Spectral analysis of θ-periodic Schrödinger operators and applications to periodic waves

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Abstract

Assuming a symmetric matrix-valued potential, we consider the associated θ-periodic Schrödinger operators H_θ on intervals $[0, P]$, where in our applications P denotes the period of a stationary periodic solution to a nonlinear evolutionary PDE. We relate the Morse index of H_θ to certain Maslov indices, and apply our theory to operators obtained when Allen-Cahn equations and systems are linearized about stationary periodic solutions.

1 Introduction

We consider boundary value problems

$$\begin{aligned} H_\theta \phi &:= -\phi'' + V(x)\phi = \lambda \phi, \\ \phi(P) &= e^{i\theta}\phi(0), \\ \phi'(P) &= e^{i\theta}\phi'(0), \end{aligned} \tag{1.1}$$

where $\phi(x;\lambda) \in \mathbb{C}^n$, $V \in C([0,P];\mathbb{R}^{n\times n})$ is a symmetric matrix-valued potential, and $\theta \in [-\pi, \pi)$. We take as our domain for H_θ

$$\mathcal{D}(H_\theta) = \{\phi \in H^2((0,P)) : (1.1) \text{ holds}\}, \tag{1.2}$$

and note that with this choice of domain, H_θ is self-adjoint. (See, for example, [19], especially Chapters 6 and 8.) In particular, the spectrum of H_θ, which we denote $\sigma(H_\theta)$, is real-valued. If $\lambda \in \mathbb{R}$ is an eigenvalue of H_θ then (by complex conjugate) λ will also be an eigenvalue of $H_{-\theta}$. In this way, we can focus on the interval $\theta \in [0, \pi]$. Finally, it is natural to scale P to 1 for analysis, but in the current setting we have found it more convenient to leave our equations unscaled so that we do not need to scale variables in our applications section.

Equations (1.1) arise naturally when a gradient reaction-diffusion system

$$u_t + F'(u) = u_{xx}; \quad u \in \mathbb{R}^n,\, x \in \mathbb{R},\, t \geq 0, \tag{1.3}$$

is linearized about a stationary P-periodic solution $\bar{u}(x)$. In this case, we obtain the perturbation equation

$$v_t + F''(\bar{u})v = v_{xx} + \mathbf{O}(v^2), \tag{1.4}$$

with associated eigenvalue problem

$$H\phi := -\phi_{xx} + V(x)\phi = \lambda\phi; \quad V(x) = F''(\bar{u}(x)). \tag{1.5}$$

By standard Floquet theory, the $L^2(\mathbb{R})$ spectrum of H is purely continuous and corresponds with the union of λ so that (1.5) admits a bounded eigenfunction of the Bloch form

$$\phi(x) = e^{i\xi x} w(x),$$

for some $\xi \in \mathbb{R}$ and P-periodic function $w(x)$. The periodicity of w allows us to write

$$\phi(0) = w(0) = w(P) = e^{-i\xi P}\phi(P),$$

and proceeding similarly for ϕ' we find that the $L^2(\mathbb{R})$ spectrum of H corresponds with the union of λ that are eigenvalues of the boundary value problem (1.5) with boundary conditions

$$\phi(P) = e^{i\xi P}\phi(0); \quad \phi'(P) = e^{i\xi P}\phi'(0), \tag{1.6}$$

for some $\xi \in \mathbb{R}$. For notational convenience, we set $\theta = \xi P$, thus obtaining the operator H_θ specified in (1.1).

Our analysis is motivated, in part, by the recent results of Jung and Zumbrun, showing that spectral stability implies nonlinear stability for a broad class of modulated reaction-diffusion periodic waves [10, 13]. In previous work, Howard has identified explicit periodic solutions for equations arising in the context of phase separation processes [4, 5], and these provide a family of applications that will be discussed in Section 6.

We are also motivated by the recent analysis of Jones, Latushkin, and Sukhtaiev, in which the authors show that the Maslov index can be used to analyze the spectrum of H_θ (see [12], and also the related analyses in [11, 16]). We utilize the framework of [12], and compute the Maslov index based on the development of Howard, Latushkin, and Sukhtayev in [7].

Our goal for this introduction is to provide an informal development of the Maslov index and to state our main results. A more systematic development of the Maslov index is provided in Section 2, and a thorough discussion can be found in [7].

As a starting point, we define what we will mean by a *Lagrangian subspace* of $\mathbb{R}^{2n}$.

Definition 1.1. *We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and*

$$(J_{2n}u, v)_{\mathbb{R}^{2n}} = 0,$$

for all $u, v \in \ell$. Here, J_{2n} denotes the standard symplectic matrix

$$J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

and $(\cdot,\cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$. We sometimes adopt standard notation for symplectic forms, $\omega(u,v) = (J_{2n}u, v)_{\mathbb{R}^{2n}}$. In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.

Any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often write $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote paths of Lagrangian subspaces $\ell_i : I \to \Lambda(n)$, for some parameter interval I. The Maslov index associated with these paths, which we will denote $\text{Mas}(\ell_1, \ell_2; I)$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (Precise definitions of what we mean in this context by *multiplicity* and *direction* will be given in Section 2.) In some cases, the Lagrangian subspaces will be defined along some path in the (α, β)-plane

$$\Gamma = \{(\alpha(t), \beta(t)) : t \in I\},$$

and when it is convenient we will use the notation $\text{Mas}(\ell_1, \ell_2; \Gamma)$.

Although cases arise for which the Maslov index can be computed analytically, our point of view is that in most applications it will be computed numerically. In particular, the general character of our theorems involves starting with a quantity that is relatively difficult to compute numerically, and expressing it in terms of one or more quantities that are relatively easy to compute numerically. Such calculations can be made via the associated frames, so the computational difficulty associated with the Maslov index is determined by the computational difficulty associated with computing the frames.

We follow the approach of [12] and reformulate (1.1) for a dependent variable $y \in \mathbb{R}^{2n}$. As a starting point, we express ϕ in terms of its real and imaginary parts, adopting from [12] the labeling convention

$$\phi_k = y_{2k-1} + i y_{2k},$$

for $k = 1, 2, \dots, n$. When expressing the resulting system for $y = (y_1, y_2, \dots, y_{2n})^t$, it is convenient to define the counterclockwise rotation matrix

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.7}$$

Also, for an $m \times n$ matrix $A = (a_{ij})_{i,j=1}^{m,n}$ and a $k \times l$ matrix $B = (b_{ij})_{i,j=1}^{k,l}$ we denote by $A \otimes B$ the *Kronecker product*, by which we mean the $mk \times nl$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}.$$

For our purposes, the two most important Kronecker products will be

$$V \otimes I_2 = \begin{pmatrix} V_{11} & 0 & V_{12} & 0 & \dots & V_{1n} & 0 \\ 0 & V_{11} & 0 & V_{12} & \dots & 0 & V_{1n} \\ V_{21} & 0 & V_{22} & 0 & \dots & V_{2n} & 0 \\ 0 & V_{21} & 0 & V_{22} & \dots & 0 & V_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ V_{n1} & 0 & V_{n2} & 0 & \dots & V_{nn} & 0 \\ 0 & V_{n1} & 0 & V_{n2} & \dots & 0 & V_{nn} \end{pmatrix},$$

and

$$I_n \otimes R_\theta = \begin{pmatrix} R_\theta & 0_2 & \dots & 0_2 \\ 0_2 & R_\theta & \dots & 0_2 \\ \vdots & \vdots & \dots & \vdots \\ 0_2 & 0_2 & \dots & R_\theta \end{pmatrix}.$$

This allows us to express our equation for y as

$$\begin{aligned} \mathcal{H}_\theta y &:= -y'' + (V(x) \otimes I_2)y = \lambda y \\ y(P) &= (I_n \otimes R_\theta)y(0) \\ y'(P) &= (I_n \otimes R_\theta)y'(0). \end{aligned} \tag{1.8}$$

In order to apply the Maslov framework to this system, we introduce two Lagrangian paths, which have been adapted from [12]. As a starting point, given any $y \in C([0, P]; \mathbb{R}^{2n})$ and any $x \in [0, P]$ we define the trace-type map

$$\mathcal{T}_x y := (y(0), y(x), -y'(0), y'(x))^t, \tag{1.9}$$

in which each vector $y(0)$, $y(x)$ etc. is viewed as a row vector so that after transposition $\mathcal{T}_x y \in \mathbb{R}^{8n}$. (For a general discussion of how the form of $\mathcal{T}_x$ is chosen, the reader is referred to [6].) We specify the set

$$\ell_1(x; \lambda) := \{\mathcal{T}_x y : -y'' + (V \otimes I_2)y = \lambda y \text{ for } x \in (0, P)\}, \tag{1.10}$$

which we verify in Section 3 is a Lagrangian subspace of $\mathbb{R}^{8n}$ (this also follows by the proof of Proposition 3.1 in [12]). In addition, we will see in Section 3 that $\ell_1(x; \lambda)$ can be continuously extended to $\ell_1(0; \lambda)$ and $\ell_1(P; \lambda)$, where $\ell_1(0; \lambda)$ is the Lagrangian subspace associated with the frame

$$\mathbf{X}_1(0; \lambda) = \begin{pmatrix} I_{2n} & 0_{2n} \\ I_{2n} & 0_{2n} \\ 0_{2n} & -I_{2n} \\ 0_{2n} & I_{2n} \end{pmatrix},$$

and we defer the specification of $\ell_1(P; \lambda)$ until Section 3.

Likewise, we specify a Lagrangian subspace associated with the boundary condition

$$\ell_2(\theta) = \{(p, (I_n \otimes R_\theta)p, -q, (I_n \otimes R_\theta)q)^t : p, q \in \mathbb{R}^{2n}\}. \tag{1.11}$$

(We verify in Section 3 that $\ell_2(\theta)$ is indeed a Lagrangian subspace for all $\theta \in [0, \pi]$.) The spaces $\ell_1(x; \lambda)$ and $\ell_2(\theta)$ have been constructed so that intersections of $\ell_1(P; \lambda)$ and $\ell_2(\theta)$ correspond with eigenvalues λ of (1.1).

Given a fixed $\theta \in [0, \pi]$ and a fixed value $\lambda_0 \in \mathbb{R}$, let $\text{Mor}(\mathcal{H}_\theta; \lambda_0)$ denote the number of eigenvalues that $\mathcal{H}_\theta$ has below λ_0, counted with multiplicity. Our first theorem relates $\text{Mor}(\mathcal{H}_\theta; \lambda_0)$ to an appropriate Maslov index.

Theorem 1.1. *Suppose $V \in C([0, P]; \mathbb{R}^{n \times n})$ is a symmetric matrix-valued potential, $\mathcal{H}_\theta$ is as specified in (1.8) for some fixed $\theta \in [0, \pi]$, and $\text{Mas}(\ell_1, \ell_2(\theta); [0, P]_{\lambda=\lambda_0})$ denotes the Maslov index for $\ell_1(x; \lambda_0)$ and $\ell_2(\theta)$ as x runs from 0 to P. Then*

$$\text{Mor}(\mathcal{H}_\theta; \lambda_0) = -\text{Mas}(\ell_1, \ell_2(\theta); [0, P]_{\lambda=\lambda_0}) - \begin{cases} 2n & \theta = 0 \\ 0 & \theta \in (0, \pi]. \end{cases}$$

Remark 1.1. *Regarding our convention for designating the Maslov index, we note that we have three variables under consideration—i.e., x, λ, and θ—and that each Maslov index is computed along a path in two dimensions—either the (λ, x)-plane or the (λ, θ)-plane (see Theorem 1.2 for the latter). Our convention is to explicitly write the variable that is not in the plane of computation. For example, this is why θ appears explicitly in the Maslov index of Theorem 1.1 (which is computed in the (λ, x)-plane) and why P (the value of x) appears explicitly in the Maslov index of Theorem 1.2 (which is computed in the (λ, θ)-plane).*

Results quite similar to Theorem 1.1 appear in [11] (Theorem 4.4) and [12] (Theorem 4.1), and we remark on the differences and connections. (See also [16] for related results in the case of Schrödinger operators on $x \in \mathbb{R}^n$.) First, the primary difference between our Theorem 1.1 and the results mentioned above is that we state our theorem in terms of a Maslov index computed on the interval $[0, P]_{\lambda=\lambda_0}$, while the referenced theorems are stated in terms of a Maslov index computed on the interval $[s, P]_{\lambda=\lambda_0}$ for some sufficiently small $s > 0$. Second, our formulation of the Maslov index is different from the formulation used in [11, 12], and consequently our proof of Theorem 1.1 proceeds along different lines than the proofs given in these references.

