Pointwise Green's Function Estimates Toward Stability for Degenerate Viscous Shock Waves

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23rd April 2004

We consider degenerate viscous shock waves arising in systems of two conservation laws, where degeneracy here describes viscous shock waves for which the asymptotic endstates are sonic to the hyperbolic system (the shock speed is equal to one of the characteristic speeds). In particular, we develop detailed pointwise estimates on the Green's function associated with the linearized perturbation equation, sufficient for establishing that spectral stability implies nonlinear stability. The analysis of degenerate viscous shock waves involves several new features, such as algebraic (non-integrable) convection coefficients, loss of analyticity of the Evans function at the leading eigenvalue, and time decay intermediate between that of the Lax case and that of the undercompressive case.

1 Introduction

We consider degenerate viscous shock waves arising in the system

$$\begin{aligned} u_t + f(u)_x &= u_{xx}, \quad u, f \in \mathbb{R}^2, \\ u(0,x) &= u_0(x); \end{aligned} \tag{1.1}$$

that is, solutions of the form $\bar{u}(x-st) = (\bar{u}_1(x-st), \bar{u}_2(x-st))^{\mathrm{tr}}$ that satisfy the Rankine–Hugoniot condition

$$s = \frac{f_k(u_1^+, u_2^+) - f_k(u_1^-, u_2^-)}{u_k^+ - u_k^+}, \quad k = 1, 2,$$

and for which $s \in \mathrm{Spectrum}(df(u_\pm)) = \{a_k^\pm\}_{k=1}^2$. Throughout the analysis, we will make the following assumptions.

(H0) $u_0(\pm\infty) = u_\pm$, $f \in C^2(\mathbb{R})$.

(H1) (Lax degeneracy) Either $a_1^- < s < a_2^-$ and $a_1^+ < s = a_2^+$ (right side degenerate) or $a_1^- = s < a_2^-$ and $a_1^+ < s = a_2^+$ (left side degenerate).

(H2) (First order degeneracy) Both $\bar{u}_1(x-st)$ and $\bar{u}_2(x-st)$ decay to the degenerate side endstate with rate $|x-st|^{-1}$(see Section 2 for detailed requirements on $u_\pm$ and f that imply (H2)).

Under these conditions we develop detailed estimates on the Green's function of the linearized perturbation equation sufficient for establishing that spectral stability implies nonlinear stability for degenerate viscous shock waves arising in (1.1).

Our interest in the degenerate case is motivated both by the physical application to detonation waves (see [H.3] and the references therein) and by its position as a boundary case between Lax and undercompressive waves. The critical new feature in the analysis of degenerate waves, necessarily absent in the case of non-degenerate waves (for which $s \notin$ Spectrum $(df(u_{\pm}))$), is the algebraic (non-integrable) decay to endstate of the degenerate wave (see hypothosis **(H2)** and the structural discussion of Section 2). As a consequence of this slow decay, the convection coefficients for the perturbation equation found by linearizing (1.1) about $\bar{u}(x)$ are also non-integrable, and the consequent asymptotic analysis is considerably more complicated than that of the non-degenerate case. More important, the Evans function associated with the linearized operator has terms of the form $\sqrt{\lambda}\log\lambda$, and hence analyticity is lost at the critical point $\lambda = 0$, around which long time behavior is determined. This loss of analyticity has two critical consequences: (1) The study of the point spectrum of the linear operator near $\lambda = 0$ is more delicate; and (2) the time decay of the linear semigroup operator e^{Lt} is reduced. While the former of these has been considered in [HZ.2], in the case of conservation laws, and in [SS], in the case of reaction–diffusion systems, the latter has not, to our knowledge been addressed. This, then, is the critical issue of the current analysis.

It is well known that solutions $u(t,x)$ of (1.1), initialized by $u(0,x)$ near a standing wave solution $\bar{u}(x)$, will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined by the amount of mass (measured by $\int_{\mathbb{R}}(u(0,x)-\bar{u}(x))dx$) carried into the shock as well as the amount carried out to the far field. In our framework, a local tracking function $\delta(t)$ will serve to approximate the shift of this translate at each time t. Following [HZ.1], we build this shift into our model by defining our perturbation $v(t,x)$ as $v(t,x) := u(t,x+\delta(t)) - \bar{u}(x)$. We will say that $\bar{u}(x)$ is stable with respect to some measure if for $v(0,x)$ sufficiently small in that measure we have $v(t,x) \to 0$ as $t \to \infty$.

Substituting $v(t,x) = u(t,x+\delta(t)) - \bar{u}(x)$ into (1.1), we obtain the perturbation equation

$$v_t = Lv + Q(v)_x + \dot{\delta}(t)(\bar{u}_x(x) + v_x), \tag{1.2}$$

where $Lv := v_{xx} - (A(x)v)_x$, $A(x) = df(\bar{u}(x))$, and $Q(v) = \mathbf{O}(v^2)$ is a smooth function of v. According to hypotheses **(H0)–(H2)**, we make the following conclusions regarding the behavior of $A(x)$ and $A_{\pm} := \lim_{x\to\pm\infty} A(x)$:

(C0) $A(x) \in C^1(\mathbb{R})$, $|\frac{\partial^k}{\partial x^k}(A(x) - A_{\pm})| = \mathbf{O}(|x|^{-k})$, $k = 1, 2$.

(C1) The eigenvalues $a_k^{\pm}$ of $A_{\pm}$ satisfy (H1).

Integrating (1.2), we have (after integration by parts on the second integral and observing that $e^{Lt}\bar{u}_x(x) = \bar{u}_x(x)$),

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds, \end{aligned}$$

where $G(t,x;y)$ represents a (matrix) Green's function for the linear part of (1.2):

$$G_t + (A(x)G)_x = G_{xx}; \quad G(0,x;y) = \delta_y(x)I. \tag{1.3}$$

The goal of this analysis is to develop pointwise estimates on $G(t,x;y)$ sufficient for establishing that an iteration on $v(t,x)$ will close. The estimates on $G(t,x;y)$ will be divided into those terms for which the x dependence is exactly $\bar{u}_x(x)$ (referred to as the *excited* terms, and denoted $e(t,y)$) and those for which the x dependence is not exactly $\bar{u}_x(x)$. Typically, the excited terms do not decay in t, and represent mass that accumlates in the shock layer, shifting the shock. Our approach will be to choose our shift $\delta(t)$ to annihilate this mass, so that we track the shock in time. Following [HZ.1], we write

$$G(t,x;y) = \tilde{G}(t,x;y) + \bar{u}_x(x)e(t,y),$$

for which we have

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy + \bar{u}_x(x)\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy + \delta(t)\bar{u}_x(x)$$
$$- \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds$$
$$-\bar{u}_x(x)\int_0^t \int_{-\infty}^{+\infty} e_y(t-s,y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds.$$

Choosing $\delta(t)$ to eliminate the excited terms, we have

$$\begin{aligned} \delta(t) := &-\int_{-\infty}^{+\infty} e(t,y)v_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(t-s,y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds. \end{aligned} \tag{1.4}$$

In order to close the analysis, we must close an iteration on $\delta(t)$ simultaneously with our iteration on $v(t,x)$ (see, for example, [?]). We mention that in the iteration on $v(t,x)$, we only require an estimate on $\dot{\delta}(t)$, so in practice we consider the time derivative of (1.4).

Typically, we analyze $G(t,x;y)$ through its Laplace transform, $G_\lambda(x,y)$, which satisfies the ODE ($t \to \lambda$)

$$G_{\lambda_{xx}} - (A(x)G_\lambda)_x - \lambda G_\lambda = -\delta_y(x)I,$$

and can be estimated by standard methods. Letting φ_1^+ and φ_2^+ represent the (necessarily) two linearly independent asymptotically decaying solutions at $+\infty$ of the eigenvalue ODE

$$L\varphi = \lambda\varphi, \tag{1.5}$$

and φ_1^- and φ_2^- similarly the two linearly independent asymptotically decaying solutions at $-\infty$, we write $G_\lambda(x,y)$ as a linear combination,

$$G_\lambda(x,y) = \begin{cases} \varphi_1^+(x)N_1^-(y) + \varphi_2^+(x)N_2^-(y), & x > y \\ \varphi_1^-(x)N_1^+(y) + \varphi_2^-(x)N_2^+(y), & x < y, \end{cases}$$

where we observe the notation

$$\varphi_k^\pm N_k^\mp = \begin{pmatrix} \varphi_{k1}^\pm \\ \varphi_{k2}^\pm \end{pmatrix} \begin{pmatrix} N_{k1}^\mp & N_{k2}^\mp \end{pmatrix} = \begin{pmatrix} \varphi_{k1}^\pm N_{k1}^\mp & \varphi_{k1}^\pm N_{k2}^\mp \\ \varphi_{k2}^\pm N_{k1}^\mp & \varphi_{k2}^\pm N_{k2}^\mp \end{pmatrix}.$$

Insisting on the continuity $G_\lambda(x,y)$ across $y = x$ and a step in $\partial_x G_\lambda(x,y)$, we have

$$\begin{aligned} (\varphi_1^+ N_1^- + \varphi_2^+ N_2^- - \varphi_1^- N_1^+ - \varphi_2^- N_2^+) =& 0 \\ ({\varphi_1^+}' N_1^- + {\varphi_2^+}' N_2^- - {\varphi_1^-}' N_1^+ - {\varphi_2^-}' N_2^+) =& -I. \end{aligned} \tag{1.6}$$

Equations (1.6) represent eight equations and eight unknowns, which decouple into two sets of four equations and four unknowns. Solving by Cramer's formula, we have, for example,

$$N_{11}^-(y;\lambda) = -\frac{\det \begin{pmatrix} \varphi_{21}^+ & \varphi_{11}^- & \varphi_{21}^- \\ \varphi_{22}^+ & \varphi_{12}^- & \varphi_{22}^- \\ {\varphi_{22}^+}' & {\varphi_{12}^-}' & {\varphi_{22}^-}' \end{pmatrix}}{\det \begin{pmatrix} \varphi_1^+, & \varphi_2^+, & \varphi_1^-, & \varphi_2^-, \\ {\varphi_1^+}' & {\varphi_2^+}' & {\varphi_1^-}' & {\varphi_2^-}' \end{pmatrix}}.$$

Clearly, then, $G_\lambda(x,y)$ will be well-behaved so long as

$$W(x;\lambda) := \det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \varphi_1^{+\prime} & \varphi_2^{+\prime} & \varphi_1^{-\prime} & \varphi_2^{-\prime} \end{pmatrix} \neq 0.$$

Following Jones et al. [AGJ, E, GZ, J, KS], we define the *Evans function* as $D(\lambda) := W(0;\lambda)$.

In order to understand the behavior of the Evans function, consider an eigenvector, $V(x;\lambda)$, of the linear operator

$$Lv = v_{xx} - (A(x)v)_x.$$

Since $V(x;\lambda)$ must decay at both $\pm\infty$, it must be a linear combination of φ_1^+ and φ_2^+ and also of φ_1^- and φ_2^-. Consequently, these four solutions must be linearly independent, and their Wronskian must be 0. In general, zeros of the Evans function correspond with eigenvalues of the operator L, an observation that has been made precise in [AGJ] in the case—pertaining to reaction diffusion equations—of isolated eigenvalues and in [ZH, GZ] in the case—pertaining to conservation laws—of nonstandard "effective" eigenvalues embedded in essential spectrum of L. (The latter correspond with resonant poles of L, as examined in the scalar context in [PW]).

In [HZ.2], the authors established that under assumptions **(H0)**–**(H2)** $D(\lambda)$ can be written as an analytic function plus a small error,

$$D(\lambda) = D_a(\lambda) + \mathbf{O}(|\lambda^{3/2}\log\lambda|), \quad \text{as } \lambda \to 0,$$

where $D_a(\lambda) = \mathbf{O}(|\lambda|)$ is analytic in a neighborhood of $\lambda = 0$. Following [HZ.2], we introduce the following stability condition $(\mathcal{D})$.

$(\mathcal{D})$: $D(\lambda)$ (and hence $D_a(\lambda)$) has precisely one zero in $\{\mathrm{Re}\lambda \geq 0\}$, necessarily at $\lambda = 0$, and $D_a'(0) \neq 0$.

While condition $(\mathcal{D})$ is generally quite difficult to verify analytically (see, for example, [D]), it can be checked numerically (see [B, OZ]). A condition that lends itself more readily to exact study is the *stability index*, typically defined as

$$\Gamma := \operatorname{sgn} D_a'(0) \times \operatorname{sgn} \lim_{\mathbb{R}\ni\lambda\to\infty} D(\lambda).$$

For $\lambda \in \mathbb{R}_+$, we have $D_a(\lambda) \in \mathbb{R}$, so that in the event that $\Gamma = -1$, $D(\lambda)$ must have a positive real root, which guarantees instability. In the case that $\Gamma = +1$ the question of stability remains undecided.

Finally, we observe that if stability condition $(\mathcal{D})$ holds, there exists a contour Γ_d defined through

$$\lambda_d(k) = -d_0 + id_1k - d_2k^2, \tag{1.7}$$

with d_0 and d_2 both positive constants, so that aside from the eigenvalue $\lambda = 0$, the point spectrum of L lies entirely to the left of Γ_d. We are now in a position to state the main result of the paper.

Theorem 1.1. Let **(H0)**–**(H2)** hold, as well as stability criterion $(\mathcal{D})$. Then for some constants C, M, $T > 1$, and $\eta > 0$, depending only on $df(\bar{u}(x))$ and the spectrum of L, the Green's function $G(t,x;y)$ described through (1.3) satisfies the following estimates.

(i) $y \leq x \leq 0$

$$\begin{aligned} G(t,x;y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_-(t,y) \\ &+ \Big[\mathbf{O}(t^{-3/4}\log t) \wedge \mathbf{O}(|x - \frac{a_1^-}{a_2^-}y - a_1^- t|^{-3/2}\log t)\Big] I_{\{|x-\frac{a_1^-}{a_2^-}y| \leq |a_1^-|t\}}. \end{aligned}$$

$$
\begin{aligned}
G_y(t,x;y) =& \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_-(t,y) \\
&+\Big[\mathbf{O}(t^{-5/4}\log t) \wedge \mathbf{O}(|x-\frac{a_1^-}{a_2^-}y - a_1^- t|^{-5/2}\log t)\Big] I_{\{|x-\frac{a_1^-}{a_2^-}y|\le |a_1^-|t\}},
\end{aligned}
$$

where

$$
\begin{aligned}
e_-(t,y) =& \mathbf{O}(1)e^{-\frac{(y+a_2^- t)^2}{Mt}} + \mathbf{O}(1)I_{\{|y|\le|a_2^-|t\}} \\
\partial_y e_-(t,y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(y+a_2^- t)^2}{Mt}} + \Big[\mathbf{O}(t^{-3/4}\log t)\wedge \mathbf{O}(|y+a_2^- t|^{-3/2}\log t)\Big] I_{\{|y|\le|a_2^-|t\}}.
\end{aligned}
$$

(ii) $x \le y \le 0$

$$
\begin{aligned}
G(t,x;y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_1^- t)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_-(t,y) \\
&+\Big[\mathbf{O}(t^{-3/4}\log t) \wedge \mathbf{O}(|x-\frac{a_1^-}{a_2^-}y - a_1^- t|^{-3/2}\log t)\Big] I_{\{|x-\frac{a_1^-}{a_2^-}y|\le |a_2^-|t\}}.
\end{aligned}
$$

$$
\begin{aligned}
G_y(t,x;y) =& \mathbf{O}(t^{-1})e^{-\frac{(x-y-a_1^- t)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_-(t,y) \\
&+\Big[\mathbf{O}(t^{-5/4}\log t)\mathbf{O}(|x-\frac{a_1^-}{a_2^-}y - a_1^- t|^{-5/2}\log t)\Big] I_{\{|x-\frac{a_1^-}{a_2^-}y|\le |a_1^-|t\}}.
\end{aligned}
$$

(iii) $x \le 0 \le y$

$$
\begin{aligned}
G(t,x;y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(x-a_1^-\int_L^y \frac{ds}{a_1(s)}-a_1^- t)^2}{Mt}} + \bar{u}_x(x)e_3(t,y) + \mathbf{O}(t^{-1/2})e^{-\frac{(x-a_1^- t)^2}{Mt}} I_{\{|x-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}} \\
&+\Big[\mathbf{O}(t^{-3/4}\log t)\wedge \mathbf{O}(|x-a_1^-\int_L^y \frac{ds}{a_1(s)} - a_1^- t|^{-3/2}\log t)\Big] I_{\{|x-a_1^-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}} \\
&+\Big[\mathbf{O}(t^{-3/4})\wedge \mathbf{O}(|x-a_1^- t|^{-3/2})e^{-\frac{y^2}{Mt}}\Big]\mathbf{O}(|y|)I_{\{|x-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}}
\end{aligned}
$$

$$
\begin{aligned}
G_y(t,x;y) =& \Big[\mathbf{O}(t^{-3/4})\wedge\mathbf{O}(t^{-1})\mathbf{O}(|y|)\Big]e^{-\frac{(x-a_1^-\int_L^y \frac{ds}{a_1(s)}-a_1^- t)^2}{Mt}} + \bar{u}_x(x)\partial_y e_3(t,y) \\
&+\Big[\mathbf{O}(t^{-5/4}\log t)\wedge \mathbf{O}(|x-a_1^-\int_L^y \frac{ds}{a_1(s)} - a_1^- t|^{-5/2}\log t)\Big] I_{\{|x-a_1^-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}} \\
&+\Big[\mathbf{O}(t^{-3/4})\wedge \mathbf{O}(t^{-1})\mathbf{O}(|y|)\wedge\mathbf{O}(|x-a_1^- t|^{-2})e^{-\frac{y^2}{Mt}}\Big]I_{\{|x-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}},
\end{aligned}
$$

where

$$
\begin{aligned}
e_3(t,y) =& \mathbf{O}(1)e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(1)I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}} + \mathbf{O}(1)e^{-\frac{y^2}{Mt}} \\
\partial_y e_3(t,y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{y^2}{Mt}} \\
&+\Big[\mathbf{O}(t^{-3/4}\log t)\wedge\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)} + t|^{-3/2}\log t)\Big] I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}.
\end{aligned}
$$

(iv) $y \le 0 \le x$

$$\begin{aligned}G(t,x;y) =& \mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\ge a_2^- t\}} + \mathbf{O}(|y+a_2^- t|^{-1/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}} I_{\{|y|\le a_2^- t\}}\\ &+\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\ge a_2^- t\}} + \mathbf{O}(|y+a_2^- t|^{-1/2}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{x^2}{Mt}} I_{\{|y|\le a_2^- t\}}\\ &+\bar u_x(x) e_4(t,y)\end{aligned}$$

$$\begin{aligned}G_y(t,x;y) =& \mathbf{O}(t^{-3/4})\mathbf{O}(|x|^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\ge a_2^- t\}} + \mathbf{O}(|y+a_2^- t|^{-3/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}} I_{\{|y|\le a_2^- t\}}\\ &+\mathbf{O}(t^{-3/4}\log t)\mathbf{O}(e^{-\eta|x|})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\ge a_2^- t\}} + \mathbf{O}(|y+a_2^- t|^{-3/2}\log t)\mathbf{O}(e^{-\eta|x|}) I_{\{|y|\le a_2^- t\}}\\ &+\bar u_x(x)\partial_y e_4(t,y),\end{aligned}$$

where

$$\begin{aligned}e_4(t,y) =& \mathbf{O}(1)e^{-\frac{(y+a_2^- t)^2}{Mt}} I_{\{|y|\ge a_2^- t\}} + \mathbf{O}(1) I_{\{|y|\le a_2^- t\}}\\ \partial_y e_4(t,y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(y+a_2^- t)^2}{Mt}}.\end{aligned}$$

(v) $0 \le y \le x$

$$\begin{aligned}G(t,x;y) =& \mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar u_x\Big(I_{\{|x-y|\le\epsilon_0\sqrt{t}\}} - I_{\{|y|\le\epsilon_0\sqrt{t}\}}\Big) + \bar u_x(x)e_+(t,y)\\ &+\mathbf{O}(t^{-1/2}(\log t)^2)\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\ &+\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}} + \mathbf{O}(t^{-1/2})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}}\end{aligned}$$

$$\begin{aligned}G_y(t,x;y) =& \mathbf{O}(t^{-1})\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-3/2}\log t)\mathbf{O}(|x|^{-1})e^{-\frac{(x-y)^2}{Mt}}\\ &+\mathbf{O}(t^{-3/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(x-y)^2}{Mt}}\\ &+\mathbf{O}(t^{-3/4}\log t)\mathbf{O}(e^{\eta|x-y|}) + \mathbf{O}(t^{-3/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\ &+\bar u_x(x)\partial_y e_+(t,y),\end{aligned}$$

where

$$\begin{aligned}e_+(t,y) =& \mathbf{O}(1)e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(1) I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}} + P_+ I_{\{|y|\le\epsilon_0\sqrt{t}\}}\\ \partial_y e_+(t,y) =& \mathbf{O}(t^{-1/2})e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}}.\end{aligned}$$

(vi) $0 \le x \le y$

$$
\begin{aligned}
G(t,x;y) =&\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar{u}_x I_{\{|x-y|\le\epsilon_0\sqrt{t}\}} + \bar{u}_x(x)e_+(t,y)\\
&+\mathbf{O}(t^{-1/2}(\log t)^2)\mathbf{O}(|x|^{-1})e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1/2})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}}\\
&+\mathbf{O}(t^{-3/4}\log t)\wedge\mathbf{O}(|\int_x^y \frac{ds}{a_1(s)} + t|^{-3/2}\log t) I_{\{|\int_x^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\
&+\mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)} + t|^{-1/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\
&+\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)} + t|^{-1/2}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{x^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}
\end{aligned}
$$

$$
\begin{aligned}
G_y(t,x;y) =&\mathbf{O}(t^{-1})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + \mathbf{O}(t^{-1})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \bar{u}_x\partial_y e_+(t,y)\\
&+\Big[\mathbf{O}(t^{-5/4}\log t)\wedge\mathbf{O}(|\int_x^y \frac{ds}{a_1(s)} + t|^{-5/2}\log t)\Big] I_{\{|\int_x^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(t^{-3/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^x \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\
&+\mathbf{O}(t^{-3/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)} + t|^{-3/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(t^{-3/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^x \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\ge t\}}\\
&+\mathbf{O}(t^{-3/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\\
&+\mathbf{O}(|x|^{-2})\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)} + t|^{-3/2})e^{-\frac{x^2}{Mt}}.
\end{aligned}
$$

For $y < L$, the integrals $\int_L^y \frac{ds}{a_1(s)}$ can be replaced by 0, and similarly for x.

