} \times [0, \tau])$, then for $t \in [\tau, \tau + T]$ we can extend δ as a function in $\mathcal{S}$.

5.3 Short-time Theory for the perturbation

Ultimately, our short-time theory has been developed so that we can avoid carrying third order derivatives of v through the iteration. In this section, we establish estimates on $D^\alpha v(x, t)$ for short times ($t \leq 1$). We note that for these values of t we already have (5.6), and the goal here is to understand decay in $\|D^\alpha v(x, t)\|_{L_{\tilde{x}}^p}$ in x_1.

For $t \leq 1$ it's useful to rearrange our perturbation equation (1.7) as

$$v_t - \mathcal{L}v = \mathcal{N}(v), \tag{5.9}$$

where

$$\begin{aligned}\mathcal{L}v &:= \nabla \cdot \Big\{ M(v + \bar{u})\Big(- D_x \Gamma \Delta v + D_u^2 F(v + \bar{u}) \mathcal{H}(x_1, t, v) v \Big) \Big\} \\ \mathcal{N}(v) &:= \bar{u}'(x_1 - \delta)\delta_t - \nabla \cdot \Big\{ M(v + \bar{u}) D_x(\Gamma \tilde{\Delta} \bar{u}) \Big\}, \\ \mathcal{H}(x_1, t, v)v &:= \Big(D^2 F(v + \bar{u}) - D^2 F(\bar{u}) \Big) D_x \bar{u},\end{aligned}$$

and in all instances $\bar{u}$ is evaluated at $x_1 - \delta(\tilde{x}, t)$. In particular, we can check that every term in $\mathcal{N}$ decays at exponential rate in x_1 and includes some combination of derivatives of δ. In order to see this we note that every term involves either δ_t or $\tilde{\Delta}\bar{u}(x_1 - \delta(\tilde{x}, t))$, and we recall

$$\tilde{\Delta}\bar{u}(x_1 - \delta(\tilde{x}, t)) = \sum_{k=2}^{n} \bar{u}'(x_1 - \delta)\delta_{x_k}^2 - \bar{u}'(x_1 - \delta)\delta_{x_k x_k}.$$

Using the fact that $\delta \in \mathcal{S}$ and v is Lipschitz-Hölder continuous we can proceed similarly as in Section 5.1 and solve (5.9) with a local Green's function

$$v(x,t) = \int_{\mathbb{R}^n} \mathcal{G}^v(x,t;\xi,0)v(\xi,0)d\xi + \int_0^t \int_{\mathbb{R}^n} \mathcal{G}^v(x,t;\xi,\tau)\Big(\mathcal{N}(\xi,\tau) + \bar{u}'(y_1 - \delta)\delta_\tau\Big)dyd\tau,$$

where $\mathcal{G}^v$ satisfies the estimates (5.2), and the superscript is intended to clarify that difference between this Green's function and the Green's function from Section 5.1. As with our analysis for u, we are justified in bringing derivatives under the integral sign, and we have

$$\begin{aligned} D^\alpha v(x,t) &= \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}^v(x,t;\xi,0)v(\xi,0)d\xi \\ &+ \int_0^t \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}^v(x,t;\xi,\tau)\Big(\mathcal{N}(\xi,\tau) + \bar{u}'(y_1 - \delta)\delta_\tau\Big)dyd\tau \\ &=: I_1 + I_2. \end{aligned}$$

For I_1, we have

$$\|v(\xi,0)\|_{L^p_{\tilde{x}}} \le \zeta(t)A_0(\xi_1,0) = \zeta(t)(1+|\xi_1|)^{-\frac{3}{2}}.$$

Adapting (4.6) we obtain an estimate by

$$\begin{aligned} \|I_1\|_{L^p_{\tilde{x}}} &\le C_1 \int_{-\infty}^{+\infty} t^{-\frac{1+|\alpha|}{4}} e^{-\frac{(x_1-\xi_1)^{4/3}}{Mt^{1/3}}} \zeta(t)(1+|\xi_1|)^{-\frac{3}{2}} d\xi_1 \\ &\le C_2\zeta(t)t^{-\frac{|\alpha|}{4}}(1+|x_1|)^{-\frac{3}{2}}. \end{aligned}$$

Likewise, for I_2 the key point about the nonlinearity $\mathcal{N}$ is that it decays at exponential rate in x_1. Since each term can be analyzed similarly, we consider $\bar{u}'(x_1 - \delta)\delta_t$, which can be expressed as

$$\bar{u}'(x_1 - \delta)\delta_t = \bar{u}'(x_1)\delta_t + \mathbf{O}(\delta e^{-\eta|x_1|})\delta_t.$$

Using our inequality

$$\|\delta_t\|_{L^p_{\tilde{x}}} \le \zeta(t)\dot{B}(t;p),$$

we see that

$$\|\bar{u}'(x_1 - \delta)\delta_t\|_{L^p_{\tilde{x}}} \le C\zeta(t)(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-1} h_{p,n}(t) e^{-\eta|x_1|}.$$