Turning to the connections, we focus on Theorem 4.4 of [11]. In this reference, the authors show that for $\theta \in (0, \pi]$, and $0 \notin \sigma(H_\theta)$, the Morse index of H_θ (i.e., $\mathrm{Mor}(H_\theta; 0)$ in our notation) is precisely the negative of a Maslov index computed on $[s, P]_{\lambda=0}$ (Item (v) in Theorem 4.4 of [11]), and that for $\theta = 0$, and $0 \notin \sigma(H_\theta)$, with also $0 \notin \sigma(V(0))$, the Morse index of H_0 is precisely the sum of a Maslov index computed on $[s, P]_{\lambda=0}$ and the Morse index of $V(0)$ (i.e., the number of negative eigenvalues of the potential matrix evaluated at $x = 0$) (Item (vii) in Theorem 4.4 of [11]). (See also [9] for a result involving $\mathrm{Mor}\, V(0)$ for Schrödinger operators with separated, self-adjoint boundary conditions.) In order to illuminate the relationship between our Theorem 1.1 and Theorem 4.4 of [11], we state the following corollary to our Theorem 1.1.

Corollary 1.1. *Let the assumptions and notation of Theorem 1.1 hold. Given any $\theta \in (0, \pi]$, there exists $s_0 > 0$ sufficiently small so that for any $s \in (0, s_0]$,*

$$\mathrm{Mor}(\mathcal{H}_\theta; \lambda_0) = -\operatorname{Mas}(\ell_1, \ell_2(\theta); [s, P]_{\lambda=\lambda_0}).$$

Moreover, for $\theta = 0$, assume $0 \notin \sigma(V(0) - \lambda_0 I_n)$ (and so $0 \notin \sigma(V(0) \otimes I_2 - \lambda_0 I_{2n})$). Then there exists $s_0 > 0$ sufficiently small so that for any $s \in (0, s_0]$,

$$\mathrm{Mor}(\mathcal{H}_0; \lambda_0) = -\operatorname{Mas}(\ell_1, \ell_2(0); [s, P]_{\lambda=\lambda_0}) + \mathrm{Mor}(V(0) \otimes I_2 - \lambda_0 I_{2n}).$$

For each fixed $\theta \in [0, \pi]$, Theorem 1.1 provides a computationally efficient way to determine the number of eigenvalues that $\mathcal{H}_\theta$ has below a fixed threshold λ_0. We will verify in Section 3 that a value $\lambda \in \mathbb{R}$ is an eigenvalue of H_θ with multiplicity m if and only if it is an eigenvalue of $\mathcal{H}_\theta$ with multiplicity $2m$. In this way, we obtain from $\mathrm{Mor}(\mathcal{H}_\theta; \lambda_0)$ a count of the number of eigenvalues H_θ has below λ_0. I.e., $\mathrm{Mor}(H_\theta; \lambda_0) = \frac{1}{2} \mathrm{Mor}(\mathcal{H}_\theta; \lambda_0)$.

Suppose we have carried out this calculation for some particular value θ_0 so that we know $\mathrm{Mor}(\mathcal{H}_\theta)$. We ask the following question: can we compute $\mathrm{Mor}(\mathcal{H}_\theta)$ for all $\theta \in [0, \pi] \backslash \{\theta_0\}$ without recomputing solutions of (1.8)? Our next theorem uses the approach of [12] to do precisely this.

Theorem 1.2. *Suppose $V \in C([0,P];\mathbb{R}^{n\times n})$ is a symmetric matrix-valued potential and $\mathcal{H}_\theta$ is as specified in (1.8). Fix values $\lambda_0 \in \mathbb{R}$ and $\theta_0, \theta_1 \in [0,\pi]$, with $\theta_0 < \theta_1$, and let $\text{Mas}(\ell_1(P;\cdot), \ell_2; [\theta_0,\theta_1]_{\lambda=\lambda_0})$ denote the Maslov index for $\ell_1(P;\lambda_0)$ and $\ell_2(\theta)$ as θ runs from θ_0 to θ_1. Then*

$$\text{Mor}(\mathcal{H}_{\theta_1};\lambda_0) = \text{Mor}(\mathcal{H}_{\theta_0};\lambda_0) - \text{Mas}(\ell_1(P;\cdot), \ell_2; [\theta_0,\theta_1]_{\lambda=\lambda_0}).$$

We emphasize that the calculation indicated in Theorem 1.2 does not require re-solving (1.8). Also, we note that Theorem 1.2 is essentially identical in content with Theorem 3.2 of our motivating reference [12]. We have chosen to state it independently because, as with Theorem 1.1, the Maslov index in our statement is formulated in a different way from the Maslov index in [12].

Last, we indicate one final result, which is based on the unitary matrix $\tilde{W}(P;\lambda,\theta)$, defined later in (3.9). Since $\tilde{W}(P;\lambda,\theta)$ is unitary, its eigenvalues will reside on the unit circle S^1. Denote by $\tilde{n}_-(P;\lambda,\theta)$ the number of eigenvalues of $\tilde{W}(P;\lambda,\theta)$ with argument on the interval $[-\pi,0)$ (i.e., on the lower semi-arc), and denote by $\tilde{n}_+(P;\lambda,\theta)$ the number of eigenvalues of $\tilde{W}(P;\lambda,\theta)$ with argument on the interval $[0,\pi)$ (i.e., on the upper semi-arc). For brevity, denote the difference in these values

$$\Delta\tilde{n}(P;\lambda,\theta) = \tilde{n}_+(P;\lambda,\theta) - \tilde{n}_-(P;\lambda,\theta). \tag{1.12}$$

Theorem 1.3. *Suppose $V \in C([0,P];\mathbb{R}^{n\times n})$ is a symmetric matrix-valued potential, H_θ is as specified in (1.1), H is as specified in (1.5), and $\tilde{W}(P;\lambda,\theta)$ is as defined in (3.9). For some fixed $\lambda_0 \in \mathbb{R}$, if there exists $\theta \in [0,\pi]$ so that $\Delta\tilde{n}(P;\lambda_0,\theta) \neq 0$, then H has $L^2(\mathbb{R})$ spectrum below λ_0. In particular, if $\Delta\tilde{n}(P;\lambda_0,\theta) \neq 0$, then we must have $\Delta\tilde{n}(P;\lambda_0,\theta) = 4\kappa$ for some $\kappa \in \mathbb{Z}\backslash\{0\}$, with the following cases: (1) if $\kappa > 0$, then H_θ has at least κ eigenvalues below λ_0; and (2) if $\kappa < 0$, then $H_{\pi-\theta}$ has at least $-\kappa$ eigenvalues below λ_0.*

To the best of our knowledge, no analogue to Theorem 1.3 appears in our references. Nonetheless, we mention that it shares the topological flavor of the analysis carried out in [8].

The paper is organized as follows. In Section 2, we briefly review standard theory associated with the Maslov index, and in Section 3 we further discuss the reformulation of (1.1) as (1.8). In Section 4 we prove Theorem 1.1 and Corollary 1.1, and in Section 5 we prove Theorems 1.2 and 1.3. Finally, in Section 6, we apply these tools to analyze the spectra associated with linear operators obtained when a single Allen-Cahn equation is linearized about a periodic solution (Section 6.1) and when two coupled Allen-Cahn equations are linearized about periodic solutions (Section 6.2). These applications serve three purposes: first, they illustrate each of our four results, and clarify the analysis; second, they demonstrate how our results, and the results in several of our references, can readily be implemented computationally; and third, they provide a complete picture of the spectrum associated with a standard model of phase separation processes.

2 The Maslov Index

In this section, we provide a short overview of the Maslov index in the current setting. Interested readers can find a more thorough discussion in [7] and the references found there.

Given any two Lagrangian subspaces ℓ_1 and ℓ_2, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we can define the complex $n \times n$ matrix

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{2.1}$$

As verified in [7], the matrices $(X_1 - iY_1)$ and $(X_2 + iY_2)$ are both invertible, and $\tilde{W}$ is unitary. We have the following theorem from [7].

Theorem 2.1. *Suppose $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ are Lagrangian subspaces, with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, and let $\tilde{W}$ be as defined in (2.1). Then*

$$\dim \ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2).$$

That is, the dimension of the eigenspace of $\tilde{W}$ associated with the eigenvalue -1 is precisely the dimension of the intersection of the Lagrangian subspaces ℓ_1 and ℓ_2.

Following [1, 3], we use Theorem 2.1, along with an approach to spectral flow introduced in [17], to define the Maslov index. Given a parameter interval $I = [a, b]$, which can be normalized to $[0, 1]$, we consider maps $\ell : I \to \Lambda(n)$, which will be expressed as $\ell(t)$. In order to specify a notion of continuity, we need to define a metric on $\Lambda(n)$, and following [3] (p. 274), we do this in terms of orthogonal projections onto elements $\ell \in \Lambda(n)$. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^t \mathbf{X}_i)^{-1}\mathbf{X}_i^t$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that $\ell : I \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Given two continuous maps $\ell_1(t), \ell_2(t)$ on a parameter interval I, we denote by $\mathcal{L}(t)$ the path

$$\mathcal{L}(t) = (\ell_1(t), \ell_2(t)).$$

In what follows, we will define the Maslov index for the path $\mathcal{L}(t)$, which will be a count, including both multiplicity and direction, of the number of times the Lagrangian paths ℓ_1 and ℓ_2 intersect. In order to be clear about what we mean by multiplicity and direction, we observe that associated with any path $\mathcal{L}(t)$ we will have a path of unitary complex matrices as described in (2.1). We have already noted that the Lagrangian subspaces ℓ_1 and ℓ_2 intersect at a value $t_0 \in I$ if and only if $\tilde{W}(t_0)$ has -1 as an eigenvalue. (We refer to the value t_0 as a *conjugate point*.) In the event of such an intersection, we define the multiplicity of the intersection to be the multiplicity of -1 as an eigenvalue of $\tilde{W}$ (since $\tilde{W}$ is unitary the algebraic and geometric multiplicities are the same). When we talk about the direction of an intersection, we mean the direction the eigenvalues of $\tilde{W}$ are moving (as t varies) along the unit circle S^1 when they cross -1 (we take counterclockwise as the positive direction). We note that we will need to take care with what we mean by a crossing in the following sense: we must decide whether to increment the Maslov index upon arrival or upon departure. Indeed, there are several different approaches to defining the Maslov index (see, for example, [2, 18]), and they often disagree on this convention.

Following [1, 3, 17] (and in particular Definition 1.5 from [1]), we proceed by choosing a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $I = [a, b]$, along with numbers $\epsilon_j \in (0, \pi)$ so that $\ker\big(\tilde{W}(t) - e^{i(\pi \pm \epsilon_j)} I\big) = \{0\}$ for $t_{j-1} \le t \le t_j$; that is, $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ and $j = 1, \dots, n$. Moreover, we notice that for each $j = 1, \dots, n$ and any $t \in [t_{j-1}, t_j]$ there are only finitely many values $\beta \in [0, \epsilon_j)$ for which $e^{i(\pi+\beta)} \in \sigma(\tilde{W}(t))$.

Fix some $j \in \{1, 2, \dots, n\}$ and consider the value

$$k(t, \epsilon_j) := \sum_{0 \le \beta < \epsilon_j} \dim \ker\big(\tilde{W}(t) - e^{i(\pi+\beta)} I\big). \tag{2.2}$$

for $t_{j-1} \le t \le t_j$. This is precisely the sum, along with multiplicity, of the number of eigenvalues of $\tilde{W}(t)$ that lie on the arc

$$A_j := \{e^{it} : t \in [\pi, \pi + \epsilon_j)\}.$$

(See Figure 1.) The stipulation that $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ asserts that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $t_{j-1} \le t \le t_j$. In this way, we see that $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that enter A_j in the counterclockwise direction (i.e., through -1) minus the number that leave in the clockwise direction (again, through -1) during the interval $[t_{j-1}, t_j]$.

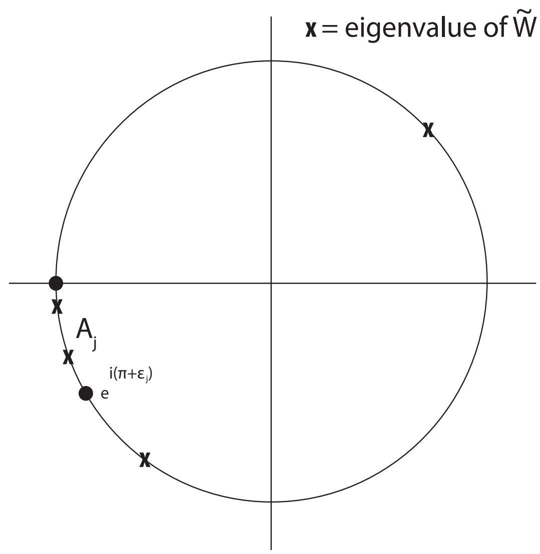

Figure 1: The arc A_j.

In dealing with the catenation of paths, it's particularly important to understand the difference $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ if an eigenvalue resides at -1 at either $t = t_{j-1}$ or $t = t_j$ (i.e., if an eigenvalue begins or ends at a crossing). If an eigenvalue moving in the counterclockwise direction arrives at -1 at $t = t_j$, then we increment the difference forward, while if the eigenvalue arrives at -1 from the clockwise direction we do not (because it was already in A_j prior to arrival). On the other hand, suppose an eigenvalue resides at -1 at $t = t_{j-1}$ and moves in the counterclockwise direction. The eigenvalue remains in A_j, and so we do not increment the difference. However, if the eigenvalue leaves in the clockwise direction then we decrement the difference. In summary, the difference increments forward upon arrivals in the counterclockwise direction, but not upon arrivals in the clockwise direction, and it decrements upon departures in the clockwise direction, but not upon departures in the counterclockwise direction.