Of particular interest are the estimates on $G(t,x;y)$ with decay rate $t^{-1/4}$. In the case $0 \le x \le y$, the ODE Green's functions associated with degenerate decay in x and non-degenerate decay in y takes the form

$$S_\lambda(x,y) = \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds}.$$

In the case $y = 0$, we can proceed here as though analyzing the heat equation to immediately recover an estimate of the form

$$\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}} I_{\{y=0\}}.$$

The difficulty arises in the limit as x goes to zero, in which case he have an estimate of the form

$$S_\lambda(x,y) = \mathbf{O}(|\lambda^{-1/2}|)e^{-\int_L^y \mu_3(s;\lambda)ds}.$$

The exponent $\int_L^y \mu_3(s;\lambda)ds$ is $\mathbf{O}(|\lambda|)$ and thus near $\lambda = 0$ gives slower decay than $-\sqrt{\lambda}y$.

2 Structure of Degenerate Viscous Shock Waves

The critical structural feature of degenerate viscous shock waves is that they decay to the degenerate side endstate at algebraic rate rather than exponential (as in the non-degenerate case). In particular, degenerate viscous shock waves decay with rate $|x|^{-1/k}$, for $k = 1, 2, ...$

Definition. We will describe degenerate viscous shock waves that decay to degenerate endstate at rate $|x|^{-1/k}$ in both coordinates as k^{th}-order degenerate.

Our focus in this paper will be on first order degeneracy, which is the generic case. It has been shown in [HZ.2] that first order degeneracy is implied by the condition

$$\begin{aligned}&\frac{a_{22}^+}{a_{11}^+ + a_{22}^+}\Big(\frac{1}{2}\partial_{u_1u_1}f_1(u_1^+,u_2^+) + \frac{1}{2}\partial_{u_2u_2}f_1(u_1^+,u_2^+)\frac{(a_{11}^+)^2}{(a_{12}^+)^2} - \partial_{u_1u_2}f_1(u_1^+,u_2^+)\frac{a_{11}^+}{a_{12}^+}\Big)\\ -&\frac{a_{12}^+}{a_{11}^+ + a_{22}^+}\Big(\frac{1}{2}\partial_{u_1u_1}f_2(u_1^+,u_2^+) + \frac{1}{2}\partial_{u_2u_2}f_2(u_1^+,u_2^+)\frac{(a_{11}^+)^2}{(a_{12}^+)^2} - \partial_{u_1u_2}f_2(u_1^+,u_2^+)\frac{a_{11}^+}{a_{12}^+}\Big) \neq 0,\end{aligned}$$

where $a_{kj}^+ = \partial_{u_j}f_k(u_1^+,u_2^+)$. For example, in the viscous p-system,

$$\begin{aligned}u_{1_t} - u_{2_x} &= u_{1_{xx}}\\ u_{2_t} + p(u_1)_x &= u_{2_{xx}},\end{aligned}$$

this reduces to the condition $p''(u_1^+) \neq 0$.

3 ODE Estimates

In this section we establish the critical estimates on $\phi_k^\pm(x;\lambda), \psi_k^\pm(x;\lambda)$ and their duals, $\tilde\phi_k^\pm(x;\lambda), \tilde\psi_k^\pm(x,\lambda)$. Our eigenvalue ODE 1.5 takes the form

$$\begin{aligned}v_{1_{xx}} - (a_{11}(x)v_1)_x - (a_{12}(x)v_2)_x &= \lambda v_1\\ v_{2_{xx}} - (a_{21}(x)v_1)_x - (a_{22}(x)v_2)_x &= \lambda v_2,\end{aligned} \tag{3.1}$$

where $a_{kj}(x) = \partial_{u_j}f_k(\bar u_1(x), \bar u_2(x))$. Writing $V_1 = v_1$,$V_2 = v_2$, $V_3 = v_{1_x}$, $V_4 = v_{2_x}$, we have the first order system

$$V' = \mathbb{M}(x;\lambda)V. \tag{3.2}$$

We will also be interested in the associated *integrated equation*

$$\begin{aligned}w_{1_{xx}} - a_{11}(x)w_{1_x} - a_{12}(x)w_{2_x} &= \lambda w_1\\ w_{2_{xx}} - a_{21}(x)w_{1_x} - a_{22}(x)w_{2_x} &= \lambda w_2.\end{aligned} \tag{3.3}$$

In particular, we will find it convenient when possible to compute growth and decay solutions of (3.1) by computing growth and decay modes of (3.3) and computing their derivatives. We stress that from this point of view no assumptions regarding integrability need be made. It is a question, rather, of scaling.

Writing (3.3) as a first order system with $W_1 = w_1$, $W_2 = w_2$, $W_3 = w_{1_x}$, $W_4 = w_{2_x}$, we have

$$W' = \mathbb{A}(x;\lambda)W; \quad ' := \partial_x,$$

where

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & a_{11}(x) & a_{12}(x) \\ 0 & \lambda & a_{21}(x) & a_{22}(x) \end{pmatrix},$$

which has four eigenvalues $\mu_k(x;\lambda)$ satisfying

$$\mu_1(x;\lambda) = \frac{a_1(x) - \sqrt{a_1(x)^2 + 4\lambda}}{2}; \quad \mu_2(x;\lambda) = \frac{a_2(x) - \sqrt{a_2(x)^2 + 4\lambda}}{2};$$
$$\mu_3(x;\lambda) = \frac{a_1(x) + \sqrt{a_1(x)^2 + 4\lambda}}{2}; \quad \mu_4(x;\lambda) = \frac{a_2(x) + \sqrt{a_2(x)^2 + 4\lambda}}{2},$$

with associated eigenvectors

$$P(x;\lambda) := (V_1, V_2, V_3, V_4) = \begin{pmatrix} r_1(x) & r_2(x) & r_1(x) & r_2(x) \\ r_1(x)\mu_1(x,\lambda) & r_2(x)\mu_2(x;\lambda) & r_1(x)\mu_3(x;\lambda) & r_2(x)\mu_4(x;\lambda) \end{pmatrix}.$$

Here, $a_1(x) \le a_2(x)$ are the eigenvalues of

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

namely

$$a_1(x) = \frac{\mathrm{tr}A - \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}; \quad a_2(x) = \frac{\mathrm{tr}A + \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2},$$

with associated eigenvectors $r_k = (1, \frac{a_k(x)-a_{11}(x)}{a_{12}(x)})^{\mathrm{tr}} = (1, \frac{a_k(x)r_{k2}(x)-a_{21}(x)}{a_{22}(x)})$. At $x = +\infty$, we have $\mu_1^+(\lambda) = \mathbf{O}(1)$, $\mu_2^+(\lambda) = \mathbf{O}(\sqrt{\lambda})$, $\mu_3(\lambda) = \mathbf{O}(\lambda)$, and $\mu_4^+(\lambda) = \mathbf{O}(\sqrt{\lambda})$, prompting our designation of μ_1 and μ_3 as *non-degenerate* modes and μ_2 and μ_4 as *degenerate* modes.

The following lemma is proven in [HZ.2].

Lemma 3.1. Under the assumptions of Theorem 1.1 there exists some constant r sufficiently small so that for $|\lambda| \le r$ $\lambda \neq \mathbb{R}_-$, we have the following estimates on degenerate solutions of the integrated equation (3.3). For $x \ge 0$, $k = 1, 2$

$$\begin{aligned}
W_{2k}^+(x;\lambda) &= e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)(1 + E_{2k}(x;\lambda)), \\
W_{2(k+2)}^+(x;\lambda) &= e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)(-\sqrt{\lambda} + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} + E_{2(k+2)}(x;\lambda) + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+}E_{2k}(x;\lambda)), \\
W_{4k}^+(x;\lambda) &= e^{+\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)(1 + E_{4k}(x;\lambda)), \\
W_{4(k+2)}^+(x;\lambda) &= e^{+\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)(+\sqrt{\lambda} + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} + E_{4(k+2)}(x;\lambda) + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+}E_{4(k-2)}(x;\lambda)),
\end{aligned}$$

where

$$E_{2k}(x;\lambda), E_{4k}(x;\lambda) = \mathbf{O}(|\lambda^{1/2}\log\lambda|) \wedge \mathbf{O}(|x|^{-1}), \quad k = 1, 2$$

($\wedge$ represents *min*), and

$$E_{2k}(x;\lambda), E_{4k}(x;\lambda) = \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1}), \quad k = 3, 4.$$

Additionally, we have for $|\lambda^{1/2}x| \leq 1$, the error estimates

$$E_{jk}(x;\lambda) = E_{jk}(0;\lambda) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|).$$

Proof. The only statement in Lemma 3.1 not established in [HZ.2] is the $|\lambda^{1/2}x| \leq 1$ estimate on $E_{jk}(x;\lambda)$. Following the proof of Lemma 4.1 in [HZ.2], we write $w_1(x)$ from equation (3.3) as $w_1(x) = (\bar{u}_1(x) - u_1^+)u_1(x)$, for which

$$\begin{aligned} u_1(x) =& e^{-\sqrt{\lambda}x}(1 + E_{21}(x;\lambda)) \\ u_{1_x}(x) =& e^{-\sqrt{\lambda}x}(-\sqrt{\lambda} + E_{23}(x;\lambda)). \end{aligned}$$

We have, then, the ODE for $E_{21}(x;\lambda)$, $\partial_x E_{21}(x;\lambda) - \sqrt{\lambda}E_{21}(x;\lambda) = E_{23}(x;\lambda)$. Integrating, we find

$$E_{21}(x;\lambda) = e^{\sqrt{\lambda}x}E_{21}(0;\lambda) + \int_0^x e^{\sqrt{\lambda}(x-\xi)}E_{23}(\xi,\lambda)d\xi,$$

from which the estimate is clear. The remaining cases are similar. □

We remark that the $\log\lambda$ behavior in Lemma 3.1 appears in the analysis of single equations as well and appears to be sharp (see [H.3, H.4, PW]).

Differentiating the estimates of Lemma 3.1 we obtain estimates on growth and decay solutions of the unintegrated equation (3.1).

Lemma 3.2. Under the assumptions of Theorem 1.1 and for L large enough so that $x \geq L$ assures $(\mathrm{tr}A)^2 - 4\det A \geq 0$, there exists some constant r sufficiently small so that for $|\lambda| \leq r$, we have the following estimates on solutions of the unintegrated equation (3.2).

(i) $x \leq 0$

$$\phi_k^-(x;\lambda) = e^{\mu_{k+2}^-(\lambda)x}(V_{k+2}^-(\lambda) + \mathbf{O}(e^{-\alpha|x|}))$$

$$\psi_k^-(x;\lambda) = e^{\mu_k^-(\lambda)x}(V_k^-(\lambda) + \mathbf{O}(e^{-\alpha|x|})),$$

where $k = 1, 2$, and $\mu_k^-(\lambda)$ and $V_k^-(\lambda)$ are eigenvalue–eigenvector pairs for the matrix $\mathbb{A}_-(\lambda) := \lim_{x\to-\infty}\mathbb{A}(x;\lambda)$.

(ii) (non-degenerate solutions) For $x \geq L$

$$\begin{aligned} (\phi_{11}^+, \phi_{12}^+)^{\mathrm{tr}} =: \Phi_1^+(x;\lambda) =& e^{\int_L^x \mu_1(s;\lambda)ds}(r_1^+ + \mathbf{O}(|x|^{-1})) \\ (\phi_{11}^{+\prime}, \phi_{12}^{+\prime})^{tr} = (\phi_{13}^+, \phi_{14}^+)^{\mathrm{tr}} =: \Phi_1^{+\prime}(x;\lambda) =& e^{\int_L^x \mu_1(s;\lambda)ds}(r_1^+\mu_1^+(\lambda) + \mathbf{O}(|x|^{-1})) \end{aligned}$$

$$\begin{aligned} (\psi_{11}^+, \psi_{12}^+)^{\mathrm{tr}} =: \Psi_1^+(x;\lambda) =& e^{\int_L^x \mu_3(s;\lambda)ds}(r_1^+ + \mathbf{O}(|x|^{-1})) \\ (\psi_{11}^{+\prime}, \psi_{12}^{+\prime})^{tr} = (\psi_{13}^+, \psi_{14}^+)^{\mathrm{tr}} =: \Psi_1^{+\prime}(x;\lambda) =& e^{\int_L^x \mu_3(s;\lambda)ds}(r_1^+\mu_3^+(\lambda) + \mathbf{O}(|x|^{-1})), \end{aligned}$$

while for $0 \leq x \leq L$ each term is $\mathbf{O}(1)$.

(iii) (degenerate solutions) For $x \geq 0$, $k = 1, 2$, and $\lambda \notin \mathbb{R}_-$

$$\phi_{2k}^+(x;\lambda) = e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(-\sqrt{\lambda} + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} + E_{2(k+2)}(x;\lambda) + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} E_{2k}(x;\lambda)\Big),$$
$$\phi_{2k}^{+\prime}(x;\lambda) = e^{-\sqrt{\lambda}x}\Big(\bar{u}_{k_{xx}} + (a_{k1}(x)\bar{u}_{1_x}E_{21} + a_{k2}(x)\bar{u}_{2_x}E_{22}) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda|)\mathbf{O}(|x|^{-1})\Big),$$

$$\psi_{2k}^+(x;\lambda) = e^{+\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(+\sqrt{\lambda} + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} + E_{4(k+2)}(x;\lambda) + \frac{\bar{u}_{k_x}(x)}{\bar{u}_k(x) - u_k^+} E_{4k}(x;\lambda)\Big),$$
$$\psi_{2k}^{+\prime}(x;\lambda) = e^{+\sqrt{\lambda}x}\Big(\bar{u}_{k_{xx}} + (a_{k1}(x)\bar{u}_{1_x}E_{41} + a_{k2}(x)\bar{u}_{2_x}E_{42}) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda|)\mathbf{O}(|x|^{-1})\Big),$$

where the $E_{kj}(x;\lambda)$ are as in Lemma 3.1.

Proof. The estimates of Lemma 3.2 on degenerate solutions follow directly from Lemma 3.1 and equation (3.3). For the non-degenerate solutions, standard theory suffices. (See, for example, [ZH].) □

Lemma 3.3. The Evans function associated with equation (3.2), defined by

$$D(\lambda) := \det\big(\ \phi_1^-(0;\lambda) \quad \phi_2^-(0;\lambda) \quad \phi_1^+(0;\lambda) \quad \phi_2^+(0;\lambda)\ \big),$$

satisfies

$$\begin{aligned} D(\lambda) &= \det\begin{pmatrix} \Phi_1^+(0;0) & \Phi_2^+(0;0) & \Phi_1^-(0;0) & \Phi_2^-(0;0) \\ \lambda\mathcal{W}_1^+(0;0) & \lambda\mathcal{W}_2^+(0;0) & -a_1^- r_1^- & \lambda\mathcal{W}_2^-(0;0) \end{pmatrix} + \mathbf{O}(|\lambda^{3/2}\log\lambda|) \\ &=: D_a(\lambda) + \mathbf{O}(|\lambda^{3/2}\log\lambda|), \end{aligned}$$

where

$$\Phi_k^\pm = (\phi_{k1}^\pm, \phi_{k2}^\pm)^{\text{tr}}.$$

Proof. Though Lemma 3.3 has been proven in [HZ.2], we restate a brief account of the argument for later reference.

Fast decay ODE solutions ($\Phi_k^\pm(x;0) = \mathbf{O}(e^{-\alpha|x|})$). The fast decay solutions in this analysis are Φ_2^- and Φ_1^+. For the first, proceeding as in [GZ] we integrate

$$\Phi_2^{-\prime\prime} - (A(x)\Phi_2^-)_x = \lambda\Phi_2^-$$

on $(-\infty, x]$ to obtain

$$\Phi_2^{-\prime}(x;\lambda) - A(x)\Phi_2^-(x;\lambda) = \lambda\int_{-\infty}^x \Phi(s;\lambda) =: \lambda\mathcal{W}_4^-(x;\lambda),$$

where $\mathcal{W}_2^-(x;\lambda)$ is analytic in λ. Similarly for Φ_1^+, we have

$$\Phi_1^{+\prime}(x;\lambda) - A(x)\Phi_1^+(x;\lambda) = \lambda\mathcal{W}_1^+(x;\lambda),$$

where $\mathcal{W}_1^+(x;\lambda)$ is analytic in λ.

Slow decay ODE solutions ($\Phi_1^+(x;0) = \mathbf{O}(1)$). The only slow decay solution in this analysis is $\Phi_1^-(x;\lambda)$. Integrating again on $(-\infty, x]$, we have

$$\Phi_1^{-\prime}(x;\lambda) - A(x)\Phi_1^-(x;\lambda) + A_- r_1^- = \lambda \mathcal{W}_3^-(x;\lambda),$$

where $\mathcal{W}_3^-(x;\lambda)$ is analytic in λ, and r_1^- is the eigenvector of A_- associated with the eigenvalue a_1^-.

Degenerate decay ODE solutions ($\Phi_2^+(x;0) = \bar{u}_x(x)$). Integrating our degenerate ODE solution on $[x, +\infty)$ we have by construction,

$$\Phi_2^{+\prime}(x;\lambda) - A(x)\Phi_2^+(x;\lambda) = \lambda \mathcal{W}_2^+(x;\lambda),$$

where by Lemma 3.1 $\mathcal{W}_2^+(0;\lambda) = \mathcal{W}_2^+(0,0) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$.