Adapting (4.6), we find

$$\begin{aligned}\|I_2\|_{L^p_{\tilde{x}}} &\le C_1 \int_0^t \int_{-\infty}^{+\infty} (t-\tau)^{-\frac{1+|\alpha|}{4}} e^{-\frac{(x_1-\xi_1)^{4/3}}{M(t-\tau)^{1/3}}} \zeta(s)(1+s)^{-\frac{n-1}{3}(1-\frac{1}{p})-1} h_{p,n}(s) e^{-\eta|\xi_1|} d\xi d\tau \\ &\le C_2 t^{1/4} e^{-\tilde{\eta}|x_1|},\end{aligned}$$

for some constant $\tilde{\eta}$.

5.4 Large time estimates for Derivatives of the Perturbation

In this section, we link derivatives $D^\alpha v$, for $|\alpha| = 3$ to first order derivatives of v. The main issue here is that if we estimate $D^\alpha v$ in terms of v (as we did in the short-time theory), we only obtain decay in t at the rate that v decays, which is not sufficient. (We recall that we expect derivatives to decay at a faster rate in t.) In light of this, we will estimate third order derivatives of v in terms of first order derivatives of v. In order to accomplish this, we begin by differentiating our perturbation equation (1.7) with respect to x_j. In component form, we obtain

$$\begin{aligned}&\partial_{x_j}(-\bar{u}_i'(x_1-\delta)\delta_t) + (\partial_{x_j} v_i)_t \\ &\quad = \nabla \cdot \Big\{ \sum_{k=1}^m DM_{ik}(\bar{u}+v)(\partial_{x_j}\bar{u}(x_1-\delta) + \partial_{x_j} v)\nabla\Big(-(\Gamma\tilde{\Delta}\bar{u}(x_1-\delta))_k - (\Gamma\Delta v)_k + \mathcal{E}_k\Big)\Big\} \\ &\quad + \nabla \cdot \Big\{ \sum_{k=1}^m M_{ik}(\bar{u}+v)\nabla\Big(-(\Gamma\tilde{\Delta}(\partial_{x_j}\bar{u}(x_1-\delta)))_k - (\Gamma\Delta v_{x_j})_k + \partial_{x_j}\mathcal{E}_k\Big)\Big\},\end{aligned} \tag{5.10}$$

where

$$\mathcal{E}_k = \mathcal{A}_k(x,t)v; \quad \mathcal{A}_k = \int_0^1 DuF_{u_k}(\bar{u}+\gamma v)d\gamma.$$

For this analysis, it's important to keep in mind how we designate inhomogeneous terms. From our previous considerations, we already understand short-time existence and qualitative behavior of v and its derivatives, and so these quantities can be incorporated as coefficients. For example, a term of the form $v_i\partial_{x_j}v_i$ would not be considered part of the inhomogeneity, because v_i serves as a coefficient for $\partial_{x_j}v_i$. On the other hand, $\partial_{x_j}(-\bar{u}_i'(x_1-\delta)\delta_t)$ certainly constituates part of the inhomogeneity. We see that this term decays at exponential rate in $|x_1|$, and indeed (as in Section 5.3) this will be the case for all inhomogeneous terms. Moreover, in light of (5.1) we see that

$$\|\partial_{x_1}(-\bar{u}_i'(x_1-\delta)\delta_t\|_{L^p_{\tilde{x}}} \le C\zeta(t)(1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-1} h_{p,n(t)} e^{-\eta|x_1|},$$

with better decay for $j = 2, 3, \ldots, n$.

We note particularly that

$$\mathcal{E}_k = \Big(\int_0^1 D_u F_{u_k}(\bar{u}(x_1-\delta)+\gamma v)d\gamma\Big)v,$$

so that

$$\partial_{x_1}\mathcal{E}_k = \int_0^1 D_u^2 F_{u_k}(\bar{u}(x_1-\delta)+\gamma v)(\bar{u}'(x_1-\delta)+\gamma v_{x_1})d\gamma v + \mathcal{A}_k(x,t)v_{x_1}.$$

We notice that the term $\mathcal{A}_k(x,t)v_{x_1}$ should not be incorporated into the inhomogeneity, while the first term should. In addition, we see that the first term decays at exponential rate in x_1, and we have

$$\|\mathcal{A}_k(x,t)v\|_{L^p_{\tilde{x}}} \le C\zeta(t)A_0(x_1,t;p)e^{-\eta|x_1|}. \tag{5.11}$$

Proceeding similarly for the remaining terms in (5.10) we find that (5.11) is the determining estimate.