We are now ready to define the Maslov index.

Definition 2.1. *Let $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. The Maslov index* $\mathrm{Mas}(\mathcal{L}; I)$ *is defined by*

$$\mathrm{Mas}(\mathcal{L}; I) = \sum_{j=1}^{n} (k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)). \tag{2.3}$$

Remark 2.1. *As we did in the introduction, we will typically refer explicitly to the individual Lagrangian subspaces with the notation* $\mathrm{Mas}(\ell_1, \ell_2; I)$*.*

Remark 2.2. *As discussed in [1], the Maslov index does not depend on the choices of $\{t_j\}_{j=0}^n$ and $\{\epsilon_j\}_{j=1}^n$, so long as these choices follow the specifications above.*

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(I)$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t, s) \in I \times [0, 1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties (see, for example, [7] in the current setting, or Theorem 3.6 in [3] for a more general result).

(P1) (Path Additivity) If $a < b < c$ then

$$\mathrm{Mas}(\mathcal{L}; [a, c]) = \mathrm{Mas}(\mathcal{L}; [a, b]) + \mathrm{Mas}(\mathcal{L}; [b, c]).$$

(P2) (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\mathrm{Mas}(\mathcal{L}; [a, b]) = \mathrm{Mas}(\mathcal{M}; [a, b]).$$

In practice, we work primarily with frames for Lagrangian subspaces, and we can use the following condition from [7] to verify that a given frame $\mathbf{X}$ is indeed the frame for a Lagrangian subspace.

Proposition 2.1. *A $2n \times n$ matrix $\mathbf{X}$ is a frame for a Lagrangian subspace if and only if the columns of $\mathbf{X}$ are linearly independent, and additionally*

$$\mathbf{X}^t J_{2n} \mathbf{X} = 0. \tag{2.4}$$

We refer to this relation as the Lagrangian property for frames.

3 Specification of $\tilde{W}$

In this section, we clarify the relationship between the spectrum of H_θ and the spectrum of $\mathcal{H}_\theta$, and also introduce frames for the Lagrangian subspaces $\ell_1(x;\lambda)$ and $\ell_2(\theta)$. These frames allow us to specify the primary object of our study, $\tilde{W}(x;\lambda,\theta)$.

We begin with the following claim.

Claim 3.1. *Let $\theta \in [-\pi,\pi)$. A value λ will be an eigenvalue of H_θ with multiplicity m if and only if λ is an eigenvalue of $\mathcal{H}_\theta$ with multiplicity $2m$.*

Proof. We only need to show that each eigenfunction of H_θ corresponds with precisely two eigenfunctions of $\mathcal{H}_\theta$. Referring to a solution y of (1.8), if we set $y_o := (y_1, y_3, \dots, y_{2n-1})^t$ and $y_e := (y_2, y_4, \dots, y_{2n})^t$, then we can write

$$\begin{aligned} -y_o'' + V(x) y_o &= \lambda y_o, \\ -y_e'' + V(x) y_e &= \lambda y_e, \end{aligned} \tag{3.1}$$

with coupling (of y_o and y_e) entirely through the boundary terms. For $\theta = -\pi$ or $\theta = 0$, there is no coupling, even in the boundary terms, and we see immediately that if $\phi(x;\lambda)$ is an eigenfunction for H_θ for some $\lambda \in \mathbb{R}$, then $\mathcal{H}_\theta$ will have two linearly independent eigenfunctions associated with the same eigenvalue: $y_o = \phi$, $y_e = 0$ and $y_o = 0$, $y_e = \phi$. Notice that in this case, if the real and imaginary parts of ϕ are linearly independent then λ will have multiplicity two as an eigenfunction of (1.1), because both the real and imaginary parts will necessarily be eigenfunctions.

If $\theta \notin \{-\pi, 0\}$ and $\phi(x;\lambda)$ is an eigenfunction for H_θ associated with λ then ϕ cannot be either purely real or purely imaginary, and we can write $\phi = y_o + i y_e$, which corresponds with an eigenfunction y of $\mathcal{H}_\theta$ (i.e., with $y = (y_1, y_2, \dots, y_{2n})^t$ constructed from y_o and y_e defined above). We now define a vector function $\tilde{y} \in \mathbb{R}^{2n}$ so that $\tilde{y}_o = y_o - y_e$ and $\tilde{y}_e = y_o + y_e$. We see from (3.1) that $-\tilde{y}'' + V(x) \otimes \tilde{y} = \lambda \tilde{y}$, and we also observe that for each $k \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} y_{2k-1}(P) &= y_{2k-1}(0)\cos\theta - y_{2k}(0)\sin\theta, \\ y_{2k}(P) &= y_{2k-1}(0)\sin\theta + y_{2k}(0)\cos\theta, \end{aligned}$$

so that

$$\begin{aligned} \tilde{y}_{2k-1}(P) &= y_{2k-1}(P) - y_{2k}(P) = (y_{2k-1}(0) - y_{2k}(0))\cos\theta - (y_{2k-1}(0) + y_{2k}(0))\sin\theta \\ &= \tilde{y}_{2k-1}(0)\cos\theta - \tilde{y}_{2k}(0)\sin\theta. \end{aligned}$$

and similarly,

$$\tilde{y}_{2k}(P) = \tilde{y}_{2k-1}(0)\sin\theta + \tilde{y}_{2k}(0)\cos\theta.$$

Derivative relations follow by identical calculations, and we see that $\tilde{y}$ is a second eigenfunction of H_θ associated with the eigenvalue λ. □

In order to construct a frame for $\ell_1(x;\lambda)$, we set $\varphi_1 = y$ and $\varphi_2 = y'$ and express our equation as a first order system

$$\varphi' = \mathbb{A}(x;\lambda)\varphi; \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I_{2n} \\ V \otimes I_2 - \lambda I_{2n} & 0 \end{pmatrix}. \tag{3.2}$$

Let $\Phi(x;\lambda)$ denote the $4n \times 4n$ fundamental matrix obtained by solving the matrix ODE

$$\Phi' = \mathbb{A}(x;\lambda)\Phi; \quad \Phi(0;\lambda) = I_{4n}, \tag{3.3}$$

which we can alternatively express in the Hamiltonian form

$$J_{4n}\Phi' = \mathbb{B}(x;\lambda)\Phi; \quad \Phi(0;\lambda) = I_{4n}, \tag{3.4}$$

where

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I_{2n} - V \otimes I_2 & 0 \\ 0 & I_{2n} \end{pmatrix}.$$

If we introduce the notation

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix},$$

for $2n \times 2n$ matrices $\{\Phi_{ij}\}_{i,j=1}^2$ then we can express a frame for $\ell_1(x;\lambda)$ as

$$\mathbf{X}_1(x;\lambda) = \begin{pmatrix} X_1(x;\lambda) \\ Y_1(x;\lambda) \end{pmatrix} = \begin{pmatrix} I_{2n} & 0_{2n} \\ \Phi_{11}(x;\lambda) & \Phi_{12}(x;\lambda) \\ 0_{2n} & -I_{2n} \\ \Phi_{21}(x;\lambda) & \Phi_{22}(x;\lambda) \end{pmatrix}. \tag{3.5}$$

Associated with (3.3), we can consider the reduced system

$$\tilde{\Phi}' = \tilde{\mathbb{A}}(x;\lambda)\tilde{\Phi}; \quad \tilde{\Phi}(0;\lambda) = I_{2n}, \tag{3.6}$$

where

$$\tilde{\mathbb{A}}(x;\lambda) = \begin{pmatrix} 0 & I_n \\ V - \lambda I_n & 0 \end{pmatrix}.$$

It is straightforward to check that $\Phi(x;\lambda) = \tilde{\Phi}(x;\lambda) \otimes I_2$, and with

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} \end{pmatrix},$$

we have $\Phi_{ij}(x;\lambda) = \tilde{\Phi}_{ij}(x;\lambda) \otimes I_2$ for $i,j \in \{1,2\}$. Although the notation is a bit unwieldy, we will find it convenient below to denote by Φ_{ij}^{kl} the entry in row k and column l of the matrix Φ_{ij}, and similarly for $\tilde{\Phi}_{ij}^{kl}$.

In order to verify that $\mathbf{X}_1(x;\lambda)$ is the frame for a Lagrangian subspace, we use Proposition 2.1, which asserts that we need to verify that

$$\mathbf{X}_1(x;\lambda)^t J_{8n} \mathbf{X}_1(x;\lambda) = 0$$

for all $(x,\lambda) \in [0,P] \times \mathbb{R}$.

First, for $x = 0$ we have

$$\mathbf{X}_1(0;\lambda) = \begin{pmatrix} I_{2n} & 0_{2n} \\ I_{2n} & 0_{2n} \\ 0_{2n} & -I_{2n} \\ 0_{2n} & I_{2n} \end{pmatrix},$$

from which we compute directly to find

$$\mathbf{X}_1(0;\lambda)^t J_{8n}\mathbf{X}_1(0;\lambda) = 0.$$

For each fixed $\lambda \in \mathbb{R}$, temporarily set

$$\mathcal{A}(x) := \mathbf{X}_1(x;\lambda)^t J_{8n}\mathbf{X}_1(x;\lambda) = \Phi(x;\lambda)^t J_{4n}\Phi(x;\lambda) - J_{4n},$$

where dependence on the fixed value λ in $\mathcal{A}$ has been suppressed, and the second equality follows from a straightforward calculation. We can now compute

$$\begin{aligned}\mathcal{A}'(x) &= \Phi'(x;\lambda)^t J_{4n}\Phi(x;\lambda) + \Phi(x;\lambda)^t J_{4n}\Phi'(x;\lambda)\\ &= -(J_{4n}\Phi'(x;\lambda))^t\Phi(x;\lambda) + \Phi(x;\lambda)^t J_{4n}\Phi'(x;\lambda)\\ &= -(\mathbb{B}(x;\lambda)\Phi(x;\lambda))^t\Phi(x;\lambda) + \Phi(x;\lambda)^t\mathbb{B}(x;\lambda)\Phi(x;\lambda) = 0,\end{aligned}$$

where the final equality follows from the symmetry of $\mathbb{B}$. We conclude that $\mathcal{A}(x) = 0$ for all $x \in [0, P]$, and since this is true for all $\lambda \in \mathbb{R}$ we conclude that $\ell_1(x;\lambda)$ is Lagrangian for all $(x,\lambda) \in [0,P] \times \mathbb{R}$.

Turning to $\ell_2(\theta)$, a natural choice of frame is

$$\mathbf{X}_2(\theta) = \begin{pmatrix} X_2(\theta) \\ Y_2(\theta) \end{pmatrix} = \begin{pmatrix} I_{2n} & 0_{2n} \\ I_n \otimes R_\theta & 0_{2n} \\ 0_{2n} & -I_{2n} \\ 0_{2n} & I_n \otimes R_\theta \end{pmatrix}. \tag{3.7}$$

Using the relation

$$(I_n \otimes R_\theta)^t (I_n \otimes R_\theta) = I_{2n}, \tag{3.8}$$

we verify by direct calculation that $\mathbf{X}_2(\theta)^t J_{8n}\mathbf{X}_2 = 0$, and we conclude from Proposition 2.1 that $\mathbf{X}_2(\theta)$ is the frame for a Lagrangian subspace.

We see immediately that

$$X_2(\theta) - iY_2(\theta) = \begin{pmatrix} I_{2n} & iI_{2n} \\ I_n \otimes R_\theta & -iI_n \otimes R_\theta \end{pmatrix},$$

and similarly for $X_2(\theta) + iY_2(\theta)$. Recalling (3.8), we readily check that

$$(X_2(\theta) + iY_2(\theta))^{-1} = \begin{pmatrix} \frac{1}{2}I_{2n} & \frac{1}{2}(I_n \otimes R_\theta)^t \\ i\frac{1}{2}I_{2n} & -i\frac{1}{2}(I_n \otimes R_\theta)^t \end{pmatrix},$$

and consequently

$$(X_2(\theta) - iY_2(\theta))(X_2(\theta) + iY_2(\theta))^{-1} = \begin{pmatrix} 0 & (I_n \otimes R_\theta)^t \\ I_n \otimes R_\theta & 0 \end{pmatrix}.$$

For notational brevity, we will denote this last matrix $\mathcal{R}_\theta$, and we note that due to (3.8) $\mathcal{R}_\theta$ will be an involutory matrix (i.e., $\mathcal{R}_\theta^2 = I_{4n}$).

Using (2.1), we can now define the primary object of our study:

$$\tilde{W}(x;\lambda,\theta) = -(X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1}\mathcal{R}_\theta. \tag{3.9}$$

It follows from our construction of $\mathcal{H}_\theta$ from H_θ, and from Claim 3.1, that if -1 is an eigenvalue of $\tilde{W}(P;\lambda,\theta)$ then it will have even multiplicity. More generally, we have the following claim.