Now, we compute

$$\begin{aligned}
D(\lambda) &= \det \begin{pmatrix} \Phi_1^+ & \Phi_2^+ & \Phi_1^- & \Phi_2^- \\ \Phi_1^{+\prime} & \Phi_2^{+\prime} & \Phi_1^{-\prime} & \Phi_2^{-\prime} \end{pmatrix} \Big|_{y=0} \\
&= \det \begin{pmatrix} \Phi_1^+ & \Phi_2^+ & \Phi_1^- & \Phi_2^- \\ A\Phi_1^+ + \lambda\mathcal{W}_1^+ & A\Phi_2^+ + \lambda\mathcal{W}_2^+ & A\Phi_1^- - a_1^- r_1^- + \lambda\mathcal{W}_3^- & A\Phi_2^- + \lambda\mathcal{W}_4^- \end{pmatrix} \Big|_{y=0} \\
&= \det \left[\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} \Phi_1^+ & \Phi_2^+ & \Phi_1^- & \Phi_2^- \\ \lambda\mathcal{W}_1^+ & \lambda\mathcal{W}_2^+ & -a_1^- r_1^- + \lambda\mathcal{W}_3^- & \lambda\mathcal{W}_2^- \end{pmatrix} \right] \Big|_{y=0} \\
&= \det \left[\begin{pmatrix} \Phi_1^+ & \Phi_2^+ & \Phi_1^- & \Phi_2^- \\ \lambda\mathcal{W}_1^+ & \lambda\mathcal{W}_2^+ & -a_1^- r_1^- + \lambda\mathcal{W}_3^- & \lambda\mathcal{W}_2^- \end{pmatrix} \right] \Big|_{y=0}.
\end{aligned}$$

From this final expression we see immediately that $D(0) = 0$. The standard approach toward gaining higher order information on the Evans function at $\lambda = 0$ involves differentiating this final expression with repect to λ (see for example, [GZ]). Since in the case of a degenerate wave, the Evans function is not analytic at $\lambda = 0$, we cannot follow this approach. As observed in [HZ.2], however, the claim of Lemma 3.3 can be established directly from our detailed ODE estimates of Lemma 3.2. □

In addition to the estimates of Lemma 3.2, we require estimates on the solutions dual to $\phi_k^\pm$ and $\psi_k^\pm$. The ODE dual to (3.1) takes the form

$$\begin{aligned}
z_{1_{xx}} + a_{11}(x) z_{1_x} + a_{21}(x) z_{2_x} &= \lambda z_1 \\
z_{2_{xx}} + a_{12}(x) z_{1_x} + a_{22}(x) z_{2_x} &= \lambda z_2,
\end{aligned} \tag{3.4}$$

whose solutions we will denote $\tilde{\phi}_k^\pm(x;\lambda), \tilde{\psi}_k^\pm(x;\lambda)$. Denoting $Z_1 = z_1$, $Z_2 = z_2$, $Z_3 = z_{1_x}$, and $Z_4 = z_{2_x}$, we have the associated first order system

$$Z' = Z\tilde{\mathbb{A}}(x;\lambda) \tag{3.5}$$

where

$$\tilde{\mathbb{A}}(x;\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & -a_{11}(x) & -a_{21}(x) \\ 0 & \lambda & -a_{12}(x) & -a_{22}(x) \end{pmatrix}$$

We will employ the following lemma from [ZH].

Lemma 3.4. Suppose $V(x;\lambda)$ satisfies (3.2). Then $Z(x;\lambda)$ is a solution of (3.5) if and only if $ZSV =$ constant, where

$$S(x) = \begin{pmatrix} -A(x) & I \\ -I & 0 \end{pmatrix}.$$

According to Lemma 3.4, we can describe dual solutions through the relations

$$\begin{aligned}
&\tilde{\phi}_j^+(x;\lambda)S(x)\psi_k^+(x;\lambda) = \tilde{\psi}_j^+(x;\lambda)S(x)\phi_k^+(x;\lambda) = 0, \quad \forall j,k = 1,2 \\
&\tilde{\phi}_j^+(x;\lambda)S(x)\phi_k^+(x;\lambda) = \tilde{\psi}_j^+(x;\lambda)S(x)\psi_k^+(x;\lambda) = 0, \quad \forall j \neq k \\
&\tilde{\phi}_1^+(x;\lambda)S(x)\phi_1^+(x;\lambda) = \tilde{\psi}_1^+(x;\lambda)S(x)\psi_1^+(x;\lambda) = 1 \\
&\tilde{\phi}_2^+(x;\lambda)S(x)\phi_2^+(x;\lambda) = \tilde{\psi}_2^+(x;\lambda)S(x)\psi_2^+(x;\lambda) = \lambda^{3/2},
\end{aligned} \tag{3.6}$$

and similarly,

$$\begin{aligned}
&\tilde{\phi}_j^-(x;\lambda)S(x)\psi_k^-(x;\lambda) = \tilde{\psi}_j^-(x;\lambda)S(x)\phi_k^-(x;\lambda) = 0, \quad \forall j,k = 1,2 \\
&\tilde{\phi}_j^-(x;\lambda)S(x)\phi_k^-(x;\lambda) = \tilde{\psi}_j^-(x;\lambda)S(x)\psi_k^-(x;\lambda) = \delta_{jk}, \quad \forall j,k = 1,2.
\end{aligned} \tag{3.7}$$

We remark that the distinguished relations

$$\tilde{\phi}_2^+(x;\lambda)S(x)\phi_2^+(x;\lambda) = \tilde{\psi}_2^+(x;\lambda)S(x)\psi_2^+(x;\lambda) = \lambda^{3/2},$$

are taken as a result of the coalescence of $\phi_2^+(x;\lambda)$ and $\psi_2^+(x;\lambda)$ at $\lambda = 0$ (see Lemma 3.2(iii)).

Accordingly, we have the following estimates on solutions of (3.4).

Lemma 3.5. Under the assumptions of Theorem 1.1, there exists some constant r sufficiently small so that for $|\lambda| \le r$ we have the following estimates on solutions of (3.4). (The order expression $\mathbf{O}_a(\cdot)$ refers to a term meromorphic in $|\lambda| \le \delta_s$.)

(i) $x \le 0$

$$\begin{aligned}
\tilde{\phi}_k^-(x;\lambda) &= e^{-\mu_{k+2}^-(\lambda)x}(\tilde{V}_{k+2}^-(\lambda) + \mathbf{O}(e^{-\alpha|x|})) \\
\tilde{\psi}_k^-(x;\lambda) &= e^{-\mu_k^-(\lambda)x}(\tilde{V}_k^-(\lambda) + \mathbf{O}(e^{-\alpha|x|})),
\end{aligned}$$

where $k = 1,2$, the $\mu_k^-(\lambda)$ are as in Lemma 3.2, and $\tilde{V}_j^-(\lambda)$ are the eigenvectors of $\lim_{x\to-\infty}\tilde{\mathbb{A}}(x;\lambda)$, scaled so that $\tilde{V}_j^\pm(\lambda)V_k^\pm(\lambda) = \delta_k^j$. Moreover, $\tilde{\psi}_{23}^-(x;\lambda), \tilde{\psi}_{24}^-(x;\lambda), \tilde{\phi}_{13}^-(x;\lambda)$, and $\tilde{\phi}_{14}^-(x;\lambda)$ are all $\mathbf{O}_a(|\lambda|)$.

(ii) (non-degenerate solutions) For $x \ge L$, $k = 1,2$

$$\begin{aligned}
\tilde{\phi}_{1k}^+(x;\lambda) &= e^{-\int_L^x \mu_1(s;\lambda)ds}[\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)] \\
\tilde{\phi}_{1k}^{+\prime}(x;\lambda) = \tilde{\phi}_{1(k+2)}^+(x;\lambda) &= e^{-\int_L^x \mu_1(s;\lambda)ds}[\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)].
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}_{1k}^+(x;\lambda) &= e^{-\int_L^x \mu_3(s;\lambda)ds}[\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)] \\
\tilde{\psi}_{1k}^{+\prime} = \tilde{\psi}_{1(k+2)}^+(x;\lambda) &= e^{-\int_L^x \mu_3(s;\lambda)ds}[\mathbf{O}_a(|\lambda|) + \mathbf{O}(|\lambda^{3/2}\log\lambda|)],
\end{aligned}$$

while for $x \le L$ each expression is $\mathbf{O}(1)$, except $\tilde{\psi}_{1k}^{+\prime}$, which is $\mathbf{O}(|\lambda|)$.

(iii) $x \ge 0$, Degenerate solutions

$$\begin{aligned}
\tilde{\phi}_{2k}^+(x;\lambda) &= e^{\sqrt{\lambda}x}\Big(\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big) \\
\tilde{\phi}_{2k}^{+\prime}(x;\lambda) = \tilde{\phi}_{2(k+1)}^+(x;\lambda) &= e^{\sqrt{\lambda}x}\mathbf{O}(|\lambda|)\mathbf{O}(|x|)
\end{aligned}$$

$$\tilde{\psi}^+_{2k}(x;\lambda) = e^{-\sqrt{\lambda}x}\Big(\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big)$$

$$\tilde{\psi}^{+\prime}_{2k}(x;\lambda) = \tilde{\psi}^+_{2(k+2)}(x;\lambda) = e^{-\sqrt{\lambda}x}\mathbf{O}(|\lambda|)\mathbf{O}(|x|),$$

where $k = 1, 2$.

Proof. The case $x \leq 0$ is non-degenerate, and since the solutions do not coalesce at $\lambda = 0$, straightforward (see, for example, [ZH]). The only new assertion regards the $\mathbf{O}(|\lambda|)$ behavior of $\tilde{\Psi}^{-\prime}_2(y;\lambda)$ and $\tilde{\Phi}^{-\prime}_1(y;\lambda)$. Such behavior is most directly observed by choosing the fast growth solution $\psi^-_1(x;\lambda)$ as the derivative of a solution to the integrated equation (3.3). In this way, we insure by construction that

$$\Psi^{-\prime}_1 - A\Psi^-_1 = \mathbf{O}(|\lambda|)e^{\mu^-_1(\lambda)x}$$

According, then, to (3.7), we must have

$$\tilde{M}^-(x;\lambda)\tilde{\psi}^-_2(x;\lambda)^{\text{tr}} = (0,0,0,1)^{\text{tr}},$$

where, omitting independent variables for brevity,

$$\tilde{M}^\pm(x;\lambda) = \begin{pmatrix} -a_{11}\phi^\pm_{11} - a_{12}\phi^\pm_{12} + \phi^\pm_{13} & -a_{21}\phi^\pm_{11} - a_{22}\phi^\pm_{12} + \phi^\pm_{14} & -\phi^\pm_{11} & -\phi^\pm_{12} \\ -a_{11}\phi^\pm_{21} - a_{12}\phi^\pm_{22} + \phi^\pm_{23} & -a_{21}\phi^\pm_{21} - a_{22}\phi^\pm_{22} + \phi^\pm_{24} & -\phi^\pm_{21} & -\phi^\pm_{22} \\ -a_{11}\psi^\pm_{11} - a_{12}\psi^\pm_{12} + \psi^\pm_{13} & -a_{21}\psi^\pm_{11} - a_{22}\psi^\pm_{12} + \psi^\pm_{14} & -\psi^\pm_{11} & -\psi^\pm_{12} \\ -a_{11}\psi^\pm_{21} - a_{12}\psi^\pm_{22} + \psi^\pm_{23} & -a_{21}\psi^\pm_{21} - a_{22}\psi^\pm_{22} + \psi^\pm_{24} & -\psi^\pm_{21} & -\psi^\pm_{22} \end{pmatrix}.$$

Applying Cramer's rule and transposing the matrices, we find

$$\tilde{\psi}^-_{23}(x;\lambda) = \frac{\det\begin{pmatrix} \phi^-_{12} & \phi^-_{22} & \psi^-_{12} \\ -a_{11}\phi^-_{11} - a_{12}\phi^-_{12} + \phi^-_{13} & -a_{11}\phi^-_{21} - a_{12}\phi^-_{22} + \phi^-_{23} & -a_{11}\psi^-_{11} - a_{12}\psi^-_{12} + \psi^-_{13} \\ -a_{21}\phi^-_{11} - a_{22}\phi^-_{12} + \phi^-_{14} & -a_{21}\phi^-_{21} - a_{22}\phi^-_{22} + \phi^-_{24} & -a_{21}\psi^-_{11} - a_{22}\psi^-_{12} + \psi^-_{14} \end{pmatrix}}{\det\begin{pmatrix} \phi^-_1 & \phi^-_2 & \psi^-_1 & \psi^-_2 \end{pmatrix}}$$

$$= \frac{\det\begin{pmatrix} \phi^-_{12} & \phi^-_{22} & \psi^-_{12} \\ -a_{11}\phi^-_{11} - a_{12}\phi^-_{12} + \phi^-_{13} & \mathbf{O}(|\lambda|) & \mathbf{O}(|\lambda|) \\ -a_{21}\phi^-_{11} - a_{22}\phi^-_{12} + \phi^-_{14} & \mathbf{O}(|\lambda|) & \mathbf{O}(|\lambda|) \end{pmatrix}}{\det\begin{pmatrix} \phi^-_1 & \phi^-_2 & \psi^-_1 & \psi^-_2 \end{pmatrix}} = \mathbf{O}(|\lambda|)e^{-\mu^-_2(\lambda)x}.$$

The calculations for $\tilde{\psi}^-_{24}(x;\lambda)$, $\tilde{\phi}^-_{13}(x;\lambda)$, and $\tilde{\phi}^-_{14}(x;\lambda)$ are similar.

For $x \geq 0$, the analysis is considerably more delicate, and we carry out a full analysis for the cases $\tilde{\phi}^+_1(x;\lambda)$ and $\tilde{\phi}^+_2(x;\lambda)$ only. For $\tilde{\phi}^+_1(x;\lambda)$, we have from (3.6) the relation

$$\tilde{M}(x;\lambda)\tilde{\phi}^+_1(x;\lambda)^{\text{tr}} = (1,0,0,0)^{\text{tr}}.$$

Applying Cramer's rule, we find

$$\tilde{\phi}^+_{13}(x;\lambda) = \frac{\det\begin{pmatrix} \phi^+_{22} & \psi^+_{12} & \psi^+_{21} \\ -a_{11}\phi^+_{21} - a_{12}\phi^+_{22} + \phi^+_{23} & -a_{11}\psi^+_{11} - a_{12}\psi^+_{12} + \psi^+_{13} & -a_{11}\psi^+_{21} - a_{12}\psi^+_{22} + \psi^+_{23} \\ -a_{21}\phi^+_{21} - a_{22}\phi^+_{22} + \phi^+_{24} & -a_{21}\psi^+_{11} - a_{22}\psi^+_{12} + \psi^+_{14} & -a_{21}\psi^+_{21} - a_{22}\psi^+_{22} + \psi^+_{24} \end{pmatrix}}{\det\begin{pmatrix} \phi^+_1 & \phi^+_2 & \psi^+_1 & \psi^+_2 \end{pmatrix}}.$$

We observe now that by our construction from the integrated equation (3.3), we have the reduced form

$$\tilde{\phi}_{13}^{+}(x;\lambda) = -\frac{\det\begin{pmatrix} \phi_{22}^{+} & \psi_{12}^{+} & \psi_{22}^{+} \\ \lambda W_{21}^{+} & W_{31}^{+} & \lambda W_{41}^{+} \\ \lambda W_{22}^{+} & W_{32}^{+} & \lambda W_{42}^{+} \end{pmatrix}}{\det\begin{pmatrix} \phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+} \end{pmatrix}}.$$

Computing directly, we observe that the numerator has the form

$$\lambda W_{31}^{+}(\phi_{22}^{+}W_{42}^{+} - \psi_{22}^{+}W_{22}^{+}) + \lambda^2\psi_{12}^{+}(W_{41}^{+}W_{22}^{+} - W_{21}^{+}W_{42}^{+}) + \lambda W_{32}^{+}(\psi_{22}^{+}W_{21}^{+} - \phi_{22}^{+}W_{41}^{+}).$$

For the terms in parentheses, we have cancellation, exemplified by the calculation,

$$\begin{aligned} \phi_{22}^{+}W_{42}^{+} - \psi_{22}^{+}W_{22}^{+} =& (\bar{u}_2(x) - u_2^{+})^2\Big[(-\sqrt{\lambda} + \frac{\bar{u}_{2_x}}{\bar{u}_2 - u_2^{+}} + E_{24} + \frac{\bar{u}_{2_x}}{\bar{u}_2 - u_2^{+}}E_{22})(1 + E_{42}) \\ & -(\sqrt{\lambda} + \frac{\bar{u}_{2_x}}{\bar{u}_2 - u_2^{+}} + E_{44} + \frac{\bar{u}_{2_x}}{\bar{u}_2 - u_2^{+}}E_{42})(1 + E_{22})\Big] \\ =& (\bar{u}_2(x) - u_2^{+})^2\Big[-2\sqrt{\lambda} + (E_{24} - E_{44}) - \sqrt{\lambda}(E_{42} + E_{22}) + (E_{42}E_{24} - E_{22}E_{44})\Big] \\ =& \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^{1/2}|) + \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda \log \lambda|), \end{aligned}$$

with similar estimates on the remaining terms. We conclude that the numerator has the form

$$e^{\int_L^x \mu_3(s;\lambda)ds}\Big[\mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^{3/2}|) + \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^2 \log \lambda|)\Big].$$

Computing directly (i.e. expanding the determinants), according to the estimates of Lemmas 3.1 and 3.2, we find

$$\det\begin{pmatrix} \phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+} \end{pmatrix} = e^{\int_L^x \mu_1(s;\lambda)+\mu_3(s;\lambda)ds}\Big(\text{ord}(\lambda^{3/2})\text{ord}(|x|^{-2}) + \text{ord}(\lambda)\text{ord}(|x|^{-3})\tilde{E}(x;\lambda)\Big),$$

where by *ord()* we mean strict order, bounded above and below (hence a term we can divide by), and by $\tilde{E}$ we mean

$$\tilde{E}(x;\lambda) = E_{41}(x;\lambda) + E_{22}(x;\lambda) - E_{42}(x;\lambda) - E_{21}(x;\lambda) = \mathbf{O}(|\lambda^{1/2}\log\lambda|) \wedge \mathbf{O}(|x|^{-1}).$$

Combining these representations, we find that

$$\tilde{\phi}_{13}^{+}(x;\lambda) = e^{-\int_L^x \mu_1(s;\lambda)ds}[\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)].$$

Integrating $\tilde{\phi}_{13}^{+}(x;\lambda)$ on $[x,+\infty)$, we obtain a similar estimate on $\tilde{\phi}_{11}^{+}(x;\lambda)$.

For $\tilde{\phi}_2^{+}(x;\lambda)$, we have

$$\tilde{M}(x;\lambda)\tilde{\phi}_2^{+}(x;\lambda)^{\text{tr}} = (0, \lambda^{3/2}, 0, 0)^{\text{tr}}.$$

Applying Cramer's rule, we find

$$\tilde{\phi}_{21}^{+}(x;\lambda) = -\lambda^{3/2}\frac{\det\begin{pmatrix} \phi_{11}^{+} & \psi_{11}^{+} & \psi_{21}^{+} \\ \phi_{12}^{+} & \psi_{12}^{+} & \psi_{22}^{+} \\ -a_{21}\phi_{11}^{+} - a_{22}\phi_{12}^{+} + \phi_{14}^{+} & -a_{21}\psi_{11}^{+} - a_{22}\psi_{12}^{+} + \psi_{14}^{+} & -a_{21}\psi_{21}^{+} - a_{22}\psi_{22}^{+} + \psi_{24}^{+} \end{pmatrix}}{\det\begin{pmatrix} \phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+} \end{pmatrix}}.$$

We observe now that by our construction from the integrated equation (3.3), we have the reduced form

$$\tilde{\phi}_{21}^+(x;\lambda) = -\lambda^{3/2} \frac{\det \begin{pmatrix} \phi_{11}^+ & \psi_{11}^+ & \psi_{21}^+ \\ \phi_{12}^+ & \psi_{12}^+ & \psi_{22}^+ \\ \lambda W_{12}^+ & W_{32}^+ & \lambda W_{42}^+ \end{pmatrix}}{\det \begin{pmatrix} \phi_1^+ & \phi_2^+ & \psi_1^+ & \psi_2^+ \end{pmatrix}}.$$

Expanding as in the case $\tilde{\phi}_{21}^+(x;\lambda)$, we have

$$\det \begin{pmatrix} \phi_{11}^+ & \psi_{11}^+ & \psi_{21}^+ \\ \phi_{12}^+ & \psi_{12}^+ & \psi_{22}^+ \\ \lambda W_{12}^+ & W_{32}^+ & \lambda W_{42}^+ \end{pmatrix} = \\ e^{\sqrt{\lambda}x + \int_L^x (\mu_1(s;\lambda)+\mu_3(x;\lambda))ds} \Big(\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda|^{1/2})\mathbf{O}(|x|^{-1}) + \mathbf{O}(|\lambda^{1/2}\log\lambda|) \Big),$$

from which our estimate on $\tilde{\phi}_{21}^+(x;\lambda)$ is immediate. The remaining cases are similar. □

Lemma 3.6. Under the assumptions of Lemma 3.5, we observe the following critical cancellation estimates, for which as in Lemma 3.5 the expression $\mathbf{O}_a(\cdot)$ refers to terms meromorphic in a neighborhood of $\lambda = 0$.