We can now form a vector W of length mn whose components are the derivatives $\partial_{x_j}v_i$, and we can express this vector in terms of an appropriate Green's function $\mathcal{G}^w$ as

$$W(x,t) = \int_{\mathbb{R}^n} \mathcal{G}^w(x,t;y,\tau)W(y,\tau)dy + \int_\tau^t \int_{\mathbb{R}^n} \mathcal{G}^w(x,t;y,s)\mathcal{N}(y,s)dyds,$$

where

$$\|\mathcal{N}(y,s)\|_{L^p_{\tilde{y}}} \le C\zeta(t)(1+s)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{2}{3}}h_{p,n}(s)e^{-\eta|y_1|}.$$

In order to estimate third derivatives of v in terms of first derivatives of v, we compute second derivatives of W,

$$\begin{aligned} D_x^\alpha W(x,t) &= \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}^w(x,t;y,\tau)W(y,\tau)dy + \int_\tau^t \int_{\mathbb{R}^n} D_x^\alpha \mathcal{G}^w(x,t;y,s)\mathcal{N}(y,s)d\xi ds \\ &= I_1 + I_2, \end{aligned}$$

for $|\alpha| = 2$. For the linear term, we employ (4.6) to see that

$$\|I_1\|_{L^p_{\tilde{x}}} \le \int_{-\infty}^{+\infty} (t-\tau)^{-\frac{3}{4}} e^{-c\frac{(x_1-\xi_1)^{4/3}}{(t-\tau)^{1/3}}} \zeta(t)A_1(\xi_1,\tau)dy_1 d\tau.$$

Since $t-\tau$ is small we have exponential decay in $|x_1-\xi_1|$, and using this we find

$$\|I_1\|_{L^p_{\tilde{x}}} \le C\zeta(t)(t-\tau)^{1/2}A_1(x_1,\tau).$$

We note in particular that since derivatives with respect to all $\{x_j\}_{j=1}^n$ appear in W, we get the bound on v_{x_1}, which is the largest.

Likewise, using (4.6) we find that

$$\begin{aligned} \|I_2\|_{L^p_{\tilde{x}}} &\le \zeta(t)\int_\tau^t \int_{-\infty}^{+\infty} (t-s)^{-\frac{3}{4}} e^{-c\frac{(x_1-y_1)^{4/3}}{(t-s)^{1/3}}} (1+s)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{2}{3}} h_{p,n}(s)e^{-\alpha|y_1|}dy_1 ds \\ &\le C_1\zeta(t)(t-\tau)^{1/2}e^{-\tilde{\eta}|x_1|}(1+\tau)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{2}{3}}h_{p,n}(\tau) \\ &= C_1\zeta(t)T^{1/2}e^{-\tilde{\eta}|x_1|}(1+\tau)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{2}{3}}h_{p,n}(\tau), \end{aligned}$$

and since $t = \tau + T$ for T chosen sufficiently small we can replace τ in this inequality with t (increasing C). We conclude that for $|\alpha| = 3$ and $t \geq 1$

$$|D_x^\alpha v(x,t)|_{L^p_{\tilde{x}}} \leq C_2 \zeta(t) A_1(x_1, t; p).$$

In this way we can complete our nonlinearity up to third order derivatives on v, noting that for all third order derivatives we have decay at the rate of an x_1 derivative, which is slower than the others.

5.5 Proof of Theorem 1.3

We now complete the proof of Theorem 1.3 by obtaining a bound on $\zeta(t)$ for all times $t \geq 0$. As a start, we claim that there exists a constant C sufficiently large so that for any $\epsilon > 0$ if

$$\|v_0(y)\|_{L^\infty_{\tilde{x}}} + \|v_0(y)\|_{L^1_{\tilde{x}}} < \frac{\epsilon}{(1+|x_1|)^{3/2}},$$

then

$$\zeta(t) \leq C(\epsilon + \zeta(t)^2). \tag{5.12}$$

To see this, let $\zeta(t)$ be as defined in (5.1), and note that from (1.19) and the definition of $\zeta(t)$ we have

$$\begin{aligned}\|v(x,t)\|_{L^p_{\tilde{x}}} &\leq \epsilon \|v_l(x,t)\|_{L^p_{\tilde{x}}} + C\zeta(t)^2 \|v_n(x,t)\|_{L^p_{\tilde{x}}} \\ &\leq C_1 \epsilon A_0(x_1,t;p) + C_2 \zeta(t)^2 A_0(x_1,t;p) \leq C_3(\epsilon + \zeta(t)^2) A_0(x_1,t;p),\end{aligned}$$

where we have used Lemmas 4.1 and 4.2). Proceeding similarly for derivatives of v and for δ and its derivatives we obtain the claim.

As verified in [8] (see Claim 4.1 on p. 799) we can conclude from (5.12) that

$$\zeta(t) \leq 2C\epsilon, \tag{5.13}$$

for all $t \geq 0$. Theorem 1.3 follows from (5.13) and our definition of ζ. □

Acknowledgments. This material is based upon work supported by the National Science Foundation under Grant Number DMS-0906370. This is a continuation of work the author began with Bongsuk Kwon, and the author is indebted to Dr. Kwon for several enlightening discussions.

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