Claim 3.2. *Each eigenvalue of $\tilde{W}(x;\lambda,\theta)$ occurs with even multiplicity.*

Proof. Suppose $e^{i\kappa}$ is an eigenvalue of $\tilde{W}(x;\lambda,\theta)$ with eigenvector r. That is,

$$-(X_1(x;\lambda)+iY_1(x;\lambda))(X_1(x;\lambda)-iY_1(x;\lambda))^{-1}\mathcal{R}_\theta r = e^{i\kappa}r.$$

Set $v=\mathcal{R}_\theta r$ and multiply on the left by $\mathcal{R}_\theta$ to obtain the equivalent problem

$$-\mathcal{R}_\theta(X_1(x;\lambda)+iY_1(x;\lambda))(X_1(x;\lambda)-iY_1(x;\lambda))^{-1}v = e^{i\kappa}v.$$

Now set $w=(X_1(x;\lambda)-iY_1(x;\lambda))^{-1}v$ so that

$$-\mathcal{R}_\theta(X_1(x;\lambda)+iY_1(x;\lambda))w = e^{i\kappa}(X_1(x;\lambda)-iY_1(x;\lambda))w. \tag{3.10}$$

Set $\hat{v}=(X_1(x;\lambda)+iY_1(x;\lambda))w$, so that (3.10) can alternatively be expressed as

$$-\mathcal{R}_\theta\hat{v} = e^{i\kappa}v, \tag{3.11}$$

and notice that we can express $\hat{v}$ as follows:

$$\begin{aligned}
\hat{v}_k &= w_k - iw_{2n+k}; \quad k\in\{1,2,\dots,2n\};\\
\hat{v}_{2n+k} &= \sum_{j=1}^n\Big\{(\tilde{\Phi}_{11}^{(\frac{k+1}{2})j}+i\tilde{\Phi}_{21}^{(\frac{k+1}{2})j})w_{2j-1}\\
&\quad+(\tilde{\Phi}_{12}^{(\frac{k+1}{2})j}+i\tilde{\Phi}_{22}^{(\frac{k+1}{2})j})w_{2(n+j)-1}\Big\}; \quad k\in\{1,3,\dots,2n-1\};\\
\hat{v}_{2n+k} &= \sum_{j=1}^n\Big\{(\tilde{\Phi}_{11}^{(\frac{k}{2})j}+i\tilde{\Phi}_{21}^{(\frac{k}{2})j})w_{2j}\\
&\quad+(\tilde{\Phi}_{12}^{(\frac{k}{2})j}+i\tilde{\Phi}_{22}^{(\frac{k}{2})j})w_{2(n+j)}\Big\}; \quad k\in\{2,4,\dots,2n\}.
\end{aligned} \tag{3.12}$$

Moreover, we can express v similarly, with the sign in front of each (explicit) appearance of i switched.

The first two equations in system (3.11) can be expressed as

$$-R_\theta^t\begin{pmatrix}\hat{v}_{2n+1}\\ \hat{v}_{2n+2}\end{pmatrix} = e^{i\kappa}\begin{pmatrix}v_1\\ v_2\end{pmatrix},$$

which can be expanded to

$$\begin{aligned}
-\cos\theta\hat{v}_{2n+1}-\sin\theta\hat{v}_{2n+2} &= e^{i\kappa}v_1,\\
\sin\theta\hat{v}_{2n+1}-\cos\theta\hat{v}_{2n+2} &= e^{i\kappa}v_2.
\end{aligned}$$

If we add these, we get

$$-\cos\theta(\hat{v}_{2n+1}+\hat{v}_{2n+1})+\sin\theta(\hat{v}_{2n+1}-\hat{v}_{2n+1}) = e^{i\kappa}(v_1+v_2),$$

and likewise if we subtract them we get

$$-\cos\theta(\hat{v}_{2n+1}-\hat{v}_{2n+1})-\sin\theta(\hat{v}_{2n+1}+\hat{v}_{2n+1}) = e^{i\kappa}(v_1-v_2).$$

We see from these calculations that the vector pair

$$\begin{pmatrix} \hat{v}_{2n+1} - \hat{v}_{2n+2} \\ \hat{v}_{2n+1} + \hat{v}_{2n+2} \end{pmatrix}; \quad \text{and} \quad \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \end{pmatrix}$$

solve the same two equations as the vector pair

$$\begin{pmatrix} \hat{v}_{2n+1} \\ \hat{v}_{2n+2} \end{pmatrix}; \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Set

$$\Box v = (v_1 - v_2, v_1 + v_2, v_3 - v_4, v_3 + v_4, \ldots, v_{4n-1} - v_{4n}, v_{4n-1} + v_{4n})^t,$$

and similarly for $\Box\hat{v}$ and $\Box w$. Proceeding similarly as above for each subsequent pair of equations in (3.11), we find that $\Box v$ and $\Box\hat{v}$ solve precisely the same equations as v and $\hat{v}$. I.e.,

$$-\mathcal{R}_\theta \hat{v} = e^{i\kappa} v \iff -\mathcal{R}_\theta \Box\hat{v} = e^{i\kappa} \Box v.$$

It follows from (3.12) and the similar relations for v that $\Box w$ is a solution to (3.10). In particular, we have from (3.12) that

$$\begin{aligned} \Box\hat{v} &= (X_1(x;\lambda) + iY_1(x;\lambda))\Box w \\ \Box v &= (X_1(x;\lambda) - iY_1(x;\lambda))\Box w. \end{aligned}$$

Then $-\mathcal{R}_\theta \Box\hat{v} = e^{i\kappa}\Box v$ can be expressed as

$$-\mathcal{R}_\theta (X_1(x;\lambda) + iY_1(x;\lambda))\Box w = e^{i\kappa}\Box v,$$

and subsequently

$$-\mathcal{R}_\theta (X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1}\Box v = e^{i\kappa}\Box v,$$

from which we see that v and $\Box v$ are both eigenvectors of $\tilde{W}(x;\lambda,\theta)$ associated with the eigenvalue $e^{\imath\kappa}$. Moreover, it's straightforward to see that v and $\Box v$ must be linearly independent, so they form a pair of two linearly independent eigenvectors for the eigenvalue $e^{i\kappa}$. This completes the proof. $\Box$

4 Proofs of Theorem 1.1 and Corollary 1.1

Suppose that for some fixed $\theta \in [-\pi, \pi)$, $\lambda \in \mathbb{R}$ is an eigenvalue of (1.1) with associated eigenvector $\phi(x;\lambda)$. If we take an $L^2([0,P])$ inner product of (1.1) with ϕ we obtain

$$\|\phi'\|^2 + \langle V(x)\phi, \phi\rangle = \lambda\|\phi\|^2,$$

and applying the Cauchy-Schwarz inequality we see that

$$\lambda \geq -\|V\|_{L^\infty([0,P])}, \tag{4.1}$$

where

$$\|V\|_{L^\infty([0,P])} = \max_{x\in[0,P]} \|V(x)\|_2,$$

with $\|\cdot\|_2$ designating the matrix norm induced by the standard Euclidean norm.

Take $\lambda_\infty > \|V\|_{L^\infty([0,P])}$ so that the spectrum of H_θ lies in $(-\lambda_\infty, \infty)$. Fix any $\lambda_0 > -\lambda_\infty$ and consider the path in the (x, λ)-plane described as follows: (1) fix $x = 0$ and let λ run from $-\lambda_\infty$ to λ_0 (the *bottom shelf*, which we will denote $[-\lambda_\infty, \lambda_0]_{x=0}$); (2) fix $\lambda = \lambda_0$ and let x run from 0 to P (the *right shelf*, $[0, P]_{\lambda=\lambda_0}$); (3) fix $x = P$ and let λ run from λ_0 to $-\lambda_\infty$ (the *top shelf*, $[\lambda_0, -\lambda_\infty]_{x=P}$); and (4) fix $\lambda = -\lambda_\infty$ and let x run from P to 0 (the *left shelf*, $[P, 0]_{\lambda=-\lambda_\infty}$). We denote by Γ the simple closed curve obtained by following each of these contours precisely once. (See Figure 2.)

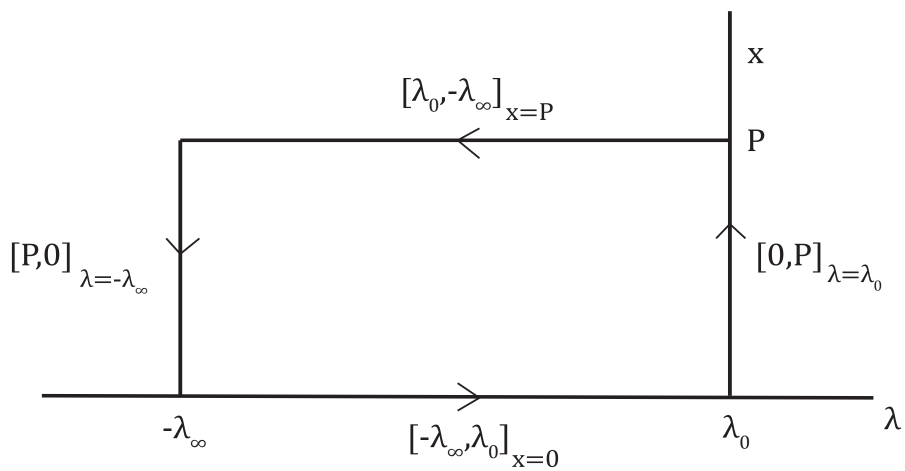

Figure 2: The (λ, x)-Maslov Box.

By catenation of paths we have

$$\begin{aligned}\operatorname{Mas}(\ell_1, \ell_2; \Gamma) &= \operatorname{Mas}(\ell_1, \ell_2; [-\lambda_\infty, \lambda_0]_{x=0}) + \operatorname{Mas}(\ell_1, \ell_2; [0, P]_{\lambda=\lambda_0}) \\ &\quad + \operatorname{Mas}(\ell_1, \ell_2; [\lambda_0, -\lambda_\infty]_{x=P}) + \operatorname{Mas}(\ell_1, \ell_2; [P, 0]_{\lambda=-\lambda_\infty}).\end{aligned} \tag{4.2}$$

For the bottom shelf, the Lagrangian subspaces $\ell_1(0; \lambda)$ and $\ell_2(\theta)$ are both independent of λ, so we have

$$\operatorname{Mas}(\ell_1, \ell_2; [-\lambda_\infty, \lambda_0]_{x=0}) = 0.$$

More precisely, we have

$$\begin{aligned}\tilde{W}(0; \lambda, \theta) &= -(X_1(0; \lambda) + iY_1(0; \lambda))(X_1(0; \lambda) - iY_1(0; \lambda))^{-1}\mathcal{R}_\theta \\ &= -\begin{pmatrix} I_n \otimes R_\theta & 0 \\ 0 & (I_n \otimes R_\theta)^t \end{pmatrix},\end{aligned}$$

where the second equality follows from the form of $\mathcal{R}_\theta$ and a straightforward calculation.

The eigenvalues of $-\tilde{W}(0;\lambda,\theta)$, which we denote here by μ, will satisfy

$$\det((I_n \otimes R_\theta - \mu I_{2n})((I_n \otimes R_\theta)^t - \mu I_{2n})) = 0,$$

and noting the relations $(I_n \otimes R_\theta)^t (I_n \otimes R_\theta) = I_{2n}$ and $(I_n \otimes R_\theta)^t + (I_n \otimes R_\theta) = 2(\cos\theta) I_{2n}$, we see that

$$\det(I_{2n}(\mu^2 - 2\mu\cos\theta + 1)) = 0.$$

We see that for any $\theta \in [0,\pi]$ the eigenvalues of $\tilde{W}(0;\lambda,\theta)$ (note the sign change) are

$$\mu = -\cos\theta \pm i \sin\theta,$$

each repeated $2n$ times. We see that for $\theta = 0$ all $4n$ eigenvalues of $\tilde{W}(0;\lambda,\theta)$ reside at -1, while for $\theta \in (0,\pi]$ none of the eigenvalues of $\tilde{W}(0;\lambda,\theta)$ reside at -1.