(i) For $k = 1, 2$,

$$\lambda^{-3/2}(E_{4k}(0;\lambda) - E_{2k}(0;\lambda)) + m(\lambda) = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)$$

(ii)

$$d_{22}^+(\lambda)\Phi_2^+(x;\lambda) - \lambda^{-3/2}\Psi_2^+(x;\lambda) \\ = e^{\sqrt{\lambda}x}\Big(\frac{P_+\bar{u}_x}{\lambda} + \mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|\lambda^{-1/2}|) \wedge \mathbf{O}(|\lambda^{-1}|)\mathbf{O}(|x|^{-1})\Big)$$

(iii)

$$d_{22}^+(\lambda)\tilde{\Psi}_2^+(x;\lambda) + \lambda^{-3/2}\tilde{\Phi}_2^+(x;\lambda) \\ = e^{\sqrt{\lambda}x}\Big(\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|) + (\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^2)) \wedge (\mathbf{O}(|\lambda^{-1}|)\mathbf{O}(|x|^{-1}))\Big)$$

(iv)

$$d_{22}^+(\lambda)\tilde{\Psi}_2^{+\prime}(x;\lambda) + \lambda^{-3/2}\tilde{\Phi}_2^{+\prime}(x;\lambda) \\ = e^{\sqrt{\lambda}x}\Big(\mathbf{O}(|\log\lambda|) + \mathbf{O}(|\lambda^{1/2}(\log\lambda)^2|)\mathbf{O}(|x|) + (\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^2)) \wedge (\mathbf{O}(|x|))\Big),$$

where $m(\lambda)$ and $d_{22}^+(\lambda)$ are defined through the relation

$$d_{22}^+(\lambda) = \lambda^{-3/2}\frac{\det\begin{pmatrix} \phi_1^+ & \psi_2^+ & \phi_1^- & \phi_2^- \end{pmatrix}}{\det\begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_1^- & \phi_2^- \end{pmatrix}} = \lambda^{-3/2} + m(\lambda), \tag{3.8}$$

for which $m(\lambda) = \mathbf{O}(|\lambda^{-1}\log\lambda|)$.

Proof. The cancellation in Lemma 3.6 arises from the coalescing of the decay and growth degenerate solutions. In the case of non-degenerate waves, independence of these solutions plays a critical role in the determination of the behavior of the scattering coefficients (see, for example, [ZH]). Here, rather, the scattering coefficients must be understood first so that the extent of the dependence can be observed.

We begin by observing that the estimate (3.8) on the scattering coefficient $d_{22}^+(\lambda)$ (and on $m(\lambda)$) is a direct result of the relationship

$$\phi_2^+(0;\lambda) = \psi_2^+(0;\lambda) + \mathbf{O}(|\lambda^{1/2}\log\lambda|),$$

which is immedate from the estimates of Lemma 2.2. In particular, since $d_{22}^+(\lambda)$ does not depend on x, we can evaluate the determinant quotient at any value of x, and we evaluate it at $x=0$.

Taking $k=1$ for definiteness in (i), we first compute

$$\begin{aligned}
d_{22}^+(\lambda)\Phi_{21}^+(x;\lambda) - \lambda^{-3/2}\Psi_{21}^+(x;\lambda) &= (\lambda^{-3/2}+m(\lambda))\Phi_{21}^+(x;\lambda) - \lambda^{-3/2}\Psi_{21}^+(x;\lambda) \\
=&(\lambda^{-3/2}+m(\lambda))e^{-\sqrt{\lambda}x}(\bar{u}_1(x)-u_1^+)\Big(-\sqrt{\lambda}+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}+E_{23}(x;\lambda)+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}E_{21}(x;\lambda)\Big) \\
&-\lambda^{-3/2}e^{\sqrt{\lambda}x}(\bar{u}_1(x)-u_1^+)\Big(\sqrt{\lambda}+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}+E_{43}(x;\lambda)+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}E_{41}(x;\lambda)\Big).
\end{aligned}$$

In the case $|\sqrt{\lambda}x|\le 1$, we have

$$\begin{aligned}
&(\lambda^{-3/2}+m(\lambda))(1+\mathbf{O}(|\sqrt{\lambda}x|))(\bar{u}_1(x)-u_1^+)\Big(-\sqrt{\lambda}+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}+E_{23}(x;\lambda)+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}E_{21}(x;\lambda)\Big) \\
&\quad-\lambda^{-3/2}(1+\mathbf{O}(|\sqrt{\lambda}x|))(\bar{u}_1(x)-u_1^+)\Big(\sqrt{\lambda}+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}+E_{43}(x;\lambda)+\frac{\bar{u}_{1_x}}{\bar{u}_1-u_1^+}E_{41}(x;\lambda)\Big) \\
&=\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-1})+\bar{u}_{1_x}(x)\Big((E_{21}(x;\lambda)-E_{41}(x;\lambda))\lambda^{-3/2}+m(\lambda)\Big) \\
&\qquad+\mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-1})+\mathbf{O}(|\lambda^{-1/2}|)\wedge\mathbf{O}(|\lambda^{-1}|)\mathbf{O}(|x|^{-1}).
\end{aligned} \tag{3.9}$$

According to Lemma 3.3, spectral stability implies

$$d_{22}^+(\lambda)\Phi_{21}^+(0;\lambda)-\lambda^{-3/2}\Psi_{21}^+(0;\lambda)=\mathbf{O}_a(|\lambda^{-1}|)+\mathbf{O}(|\lambda^{-1/2}\log\lambda|).$$

Combining these last two equalities, we conclude (i). Recalling from Lemma 3.1 the relations $E_{jk}(x;\lambda) = E_{jk}(0;\lambda)+\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|)$, we have

$$\begin{aligned}
&\bar{u}_{1_x}(x)\Big((E_{21}(x;\lambda)-E_{41}(x;\lambda))\lambda^{-3/2}+m(\lambda)\Big) \\
&=\bar{u}_{1_x}(x)\Big((E_{21}(0;\lambda)-E_{41}(0;\lambda))\lambda^{-3/2}+m(\lambda)+\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|)\lambda^{-3/2}\Big) \\
&=\bar{u}_{1_x}(x)\Big(\mathbf{O}_a(|\lambda^{-1}|)+\mathbf{O}(|\lambda^{-1/2}\log\lambda|)+\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|)\Big),
\end{aligned}$$

from which we conclude (ii). In the case $|\sqrt{\lambda}x|\ge 1$, we observe that $|x|^{-1}\le\sqrt{\lambda}$ from which (ii) follows immediately from (3.9).

For estimate (iii), we observe as in the proof of Lemma 3.5 that

$$\begin{aligned}
\tilde{\phi}_{21}^+(x;\lambda) =& -\lambda^{3/2}\frac{\det\begin{pmatrix}\phi_{11}^+ & \psi_{11}^+ & \psi_{21}^+\\ \phi_{12}^+ & \psi_{12}^+ & \psi_{22}^+\\ \lambda W_{12}^+ & W_{32}^+ & \lambda W_{42}^+\end{pmatrix}}{\det\begin{pmatrix}\phi_1^+ & \phi_2^+ & \psi_1^+ & \psi_2^+\end{pmatrix}} \\
=&\frac{-\lambda^{3/2}}{\det\begin{pmatrix}\phi_1^+ & \phi_2^+ & \psi_1^+ & \psi_2^+\end{pmatrix}}\Big[W_{32}^+(\psi_{21}^+\phi_{12}^+-\phi_{11}^+\psi_{22}^+)+\mathbf{O}(|\lambda|)\Big],
\end{aligned}$$

with also

$$\tilde{\psi}_{21}^{+}(x;\lambda) = -\lambda^{3/2}\frac{\det\begin{pmatrix}\phi_{11}^{+} & \phi_{21}^{+} & \psi_{11}^{+}\\ \phi_{12}^{+} & \phi_{22}^{+} & \psi_{12}^{+}\\ \lambda W_{12}^{+} & \lambda W_{22}^{+} & W_{32}^{+}\end{pmatrix}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}$$
$$= \frac{-\lambda^{3/2}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[W_{32}^{+}(\phi_{11}^{+}\phi_{22}^{+} - \phi_{21}^{+}\phi_{12}^{+}) + \mathbf{O}(|\lambda|)\Big].$$

We have, then,

$$\begin{aligned}&d_{22}^{+}(\lambda) + \tilde{\psi}_2^{+}(x;\lambda) + \lambda^{-3/2}\tilde{\phi}_2^{+}(x;\lambda)\\ &\quad = -\frac{\lambda^{3/2}(\psi_{11}^{+} - a_{11}(x)\psi_{11}^{+} - a_{12}(x)\psi_{12}^{+})}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[\phi_{12}^{+}(d_{22}(\lambda)\psi_{21}^{+} - \lambda^{-3/2}\phi_{21}^{+}) - \phi_{11}^{+}(d_{22}(\lambda)\psi_{22}^{+} - \lambda^{-3/2}\phi_{22}^{+})\Big]\\ &\quad +\mathbf{O}(|\lambda|).\end{aligned}$$

Combining (ii) with the estimate

$$\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix} = e^{\int_L^x \mu_1(s;\lambda)+\mu_3(s;\lambda)ds}\Big(\mathrm{ord}(\lambda^{3/2})\mathrm{ord}(|x|^{-2}) + \mathrm{ord}(\lambda)\mathrm{ord}(|x|^{-3})\tilde{E}\Big),$$

and repeating the analysis for $\tilde{\phi}_{22}^{+}$ and $\tilde{\psi}_{22}^{+}$ we obtain (iii).

Finally, for estimate (iv) we note that

$$\tilde{\phi}_{21}^{+\prime}(x;\lambda) = \lambda^{3/2}\frac{\det\begin{pmatrix}\phi_{12}^{+} & \psi_{12}^{+} & \psi_{22}^{+}\\ \lambda W_{11}^{+} & W_{31}^{+} & \lambda W_{41}^{+}\\ \lambda W_{12}^{+} & W_{32}^{+} & \lambda W_{42}^{+}\end{pmatrix}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}$$
$$= \frac{\lambda^{3/2}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[(\lambda\phi_{12}^{+}W_{42}^{+} - \lambda\psi_{22}^{+}W_{12}^{+})W_{31}^{+} + (\lambda\psi_{22}^{+}W_{11}^{+} - \lambda\phi_{12}^{+}W_{41}^{+})W_{32}^{+} + \mathbf{O}(|\lambda^2|)\Big],$$

with also

$$\tilde{\psi}_{21}^{+\prime}(x;\lambda) = \lambda^{3/2}\frac{\det\begin{pmatrix}\phi_{12}^{+} & \phi_{22}^{+} & \psi_{12}^{+}\\ \lambda W_{11}^{+} & \lambda W_{21}^{+} & W_{31}^{+}\\ \lambda W_{12}^{+} & \lambda W_{22}^{+} & W_{32}^{+}\end{pmatrix}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}$$
$$= \frac{\lambda^{3/2}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[(\lambda\phi_{12}^{+}W_{21}^{+} - \lambda\phi_{22}^{+}W_{11}^{+})W_{32}^{+} + (\lambda\phi_{22}^{+}W_{12}^{+} - \lambda\phi_{12}^{+}W_{22}^{+})W_{31}^{+} + \mathbf{O}(|\lambda^2|)\Big].$$

Combining, we have

$$
\begin{aligned}
d_{22}^{+}(\lambda)\tilde{\psi}_2^{+\prime}(x;\lambda) + \lambda^{-3/2}\tilde{\phi}_2^{+\prime}(x;\lambda)&\\
=&\frac{\lambda^{5/2}W_{31}^{+}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[d_{22}(\lambda)(\phi_{22}^{+}W_{12}^{+} - \phi_{12}^{+}W_{22}^{+}) + \lambda^{-3/2}(\phi_{12}^{+}W_{42}^{+} - \psi_{22}^{+}W_{12}^{+}) + \mathbf{O}(|\lambda|)\Big]\\
&-\frac{\lambda^{5/2}W_{31}^{+}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[d_{22}(\lambda)(\phi_{12}^{+}W_{21}^{+} - \phi_{22}^{+}W_{11}^{+}) + \lambda^{-3/2}(\psi_{22}^{+}W_{11}^{+} - \phi_{12}^{+}W_{41}^{+}) + \mathbf{O}(|\lambda|)\Big]\\
=&\frac{\lambda^{5/2}W_{31}^{+}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[W_{12}^{+}(d_{22}(\lambda)\phi_{22}^{+} - \lambda^{-3/2}\psi_{22}^{+}) - \phi_{12}^{+}(d_{22}(\lambda)W_{22}^{+} - \lambda^{-3/2}W_{42}^{+}) + \mathbf{O}(|\lambda|)\Big]\\
&-\frac{\lambda^{5/2}W_{32}^{+}}{\det\begin{pmatrix}\phi_1^{+} & \phi_2^{+} & \psi_1^{+} & \psi_2^{+}\end{pmatrix}}\Big[W_{11}^{+}(d_{22}(\lambda)\phi_{22}^{+} - \lambda^{-3/2}\psi_{22}^{+}) - \phi_{12}^{+}(d_{22}(\lambda)W_{21}^{+} - \lambda^{-3/2}W_{41}^{+}) + \mathbf{O}(|\lambda|)\Big],
\end{aligned}
$$

from which we have (iv). □

4 Estimates on $G_\lambda(x;y)$

In this section we develop estimates on $G_\lambda(x,y)$, first for the critical case of λ near 0 (which will correspond with large t behavior).

Lemma 4.1. (Small $|\lambda|$ $G_\lambda(x,y)$ estimates.) Let hypotheses **(H0)–(H2)** hold, as well as condition $(\mathcal{D})$. Then for $|\lambda| \le r$, some r sufficiently small, we have the following estimates, for which terms containing $\mathbf{O}_a(\cdot)$ are meromorphic to the right of Γ_d, while the remaining terms are analytic to the right of Γ_d and away from the negative real axis.

(i) $y \le x \le 0$

$$
\begin{aligned}
G_\lambda(x,y) =&\mathbf{O}_a(1)e^{\mu_2^-(\lambda)(x-y)} + \mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}\\
&+\bar{u}_x(x)\Big(\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =&\mathbf{O}_a(|\lambda|)e^{\mu_2^-(\lambda)(x-y)} + \mathbf{O}_a(|\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \mathbf{O}_a(1)e^{\mu_1^-(\lambda)(x-y)}\\
&+\mathbf{O}(|\lambda^{3/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \bar{u}_x(x)\Big(\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

(ii) $x \le y \le 0$

$$
\begin{aligned}
G_\lambda(x,y) =&\mathbf{O}_a(1)e^{\mu_3^-(\lambda)(x-y)} + \mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}\\
&+\bar{u}_x(x)\Big(\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =&\mathbf{O}_a(|\lambda|)e^{\mu_3^-(\lambda)(x-y)} + \mathbf{O}_a(|\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \mathbf{O}_a(1)e^{\mu_4^-(\lambda)(x-y)}\\
&+\mathbf{O}(|\lambda^{3/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} + \bar{u}_x(x)\Big(\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

(iii) $x \le 0 \le y$

$$
\begin{aligned}
G_\lambda(x,y) =&\mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\int_L^y \mu_3(s;\lambda)ds} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\int_L^y \mu_3(s;\lambda)ds} + \mathbf{O}(1)e^{\mu_3^-(\lambda)x-\sqrt{\lambda}y}\\
&+\bar{u}_x(x)\Big(\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{\int_L^y \mu_3(s;\lambda)ds}\\
&+\bar{u}_x(x)\Big(\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{-\sqrt{\lambda}y}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =& \mathbf{O}_a(|\lambda|)e^{\mu_3^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} + \mathbf{O}(|\lambda^{3/2}\log\lambda|)e^{\mu_3^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} + \mathbf{O}(|\lambda|)\mathbf{O}(|y|)e^{\mu_3^-(\lambda)x - \sqrt{\lambda}y} \\
&+ \bar{u}_x(x)\Big(\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big)e^{-\int_L^y \mu_3(s;\lambda)ds} + \bar{u}_x(x)\mathbf{O}(|y|)e^{-\sqrt{\lambda}y}
\end{aligned}
$$

(iv) $y \le 0 \le x$

$$
\begin{aligned}
G_\lambda(x,y) =& \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x - \mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}x - \mu_2^-(\lambda)y} \\
&+ \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =& \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x - \mu_2^-(\lambda)y} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_L^x \mu_1(s;\lambda)ds - \mu_2^-(\lambda)y} \\
&+ \bar{u}_x(x)\mathbf{O}_a(1)e^{-\mu_2^-(\lambda)y}
\end{aligned}
$$

(v) $0 \le y \le x$

$$
\begin{aligned}
G_\lambda(x,y) =& e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|\lambda^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|) + \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)\Big] \\
&+ e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds}\Big[\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big] \\
&+ \mathbf{O}_a(1)e^{\int_y^x \mu_1(s;\lambda)ds} + \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\int_L^y \mu_3(s;\lambda)ds}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =& e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|x|^{-2})\mathbf{O}(|y|) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\mathbf{O}(|x|^{-1}) \\
&+ \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|) + \mathbf{O}(|\log\lambda|)\mathbf{O}(|x|^{-2})\Big] \\
&+ \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds} + \bar{u}_x(x)\mathbf{O}_a(1)e^{-\int_L^y \mu_3(s;\lambda)ds} \\
&+ \mathbf{O}_a(1)e^{\int_y^x \mu_1(s;\lambda)ds} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_y^x \mu_1(s;\lambda)ds}.
\end{aligned}
$$

(vi) $0 \le x \le y$

$$
\begin{aligned}
G_\lambda(x,y) =& e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|) + \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-1})\Big] \\
&+ e^{-\int_x^y \mu_3(s;\lambda)ds}\Big[\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)\Big] \\
&+ e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds}\Big[\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big] \\
&+ \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\int_L^y \mu_3(s;\lambda)ds}
\end{aligned}
$$

$$
\begin{aligned}
\partial_y G_\lambda(x,y) =& e^{-\sqrt{\lambda}|x-y|}\mathbf{O}(|x|^{-1})\mathbf{O}(|y|) + \mathbf{O}_a(|\lambda|)e^{-\int_x^y \mu_3(s;\lambda)ds} \\
&+ \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds} + \bar{u}_x(x)\mathbf{O}_a(1)e^{-\int_L^y \mu_3(s;\lambda)ds} \\
&+ \mathbf{O}(|\lambda^{3/2}\log\lambda|)e^{-\int_x^y \mu_3(s;\lambda)ds}
\end{aligned}
$$

For $y \le L$, the expressions $e^{\int_L^y \mu_3(s;\lambda)ds}$ can be replaced by $\mathbf{O}(1)$ and similarly for x.

Remarks on Lemma 4.1. The fundamentally new aspect of the estimates of Lemma 4.1 with respect to analogous results for non-degenerate waves is the loss of analyticity in a neighborhood of $\lambda = 0$. On the degenerate side ($x > 0$), our linear equation behaves like a heat equation, and we expect such loss of analyticity, at least with terms of the form $\sqrt{\lambda}$. Even on the non-degenerate side, however, the Green's function is constructed from solutions that decay at both $\pm\infty$ and our expansion coefficients carry $\sqrt{\lambda}$ and $\sqrt{\lambda}\log\lambda$ behavior.

Proof of Lemma 4.1. Following [ZH], we establish estimates on $G_\lambda(x,y)$ through the useful representation

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \begin{cases} \Phi^+(x;\lambda)M^+(\lambda)\tilde\Psi^-(y;\lambda), & x>y \\ \Phi^-(x;\lambda)M^-(\lambda)\tilde\Psi^+(y;\lambda), & x<y, \end{cases}$$

where

$$\Phi^\pm = \begin{pmatrix} \Phi_1^\pm & \Phi_2^\pm \\ \Phi_1^{\pm\prime} & \Phi_2^{\pm\prime} \end{pmatrix} \text{ and } \tilde\Psi^\pm = \begin{pmatrix} \tilde\Psi_1^\pm & \tilde\Psi_1^{\pm\prime} \\ \tilde\Psi_2^\pm & \tilde\Psi_2^{\pm\prime} \end{pmatrix}.$$

According to [ZH] and the methods described there, we have the additional representations,

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \begin{cases} -(0,\Phi^-(x;\lambda))(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}, & x\le y \\ (\Phi^+(x;\lambda),0)(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}, & x\ge y \end{cases} \tag{4.1}$$

and

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \begin{cases} -S(x)^{-1}\begin{pmatrix} \tilde\Psi^-(x;\lambda) \\ \tilde\Psi^+(x;\lambda) \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \tilde\Psi^+(y;\lambda) \end{pmatrix}, & x\le y \\ S(x)^{-1}\begin{pmatrix} \tilde\Psi^-(x;\lambda) \\ \tilde\Psi^+(x;\lambda) \end{pmatrix}^{-1}\begin{pmatrix} \tilde\Psi^-(y;\lambda) \\ 0 \end{pmatrix}, & x\ge y. \end{cases} \tag{4.2}$$

In order to see the validity of these relationships, we briefly recall the development of [ZH]. Observing the jump condition,

$$\begin{bmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{bmatrix} = \begin{pmatrix} 0 & -I \\ I & -A \end{pmatrix} =: S^{-1},$$

we have

$$\Phi^+(y;\lambda)M^+(\lambda)\tilde\Psi^+(y;\lambda) - \Phi^-(y;\lambda)M^-(\lambda)\tilde\Psi^+(y;\lambda) = S(y)^{-1},$$

or in matrix form

$$(\Phi^+(y;\lambda),\Phi^-(y;\lambda))\begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix}\begin{pmatrix} \tilde\Psi^-(y;\lambda) \\ \tilde\Psi^+(y;\lambda) \end{pmatrix} = S(y)^{-1},$$

from which we conclude the critical relation

$$\begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} = (\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}\begin{pmatrix} \tilde\Psi^-(y;\lambda) \\ \tilde\Psi^+(y;\lambda) \end{pmatrix}^{-1}.$$

In order to establish (4.1) we re-write this last relationship as

$$(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1} = \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix}\begin{pmatrix} \tilde\Psi^-(y;\lambda) \\ \tilde\Psi^+(y;\lambda) \end{pmatrix},$$

and for $x>y$ multiply on both sides by $(\Phi^+(x;\lambda),0)$ and for $x<y$ multiply on both sides by $-(0,\Phi^-(y;\lambda))$. The case of (4.2) is similar.

Case (i) $y\le x\le 0$. In the case $y\le x\le 0$, we compare representations

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k=1}^{2}\phi_k^-(x;\lambda)d_{kj}^-(\lambda)\tilde\psi_j^-(y;\lambda) + \sum_{j,k=1}^{2}\psi_k^-(x;\lambda)e_{kj}^-(\lambda)\tilde\psi_j^-(y;\lambda)$$

and

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = S(x)^{-1} \begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^-(y;\lambda) \\ 0 \end{pmatrix}$$
$$= \sum_{j=1}^{2} S(x)^{-1} \begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix}^{-1} E_j \tilde{\psi}_j^-(y;\lambda)$$

to obtain the relation

$$\sum_{k=1}^{2} \phi_k^-(x;\lambda) d_{kj}^-(\lambda) + \sum_{k=1}^{2} \psi_k^-(x;\lambda) e_{kj}^-(\lambda) = S(x)^{-1} \begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix}^{-1} E_j. \tag{4.3}$$

Multiplying (4.3) on the left by $\tilde{\psi}_l^-(x;\lambda)S(x)$ and recalling relations (3.7) we find

$$e_{lj}^-(\lambda) = \tilde{\psi}_l^-(x;\lambda) \begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix}^{-1} E_j,$$

for which $e_{lj}^-(\lambda)$ is the j^{th} component of the vector J satisfying

$$\begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix} J = \tilde{\psi}_l^-(x;\lambda).$$

Proceeding by Cramer's rule, we have for $j = 1$

$$e_{l1}^-(\lambda) = \frac{\det \begin{pmatrix} \tilde{\psi}_l^-(x;\lambda)^{\text{tr}} & \tilde{\psi}_2^-(x;\lambda)^{\text{tr}} & \tilde{\psi}_1^+(x;\lambda)^{\text{tr}} & \tilde{\psi}_2^+(x;\lambda)^{\text{tr}} \end{pmatrix}}{\det \begin{pmatrix} \tilde{\psi}_1^-(x;\lambda)^{\text{tr}} & \tilde{\psi}_2^-(x;\lambda)^{\text{tr}} & \tilde{\psi}_1^+(x;\lambda)^{\text{tr}} & \tilde{\psi}_2^+(x;\lambda)^{\text{tr}} \end{pmatrix}} = \begin{cases} 1, & l = 1 \\ 0, & l = 2, \end{cases}$$

and similarly for $j = 2$, so that $e_{lj}^-(\lambda) = \delta_l^j$, where δ_l^j represents the Kronecker delta.

Alternatively, multiplying (4.3) on the left by $\tilde{\phi}_l^-(x;\lambda)S(x)$ we find

$$d_{lj}^-(\lambda) = \tilde{\phi}_l^-(x;\lambda) \begin{pmatrix} \tilde{\Psi}^-(x;\lambda) \\ \tilde{\Psi}^+(x;\lambda) \end{pmatrix}^{-1} E_j.$$

Proceeding again by Cramer's rule and observing the estimates of Lemma 3.5 $\tilde{\Phi}_1^{-\prime}(x;\lambda) = \mathbf{O}(|\lambda|)$ and $\tilde{\Psi}_2^{-\prime}(x;\lambda) = \mathbf{O}(|\lambda|)$, we find the estimates $d_{11}^-(\lambda) = \mathbf{O}_a(|\lambda|) + \mathbf{O}(|\lambda^{3/2}\log\lambda|)$, $d_{12}^-(\lambda) = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$, $d_{21}(\lambda) = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$, and $d_{22}^-(\lambda) = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)$. Collecting these observations, we have

$$\begin{pmatrix} \Psi_1^-(x;\lambda)\tilde{\Psi}_1^-(y;\lambda) \\ \Psi_1^-(x;\lambda)\tilde{\Psi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} \mathbf{O}_a(1)e^{\mu_1^-(\lambda)(x-y)} \\ \mathbf{O}_a(1)e^{\mu_1^-(\lambda)(x-y)} \end{pmatrix}$$

$$\begin{pmatrix} \Psi_2^-(x;\lambda)\tilde{\Psi}_2^-(y;\lambda) \\ \Psi_2^-(x;\lambda)\tilde{\Psi}_2^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} \mathbf{O}_a(1)e^{\mu_2^-(\lambda)(x-y)} \\ \mathbf{O}_a(|\lambda|)e^{\mu_2^-(\lambda)(x-y)} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1^-(x;\lambda)d_{11}^-(\lambda)\tilde{\Psi}_1^-(y;\lambda) \\ \Phi_1^-(x;\lambda)d_{11}^-(\lambda)\tilde{\Psi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}_a(|\lambda|)+\mathbf{O}(|\lambda^{3/2}\log\lambda|))e^{\mu_3^-(\lambda)x-\mu_1^-(\lambda)y} \\ (\mathbf{O}_a(|\lambda|)+\mathbf{O}(|\lambda^{3/2}\log\lambda|))e^{\mu_3^-(\lambda)x-\mu_1^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1^-(x;\lambda)d_{12}^-(\lambda)\tilde{\Psi}_2^-(y;\lambda) \\ \Phi_1^-(x;\lambda)d_{12}^-(\lambda)\tilde{\Psi}_2^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}_a(1)+\mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} \\ (\mathbf{O}_a(|\lambda|)+\mathbf{O}(|\lambda^{3/2}\log\lambda|))e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_2^-(x;\lambda)d_{21}^-(\lambda)\tilde{\Psi}_1^-(y;\lambda) \\ \Phi_2^-(x;\lambda)d_{21}^-(\lambda)\tilde{\Psi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}_a(1)+\mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_4^-(\lambda)x-\mu_1^-(\lambda)y} \\ (\mathbf{O}_a(1)+\mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_4^-(\lambda)x-\mu_1^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_2^-(x;\lambda)d_{22}^-(\lambda)\tilde{\Psi}_2^-(y;\lambda) \\ \Phi_2^-(x;\lambda)d_{22}^-(\lambda)\tilde{\Psi}_2^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}_a(|\lambda^{-1}|)+\mathbf{O}(|\lambda^{-1/2}\log\lambda|))e^{\mu_4^-(\lambda)x-\mu_2^-(\lambda)y} \\ (\mathbf{O}_a(1)+\mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_4^-(\lambda)x-\mu_2^-(\lambda)y} \end{pmatrix}.$$

Finally, observing that $\Phi_1^-(x;0)$ does not decay as $x\to-\infty$, we conclude that

$$\Phi_2^-(x;0) = P\bar{u}_x(x),$$

for some *projection constant* P, from which the estimates of Case (i) are immediate.

Case (ii) $x\le y\le 0$. The analysis of Case (ii) is almost identical to the analysis of Case (i). The only new terms are $\Phi_1^-(x;\lambda)\tilde{\Phi}_1^-(x;\lambda)$ and $\Phi_2^-(x;\lambda)\tilde{\Phi}_2^-(x;\lambda)$, for which we find

$$\begin{pmatrix} \Phi_1^-(x;\lambda)\tilde{\Phi}_1^-(y;\lambda) \\ \Phi_1^-(x;\lambda)\tilde{\Phi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} \mathbf{O}_a(1)e^{\mu_3^-(\lambda)(x-y)} \\ \mathbf{O}_a(|\lambda|)e^{\mu_3^-(\lambda)(x-y)} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_2^-(x;\lambda)\tilde{\Phi}_2^-(y;\lambda) \\ \Phi_2^-(x;\lambda)\tilde{\Phi}_2^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} \mathbf{O}_a(1)e^{\mu_4^-(\lambda)(x-y)} \\ \mathbf{O}_a(1)e^{\mu_4^-(\lambda)(x-y)} \end{pmatrix}.$$

Case (iii) $x\le 0\le y$. In the case $x\le 0\le y$, we compare the representations

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k=1}^{2} \phi_k^-(x;\lambda)d_{kj}(\lambda)\tilde{\psi}_j^+(y;\lambda) \tag{4.4}$$

and

$$\begin{aligned} \begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} &= -(0,\Phi^-(x;\lambda))(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1} \\ &= -\sum_{k=1}^{2}\phi_k^-(x;\lambda)E_{2+k}^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1} \end{aligned}$$

to obtain the relation

$$\sum_{j=1}^{2} d_{kj}^-(\lambda)\tilde{\psi}_j^+(y;\lambda) = -E_{2+k}^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}.$$

(We observe that the scattering coefficients $d_{kj}(\lambda)$ here are not necessarily the same as the coefficients $d^-_{kj}(\lambda)$ from Cases (i) and (ii).) Multiplying this last equation on the right by $S(y)\psi_l^+(y;\lambda)$, we find

$$d_{kl}(\lambda)\tilde{\psi}_l^+(y;\lambda)S(y)\psi_l^+(y;\lambda) = -E_{2+k}^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}\psi_l^+(y;\lambda).$$

Proceeding as in Case (i) and the proof of Lemma 3.3, we find

$$\begin{aligned} d_{11}(\lambda) =& \frac{\det(\phi_1^+,\phi_2^+,\psi_1^+,\phi_2^-)}{\det(\phi_1^+,\phi_2^+,\phi_1^-,\phi_2^-)} = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|) \\ d_{21}(\lambda) =& \frac{\det(\phi_1^+,\phi_2^+,\phi_1^-,\psi_1^+)}{\det(\phi_1^+,\phi_2^+,\phi_1^-,\phi_2^-)} = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|). \end{aligned}$$

On the other hand, for $d_{12}(\lambda)$ and $d_{22}(\lambda)$ we have, due to our scaling (3.6),

$$\begin{aligned} d_{12}(\lambda) =& \lambda^{-3/2}\frac{\det(\phi_1^+,\phi_2^+,\psi_2^+,\phi_2^-)}{\det(\phi_1^+,\phi_2^+,\phi_1^-,\phi_2^-)} \\ d_{21}(\lambda) =& \lambda^{-3/2}\frac{\det(\phi_1^+,\phi_2^+,\phi_1^-,\psi_2^+)}{\det(\phi_1^+,\phi_2^+,\phi_1^-,\phi_2^-)}. \end{aligned}$$

In this case, we observe that the fast growth solution ψ_2^+ can without loss of generality be constructed by determining the fast growth solution to the integrated equation (3.3) and computing its derivative. Constructed in this way, ψ_2^+ clearly satisfies

$$({\Psi_2^+}'' - A(x)\Psi_2^+)\Big|_{\lambda=0} = 0,$$

according to which

$$\det(\phi_1^+,\phi_2^+,\psi_2^+,\phi_2^-) = \det\begin{pmatrix} \Phi_1^+ & \Phi_2^+ & \Psi_2^+ & \Phi_2^- \\ \lambda\mathcal{W}_1^+ & \lambda\mathcal{W}_2^+ & \lambda\mathcal{W}_4^+ & \lambda\mathcal{W}_2^- \end{pmatrix}.$$

Proceeding as in Lemma 3.3 and observing cancellation between the degenerate solutions Φ_2^+ and Ψ_2^+as in Lemma 3.6, we conclude

$$d_{12}(\lambda) = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|),$$

and similarly,

$$d_{22}(\lambda) = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|).$$

Gathering these observations, we have the estimates,

$$\begin{aligned} \begin{pmatrix} \Phi_1^-(x;\lambda)d_{11}(\lambda)\tilde{\Psi}_1^+(y;\lambda) \\ \Phi_1^-(x;\lambda)d_{11}(\lambda)\tilde{\Psi}_1^{+\prime}(y;\lambda) \end{pmatrix} &= \begin{pmatrix} (\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_3^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} \\ (\mathbf{O}_a(|\lambda|) + \mathbf{O}(|\lambda^{3/2}\log\lambda|))e^{\mu_3^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} \end{pmatrix} \\ \begin{pmatrix} \Phi_1^-(x;\lambda)d_{12}(\lambda)\tilde{\Psi}_2^+(y;\lambda) \\ \Phi_1^-(x;\lambda)d_{12}(\lambda)\tilde{\Psi}_2^{+\prime}(y;\lambda) \end{pmatrix} &= \begin{pmatrix} \mathbf{O}(1)e^{\mu_3^-(\lambda)x - \sqrt{\lambda}y} \\ \mathbf{O}(|\lambda|)\mathbf{O}(|y|)e^{\mu_3^-(\lambda)x - \sqrt{\lambda}y} \end{pmatrix} \\ \begin{pmatrix} \Phi_2^-(x;\lambda)d_{21}(\lambda)\tilde{\Psi}_1^+(y;\lambda) \\ \Phi_2^-(x;\lambda)d_{21}(\lambda)\tilde{\Psi}_1^{+\prime}(y;\lambda) \end{pmatrix} &= \begin{pmatrix} (\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|))e^{\mu_4^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} \\ (\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\mu_4^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds} \end{pmatrix} \\ \begin{pmatrix} \Phi_2^-(x;\lambda)d_{22}(\lambda)\tilde{\Psi}_2^+(y;\lambda) \\ \Phi_2^-(x;\lambda)d_{22}(\lambda)\tilde{\Psi}_2^{+\prime}(y;\lambda) \end{pmatrix} &= \begin{pmatrix} (\mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|))e^{\mu_4^-(\lambda)x - \sqrt{\lambda}y} \\ \mathbf{O}(|y|)e^{\mu_4^-(\lambda)x - \sqrt{\lambda}y} \end{pmatrix}. \end{aligned}$$

Case (iv) $y \le 0 \le x$. In the case $y \le 0 \le x$, we compare representations

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k=1}^{2} \phi_k^+(x;\lambda)\bar{d}_{kj}(\lambda)\tilde{\psi}_j^-(y;\lambda)$$

and

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} =(\Phi^+(x;\lambda),0)(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}$$
$$=\sum_{k=1}^{2} \phi_k^+(x;\lambda)E_k^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1},$$

to obtain the relation

$$\sum_{j=1}^{2} \bar{d}_{kj}(\lambda)\tilde{\psi}_j^-(y;\lambda) = E_k^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}. \tag{4.5}$$

Multiplying equation (4.5) by $S(y)\psi_l^-(y;\lambda)$ on the right, we find

$$\bar{d}_{kl}(\lambda) = E_k^{\mathrm{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}\psi_l^-(y;\lambda).$$

Proceeding as in Case (iii), we determine $\bar{d}_{11}(\lambda) = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$, $\bar{d}_{21}(\lambda) = \mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$, and $\bar{d}_{22}(\lambda) = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)$. Morever, for the critical term $\bar{d}_{12}(\lambda)$, we have

$$\bar{d}_{12}(\lambda) = \frac{\det(\psi_2^-,\phi_2^+,\phi_1^-,\phi_2^-)}{\det(\phi_1^+,\phi_2^+,\phi_1^-,\phi_2^-)}.$$

According to our scaling, $\phi_2^+(x;0)$ and $\phi_2^-(x;0)$ are both proportional to $\bar{u}_x(x)$, and the numerator vanishes at $\lambda = 0$, giving

$$\bar{d}_{12}(\lambda) = \mathbf{O}(|\lambda^{-1/2}\log\lambda|) + \mathbf{O}(|\log\lambda|).$$

Collecting these observations, we find

$$\begin{pmatrix} \Phi_1^+(x;\lambda)\bar{d}_{11}(\lambda)\tilde{\Psi}_1^-(y;\lambda) \\ \Phi_1^+(x;\lambda)\bar{d}_{11}(\lambda)\tilde{\Psi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\int_L^x \mu_1(s;\lambda)ds - \mu_1^-(\lambda)y} \\ (\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|))e^{\int_L^x \mu_1(s;\lambda)ds - \mu_1^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1^+(x;\lambda)\bar{d}_{12}(\lambda)\tilde{\Psi}_2^-(y;\lambda) \\ \Phi_1^+(x;\lambda)\bar{d}_{12}(\lambda)\tilde{\Psi}_2^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}(|\lambda^{-1/2}\log\lambda|) + \mathbf{O}(|\log\lambda|))e^{\int_L^x \mu_1(s;\lambda)ds - \mu_2^-(\lambda)y} \\ (\mathbf{O}_a(1) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_L^x \mu_1(s;\lambda)ds - \mu_2^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_2^+(x;\lambda)\bar{d}_{21}(\lambda)\tilde{\Psi}_1^-(y;\lambda) \\ \Phi_2^+(x;\lambda)\bar{d}_{21}(\lambda)\tilde{\Psi}_1^{-\prime}(y;\lambda) \end{pmatrix} = \begin{pmatrix} (\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2}))e^{-\sqrt{\lambda}x - \mu_1^-(\lambda)y} \\ (\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2}))e^{-\sqrt{\lambda}x - \mu_1^-(\lambda)y} \end{pmatrix}$$

$$\begin{pmatrix} \Phi_2^+(x;\lambda)\bar{d}_{22}(\lambda)\tilde{\Psi}_2^-(y;\lambda) \\ \Phi_2^+(x;\lambda)\bar{d}_{22}(\lambda)\tilde{\Psi}_2^{-\prime}(y;\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} \Big[\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|\lambda^{-1/2}\log\lambda|)\mathbf{O}(|x|^{-2})\Big]e^{-\sqrt{\lambda}x - \mu_2^-(\lambda)y} \\ \Big[\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2})\Big]e^{-\sqrt{\lambda}x - \mu_2^-(\lambda)y} \end{pmatrix}$$