For the top shelf, and indeed for any intermediate horizontal shelf $[\lambda_0, -\lambda_\infty]_{x=s}$, with $s \in (0,P]$, we can check that the eigenvalues of $\tilde{W}(x;\lambda,\theta)$ rotate monotonically counterclockwise as λ decreases. In order to see this, we use Lemma 4.2 of [7], which asserts that monotonicity will be determined by the nature of

$$\Omega(x;\lambda) := -\mathbf{X}_1(x;\lambda)^t J_{8n} \partial_\lambda \mathbf{X}_1(x;\lambda) = -\Phi(x;\lambda)^t J_{4n} \partial_\lambda \Phi(x;\lambda),$$

where the second equality follows from a straightforward calculation. In order to get a sign for this matrix we compute

$$\begin{aligned}
\Omega'(x;\lambda) &= -\Phi'(x;\lambda)^t J_{4n} \partial_\lambda \Phi(x;\lambda) - \Phi(x;\lambda)^t J_{4n} \partial_\lambda \Phi'(x;\lambda) \\
&= (J_{4n}\Phi'(x;\lambda))^t \partial_\lambda \Phi(x;\lambda) - \Phi(x;\lambda)^t \partial_\lambda J_{4n} \Phi'(x;\lambda) \\
&= (\mathbb{B}(x;\lambda)\Phi(x;\lambda))^t \partial_\lambda \Phi(x;\lambda) - \Phi(x;\lambda)^t \partial_\lambda (\mathbb{B}(x;\lambda)\Phi(x;\lambda)) \\
&= \Phi(x;\lambda)^t \mathbb{B}(x;\lambda) \partial_\lambda \Phi(x;\lambda) - \Phi(x;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda) - \Phi(x;\lambda)^t \mathbb{B}(x;\lambda)\partial_\lambda \Phi(x;\lambda) \\
&= -\Phi(x;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda).
\end{aligned}$$

Integrating, and noting that $\Omega(0;\lambda) = 0$ (because $\Phi(0;\lambda)$ is independent of λ), we obtain

$$\Omega(x;\lambda) = -\int_0^x \Phi(y;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\Phi(y;\lambda) dy.$$

We have

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} I_{2n} & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\Omega(x;\lambda) = -\int_0^x \Phi_1(y;\lambda)^t \Phi_1(y;\lambda) dy.$$

This matrix is clearly non-positive, and moreover it cannot have 0 as an eigenvalue because the associated eigenvector $v = v(x;\lambda) \in \mathbb{R}^{4n}$ would necessarily satisfy $\Phi_1(y;\lambda)v(x;\lambda) = 0$ for all $y \in (0,x)$, and this would contradict linear independence of the columns of Φ_1 (as solutions of $-y'' + (V \otimes I_2) = \lambda y$).

Lemma 4.2 of [7] asserts that if $\Omega(x;\lambda)$ is negative definite then as λ decreases from λ_0 to $-\lambda_\infty$ the eigenvalues of $\tilde{W}(x;\lambda,\theta)$ will rotate monotonically counterclockwise. For the top shelf, we have $x = P$, and so each conjugate point λ corresponds with an intersection between $\ell_1(P;\lambda)$ and $\ell_2(\theta)$, and so with an eigenvalue of $\mathcal{H}_\theta$. By monotonicity, the Maslov index along the top shelf will simply be a count, including multiplicity, of these eigenvalues, and so will be precisely the number of eigenvalues that $\mathcal{H}_\theta$ has between $-\lambda_\infty$ and λ_0. But since $\mathcal{H}_\theta$ has no eigenvalues below $-\lambda_\infty$ this count is precisely the number of eigenvalues, counted with multiplicity, that $\mathcal{H}_\theta$ has below λ_0. We denote this count $\mathrm{Mor}(\mathcal{H}_\theta;\lambda_0)$. I.e.,

$$\mathrm{Mas}(\ell_1,\ell_2;[\lambda_0,-\lambda_\infty]_{x=P}) = \mathrm{Mor}(\mathcal{H}_\theta;\lambda_0).$$

For the left shelf, conjugate points $x > 0$ correspond with solutions to the boundary value problem

$$\begin{aligned} H_\theta^x &:= -\phi'' + V(y)\phi = -\lambda_\infty\phi \\ \phi(x) &= e^{i\theta}\phi(0) \\ \phi'(x) &= e^{i\theta}\phi'(0). \end{aligned}$$

However, proceeding precisely as with H_θ we find that the eigenvalues of H_θ^x satisfy

$$\lambda \geq -\|V\|_{L^\infty([0,x])} > -\lambda_\infty,$$

from which we conclude that there are no conjugate points $x > 0$ on the left shelf.

The case $x = 0$ is not covered in this calculation, but can be understood from our analysis of the bottom shelf. First, we have seen that for $\theta \in (0,\pi]$ none of the eigenvalues of $\tilde{W}(0;\lambda,\theta)$ will reside at -1, and we can conclude that

$$\mathrm{Mas}(\ell_1,\ell_2(\theta);[P,0]_{\lambda=-\lambda_\infty}) = 0$$

in these cases. For $\theta = 0$ we know that all $4n$ eigenvalues of $\tilde{W}(0;-\lambda_\infty,\theta)$ reside at -1, so in order to compute $\mathrm{Mas}(\ell_1,\ell_2(\theta);[P,0]_{\lambda=-\lambda_\infty})$ we must determine the rotation of these eigenvalues as $x \to 0^+$.

According to Lemma 4.2 of [7], rotation of the eigenvalues of $\tilde{W}(x;-\lambda_\infty,\theta)$ as x varies is determined by the nature of

$$\begin{aligned} \tilde{\Omega}(x;-\lambda_\infty) &:= -\mathbf{X}_1(x;-\lambda_\infty)^t J_{8n}\mathbf{X}_1'(x;-\lambda_\infty) \\ &= -\Phi(x;-\lambda_\infty)^t J_{4n}\Phi'(x;-\lambda_\infty) = -\Phi(x;-\lambda_\infty)^t\mathbb{B}(x;-\lambda_\infty)\Phi(x;-\lambda_\infty). \end{aligned}$$

Since $\Phi(0;-\lambda_\infty) = I_{4n}$, we see that

$$\tilde{\Omega}(0;-\lambda_\infty) = -\mathbb{B}(0;-\lambda_\infty) = \begin{pmatrix} V(0)\otimes I_2 + \lambda_\infty I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

For $\lambda_\infty > 0$ sufficiently large (in particular, for $-\lambda_\infty < -\|V\|_{L^\infty([0,P])}$), half the eigenvalues of $\tilde{\Omega}(0;-\lambda_\infty)$ will be positive and half will be negative. It follows (see the proof of Lemma 3.11 in [9]) that half of the $4n$ eigenvalues of $\tilde{W}(x;-\lambda_\infty,0)$ will arrive at -1 from the counterclockwise direction as $x \to 0^+$, and half will arrive from the clockwise direction. We conclude that

$$\mathrm{Mas}(\ell_1,\ell_2(0);[P,0]_{\lambda=-\lambda_\infty}) = 2n.$$

Theorem 1.1 now follows immediately from (4.2) and the four established identities:

$$\begin{aligned}
\operatorname{Mas}(\ell_1, \ell_2; \Gamma) &= 0 \\
\operatorname{Mas}(\ell_1, \ell_2; [-\lambda_\infty, \lambda_0]_{x=0}) &= 0 \\
\operatorname{Mas}(\ell_1, \ell_2; [\lambda_0, -\lambda_\infty]_{x=P}) &= \operatorname{Mor}(\mathcal{H}_\theta; \lambda_0) \\
\operatorname{Mas}(\ell_1, \ell_2(\theta); [P, 0]_{\lambda=-\lambda_\infty}) &= \begin{cases} 2n & \theta = 0 \\ 0 & \theta \in (0, \pi]. \end{cases}
\end{aligned}$$

□

We conclude this section with a proof of Corollary 1.1.

Proof of Corollary 1.1. First, for $\theta \in (0, \pi]$, we have seen during the proof of Theorem 1.1 that $-1 \notin \sigma(\tilde{W}(0; \lambda_0, \theta))$. By continuity, there exists $s_0 > 0$ sufficiently small so that $-1 \notin \sigma(\tilde{W}(x; \lambda_0, \theta))$ for any $x \in [0, s_0]$. Consequently, we see that for any $s \in (0, s_0]$,

$$\operatorname{Mas}(\ell_1, \ell_2(\theta); [0, s]_{\lambda=\lambda_0}) = 0.$$

But then by path catenation,

$$\begin{aligned}
\operatorname{Mas}(\ell_1, \ell_2(\theta); [0, P]_{\lambda=\lambda_0}) &= \operatorname{Mas}(\ell_1, \ell_2(\theta); [0, s]_{\lambda=\lambda_0}) + \operatorname{Mas}(\ell_1, \ell_2(\theta); [s, P]_{\lambda=\lambda_0}) \\
&= \operatorname{Mas}(\ell_1, \ell_2(\theta); [s, P]_{\lambda=\lambda_0}),
\end{aligned}$$

and the claim of Corollary 1.1 for $\theta \in (0, \pi]$ now follows immediately from Theorem 1.1.

For $\theta = 0$, we have seen during the proof of Theorem 1.1 that all $4n$ eigenvalues of $\tilde{W}(0; \lambda_0, 0)$ reside at -1. Similarly as in our discussion of the rotation of these eigenvalues as x arrives at 0 for $\lambda = -\lambda_\infty$ (during the proof of Theorem 1.1), rotation of the eigenvalues of $\tilde{W}(x; \lambda_0, 0)$ as x increases from 0 is determined by the eigenvalues of

$$-\mathbb{B}(0; \lambda_0) = \begin{pmatrix} V(0) \otimes I_2 - \lambda_0 I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

In particular, each negative eigenvalue of this matrix corresponds with an eigenvalue of $\tilde{W}(x; \lambda_0, 0)$ that will rotate away from -1 in the clockwise direction as x increases from 0 (thus decrementing the Maslov index), and likewise, each positive eigenvalue of this matrix corresponds with an eigenvalue of $\tilde{W}(x; \lambda_0, 0)$ that will rotate away from -1 in the counter-clockwise direction as x increases from 0 (thus leaving the Maslov index unchanged). (We recall that for $\theta = 0$, Corollary 1.1 assumes 0 is not an eigenvalue of $V(0) - \lambda_0 I_n$, and it follows immediately that 0 is not an eigenvalue of $-\mathbb{B}(0; \lambda_0)$. We use here the simple observation that μ is an eigenvalue of $V(0) - \lambda_0 I_n$ with multiplicity m iff μ is an eigenvalue of $V(0) \otimes I_2 - \lambda_0 I_{2n}$ with multiplicity $2m$.) Since $-\mathbb{B}(0; \lambda_0)$ is block diagonal, its eigenvalues will be the union of the eigenvalues of $V(0) \otimes I_2 - \lambda_0 I_{2n}$ and the eigenvalues of $-I_{2n}$ (clearly, all -1 for the latter). The total number of negative eigenvalues in this union will correspond with the number of eigenvalues of $\tilde{W}(x; \lambda_0, 0)$ that rotate in the clockwise direction as x increases from 0, and hence will correspond with the decrement count of the Maslov index as x increases from 0. If we let $\operatorname{Mor}(V(0) \otimes I_2 - \lambda_0 I_{2n})$ denote the number of negative eigenvalues of $V(0) \otimes I_2 - \lambda_0 I_{2n}$, then we can conclude by continuity that there exists $s_0 > 0$ sufficiently small so that for any $s \in (0, s_0]$, we have

$$\operatorname{Mas}(\ell_1, \ell_2(\theta); [0,s]_{\lambda=\lambda_0}) = -2n - \operatorname{Mor}(V(0) \otimes I_2 - \lambda_0 I_{2n}).$$

Using catenation of paths, we find

$$\begin{aligned}\operatorname{Mas}(\ell_1, \ell_2(\theta); [0,P]_{\lambda=\lambda_0}) &= \operatorname{Mas}(\ell_1, \ell_2(\theta); [0,s]_{\lambda=\lambda_0}) + \operatorname{Mas}(\ell_1, \ell_2(\theta); [s,P]_{\lambda=\lambda_0}) \\ &= -2n - \operatorname{Mor}(V(0) \otimes I_2 - \lambda_0 I_{2n}) + \operatorname{Mas}(\ell_1, \ell_2(\theta); [s,P]_{\lambda=\lambda_0}),\end{aligned}$$

and the claim of Corollary 1.1 for $\theta = 0$ now follows immediately from Theorem 1.1. □

5 Proofs of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3.

5.1 Proof of Theorem 1.2

For each fixed $\theta \in [0, \pi]$, Theorem 1.1 provides a computationally efficient way to determine the number of eigenvalues $\mathcal{H}_\theta$ has below a fixed threshold λ_0. Suppose we have carried out this calculation for some particular value of θ_0 so that we know $\operatorname{Mor}(\mathcal{H}_{\theta_0}; \lambda_0)$. Theorem 1.2 allows us to compute $\operatorname{Mor}(\mathcal{H}_\theta)$ for all $\theta \in [0, \pi] \backslash \{\theta_0\}$ without recomputing solutions of (1.8).

In order to prove Theorem 1.2 we let $\mathbf{X}_1(P; \lambda) = \binom{X_1(P;\lambda)}{Y_1(P;\lambda)}$ denote our frame for $\ell_1(x; \lambda)$ evaluated at $x = P$ (recall (3.5)), and we let $\mathbf{X}_2(\theta) = \binom{X_2(\theta)}{Y_2(\theta)}$ denote our frame for $\ell_2(\theta)$ (recall (3.7)). We calculate the Maslov index along a path in the (λ, θ)-plane via the unitary matrix $\tilde{W}(P; \lambda, \theta)$.

We have already seen that we can choose λ_∞ sufficiently large so that H_θ will not have any eigenvalues $\lambda \leq -\lambda_\infty$ for any $\theta \in [0, \pi]$. (The calculation at the beginning of Section 4 was independent of θ.)