Case (v) $0 \le y \le x$. In the case $0 \le y \le x$, we compare representations

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \sum_{j,k=1}^{2} \phi_k^+(x;\lambda)d_{kj}^+(\lambda)\tilde{\psi}_j^+(y;\lambda) + \sum_{j,k=1}^{2} \phi_k^+(x;\lambda)e_{kj}^+(\lambda)\tilde{\phi}_j^+(y;\lambda). \tag{4.6}$$

and

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = (\Phi^+(x;\lambda),0)(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}$$
$$= \sum_{k=1}^{2} \phi_k^+(x;\lambda)E_k^{\text{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}, \tag{4.7}$$

to obtain the relation

$$\sum_{j=1}^{2} d_{kj}^+(\lambda)\tilde{\psi}_j^+(y;\lambda) + \sum_{j=1}^{2} e_{kj}^+(\lambda)\tilde{\phi}_j^+(y;\lambda) = E_k^{\text{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}S(y)^{-1}. \tag{4.8}$$

Multiplying equation (4.8) by $S(y)\phi_l^+(y;\lambda)$ on the right and employing relations (3.6), we obtain

$$e_{kl}^+(\lambda)\tilde{\phi}_l^+(y;\lambda)S(y)\phi_l^+(y;\lambda) = E_k^{\text{tr}}(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}\phi_l^+(y;\lambda).$$

In the case $k = l = 2$, we have

$$\lambda^{3/2}e_{22}^+(\lambda) = (0,1,0,0)(\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}\phi_2^+(y;\lambda).$$

Observing that $J = (\Phi^+(y;\lambda),\Phi^-(y;\lambda))^{-1}\phi_2^+(y;\lambda)$ is a matrix equation, we apply Cramer's rule to determine

$$e_{22}^+(\lambda) = \lambda^{-3/2}J_2(\lambda) = \lambda^{-3/2}\frac{\det\begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_1^- & \phi_2^- \end{pmatrix}}{\det\begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_1^- & \phi_2^- \end{pmatrix}} = \lambda^{-3/2}.$$

Similarly, $e_{12}^+(\lambda) = e_{21}^+(\lambda) = 0$, and $e_{11}^+(\lambda) = 1$. We analyze the $d_{kj}^+(\lambda)$ similarly, arriving at $d_{11}^+(\lambda) = \mathbf{O}(|\lambda^{-1/2}\log\lambda|)$, $d_{12}^+(\lambda) = \mathbf{O}(|\lambda^{-1/2}\log\lambda|)$, $d_{21}^+(\lambda) = \mathbf{O}_a(|\lambda^{-1}|) + \mathbf{O}(|\lambda^{1/2}\log\lambda|)$, and $d_{22}^+(\lambda) = \lambda^{-3/2} + m(\lambda)$, where $m(\lambda) = \mathbf{O}(|\lambda^{-1}\log\lambda|)$.

Combining these observations and expanding $G_\lambda(x;y)$ in detail, we have (for $0 \le y \le x$)

$$\begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \phi_1^+(x;\lambda)d_{11}^+(\lambda)\tilde{\psi}_1^+(y;\lambda) + \phi_1^+(x;\lambda)d_{12}^+(\lambda)\tilde{\psi}_2^+(y;\lambda) + \phi_2^+(x;\lambda)d_{21}^+(\lambda)\tilde{\psi}_1^+(y;\lambda)$$
$$+\phi_2^+(x;\lambda)d_{22}^+(\lambda)\tilde{\psi}_2^+(y;\lambda) + \phi_1^+(x;\lambda)\tilde{\phi}_1^+(y;\lambda) + \lambda^{-3/2}\phi_2^+(x;\lambda)\tilde{\phi}_2^+(y;\lambda).$$

Computing directly from Lemma 3.2 and from Lemma 3.6, we have

$$\Phi_1^+(x;\lambda)d_{11}^+(\lambda)\tilde{\Psi}_1^+(y;\lambda) = \Big(\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{\int_L^x \mu_1(s;\lambda)ds - \int_L^y \mu_3(s;\lambda)ds}$$
$$\Phi_1^+(x;\lambda)d_{11}^+(\lambda)\tilde{\Psi}_1^{+\prime}(y;\lambda) = \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_L^x \mu_1(s;\lambda)ds - \int_L^y \mu_3(s;\lambda)ds}$$
$$\Phi_1^+(x;\lambda)d_{12}^+(\lambda)\tilde{\Psi}_2^+(y;\lambda) = \Big(\mathbf{O}(|\lambda^{-1/2}\log\lambda|) + \mathbf{O}(|y|)\mathbf{O}(|\log\lambda|)\Big)e^{\int_L^x \mu_1(s;\lambda)ds - \sqrt{\lambda}y}$$
$$\Phi_1^+(x;\lambda)d_{12}^+(\lambda)\tilde{\Psi}_2^{+\prime}(y;\lambda) = \mathbf{O}(|\lambda^{1/2}\log\lambda|)\mathbf{O}(|y|)e^{\int_L^x \mu_1(s;\lambda)ds - \sqrt{\lambda}y}$$
$$\Phi_1^+(x;\lambda)\tilde{\Phi}_1^+(y;\lambda) = \mathbf{O}_a(1)e^{\int_y^x \mu_1(s;\lambda)ds} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_y^x \mu_1(s;\lambda)ds}$$
$$\Phi_1^+(x;\lambda)\tilde{\Phi}_1^{+\prime}(y;\lambda) = \mathbf{O}_a(1)e^{\int_y^x \mu_1(s;\lambda)ds} + \mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\int_y^x \mu_1(s;\lambda)ds}$$
$$\Phi_2^+(x;\lambda)d_{21}^+(\lambda)\tilde{\Psi}_1^+(y;\lambda) =$$
$$\Big(\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1}) + \mathbf{O}(|x|^{-2})\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds}$$
$$\Phi_2^+(x;\lambda)d_{21}^+(\lambda)\tilde{\Psi}_1^{+\prime}(y;\lambda) = \Big(\mathbf{O}(|x|^{-2}) + \mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})\Big)e^{-\sqrt{\lambda}x - \int_L^y \mu_3(s;\lambda)ds},$$

and finally, according to Lemma 3.6,

$$\begin{aligned}&\Phi_2^+(x;\lambda)\Big(d_{22}^+(\lambda)\tilde\Psi_2^+(y;\lambda)+\lambda^{-3/2}\tilde\Phi_2^+(y;\lambda)\Big)\\&\quad=e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)+\frac{P_1^+\bar u_x}{\lambda}+\mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)\Big]\\&\Phi_2^+(x;\lambda)\Big(d_{22}^+(\lambda)\tilde\Psi_2^{+\prime}(y;\lambda)+\lambda^{-3/2}\tilde\Phi_2^{+\prime}(y;\lambda)\Big)\\&\quad=e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)+\mathbf{O}(|\lambda^{1/2}\log\lambda|)\mathbf{O}(|x|^{-1})+\mathbf{O}(|\lambda^{1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)\\&\quad+\mathbf{O}(|\log\lambda|)\mathbf{O}(|x|^{-2})\Big].\end{aligned}$$

Case (vi) $0\le x\le y$. The case $0\le x\le y$ is similar to Case (v), with only two new terms, namely,

$$\begin{aligned}\Psi_1^+(x;\lambda)\tilde\Psi_1^+(y;\lambda)&=\mathbf{O}(1)e^{-\int_x^y\mu_3(s;\lambda)ds}+\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{-\int_x^y\mu_3(s;\lambda)ds}\\\Psi_1^+(x;\lambda)\tilde\Psi_1^{+\prime}(y;\lambda)&=\mathbf{O}_a(|\lambda|)e^{\int_x^y\mu_3(s;\lambda)ds}+\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{-\int_x^y\mu_3(s;\lambda)ds},\end{aligned}$$

and, according to Lemma 3.6,

$$\begin{aligned}&\Big(\Phi_2^+(x;\lambda)d_{22}(\lambda)-\lambda^{-3/2}\Psi_2^+(x;\lambda)\Big)\tilde\Psi_2^+(y;\lambda)=e^{-\sqrt{\lambda}|x-y|}\Big[\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)+\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2})\\&\qquad+\mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-1})\Big],\\&\Big(\Phi_2^+(x;\lambda)d_{22}(\lambda)-\lambda^{-3/2}\Psi_2^+(x;\lambda)\Big)\tilde\Psi_2^{+\prime}(y;\lambda)=e^{-\sqrt{\lambda}|x-y|}\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)\end{aligned}$$

□

Lemma 4.2. (Large $|\lambda|$ $G_\lambda(x,y)$ estimates.) For λ bounded to the right of Γ_d and for $|\lambda|\ge M$, some constant M sufficiently large, there exists constants C,$\beta>0$ so that

$$\begin{aligned}|G_\lambda(x,y)|&\le C|\lambda^{-1/2}|e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|}\\|\partial_yG_\lambda(x,y)|&\le Ce^{-\beta^{-1/2}|\lambda^{1/2}||x-y|}.\end{aligned}$$

Proof. The large $|\lambda|$ behavior of $G_\lambda(x,y)$ depends on viscosity rather than convection and can be developed exactly as in the nondegenerate case. See in particular [ZH], p. 806. □

Lemma 4.3. (Medium $|\lambda|$ $G_\lambda(x,y)$ estimates.) For λ bounded to the right of Γ_d and for $r\le|\lambda|\le R$, any fixed constants $R>r>0$, there exists $C>0$ so that

$$\begin{aligned}|G_\lambda(x,y)|&\le C\\|\partial_yG_\lambda(x,y)|&\le C.\end{aligned}$$

Proof. The medium $|\lambda|$ estimates follow from decay properties rather than decay rates and can be developed exactly as in the nondegenerate case. See in particular [ZH] p. 805. □

5 Estimates on $G(t,x;y)$

We now employ the estimates of Lemmas 4.1, 4.2, and 4.3 to derive estimates on the Green's function $G(t,x;y)$ through the inverse Laplace transform representation

$$G(t,x;y)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}G_\lambda(x,y)d\lambda,$$

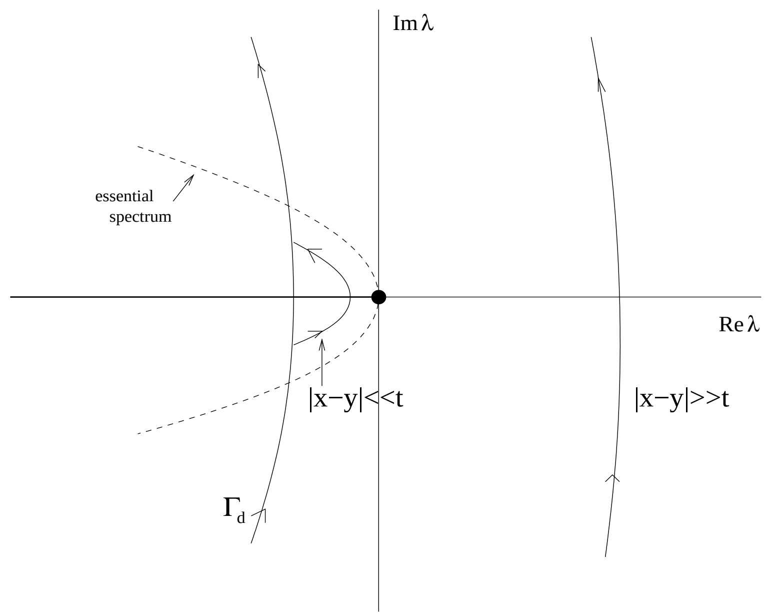

Figure 5.1: Contours in the case of analyticity at $\lambda = 0$.

where the contour of integration Γ must encircle the poles of G_λ (which occur at point spectrum for the operator L).

Before beginning the detailed proof of Theorem 1.1, we give a brief overview of the approach taken and set some notation. In each case of Lemma 4.1, the estimate on $G_\lambda(x,y)$ is divided into a number of terms that can be integrated separately against $e^{\lambda t}$. For every such term, the contour of integration, Γ, will be chosen to depend on t, x, and y. In the event that $|x-y| \gg t$, it will typically be advantageous to take a (parabolic) contour that crosses the real axis far to the right of the imaginary axis, while for $|x-y| \ll t$, it will typically be advantageous to take a contour to the left of the imaginary axis (see Figure 5.1). In either case, we only follow our contour of choice until it strikes the contour Γ_d, defined in (1.7), which aside from the point $\lambda = 0$ lies to the right of the point spectrum of L. Throughout the analysis, then, for chosen contour Γ we will use the notation Γ^* to indicate the truncated portion of the contour that stops at Γ_d.

While there are a great many terms in $G_\lambda(x,y)$ to analyze, the analysis of several are similar. Two particularly important terms are the scattering and excited terms from the case $0 \le x \le y$ ($|x-y| \le Kt$),

$$S_\lambda(x,y) = \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\sqrt{\lambda}|x-y|}, \quad \text{and} \quad E_\lambda(x,y) = \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\sqrt{\lambda}|x-y|},$$

Though several estimates will be more technical than these, the fundamental ideas are all contained here. For the first we take the heat-equation-like contour defined through

$$\sqrt{\lambda(k)} = \frac{|x-y|}{Lt} + ik,$$

for which

$$\lambda(k) = \frac{|x-y|^2}{L^2t^2} + 2ik\frac{|x-y|}{Lt} - k^2; \quad d\lambda = (2i\frac{|x-y|}{Lt} - 2k)dk,$$

where L is chosen large enough so that for k sufficiently small, this contour remains in a small neighborhood

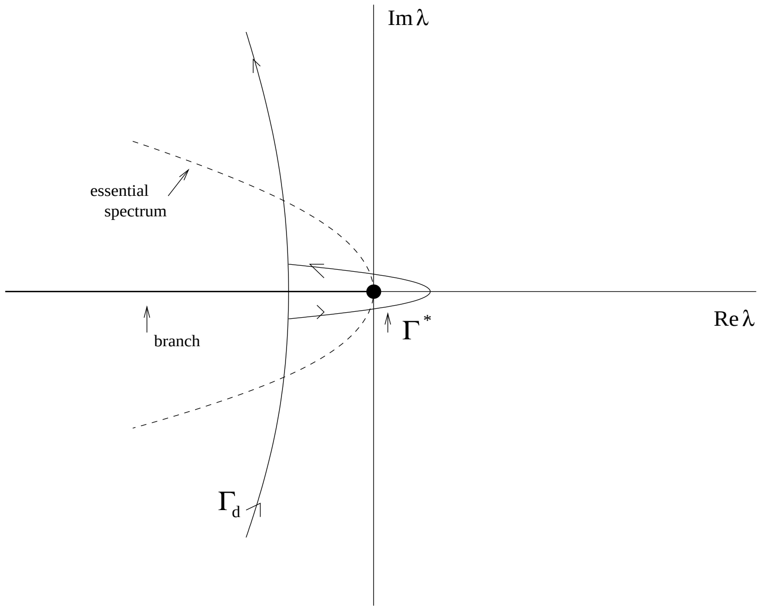

Figure 5.2: Contours in the case of a branch along Re$\lambda < 0$.

of the origin. We follow this contour until it strikes Γ_d, and then follow Γ_d out to the point at ∞ (see Figure 5.2).

Observe in particular that though our contour of choice in this case always crosses the real axis to the right of the imaginary axis, it moves rapidly into essential spectrum. Letting $\pm k^*$ represent the values of k for which we strike Γ_d, we have

$$\begin{aligned}\Big|\int_{\Gamma^*} e^{\lambda t} S_\lambda(x,y)d\lambda\Big| =&\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)\int_{-k^*}^{+k^*} e^{\lambda(k)t}\mathbf{O}(|\lambda(k)|^{-1/2})e^{-\sqrt{\lambda(k)}|x-y|}d\lambda\\ =&\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)\int_{-k^*}^{+k^*} e^{\frac{|x-y|^2}{L^2t^2}t-k^2t-\frac{|x-y|^2}{Lt}}\Big|\frac{2i\frac{|x-y|}{Lt}-2k}{\frac{|x-y|}{Lt}+ik}\Big|dk\\ =&\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}}.\end{aligned}$$

Finally, along Γ_d, we observe that Re$\sqrt{\lambda} \geq 0$ and $\mathbf{O}(|\lambda(k)^{-1/2}|) = \mathbf{O}(1)$, so that our integral decays at exponential rate in time, $e^{-\eta t}$, $\eta > 0$. Since $|x-y| \leq Kt$, we have

$$e^{-\eta t} \leq e^{-\eta t\frac{|x-y|^2}{K^2t^2}} = e^{-\frac{\eta}{K^2}\frac{(x-y)^2}{t}},$$

which leads to our final estimate on this term

$$\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

For $E_\lambda(x,y)$, we again begin along the contour defined through $\sqrt{\lambda(k)} = \frac{|x-y|}{Lt} + ik$, along which we have

$$\Big|\bar{u}_x(x)\int_{\Gamma^*}\mathbf{O}_a(|\lambda^{-1}|)e^{\lambda t-\sqrt{\lambda}|x-y|}d\lambda\Big| = \bar{u}_x(x)\int_{-k^*}^{+k^*} e^{-\frac{(x-y)^2}{Mt}-k^2 t}\frac{|2i\frac{|x-y|}{Lt}-2k|}{|\frac{|x-y|^2}{L^2t^2}+ik\frac{|x-y|}{t}-k^2|}|dk|$$
$$=\mathbf{O}(|x|^{-2})\mathbf{O}(\frac{t}{|x-y|})\mathbf{O}(t^{-1/2})e^{-\frac{(x-y)^2}{Mt}}.$$

In the event that $|x-y|\geq \epsilon_0\sqrt{t}$, we observe that $\frac{t^{1/2}}{|x-y|}\leq\frac{1}{\epsilon_0}$, so that we have an estimate by (recalling that we are considering the case $0\leq x\leq y$)

$$\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-2})\mathbf{O}(|x-y|)e^{-\frac{(x-y)^2}{Mt}} = \mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}}.$$

In the event that $|x-y|\leq\epsilon_0\sqrt{t}$, we proceed as in [H.3, H.4] and divide the integrand into an analytic term and an error, as

$$\frac{1}{2\pi i}\int_{\Gamma^*}\frac{P_+\bar{u}_x(x)e^{\lambda t-\sqrt{\lambda}|x-y|}}{\lambda}d\lambda = \frac{1}{2\pi i}\int_{\Gamma^*}\frac{P_+\bar{u}_x(x)e^{\lambda t}}{\lambda}d\lambda + \frac{1}{2\pi i}\int_{\Gamma^*}\frac{P_+\bar{u}_x(x)e^{\lambda t}(e^{-\sqrt{\lambda}|x-y|}-1)}{\lambda}d\lambda. \quad (5.1)$$

Here, we have observed that for $\mathbf{O}_a(|\lambda^{-1}|)$ meromorphic in λ in a neighborhood of the origin, we have

$$\mathbf{O}_a(|\lambda^{-1}|) = \frac{P_+}{\lambda} + \mathbf{O}_a(1),$$

whence

$$E_\lambda(x,y) = \bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\sqrt{\lambda}|x-y|} = \frac{P_+\bar{u}_x(x)}{\lambda}e^{-\sqrt{\lambda}|x-y|} + \mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}|x-y|},$$

the second of which can be subsumed into $S_\lambda(x,y)$. For the first integral on the right-hand side of equation (5.1), we employ analyticity of numerator and denominator to proceed similarly as in the nondegenerate analysis of [ZH] and shift our contour to the left of the imaginary axis, using Cauchy's integral formula to compute the residue. Our estimate on this term becomes

$$P_+\bar{u}_x(x)I_{\{|x-y|\leq\epsilon_0\sqrt{t}\}}.$$

For the second integral, we first consider the strip of Γ^* over which $|\sqrt{\lambda}(x-y)|\leq 1$, for which we have

$$\Big|\int_{\Gamma^*}\frac{\bar{u}_x(x)e^{\lambda t}(e^{-\sqrt{\lambda}|x-y|}-1)}{\lambda}d\lambda\Big| = \mathbf{O}(|x|^{-2})\Big|\int_{\Gamma^*}\frac{e^{\lambda t}\mathbf{O}(|\sqrt{\lambda}(x-y)|)}{\lambda}d\lambda\Big|$$
$$=\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)\int_{\Gamma^*}\mathbf{O}(|\lambda^{-1/2}|)e^{\lambda t}d\lambda = \mathbf{O}(|x|^{-2})\mathbf{O}(|y|)\mathbf{O}(t^{-1/2})e^{-\frac{(x-y)^2}{Lt}},$$

where we have made use of the observation that for $|x-y|\leq\epsilon_0\sqrt{t}$, $1\leq Ce^{-\frac{(x-y)^2}{Lt}}$. On the other hand, for $|\sqrt{\lambda}(x-y)|>1$, we have $|\lambda^{-1/2}| = \mathbf{O}(|x-y|)$ and consequently the estimate

$$\Big|\int_{\Gamma^*}\frac{\bar{u}_x(x)e^{\lambda t}(e^{-\sqrt{\lambda}|x-y|}-1)}{\lambda}d\lambda\Big| = \mathbf{O}(|x|^{-2})\Big|\int_{\Gamma^*}e^{\lambda t}\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x-y|)d\lambda\Big|,$$

as above. Our final estimate becomes

$$\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar{u}_x(x)I_{\{|x-y|\leq\epsilon_0\sqrt{t}\}}.$$

Note in particular that in this analysis we have determined critical tracking information regarding how rapidly mass in the far field contributes to a shift of the shock wave. According to our choice of shift $\delta(t)$ in equation (1.4), we will have a linear term of the form

$$\delta(t) \sim P_+ \int_0^{\epsilon_0\sqrt{t}} v_0(y)dy,$$

for which we observe that the window of mass that has accumulated in the shock layer at time t is $[0, \epsilon_0\sqrt{t}]$.

Proof of Theorem 1.1. In the *small time* case $|x-y| \geq Kt$, some K sufficiently large, we can proceed via the large $|\lambda|$ estimates of Lemma 4.2, which are exactly the same estimates as in [ZH]. For the case $|x-y| \leq Kt$, we proceed in a number of subcases.

Case (i) $y \leq x \leq 0$. According to Lemma 4.1, we have five integrals to evaluate in the case $y \leq x \leq 0$, beginning with the integrands $\mathbf{O}_a(1)e^{\mu_2^-(\lambda)(x-y)}$, $\mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}$, and $\bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\mu_2^-(\lambda)y}$. Each of these arises in the case of non-degenerate waves and can be analyzed as in [ZH]. Summarizing, we have

$$\begin{aligned}
\int_\Gamma \mathbf{O}_a(1)e^{\lambda t+\mu_2^-(\lambda)(x-y)}d\lambda =& \mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_2^-t)^2}{Mt}}\\
\int_\Gamma \mathbf{O}_a(1)e^{\lambda t+\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}d\lambda =& \mathbf{O}(t^{-1/2})e^{-\frac{(x-\frac{a_1^-}{a_2^-}y-a_1^-t)^2}{Mt}}\\
\bar{u}_x(x)\int_\Gamma \mathbf{O}_a(|\lambda^{-1}|)e^{\lambda t-\mu_2^-(\lambda)y}d\lambda =& \bar{u}_x(x)\Big(\mathbf{O}(1)e^{-\frac{(y+a_2^-t)^2}{Mt}} + \mathbf{O}(1)I_{\{|y|\leq|a_2^-|t\}}\Big).
\end{aligned} \tag{5.2}$$

For the term $\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}$, we must alter the analysis employed above so as to avoid $\lambda \in \mathbb{R}_-$, where we lose analyticity. Expanding $\mu_2^-(\lambda)$ and $\mu_3^-(\lambda)$, we have

$$\begin{aligned}
\mu_2^-(\lambda) =& \frac{a_2^- - \sqrt{(a_2^-)^2+4\lambda}}{2} = -\frac{1}{a_2^-}\lambda + \frac{1}{(a_2^-)^3}\lambda^2 + \mathbf{O}(\lambda^3), \quad \text{(Recall: } a_2^- > 0)\\
\mu_3^-(\lambda) =& \frac{a_1^- + \sqrt{(a_1^-)^2+4\lambda}}{2} = -\frac{1}{a_1^-}\lambda + \frac{1}{(a_1^-)^3}\lambda^2 + \mathbf{O}(\lambda^3), \quad \text{(Recall: } a_1^- < 0)
\end{aligned}$$

Our principal contour will be chosen as in [ZH] by the relation

$$\begin{aligned}
&\Big(-\frac{x}{a_1^-}+\frac{y}{a_2^-}\Big)\lambda(k) + \Big(\frac{x}{(a_1^-)^3}-\frac{y}{(a_2^-)^3}\Big)\lambda(k)^2\\
=&\Big(-\frac{x}{a_1^-}+\frac{y}{a_2^-}\Big)\lambda_R + \Big(\frac{x}{(a_1^-)^3}-\frac{y}{(a_2^-)^3}\Big)\lambda_R^2 + i\Big(-\frac{x}{a_1^-}+\frac{y}{a_2^-}\Big)k,
\end{aligned}$$

where λ_R is the critical point where the contour crosses the real axis. Expanding $\lambda(k) = \lambda_R + \lambda_1 k + \lambda_2 k^2 + ...$, we immediately find $\lambda_1 = i(1+\mathbf{O}(\lambda_R))$ and

$$\lambda_2 = \frac{\frac{x}{(a_1^-)^3}-\frac{y}{(a_2^-)^3}}{-\frac{x}{a_1^-}+\frac{y}{a_2^-}}(1+\mathbf{O}(\lambda_R)),$$

for which $\lambda_2 \leq -\gamma < 0$, some fixed γ. We choose λ_R optimally by minimizing the exponent function

$$g(\lambda_R) = \lambda_R t + \Big(-\frac{x}{a_1^-}+\frac{y}{a_2^-}\Big)\lambda_R + \Big(\frac{x}{(a_1^-)^3}-\frac{y}{(a_2^-)^3}\Big)\lambda_R^2,$$

for

$$\lambda_R = \frac{\bar{\alpha}_-}{\bar{p}_-}; \quad \bar{\alpha}_- := \frac{x - \frac{a_1^-}{a_2^-} y - a_1^- t}{2t}; \quad \bar{p}_- := \Big(\frac{x}{(a_1^-)^2 t} - \frac{a_1^- y}{(a_2^-)^3 t} \Big) \le 0.$$

Note in particular that our contour of choice is entirely determined by our choice of λ_R. Following [ZH] in the case $\frac{\bar{\alpha}_-}{\bar{p}_-} \ge 0$ (that is, for $|x - \frac{a_1^-}{a_2^-} y| \ge |a_1^-|t)$, we take

$$\lambda_R = \begin{cases} +\epsilon, & \epsilon \le \frac{\bar{\alpha}_-}{\bar{p}_-} \le M, \\ \frac{\bar{\alpha}_-}{\bar{p}_-}, & t^{-1/2} \le \frac{\bar{\alpha}_-}{\bar{p}_-} \le \epsilon \\ t^{-1/2}, & 0 \le \frac{\bar{\alpha}_-}{\bar{p}_-} \le t^{-1/2}, \end{cases}$$

to obtain an estimate by

$$\mathbf{O}(t^{-3/4} \log t) e^{-\frac{(x - \frac{a_1^-}{a_2^-} y - a_1^- t)^2}{Mt}},$$

which can be subsumed into estimates (5.2). The difficulty arises in the case $\frac{\bar{\alpha}_-}{\bar{p}_-} < 0$ (that is, $|x - \frac{a_1^-}{a_2^-} y| \le |a_1^-|t)$, when the non-degenerate analysis would select $\lambda_R < 0$, according to

$$\lambda_R = \begin{cases} -\epsilon, & -\epsilon \ge \frac{\bar{\alpha}_-}{\bar{p}_-}, \\ \frac{\bar{\alpha}_-}{\bar{p}_-}, & -t^{-1/2} \ge \frac{\bar{\alpha}_-}{\bar{p}_-} \ge -\epsilon \\ -t^{-1/2}, & 0 \ge \frac{\bar{\alpha}_-}{\bar{p}_-} \ge -t^{-1/2}, \end{cases}$$

which is precluded by our branch along $\lambda \in \mathbb{R}_-$. At this point, we arrive at the primary new feature of the degenerate-wave contour analysis. The idea in the non-degenerate case was, for $t \gg |x - y|$, to take contours that remain entirely in the negative real half-plane and thus take advantage of exponential time decay, which dominates the exponential growth in $|x|$ and $|y|$. In the degenerate case, we cannot avoid passing our contours through the positive real axis, and our new approach will be to move our contours quickly into the negative real half-plane.

We first observe that we can integrate along essential spectrum ($\Gamma = \{\lambda :$ Either $\mathrm{Re}\mu_2^-(\lambda) = 0$ or $\mathrm{Re}\mu_3^-(\lambda) = 0\}$) to obtain an estimate

$$\mathbf{O}(t^{-3/4} \log t) I_{\{|x - \frac{a_1^-}{a_2^-} y| \le |a_1^-|t\}}.$$

Alternatively, we can move more rapidly into essential spectrum by taking the heat-equation-like contour

$$\sqrt{\lambda(k)} = t^{-1} + ik,$$

which we denote Γ_D, until it strikes the non-degenerate contour described above. Along Γ_D we have

$$\lambda(k) = t^{-2} + 2ikt^{-1} - k^2; \quad d\lambda = (2it^{-1} - 2k)dk; \quad \text{and} \quad \mathrm{Re}\lambda(k)^2 = t^{-4} + k^4 - 6k^2 t^{-2},$$

so that

$$\begin{aligned} &\Big| \int_{\Gamma_D^*} \mathbf{O}(|\lambda^{1/2} \log \lambda|) e^{\lambda t + \mu_3^-(\lambda) x - \mu_2^-(\lambda) y} d\lambda \Big| \\ &\le C \int_{-k^*}^{+k^*} |2it^{-1} - 2k| |(t^{-1} + ik) \log(t^{-1} + ik)| e^{-k^2 (t - \frac{x}{a_1^-} + \frac{y}{a_2^-}) + k^4 (\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3})} dk. \end{aligned}$$

We observe here that for $|k|$ sufficiently large, we cannot get a good estimate along this contour. The index k^*, then, where me cross the non-degenerate conrou is critical. Indexing our non-degenerate contour by l, we have

$$\lambda(l) = \lambda_R + il(1 + \mathbf{O}(\lambda_R)) + \frac{\Big(\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}\Big)}{\Big(-\frac{x}{a_1^-} + \frac{y}{a_2^-}\Big)} l^2(1 + \mathbf{O}(\lambda_R)) + \mathbf{O}(l^3),$$

for which the intersection with Γ_D occurs for

$$k^2 = \frac{t^{-2} - \lambda_R}{1 + \mathbf{O}(t^{-2})},$$

for which we recall that in this case $\lambda_R < 0$. In the first case, $t^{-1/2} \le \frac{\bar{\alpha}_-}{\bar{p}_-} \le 0$, we choose $\lambda_R = -t^{-1/2}$, for which (for $t \ge 1$) we have $|k| \le Ct^{-1/4}$. In this case, since $|x - \frac{a_1^-}{a_2^-} y| \le |a_1^-|t$, the exponent $k^4(\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3})$ is bounded, and we can integrate over $|k| \le Ct^{-1/4}$ to obtain an estimate by

$$\mathbf{O}(|x - \frac{a_1^-}{a_2^-} y - a_1^- t|^{-3/2} \log t) I_{\{|x - \frac{a_1^-}{a_2^-} y| \le |a_1^-|t\}}.$$

In the case $-\epsilon \le \frac{\bar{\alpha}_-}{\bar{p}_-} \le -t^{-1/2}$, we choose

$$\lambda_R = \frac{\bar{\alpha}_-}{\bar{p}_-} = -\frac{1}{2} \frac{(t - \frac{x}{a_1^-} + \frac{y}{a_2^-})}{\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}},$$

for which our degenerate contour intersects with our non-degenerate contour for

$$k^2 \le \frac{1}{2} \frac{(t - \frac{x}{a_1^-} + \frac{y}{a_2^-})}{\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}} + \mathbf{O}(t^{-2}).$$

For this range of k, we have

$$\begin{aligned} -k^2(t - \frac{x}{a_1^-} + \frac{y}{a_2^-}) + k^4(\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}) &= -\frac{k^2}{\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}} \Big(\frac{t - \frac{x}{a_1^-} + \frac{y}{a_2^-}}{\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}} - k^2 \Big) \\ &\le -\frac{1}{2} k^2 (t - \frac{x}{a_1^-} + \frac{y}{a_2^-}) + \mathbf{O}(t^{-1}), \end{aligned}$$

for which we can integrate over k as above. In the final case, $\frac{\bar{\alpha}_-}{\bar{p}_-} \le -\epsilon$, we observe that

$$\mathrm{Re}\Big(-k^2(t - \frac{x}{a_1^-} + \frac{y}{a_2^-}) + k^4(\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}) \Big) = -\frac{k^2}{\frac{x}{(a_1^-)^3} - \frac{y}{(a_2^-)^3}} \Big(-2\frac{\bar{\alpha}_-}{\bar{p}_-} - k^2 \Big),$$

for which we have decay as in the previous cases for $k^2 \le \epsilon$. We can choose d_0 (for the contour Γ_d defined in (1.7)) sufficiently small so that we strike Γ_d for $k^2 \le \epsilon$. The non-degenerate analysis of [ZH] applies along Γ_d, and we obtain an estimate that can be subsumed into those above.

The remaining term in this case, $\mathbf{O}(|\lambda^{-1/2} \log \lambda|) e^{-\mu_2^-(\lambda) y}$, can be analyzed as in the case $\mathbf{O}(\lambda^{1/2} \log \lambda) e^{\mu_3^-(\lambda)x - \mu_2^-(\lambda)y}$, with x set to 0. The estimates on $G_y(t, x; y)$ follow similarly.

Case (ii) $x \le y \le 0$. According to Lemma 4.1, we have five integrands to evaluate in the case $x \le y \le 0$, though the only integrand not examined in the analysis of Case (i) is $\mathbf{O}_a(1)e^{\mu_3^-(\lambda)(x-y)}$, which arises in the non-degenerate case and has been analyzed in [ZH], with an estimate by

$$\mathbf{O}(t^{-1/2})e^{-\frac{(x-y-a_1^- t)^2}{Mt}}.$$

Estimates on $G_y(t,x;y)$ in this case follow similarly.

Case (iii) $x \le 0 \le y$. According to Lemma 4.1, we have seven integrals to evaluate in the case $x \le 0 \le y$, beginning with the integrand $\mathbf{O}_a(1)e^{\mu_3^-(\lambda)x - \int_L^y \mu_3(s;\lambda)ds}$. Since $\mu_3(x;\lambda)$ is analytic in λ in a neighborhood of $\lambda = 0$, this integral can be analyzed similarly as in the non-degenerate case. We must, however, keep track of the y-dependence in μ_3 through the Taylor expansion

$$\int_L^y \mu_3(s;\lambda)ds = -\lambda \int_L^y \frac{ds}{a_1(s)} + \lambda^2 \int_L^y \frac{ds}{a_1(s)^3} + \mathbf{O}(\lambda^3),$$

for which we take $y \ge L$. (In the event $y \le L$, we can subsume y behavior into $\mathbf{O}_a(1)$ and proceed with an estimate for the case $y = 0$.) Consequently, we choose our contour $\lambda(k)$ through

$$\Big(-\frac{x}{a_1^-} + \int_L^y \frac{ds}{a_1(s)}\Big)\lambda(k) + \Big(\frac{x}{(a_1^-)^3} - \int_L^y \frac{ds}{a_1(s)^3}\Big)\lambda(k)^2 = \Big(-\frac{x}{a_1^-} + \int_L^y \frac{ds}{a_1(s)}\Big)\lambda_R$$
$$+\Big(\frac{x}{(a_1^-)^3} - \int_L^y \frac{ds}{a_1(s)^3}\Big)\lambda_R^2 + ik\Big(-\frac{x}{a_1^-} + \int_L^y \frac{ds}{a_1(s)}\Big),$$

for which

$$\lambda(k) = \lambda_R + ik(1 + \mathbf{O}(\lambda_R)) + \gamma(x,y)k^2(1 + \mathbf{O}(\lambda_R)) + \mathbf{O}(k^3),$$

where

$$\gamma(x,y) = \frac{\big(\frac{x}{(a_1^-)^3} - \int_L^y \frac{ds}{a_1(s)^3}\big)}{\big(-\frac{x}{a_1^-} + \int_L^y \frac{ds}{a_1(s)}\big)} \le \gamma_0 < 0.$$

As usual, we choose our principal value of λ_R to minimize

$$g(\lambda_R) = \lambda_R t + \Big(-\frac{x}{a_1^-} + \int_L^y \frac{ds}{a_1(s)}\Big)\lambda_R + \Big(\frac{x}{(a_1^-)^3} - \int_L^y \frac{ds}{a_1(s)^3}\Big)\lambda_R^2,$$

so that

$$\lambda_R = \frac{\tilde{\alpha}}{\tilde{p}}; \quad \tilde{\alpha} := \frac{x - a_1^- \int_L^y \frac{ds}{a_1(s)} - a_1^- t}{2t}; \quad \tilde{p} := \frac{\Big(\frac{x}{(a_1^-)^2} - a_1^- \int_L^y \frac{ds}{a_1(s)^3}\Big)}{t}.$$

We can now proceed as in Case (i) with

$$\lambda_R = \begin{cases} \pm\epsilon, & \pm\epsilon \lesseqgtr \frac{\tilde{\alpha}}{\tilde{p}} \lesseqgtr \pm M \\ \frac{\tilde{\alpha}}{\tilde{p}}, & t^{-1/2} \le |\frac{\tilde{\alpha}}{\tilde{p}}| \le \epsilon \\ \pm t^{-1/2}, & 0 \lesseqgtr \frac{\tilde{\alpha}}{\tilde{p}} \lesseqgtr \pm t^{-1/2} \end{cases}.$$

The final estimate becomes

$$\mathbf{O}(t^{-1/2})e^{-\frac{(x - a_1^- \int_L^y \frac{ds}{a_1(s)} - a_1^- t)^2}{Mt}}.$$

For the integrand $\bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\int_L^y \mu_3(s;\lambda)ds}$, we can proceed again as in the case of non-degenerate waves, with $\mu_3(x;\lambda)$ treated as above. We find

$$\int_\Gamma \mathbf{O}_a(|\lambda^{-1}|)\bar{u}_x(x)e^{\lambda t-\int_L^y \mu_3(s;\lambda)ds}d\lambda = \bar{u}_x(x)\Big(\mathbf{O}(1)e^{-\frac{(\int_L^y \frac{ds}{a_1(s)}+a_1^- t)^2}{Mt}} + \mathbf{O}(1)I_{\{|\int_L^y \frac{ds}{a_1(s)}|\le t\}}\Big).$$

For the integrands $\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\int_L^y \mu_3(s;\lambda)ds}$ and $\bar{u}_x(x)\mathbf{O}(|\lambda^{-1/2}\log\lambda|)e^{-\int_L^y \mu_3(s;\lambda)ds}$ we proceed as in the analysis of $\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}$ in Case (i). We obtain estimates that can be subsumed into those above, as well as the additional estimate

$$\Big[\mathbf{O}(t^{-3/4}\log t)\wedge\mathbf{O}(|x-a_1^-\int_L^y \frac{ds}{a_1(s)} - a_1^- t|^{-3/2}\log t)\Big]I_{\{|x-a_1^-\int_L^y \frac{ds}{a_1(s)}|\le|a_1^-|t\}}.$$