Now let (θ_0, λ_0) denote a pair of values $0 \leq \theta_0 \leq \pi$, $\lambda_0 \in \mathbb{R}$, with $\lambda_0 > -\lambda_\infty$, and fix any $\theta_1 \in (\theta_0, \pi]$. We consider a rectangle in the (λ, θ)-plane determined by the following four contours: (1) for $\theta = \theta_0$, let λ run from $-\lambda_\infty$ to λ_0 (the bottom shelf, $[-\lambda_\infty, \lambda_0]_{\theta=\theta_0}$); (2) for $\lambda = \lambda_0$, let θ run from θ_0 to θ_1 (the right shelf, $[\theta_0, \theta_1]_{\lambda=\lambda_0}$); (3) For $\theta = \theta_1$ let λ run from λ_0 to $-\lambda_\infty$ (the top shelf, $[\lambda_0, -\lambda_\infty]_{\theta=\theta_1}$); and (4) for $\lambda = -\lambda_\infty$, let θ run from θ_1 to θ_0 (the left shelf, $[\theta_1, \theta_0]_{\lambda=-\lambda_\infty}$). (See Figure 3.)

We immediately see from (4.1) that if we take $\lambda_\infty > \|V\|_{L^\infty(\mathbb{R})}$ then we will have

$$\operatorname{Mas}(\ell_1(P, \cdot), \ell_2; [\theta_1, \theta_0]_{\lambda=-\lambda_\infty}) = 0. \tag{5.1}$$

For the horizontal shelves $[-\lambda_\infty, \lambda_0]_{\theta=\theta_0}$ and $[\lambda_0, -\lambda_\infty]_{\theta=\theta_1}$ we have already seen that the eigenvalues of $\tilde{W}(P; \lambda, \theta)$ rotate monotonically counterclockwise as λ decreases. We conclude immediately that

$$\begin{aligned}\operatorname{Mas}(\ell_1(P, \cdot), \ell_2; [-\lambda_\infty, \lambda_0]_{\theta=\theta_0}) &= -\operatorname{Mor}(\mathcal{H}_{\theta_0}; \lambda_0) \\ \operatorname{Mas}(\ell_1(P, \cdot), \ell_2; [\lambda_0, -\lambda_\infty]_{\theta=\theta_1}) &= \operatorname{Mor}(\mathcal{H}_{\theta_1}; \lambda_0).\end{aligned}$$

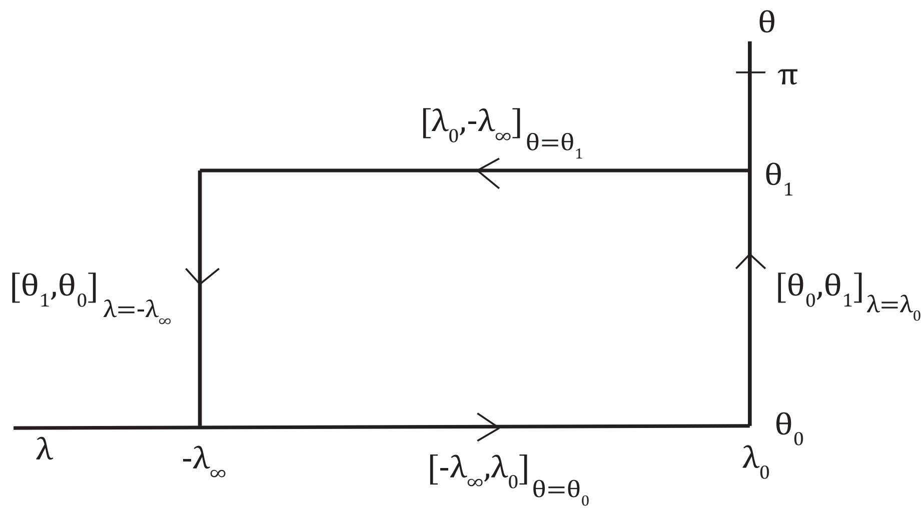

Figure 3: The (λ, θ)-Maslov Box.

Using path additivity and homotopy invariance, we see that

$$\operatorname{Mor}(\mathcal{H}_{\theta_1}; \lambda_0) = \operatorname{Mor}(\mathcal{H}_{\theta_0}; \lambda_0) - \operatorname{Mas}(\ell_1(P, \cdot), \ell_2; [\theta_0, \theta_1]_{\lambda=\lambda_0}).$$

This is Theorem 1.2.

In order to characterize our results in terms of standard dispersion relations, we recall the following definition (see, e.g., [14]).

Definition 5.1. *The dispersion relation (or the Bloch variety $\mathcal{B}_H$) of H is defined as*

$$\mathcal{B}_H = \{(\xi, \lambda) \in \mathbb{R}^2 : H\phi = \lambda\phi \text{ has a nontrivial Floquet-Bloch solution } \phi(x) = e^{i\xi x} w(x)\}.$$

Here, $w(x)$ has period P. Since $\xi = \theta/P$, dispersion relations can be expressed in terms of (θ, λ) with no qualitative difference.

The Bloch variety can be characterized as the set of points $(\theta, \lambda) \in \mathbb{R}^2$ so that $\tilde{W}(P; \lambda, \theta)$ has -1 as an eigenvalue. In this way, intersections along the Maslov Box relate the number of times $\mathcal{B}_H$ crosses the axes $\theta = 0$ and $\theta = \pi$ to the number of times it crosses $\lambda = \lambda_0$. (See Figures 6 and 10 for examples.)

5.2 Proof of Theorem 1.3

In this section we establish Theorem 1.3, which gives a lower bound on the number of eigenvalues either H_θ or $H_{\pi-\theta}$ has below some fixed threshold $\lambda_0 \in \mathbb{R}$. Our starting point is to consider the nature of eigenvalues of $\tilde{W}(P; \lambda, \theta)$ as $\lambda \to -\infty$. Since ℓ_2 does not vary with λ, this behavior will be determined by ℓ_1, for which the λ dependence is determined by solutions to the matrix ODE

$$\Phi'(x;\lambda) = \mathbb{A}(x;\lambda)\Phi(x;\lambda); \quad \Phi(0;\lambda) = I_{4n}; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I_{2n} \\ V \otimes I_2 - \lambda I_{2n} & 0 \end{pmatrix}. \tag{5.2}$$

In order to understand the behavior of $\Phi(x;\lambda)$ for large values of $-\lambda$, we write $\Phi = \binom{\Phi_1}{\Phi_2}$ and introduce the scaling

$$\xi = \sqrt{-\lambda}x; \quad \Psi_1(\xi;\lambda) = \Phi_1(x;\lambda); \quad \Psi_2(\xi;\lambda) = \frac{1}{\sqrt{-\lambda}}\Phi_2(x;\lambda),$$

for which

$$\Psi'(\xi;\lambda) = \tilde{\mathbb{A}}(\xi;\lambda)\Psi(\xi;\lambda); \quad \tilde{\mathbb{A}}(\xi;\lambda) = \begin{pmatrix} 0_{2n} & I_{2n} \\ I_{2n} - \frac{1}{\lambda}V(\frac{\xi}{\sqrt{-\lambda}}) \otimes I_2 & 0_{2n} \end{pmatrix}$$
$$\Psi(0;\lambda) = \begin{pmatrix} I_{2n} & 0_{2n} \\ 0_{2n} & \frac{1}{\sqrt{-\lambda}}I_{2n} \end{pmatrix}.$$

We can express this equation as

$$\Psi' = \tilde{\mathbb{A}}_0\Psi + \mathbb{E}\Psi, \tag{5.3}$$

where

$$\tilde{\mathbb{A}}_0 = \begin{pmatrix} 0_{2n} & I_{2n} \\ I_{2n} & 0_{2n} \end{pmatrix}; \quad \mathbb{E}(\xi;\lambda) = \begin{pmatrix} 0_{2n} & 0_{2n} \\ -\frac{1}{\lambda}V(\frac{\xi}{\sqrt{-\lambda}}) \otimes I_2 & 0_{2n} \end{pmatrix}.$$

We proceed by looking for (matrix) solutions to (5.3) of the form $\Psi(\xi;\lambda) = e^{\xi}Z(\xi;\lambda)$, for which

$$Z' = (\tilde{\mathbb{A}}_0 - I_{4n})Z + \mathbb{E}Z. \tag{5.4}$$

Integrating, we find

$$Z(\xi;\lambda) = e^{(\tilde{\mathbb{A}}_0 - I_{4n})\xi}Z(0;\lambda) + \int_0^{\xi} e^{(\tilde{\mathbb{A}}_0 - I_{4n})(\xi-\zeta)}\mathbb{E}(\zeta;\lambda)Z(\zeta;\lambda)d\zeta. \tag{5.5}$$

Since $\tilde{\mathbb{A}}_0$ is a constant matrix, we can directly compute

$$e^{(\tilde{\mathbb{A}}_0 - I_{4n})(\xi-\zeta)} = e^{-(\xi-\zeta)}\begin{pmatrix} \cosh(\xi-\zeta)I_{2n} & \sinh(\xi-\zeta)I_{2n} \\ \sinh(\xi-\zeta)I_{2n} & \cosh(\xi-\zeta)I_{2n} \end{pmatrix},$$

which is uniformly bounded for $\zeta \in [0,\xi]$. Fix a matrix norm $|\cdot|$ and constant C (in particular, independent of λ) so that

$$\left|e^{(\tilde{\mathbb{A}}_0 - I_{4n})\xi}Z(0;\lambda)\right| \le C.$$

In addition, there exists a constant K (independent of λ) so that

$$\left|e^{(\tilde{\mathbb{A}}_0 - I_{4n})(\xi-\zeta)}\mathbb{E}(\zeta;\lambda)\right| \le \frac{K}{|\lambda|}|V(\frac{\zeta}{\sqrt{-\lambda}})|.$$

It follows that

$$\int_0^{\sqrt{-\lambda}P} \left| e^{(\tilde{\mathbb{A}}_0 - I_{4n})(\xi-\zeta)} \mathbb{E}(\zeta;\lambda) \right| d\zeta \le \frac{K}{|\lambda|} \int_0^{\sqrt{-\lambda}P} |V(\frac{\zeta}{\sqrt{-\lambda}})| d\zeta. \tag{5.6}$$

Setting $y = \zeta/\sqrt{-\lambda}$ we see that this integral is bounded by

$$\frac{K}{|\lambda|} \int_0^P |V(y)| \sqrt{-\lambda} dy \le \tilde{K}|\lambda|^{-1/2},$$

for some constant $\tilde{K}$, independent of λ.

Now take any $\lambda < 0$ with $|\lambda|$ sufficiently large so that

$$\tilde{K}|\lambda|^{-1/2} \le \frac{1}{2}.$$

Let $\mathcal{T}Z$ denote the right-hand side of (5.5), and consider $\mathcal{T}$ as a map on the space

$$\begin{aligned} \mathcal{S}_\lambda := \Big\{ Z(\cdot,\lambda) \in C([0,\sqrt{-\lambda}P]; \mathbb{R}^{4n\times 4n}) : Z(0;\lambda) = \begin{pmatrix} I_{2n} & 0_{2n} \\ 0_{2n} & \frac{1}{\sqrt{-\lambda}} I_{2n} \end{pmatrix}, \\ \|Z(\cdot;\lambda)\|_{C([0,\sqrt{-\lambda}P])} \le 2C \Big\}, \end{aligned}$$

where

$$\|Z(\cdot;\lambda)\|_{C([0,\sqrt{-\lambda}P])} := \max_{\zeta\in[0,\sqrt{-\lambda}P]} |Z(\zeta;\lambda)|,$$

with $|\cdot|$ designating the matrix norm induced by the Euclidean inner product.