The fundamentally new terms in this case are $\mathbf{O}(1)e^{\mu_3^-(\lambda)x-\sqrt{\lambda}y}$ and $\bar{u}_x(x)\Big(\mathbf{O}_a(|\lambda^{-1}|)+\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\Big)e^{-\sqrt{\lambda}y}$. In both cases, we first observe that for $\frac{\tilde{\alpha}}{\tilde{p}}\ge 0$, we may take advantage of the observation that along the contour chosen for the integrand $\mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\int_L^y \mu_3(s;\lambda)}$, and for $|\lambda|$ sufficiently small, we have $\mathrm{Re}\sqrt{\lambda}y\ge \mathrm{Re}\int_L^y \mu_3(s;\lambda)ds$, so that the estimates obtained can be subsumed into those above. For $\frac{\tilde{\alpha}}{\tilde{p}}<0$, we begin with the integrand $\mathbf{O}(1)e^{\mu_3^-(\lambda)x-\sqrt{\lambda}y}$, for which we proceed as in the analysis of $\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}$ in Case (i), by taking the degenerate contour defined through $\sqrt{\lambda(k)}=\frac{y}{Lt}+ik$ (denoted, again, Γ_D), until it strikes the non-degenerate contour defined above for the term $\mathbf{O}_a(1)e^{\mu_3^-(\lambda)x-\int_L^y \mu_3(s;\lambda)ds}$. Along Γ_D, we have

$$\lambda(k)=\frac{y^2}{L^2t^2}+2ik\frac{y}{Lt}-k^2;\quad \mathrm{Re}\lambda(k)^2=\frac{y^4}{L^4t^4}-6\frac{y^2}{L^2t^2}k^2+k^4,$$

and consequently

$$\mathrm{Re}\ (\lambda t+\mu_3^-(\lambda)x-\sqrt{\lambda}y)\le-\frac{y^2}{Mt}-k^2(t-\frac{x}{a_1^-})+\frac{x}{(a_1^-)^3}k^4.$$

In this case, our contours Γ_D and Γ_{ND} intersect for

$$k^2=\frac{\frac{y^2}{L^2t^2}-\lambda_R}{1+\mathbf{O}(\frac{y^2}{L^2t^2})}.$$

In the case $-t^{-1/2}\le\frac{\tilde{\alpha}}{\tilde{p}}\le 0$, we have $\lambda_R=-t^{-1/2}$, and the growth term $\frac{x}{(a_1^-)^3}k^4$ remains bounded. For the second case $-\epsilon\le\frac{\tilde{\alpha}}{\tilde{p}}\le -t^{-1/2}$, we choose

$$\lambda_R=\frac{\tilde{\alpha}}{\tilde{p}}=\frac{1}{2}\frac{\frac{x}{a_1^-}-\int_L^y\frac{ds}{a_1(s)}-t}{\frac{x}{(a_1^-)^3}-\int_L^y\frac{ds}{a_1(s)^3}},$$

for which Γ_D and Γ_{ND} intersect for

$$k^2=-\frac{1}{2}\frac{\frac{x}{a_1^-}-\int_L^y\frac{ds}{a_1(s)}-t}{\frac{x}{(a_1^-)^3}-\int_L^y\frac{ds}{a_1(s)^3}}+\mathbf{O}(\frac{y^2}{L^2t^2}).$$

We have, then,

$$\begin{aligned}-\frac{y^2}{Mt}-k^2(t-\frac{x}{a_1^-})+\frac{x}{(a_1^-)^3}k^4 &\le -\frac{y^2}{Mt}-k^2\frac{x}{(a_1^-)^3}(\frac{t-\frac{x}{a_1^-}}{\frac{x}{(a_1^-)^3}}-k^2)\\ &\le -\frac{y^2}{Mt}-\frac{1}{2}k^2(t-\frac{x}{a_1^-}).\end{aligned}$$

Our estimate becomes

$$\mathbf{O}(t^{-1/2}) \wedge \mathbf{O}(|x - a_1^- t|^{-1}) e^{-\frac{y^2}{Mt}} I_{\{|x - \int_L^y \frac{ds}{a_1(s)}| \le |a_1^- |t\}},$$

where the time decay $\mathbf{O}(t^{-1/2})$ can be obtained in the usual way by integrating along essential spectrum. Finally, we observe that for $y > 0$

$$e^{-\sqrt{\lambda} y} = 1 + \mathbf{O}(|\sqrt{\lambda} y|),$$

for which we have

$$\mathbf{O}_a(1) e^{\mu_3^-(\lambda) x - \sqrt{\lambda} y} = \mathbf{O}_a(1) e^{\mu_3^-(\lambda) x} + \mathbf{O}(|\sqrt{\lambda} y|) e^{\mu_3^-(\lambda) x}.$$

Proceeding through the non-degenerate analysis with $\mathbf{O}_a(1) e^{\mu_3^-(\lambda) x}$, and proceeding as above for $\mathbf{O}(|\sqrt{\lambda} y|) e^{\mu_3^-(\lambda) x}$, we obtain an alternative estimate of

$$\mathbf{O}(t^{-1/2}) e^{-\frac{(x - a_1^- t)^2}{Mt}} + \Big[\mathbf{O}(t^{-3/4}) \wedge \Big(\mathbf{O}(|y|) \mathbf{O}(|x - a_1^- t|^{-3/2}) \Big) \Big].$$

We next consider the term $\bar{u}_x(x) \mathbf{O}_a(|\lambda|^{-1}) e^{-\sqrt{\lambda} y}$ in the case $\frac{\tilde{\alpha}}{\tilde{p}} < 0$, for which we take the contour defined through $\sqrt{\lambda(k)} = \frac{y}{Lt} + ik$. Following [H.4], we divide the integrand into an analytic piece plus a non-analytic error,

$$\int_{\Gamma^*} \frac{e^{\lambda t - \sqrt{\lambda} y}}{\lambda} d\lambda = \int_{\Gamma^*} \frac{e^{\lambda t}}{\lambda} d\lambda + \int_{\Gamma^*} \frac{e^{\lambda t}(e^{-\sqrt{\lambda} y} - 1)}{\lambda} d\lambda.$$

For the first we proceed as in the non-degenerate case by taking a contour that passes to the left of the imaginary axis (and to the right of Γ_d). By Cauchy's integral formula, we obtain an estimate by

$$\bar{u}_x(x) I_{\{|x - \int_L^y \frac{ds}{a_1(s)}| \le |a_1^-| t\}} + \bar{u}_x(x) \mathbf{O}(e^{-\eta t}).$$

For the second, we observe that in the case $|\sqrt{\lambda} y| < 1$, we have

$$\Big| \int_{\Gamma^*} \frac{e^{\lambda t}(e^{-\sqrt{\lambda} y} - 1)}{\lambda} d\lambda \Big| = \Big| \int_{\Gamma^*} \frac{e^{\lambda t} \mathbf{O}(|\sqrt{\lambda} y|)}{\lambda} d\lambda \Big| = \mathbf{O}(t^{-1/2}) \mathbf{O}(|y|),$$

while for $|\sqrt{\lambda} y| \ge 1$, we have $|\frac{1}{\sqrt{\lambda}}| \le |y|$, so that

$$\Big| \int_{\Gamma^*} \frac{e^{\lambda t}(e^{-\sqrt{\lambda} y} - 1)}{\lambda} d\lambda \Big| = \Big| \int_{\Gamma^*} \frac{e^{\lambda t}(e^{-\sqrt{\lambda} y} - 1) \mathbf{O}(|y|)}{\sqrt{\lambda}} d\lambda \Big| = \mathbf{O}(t^{-1/2}) \mathbf{O}(|y|).$$

The final term in this case is $\bar{u}_x(x) \mathbf{O}(|\lambda^{-1/2} \log \lambda|) e^{-\sqrt{\lambda} y}$, for which we again take the contour defined through $\sqrt{\lambda} = \frac{y}{Lt} + ik$ to determine an estimate by

$$\bar{u}_x(x) \mathbf{O}(t^{-1/2} \log t) e^{-\frac{y^2}{Lt}}.$$

Derivative estimates follow similarly.

Case (iv) $y \le 0 \le x$. According to Lemma 4.1, we have three integrands to evaluate in the case $y \le 0 \le x$, beginning with $\mathbf{O}(|\lambda^{-1/2}|) \mathbf{O}(|x|^{-1}) e^{-\sqrt{\lambda} x - \mu_2^-(\lambda) y}$, where we recall

$$\mu_2^-(\lambda) = \frac{a_2^- - \sqrt{(a_2^-)^2 + 4\lambda}}{2} = -\frac{1}{a_2^-} \lambda + \frac{1}{(a_2^-)^3} \lambda^2 + \mathbf{O}(\lambda^3).$$

For $|y| \geq a_2^- t$, we expect the kernel to propagate to the right with speed a_2^-, remaining to the right of the shock layer. Observing that along the contours employed in Case (i), we have $\mathrm{Re}(-\sqrt{\lambda}) \leq \mathrm{Re}\mu_2^-(\lambda)$, we conclude that by making this estimate we can obtain the initial bound,

$$\mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\geq a_2^- t\}}.$$

For $|y| < a_2^- t$, we expect the kernel to cross the shock layer and begin decaying in x according to the degenerate rate $e^{-\frac{x^2}{Mt}}$. In this case, we take the contour defined through $\sqrt{\lambda} = \frac{x}{Lt} + ik$ (where $\lambda(k) = \frac{x^2}{L^2t^2} + 2ik\frac{x}{Lt} - k^2$, $\mathrm{Re}\lambda(k)^2 = \frac{x^4}{L^4t^4} - \frac{6x^2}{L^2t^2}k^2 + k^4$), where L is assumed sufficiently large so that $|\lambda(k)| \leq r$ prior to striking the non-degenerate contour. Along the degenerate contour, we have

$$\begin{aligned}
&\left|\mathbf{O}(|x|^{-1})\int_{\Gamma^*}\mathbf{O}(|\lambda^{-1/2}|)e^{\lambda t-\sqrt{\lambda}x-\mu_2^-(\lambda)y}d\lambda\right| \\
&=\mathbf{O}(|x|^{-1})\int_{-k^*}^{+k^*} e^{\frac{x^2}{L^2t}-k^2t-\frac{x^2}{Lt}+\frac{y}{a_2^-}(\frac{x^2}{L^2t^2}-k^2)-\frac{y}{(a_2^-)^3}(\frac{x^4}{L^4t^4}-\frac{6x^2}{L^2t^2}k^2+k^4)}dk \\
&=\mathbf{O}(|x|^{-1})\mathbf{O}(|y+a_2^- t|^{-1/2})e^{-\frac{x^2}{Lt}} I_{\{|y|\leq a_2^- t\}},
\end{aligned}$$

where as in the analysis of Case (i), our contours Γ_D and Γ_{ND} intersect for k sufficiently small so that the growth terms are subsumed. Finally, we observe that for $|y| \geq a_2^- t$, we have

$$\begin{aligned}
t^{-1/4}e^{-\frac{(x-y-a_2^- t)^2}{Mt}} &= \frac{|x-y-a_2^- t|^{1/2}}{t^{1/4}}|x-y-a_2^- t|^{-1/2}e^{-\frac{(x-y-a_2^- t)^2}{Mt}} \\
&\leq C|x-y-a_2^- t|^{-1/2}e^{-\frac{(x-y-a_2^- t)^2}{2Mt}} \leq C|y+a_2^- t|^{-1/2}e^{-\frac{(x-y-a_2^- t)^2}{Mt}}.
\end{aligned}$$

The analysis of our second term from Lemma 4.1,

$$\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}x-\mu_2^-(\lambda)y},$$

is almost identical and can be estimated by

$$\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(x-y-a_2^- t)^2}{Mt}} I_{\{|y|\leq a_2^- t\}} + \mathbf{O}(|y+a_2^- t|^{-1/2})e^{-\frac{x^2}{Mt}} I_{\{|y|\leq a_2^- t\}}.$$

The excited term, $\bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\mu_2^-(\lambda)y}$, can be analyzed as in its counterpart from Case (ii). Derivative estimates follow similarly.

Case (v) $0 \leq y \leq x$. According to Lemma 4.1, we have eight integrands to evaluate in the case $0 \leq y \leq x$, beginning with $\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\sqrt{\lambda}|x-y|}$ and $\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}|x-y|}$, which were both analyzed in the beginning of this section, with combined estimate

$$\mathbf{O}(t^{-1/2})\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}} + P_+\bar{u}_x(x)I_{\{|x-y|\leq\epsilon_0\sqrt{t}\}}.$$

The third integrand, $\mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\sqrt{\lambda}|x-y|}$, can be analyzed similarly to obtain an estimate by $\mathbf{O}(t^{-1/2}(\log t)^2)\mathbf{O}(|x|^{-2})\mathbf{O}(|y|)e^{-\frac{(x-y)^2}{Mt}}$.

We next consider the integrand

$$\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}x-\int_L^y \mu_3(s;\lambda)ds}.$$

We observe that for $|\int_L^y \frac{ds}{a_1(s)}| \geq t$, we may use the observation that $\mathrm{Re}(-\sqrt{\lambda}x) \leq \mathrm{Re}(-\int_L^x \mu_3(s;\lambda)ds)$ and take a contour as in the non-degenerate analysis. In this way we obtain an estimate of the form

$$\mathbf{O}(t^{-1/4})\mathbf{O}(|x|^{-1})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} I_{\{|\int_L^y \frac{ds}{a_1(s)}|\geq t\}}.$$

In the case $|\int_L^y \frac{ds}{a_1(s)}| < t$, we take the contour defined through $\sqrt{\lambda(k)} = \frac{x}{Lt} + ik$, with

$$\lambda(k) = \frac{x^2}{L^2t^2} + 2ik\frac{x}{Lt} - k^2; \quad \mathrm{Re}\lambda(k)^2 = \frac{x^4}{L^4t^4} - \frac{6x^2}{L^2t^2}k^2 + k^4$$

until it intersects the non-degenerate contour. We have, along Γ_D,

$$\begin{aligned}
&\Big|\mathbf{O}(|x|^{-1})\int_{\Gamma^*}\mathbf{O}(|\lambda^{-1/2}|)e^{\lambda t-\sqrt{\lambda}x-\int_L^y \mu_3(s;\lambda)ds}d\lambda\Big| \\
&=\Big|\mathbf{O}(|x|^{-1})\int_{\Gamma^*}\mathbf{O}(|\lambda^{-1/2}|)e^{\lambda t-\sqrt{\lambda}x+\lambda\int_L^y \frac{ds}{a_1(s)}-\lambda^2\int_L^y \frac{ds}{a_1(s)^2}+\mathbf{O}(|\lambda|^3)}d\lambda\Big| \\
&=\mathbf{O}(|x|^{-1})\int_{-k^*}^{+k^*} e^{\frac{x^2}{L^2t}-k^2t-\frac{x^2}{Lt}+(\frac{x^2}{L^2t^2}-k^2)\int_L^y \frac{ds}{a_1(s)}-(\frac{x^4}{L^4t^4}-\frac{6x^2}{L^2t^2}k^2+k^4)\int_L^y \frac{ds}{a_1(s)^3}ds+\mathbf{O}(|\lambda(k)|^3)}dk \\
&=\mathbf{O}(|\int_L^y \frac{ds}{a_1(s)}+t|^{-1/2})\mathbf{O}(|x|^{-1})e^{-\frac{x^2}{Mt}}I_{\{|\int_L^y \frac{ds}{a_1(s)}|\leq t\}}.
\end{aligned}$$

Similarly, the summand $\mathbf{O}(|\lambda^{-1/2}\log\lambda|)\mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}x-\int_L^y \mu_3(s;\lambda)ds}$ leads to an estimate by

$$\mathbf{O}(t^{-1/4}\log t)\mathbf{O}(|x|^{-2})e^{-\frac{(\int_L^x \frac{ds}{a_1(s)}+\int_L^y \frac{ds}{a_1(s)}+t)^2}{Mt}} + \mathbf{O}(|\int_L^y \frac{ds}{a_1(s)}+t|\log t)\mathbf{O}(|x|^{-2})e^{-\frac{x^2}{Mt}}I_{\{|\int_L^y \frac{ds}{a_1(s)}|\leq t\}}.$$

The summand $\bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\int_L^y \mu_3(s;\lambda)ds}$, has been evaluated in Case (iii), and summand

$$\bar{u}_x(x)\mathbf{O}_a(|\lambda^{-1}|)e^{-\int_L^y \mu_3(s;\lambda)ds}$$

can be analyzed by non-degenerate methods to obtain an exponentially decaying excited term plus further estimates, that due to their exponential rate of decay in x can be subsumed. The integrand $\mathbf{O}_a(1)e^{\int_y^x \mu_1(s;\lambda)ds}$ can also be treated by non-degenerate methods, which lead to an estimate that can be subsumed.

Case (vi) $0 \leq x \leq y$. According to Lemma 4.1, we have eight integrals to evaluate in the case $y \leq x \leq 0$, beginning with the summands $\mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(|x|^{-1})\mathbf{O}(|y|)e^{-\sqrt{\lambda}|x-y|}$ and $\mathbf{O}_a(|\lambda^{-1}|)\mathbf{O}(|x|^{-2})e^{-\sqrt{\lambda}|x-y|}$, which were analyzed in the introduction of this section. Following the arguments introduced there, we also find that the summand $\mathbf{O}(|\lambda^{-1/2}(\log\lambda)^2|)\mathbf{O}(|x|^{-1})e^{-\sqrt{\lambda}|x-y|}$ leads to an estimate by $\mathbf{O}(t^{-1/2}(\log t)^2)\mathbf{O}(|x|^{-1})e^{-\frac{(x-y)^2}{Mt}}$.

For the summand $\mathbf{O}_a(1)e^{-\int_x^y \mu_3(s;\lambda)ds}$, $x \geq L$, we can proceed by taking contours from the non-degenerate case to obtain an estimate by

$$\mathbf{O}(t^{-1/2})e^{-\frac{(\int_x^y \frac{ds}{a_1(s)}+t)^2}{Mt}}.$$

For the summand $e^{-\int_x^y \mu_3(s;\lambda)ds}\mathbf{O}(|\lambda^{1/2}\log\lambda|)$, we first consider the case $|\int_y^x \frac{ds}{a_1(s)}| \geq t$, for which the non-degenerate contours do not cross the negative real axis, and taking them, we conclude an estimate of $\mathbf{O}(t^{-3/4}\log t)e^{-\frac{(\int_y^x \frac{ds}{a_1(s)}+t)^2}{Mt}}$. For the case $|\int_x^y \frac{ds}{a_1(s)}| < t$, we proceed as in the analysis of the summand $\mathbf{O}(|\lambda^{1/2}\log\lambda|)e^{\mu_3^-(\lambda)x-\mu_2^-(\lambda)y}$ in Case (i) and take the contour defined through $\sqrt{\lambda} = t^{-1} + ik$ until it strikes the non-degenerate contour defined through

$$-\lambda\int_x^y \frac{ds}{a_1(s)} + \lambda^2\int_x^y \frac{ds}{a_1(s)^3} = -\lambda_R\int_x^y \frac{ds}{a_1(s)} + \lambda_R^2\int_x^y \frac{ds}{a_1(s)^3} + il\int_x^y \frac{ds}{a_1(s)},$$

with λ_R chosen according to

$$\lambda_R = \begin{cases} -\epsilon, & -\epsilon \geq \frac{\bar{\alpha}_+}{\bar{p}_+}, \\ \frac{\bar{\alpha}_+}{\bar{p}_+}, & -t^{-1/2} \geq \frac{\bar{\alpha}_+}{\bar{p}_+} \geq -\epsilon \\ -t^{-1/2}, & 0 \geq \frac{\bar{\alpha}_+}{\bar{p}_+} \geq -t^{-1/2}, \end{cases}$$

where

$$\lambda_R = \frac{\bar{\alpha}_+}{\bar{p}_+}; \quad \bar{\alpha}_+ := \frac{\int_x^y \frac{ds}{a_1(s)} + t}{2t}; \quad \bar{p}_+ := \int_x^y \frac{ds}{a_1(s)^3} \leq 0.$$

We determine an estimate by

$$\mathbf{O}(t^{-3/4} \log t) \wedge \mathbf{O}(|t + \int_x^y \frac{ds}{a_1(s)}|^{-3/2} \log t).$$

The remaining cases were analyzed in Case (v). □

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