Noting that for any $Z \in \mathcal{S}_\lambda$

$$\begin{aligned} |\mathcal{T}Z| &\le C + \|Z\|_{C([0,\sqrt{-\lambda}P])} \tilde{K}|\lambda|^{-1/2} \\ &\le C + 2C\tilde{K}|\lambda|^{-1/2} \le 2C, \end{aligned}$$

we see that $\mathcal{T}$ is invariant on $\mathcal{S}_\lambda$. Likewise, we find that for any $Z_1, Z_2 \in \mathcal{S}_\lambda$

$$\|\mathcal{T}(Z_1 - Z_2)\|_{C([0,\sqrt{-\lambda}P])} \le \tilde{K}|\lambda|^{-1/2} \|Z_1 - Z_2\|_{C([0,\sqrt{-\lambda}P])} \le \frac{1}{2} \|Z_1 - Z_2\|_{C([0,\sqrt{-\lambda}P])},$$

verifying that $\mathcal{T}$ is a contraction. We conclude that our matrix equation (5.4) has a unique solution in $\mathcal{S}_\lambda$, and that this solution can be expressed as

$$Z(\xi;\lambda) = e^{(\tilde{\mathbb{A}}_0 - I_{4n})\xi} Z(0;\lambda) + \mathbf{O}(|\lambda|^{-1/2}). \tag{5.7}$$

We can now substitute $Z(\xi;\lambda)$ back into (5.5) to get a refinement of this estimate. For this, we notice that by direct calculation

$$e^{(\tilde{\mathbb{A}}_0 - I_{4n})(\xi-\zeta)} \mathbb{E}(\zeta;\lambda) = e^{-(\xi-\zeta)} \begin{pmatrix} -\frac{\sinh(\xi-\zeta)}{\lambda} V(\frac{\zeta}{\sqrt{-\lambda}}) \otimes I_2 & 0 \\ -\frac{\cosh(\xi-\zeta)}{\lambda} V(\frac{\zeta}{\sqrt{-\lambda}}) \otimes I_2 & 0 \end{pmatrix},$$

so that

$$e^{(\tilde{\mathbb{A}}_0-I_{4n})(\xi-\zeta)}\mathbb{E}(\zeta;\lambda)e^{(\tilde{\mathbb{A}}_0-I_{4n})\zeta}Z(0;\lambda)$$
$$= e^{-\xi}\begin{pmatrix} \frac{\sinh(\xi-\zeta)\cosh\zeta}{-\lambda}V(\frac{\zeta}{\sqrt{-\lambda}})\otimes I_2 & \frac{\sinh(\xi-\zeta)\sinh\zeta}{(-\lambda)^{3/2}}V(\frac{\zeta}{\sqrt{-\lambda}})\otimes I_2 \\ \frac{\cosh(\xi-\zeta)\cosh\zeta}{-\lambda}V(\frac{\zeta}{\sqrt{-\lambda}})\otimes I_2 & \frac{\cosh(\xi-\zeta)\sinh\zeta}{(-\lambda)^{3/2}}V(\frac{\zeta}{\sqrt{-\lambda}})\otimes I_2 \end{pmatrix}.$$

Integrating as in (5.6), we find that

$$\int_0^{\xi} e^{(\tilde{\mathbb{A}}_0-I_{4n})(\xi-\zeta)}\mathbb{E}(\zeta;\lambda)e^{(\tilde{\mathbb{A}}_0-I_{4n})\zeta}Z(0;\lambda)d\zeta = \begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \end{pmatrix},$$

where the order relations are uniform for $\xi \in [0, \sqrt{-\lambda}P]$. Likewise, the second summand in $Z(\xi;\lambda)$ (i.e., the error term $\mathbf{O}(|\lambda|^{-1/2})$) leads to a slightly better error matrix with $\mathbf{O}(|\lambda|^{-1})$ in every entry. This provides a slight refinement of our estimate (5.7), which we can now express as

$$\begin{aligned} Z(\xi;\lambda) &= \begin{pmatrix} Z_{11}(\xi;\lambda) & Z_{12}(\xi;\lambda) \\ Z_{21}(\xi;\lambda) & Z_{22}(\xi;\lambda) \end{pmatrix} \\ &= \begin{pmatrix} e^{-\xi}\cosh\xi I_{2n} & e^{-\xi}\frac{\sinh\xi}{\sqrt{-\lambda}}I_{2n} \\ e^{-\xi}\sinh\xi I_{2n} & e^{-\xi}\frac{\cosh\xi}{\sqrt{-\lambda}}I_{2n} \end{pmatrix} + \begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \end{pmatrix}. \end{aligned} \tag{5.8}$$

Returning to original variables, we can express this as

$$\begin{aligned} \Phi(x;\lambda) &= \begin{pmatrix} \Phi_{11}(x;\lambda) & \Phi_{12}(x;\lambda) \\ \Phi_{21}(x;\lambda) & \Phi_{22}(x;\lambda) \end{pmatrix} \\ &= e^{\sqrt{-\lambda}x}\begin{pmatrix} e^{-\sqrt{-\lambda}x}\cosh(\sqrt{-\lambda}x)I_{2n} & e^{-\sqrt{-\lambda}x}\frac{\sinh(\sqrt{-\lambda}x)}{\sqrt{-\lambda}}I_{2n} \\ \sqrt{-\lambda}e^{-\sqrt{-\lambda}x}\sinh(\sqrt{-\lambda}x)I_{2n} & e^{-\sqrt{-\lambda}x}\cosh(\sqrt{-\lambda}x)I_{2n} \end{pmatrix} \\ &+ e^{\sqrt{-\lambda}x}\begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(1) & \mathbf{O}(|\lambda|^{-1/2}) \end{pmatrix}. \end{aligned} \tag{5.9}$$

We observe that

$$e^{-\sqrt{-\lambda}P}\cosh(\sqrt{-\lambda}P) = \frac{1}{2}(1 + e^{-2\sqrt{-\lambda}P}),$$

where the exponential is clearly transcendentally small. Proceeding similarly for

$$e^{-\sqrt{-\lambda}P}\sinh(\sqrt{-\lambda}P),$$

we can write

$$\Phi(P;\lambda) = e^{\sqrt{-\lambda}P}\begin{pmatrix} \frac{1}{2}I_{2n} & \frac{1}{2\sqrt{-\lambda}}I_{2n} \\ \frac{\sqrt{-\lambda}}{2}I_{2n} & \frac{1}{2}I_{2n} \end{pmatrix} + e^{\sqrt{-\lambda}P}\begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(1) & \mathbf{O}(|\lambda|^{-1/2}) \end{pmatrix}. \tag{5.10}$$

Recalling (3.5) we see that

$$X_1(P;\lambda) - iY_1(P;\lambda) = \begin{pmatrix} I_{2n} & iI_{2n} \\ e^{\sqrt{-\lambda}P}(-i\frac{\sqrt{-\lambda}}{2}I_{2n} + \mathbf{O}(1)) & e^{\sqrt{-\lambda}P}(-i\frac{1}{2}I_{2n} + \mathbf{O}(|\lambda|^{-1/2})) \end{pmatrix},$$

and likewise for $(X_1(P;\lambda) + iY_1(P;\lambda))$.

In particular, we can write $X_1(P;\lambda) - iY_1(P;\lambda) = R(\lambda) + E(\lambda)$, where

$$R(\lambda) = \begin{pmatrix} I_{2n} & iI_{2n} \\ -\frac{i\sqrt{-\lambda}}{2}e^{\sqrt{-\lambda}P}I_{2n} & -\frac{i}{2}e^{\sqrt{-\lambda}P}I_{2n} \end{pmatrix},$$

and

$$E(\lambda) = \begin{pmatrix} 0_{2n} & 0_{2n} \\ e^{\sqrt{-\lambda}P}\mathbf{O}(1) & e^{\sqrt{-\lambda}P}\mathbf{O}(|\lambda|^{-1/2}) \end{pmatrix}.$$

We see that

$$R(\lambda)^{-1} = \frac{1}{\Delta}\begin{pmatrix} -\frac{i}{2}e^{\sqrt{-\lambda}P}I_{2n} & -iI_{2n} \\ \frac{i\sqrt{-\lambda}}{2}e^{\sqrt{-\lambda}P}I_{2n} & I_{2n} \end{pmatrix},$$

where

$$\Delta = e^{\sqrt{-\lambda}P}(-\frac{i}{2} - \frac{\sqrt{-\lambda}}{2}).$$

This allows us to write

$$X_1(P;\lambda) - iY_1(P;\lambda) = R(\lambda)(I + R(\lambda)^{-1}E(\lambda)),$$

and subsequently

$$(X_1(P;\lambda) - iY_1(P;\lambda))^{-1} = (I + R(\lambda)^{-1}E(\lambda))^{-1}R(\lambda)^{-1}.$$

By direct calculation, we see that

$$R(\lambda)^{-1}E(\lambda) = \begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \end{pmatrix}.$$

It follows by Neumann expansion that

$$(I + R(\lambda)^{-1}E(\lambda))^{-1} = I + \tilde{E}(\lambda),$$

where

$$\tilde{E}(\lambda) = \begin{pmatrix} \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \\ \mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1}) \end{pmatrix}.$$

Combining these observations, we see that

$$(X_1(P;\lambda) - iY_1(P;\lambda))^{-1} = (I + \tilde{E}(\lambda))R(\lambda)^{-1} = R(\lambda)^{-1} + \tilde{E}(\lambda)R(\lambda)^{-1},$$

where

$$\tilde{E}(\lambda)R(\lambda)^{-1} = \frac{1}{\Delta}\begin{pmatrix} e^{\sqrt{-\lambda}P}\mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1/2}) \\ e^{\sqrt{-\lambda}P}\mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1/2}) \end{pmatrix}.$$

We now have

$$\begin{aligned}(X_1(P;\lambda) &+ iY_1(P;\lambda))(X_1(P;\lambda) - iY_1(P;\lambda))^{-1} \\ &= (X_1(P;\lambda) + iY_1(P;\lambda))R(\lambda)^{-1} + (X_1(P;\lambda) + iY_1(P;\lambda))\tilde{E}(\lambda)R(\lambda)^{-1},\end{aligned}$$

and we consider each summand on the right-hand side in turn. First,

$$(X_1(P;\lambda)+iY_1(P;\lambda))R(\lambda)^{-1}=\frac{1}{\Delta}\begin{pmatrix} e^{\sqrt{-\lambda}P}(\frac{\sqrt{-\lambda}}{2}-\frac{i}{2})I_{2n} & -2iI_{2n} \\ e^{2\sqrt{-\lambda}P}\mathbf{O}(1) & e^{\sqrt{-\lambda}P}(\frac{\sqrt{-\lambda}}{2}I_{2n}+\mathbf{O}(1)) \end{pmatrix}.$$

Likewise,

$$(X_1(P;\lambda)-iY_1(P;\lambda))\tilde{E}(\lambda)R(\lambda)^{-1}=\frac{1}{\Delta}\begin{pmatrix} e^{\sqrt{-\lambda}P}\mathbf{O}(|\lambda|^{-1/2}) & \mathbf{O}(|\lambda|^{-1/2}) \\ e^{2\sqrt{-\lambda}P}\mathbf{O}(1) & e^{\sqrt{-\lambda}P}\mathbf{O}(1) \end{pmatrix}.$$

Combining these observations, we conclude

$$\begin{aligned}(X_1(P;\lambda)&+iY_1(P;\lambda))(X_1(P;\lambda)-iY_1(P;\lambda))^{-1} \\ &=\frac{1}{\Delta}\begin{pmatrix} e^{\sqrt{-\lambda}P}(\frac{\sqrt{-\lambda}}{2}I_{2n}+\mathbf{O}(1)) & \mathbf{O}(1) \\ e^{2\sqrt{-\lambda}P}\mathbf{O}(1) & e^{\sqrt{-\lambda}P}(\frac{\sqrt{-\lambda}}{2}I_{2n}+\mathbf{O}(1)) \end{pmatrix}.\end{aligned} \tag{5.11}$$

This matrix is unitary, and we temporarily express it as

$$U=\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

where each $2n\times 2n$ block U_{ij} can be identified from (5.11). In particular, we must have $U_{11}^*U_{11}+U_{21}^*U_{21}=I_{2n}$. Here,

$$U_{11}=\frac{1}{\Delta}e^{\sqrt{-\lambda}P}(\frac{\sqrt{-\lambda}}{2}I_{2n}+\mathbf{O}(1))=\frac{1}{-\frac{i}{2}-\frac{\sqrt{-\lambda}}{2}}(\frac{\sqrt{-\lambda}}{2}I_{2n}+\mathbf{O}(1)).$$

We see that

$$\lim_{\lambda\to-\infty}U_{11}^*(\lambda)U_{11}(\lambda)=I_{2n},$$

and it follows that

$$\lim_{\lambda\to-\infty}U_{21}^*(\lambda)U_{21}(\lambda)=0_{2n}\implies\lim_{\lambda\to-\infty}U_{21}(\lambda)=0_{2n}.$$

Proceeding more directly with the other components of U, we find that

$$\lim_{\lambda\to-\infty}(X_1(P;\lambda)+iY_1(P;\lambda))(X_1(P;\lambda)-iY_1(P;\lambda))^{-1}=-I_{4n}.$$

It follows immediately that

$$\lim_{\lambda\to-\infty}\tilde{W}(P;\lambda,\theta)=\begin{pmatrix} 0_{2n} & (I_n\otimes R_\theta)^t \\ I_n\otimes R_\theta & 0_{2n} \end{pmatrix}.$$

Recalling that $(I_n\otimes R_\theta)^t(I_n\otimes R_\theta)=I_{2n}$, we find that the eigenvalues μ of this last equation satisfy

$$\det(\mu^2 I_{2n}-I_{2n})=0, \tag{5.12}$$

from which we conclude that the eigenvalues of $\tilde{W}(P;\lambda,\theta)$ approach ± 1 as λ approaches $-\infty$, each with multiplicity $2n$.

We summarize these observations into a lemma.

Lemma 5.1. *Suppose $V \in C([0,P];\mathbb{R}^{n\times n})$ is a symmetric matrix-valued potential, and $\tilde{W}(P;\lambda,\theta)$ is defined as in (3.9). Given any $\epsilon > 0$ there exists $N > 0$ sufficiently large so that if $\lambda < -N$ then $\tilde{W}(P;\lambda,\theta)$ has $2n$ eigenvalues μ that satisfy $|\mu + 1| < \epsilon$ and $2n$ eigenvalues μ that satisfy $|\mu - 1| < \epsilon$.*

In order to see that Theorem 1.3 follows from these considerations, we begin by supposing that for some fixed $\lambda_0 \in \mathbb{R}$ and some $\theta \in [0,\pi]$, the matrix $\tilde{W}(P;\lambda_0,\theta)$ has $2n+2\kappa$ eigenvalues (counted with multiplicity; $\kappa \in \{1,2,\dots,n\}$) with arguments on the interval $[0,\pi)$ (i.e., on the upper semi-arc; we will denote the number of eigenvalues of $\tilde{W}(P;\lambda_0,\theta)$ on this arc by $\tilde{n}_+(P;\lambda_0,\theta)$). Notice that since the eigenvalues of $\mathcal{H}_\theta$ all occur with multiplicity 2, $\tilde{n}_+(P;\lambda_0,\theta)$ must be an even number. As λ decreases from λ_0 toward $-\lambda_\infty$ the eigenvalues of $\tilde{W}(P;\lambda,\theta)$ will rotate monotonically counterclockwise, and we know from Lemma 5.1 that for λ_∞ sufficiently large $2n$ of these will end at -1 and $2n$ will end at $+1$ (possibly after multiple full rotations about S^1). We conclude that at least 2κ eigenvalues must cross -1, and each of these crossings will correspond with a multiplicity-2 eigenvalue of $\mathcal{H}_\theta$. In this way, we see that H_θ has at least κ eigenvalues below λ_0. We note particularly that if an eigenvalue resides at the point $(1,0)$ when $\lambda = \lambda_0$ then by strict monotonicity it cannot remain at $(1,0)$ as λ decreases toward $-\lambda_\infty$.

On the other hand, suppose that for some $\theta \in [0,\pi]$ the matrix $\tilde{W}(P;\lambda_0,\theta)$ has $2n+2\kappa$ eigenvalues ($\kappa \in \{1,2,\dots,n\}$) with arguments on the interval $[-\pi,0)$ (i.e., on the lower semi-arc; we will denote the number of eigenvalues of $\tilde{W}(P;\lambda_0,\theta)$ on this arc by $\tilde{n}_-(P;\lambda_0,\theta)$). Noting that for any $\theta \in [0,\pi]$ (and indeed for any $\theta \in \mathbb{R}$), $\mathcal{R}_{\theta-\pi} = -\mathcal{R}_\theta$, we see that the matrix $\tilde{W}(P;\lambda_0,\theta-\pi)$ will have $2n+2\kappa$ eigenvalues on the upper semi-arc. We conclude precisely as before that $H_{\theta-\pi}$ will have at least κ eigenvalues below λ_0. As discussed in the introduction, $H_{\theta-\pi}$ has precisely the same eigenvalues as $H_{\pi-\theta}$, and so we can conclude that $H_{\pi-\theta}$ has at least κ eigenvalues below λ_0.

Theorem 1.3 follows immediately from these observations.

6 Applications

In this section we apply the preceding theory to specific linear operators obtained when gradient systems are linearized about stationary periodic solutions.

6.1 Allen-Cahn Equations

Consider the single Allen-Cahn equation

$$u_t = u_{xx} - F'(u), \tag{6.1}$$

where

$$F(u) = \frac{1}{4}\alpha u^4 - \frac{1}{2}\beta u^2, \tag{6.2}$$

for some positive constants α and β. Such equations arise naturally in the context of non-conservative phase separation processes, and the family of double-well functions (6.2) is taken from [15]. As discussed in [4, 5], given any amplitude $u_* \in (0, \sqrt{\beta/\alpha})$ there exists a periodic solution $\bar{u}(x; u_*)$ that can be expressed in terms of a Jacobi elliptic function as

$$\bar{u}(x; u_*) = u_* \text{sn}\,(\frac{\sqrt{-2F(u_*)}}{u_*} x; k); \quad k^2 = -\frac{\alpha u_*^4}{4F(u_*)}. \tag{6.3}$$

Here, $\text{sn}(y; k)$ is defined so that

$$\text{sn}(y; k) = \sin \phi; \quad \text{where} \quad y = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

In practice, such waves can be computed with MATLAB's built-in function *ellipje*, and an example implementation looks as follows (with *ubar* designating $\bar{u}(x; u^*)$, *alph* designating α, A designating amplitude u_*, and F denoting a MATLAB anonymous function specifying 6.2):

```
ubar = A*ellipj((sqrt((-2*F(A)))/A)*x,-alph*A∧4/(4*F(A)));
```

Likewise, the period for $\bar{u}(x; u_*)$ can be computed as

$$P(u_*) = \frac{4u_*}{\sqrt{-2F(u_*)}} K(k), \tag{6.4}$$

where K denotes the complete elliptic integral

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - k^2 t^2)}},$$

which can be computed with MATLAB's built-in function *ellipke*. An example implementation looks as follows:

```
P = (4*A/sqrt(-2*F(A)))*ellipke(-alph*A∧4/(4*F(A)));
```

Linearizing (6.1) about (6.3), we obtain the eigenvalue problem

$$H_\theta \phi = -\phi'' + V(x)\phi = \lambda \phi; \quad V(x) = F''(\bar{u}(x; u_*)), \tag{6.5}$$

to which we can associate boundary conditions

$$\phi(P) = e^{i\theta} \phi(0); \quad \phi'(P) = e^{i\theta} \phi'(0). \tag{6.6}$$

As discussed in the general development, we will carry out our calculations in the context of the related equation

$$\begin{aligned} \mathcal{H}_\theta y &= -y'' + (V(x) \otimes I_2) y = \lambda y \\ y(P) &= (I_n \otimes R_\theta) y(0) \\ y'(P) &= (I_n \otimes R_\theta) y'(0). \end{aligned}$$

In order to carry out the numerical calculations below, we fix $\alpha = 1$ and $\beta = 1$. In addition, we fix $u_* = .5$, except for the calculation in which we determine how the leading eigenvalue varies as u_* varies.

Lower bound computation. We start by showing that $H := -\partial_x^2 + V(x)$ has $L^2(\mathbb{R})$ spectrum below $\lambda_0 = 0$. This is an efficient calculation, only requiring the evolution of (3.3) for the single value $\lambda = 0$, and evaluation of the eigenvalues of $\tilde{W}(P; 0, 0)$.

When $\theta = 0$, $\bar{u}_x$ is an eigenfunction associated with $\lambda = 0$, and consequently $\tilde{W}(P; 0, 0)$ will certainly have -1 as an eigenvalue, with multiplicity at least two. According to Theorem 1.3, we will be able to conclude that H has spectrum below $\lambda_0 = 0$ so long as $\Delta\tilde{n}(P; 0, 0)) \neq 0$.

The spectrum of $\tilde{W}(P; 0, 0)$, generated numerically, is depicted in Figure 4, from which we see that

$$\Delta\tilde{n}(P; 0, 0) = \tilde{n}_+(P; 0, 0) - \tilde{n}_-(P; 0, 0) = -4.$$

We emphasize that we know that -1 is an eigenvalue for $\tilde{W}(P; 0, 0)$ with multiplicity at least two, and so there is no issue with numerical error for that value. The other eigenvalue of $\tilde{W}(P; 0, 0)$ is located at approximately $-.0391 - .9992i$, and while there is certainly numerical error associated with this value it will not be enough to move the value into the upper semi-arc (though we do not prove this assertion here). We conclude from Theorem 1.3 that H_π has at least one eigenvalue below $\lambda = 0$.

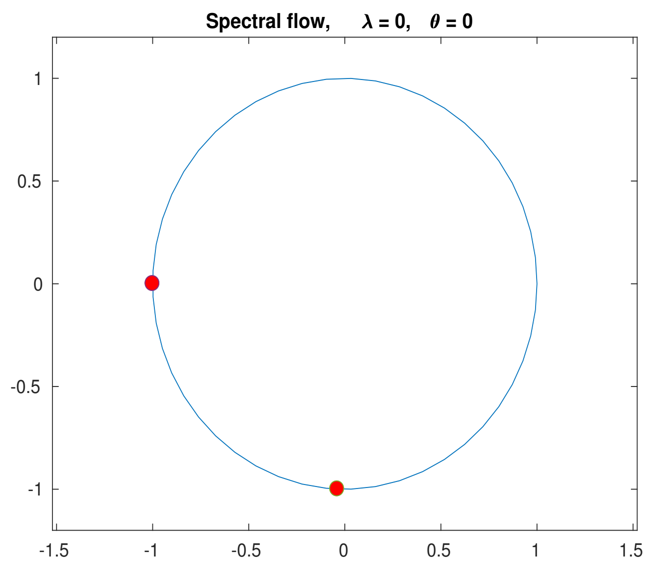

Figure 4: Eigenvalues of $\tilde{W}(P; 0, 0)$ for Example 1.

Morse index for $\theta = 0$. We next use Theorem 1.1 to compute the number of eigenvalues below $\lambda_0 = 0$ for $\theta = 0$. This requires the numerical calculation of $\text{Mas}(\ell_1, \ell_2(0); [0, P]_{\lambda=0})$, and the flow runs as follows: the four eigenvalues of $\tilde{W}(0; 0, 0)$ all reside at -1, and as x runs from 0 to P, they rotate monotonically clockwise, splitting apart into two separated multiplicity-2 eigenvalues. As noted in our lower bound computation, one of these stops at -1 (when $x = P$) and the other stops at approximately $-.0391 - .9992i$. (The additional thing we had to observe here is that there were no intermediate conjugate points.) We conclude that

$$\text{Mas}(\ell_1, \ell_2(0); [0, P]_{\lambda=0}) = -4.$$

According then to Theorem 1.1 we have

$$\mathrm{Mor}(\mathcal{H}_0) = 4 - 2 = 2.$$

We conclude that $\mathcal{H}_0$ has precisely one multiplicity-2 eigenvalue below $\lambda_0 = 0$ and H_0 has precisely one (necessarily simple) eigenvalue below $\lambda_0 = 0$.

Although Corollary 1.1 was not used in this calculation, we remark on how it fits in with the observed dynamics. The proof of Corollary 1.1 in the case $\theta = 0$ hinges on the observation that the number of eigenvalues of $\tilde{W}(x; \lambda_0, 0)$ rotating away from -1 in the clockwise direction as x increases from 0 is precisely the number of negative eigenvalues of the matrix

$$-\mathbb{B}(0; \lambda_0) = \begin{pmatrix} V(0) \otimes I_2 - \lambda_0 I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

In the current example, we have $n = 1$, $\lambda_0 = 0$, and

$$V(0) = F''(\bar{u}(0; u_*)) = F''(0) = -\beta = -1.$$

We conclude that the matrix $-\mathbb{B}(0; 0)$ has four negative eigenvalues, and these correspond with the four eigenvalues in our example that rotate from -1 in the clockwise direction as x increases from 0.

The full (λ, x)-Maslov box. For purposes of illustration, we depict the full (λ, x)-Maslov box in Figure 5. In this case, $\theta = 0$, so as discussed in the proof of Theorem 1.1 the eigenvalues of $\tilde{W}(0; \lambda, 0)$ will all be -1 for all $\lambda \in \mathbb{R}$. I.e., the entire bottom shelf is conjugate. This is not depicted in the figure.

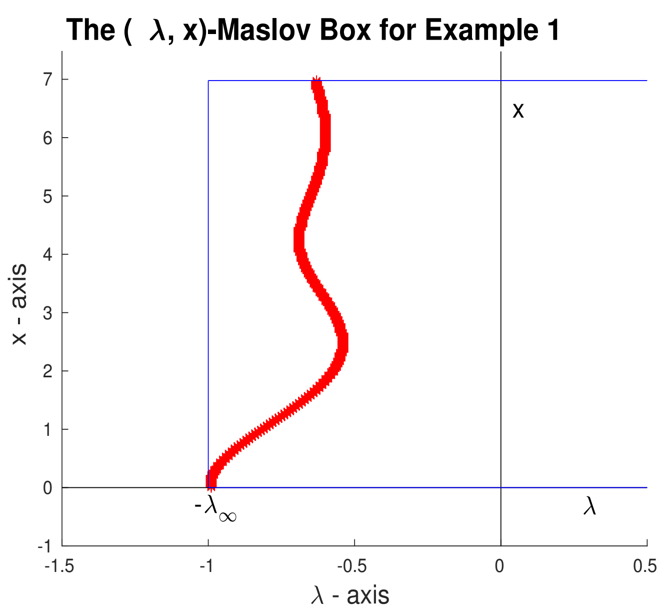

Figure 5: The (λ, x)-Maslov Box with $\theta = 0$ for Example 1.

Morse index for $\theta \in (0, \pi]$. Having computed $\mathbf{X}_1(P; 0)$, we can now evaluate the Maslov index $\mathrm{Mas}(\ell_1(P, \cdot), \ell_2; [0, \theta]_{\lambda=0})$ for all $\theta \in (0, \pi]$ by evolving the matrix $\mathcal{R}_\theta$ and following the eigenvalues of $\tilde{W}(P;0,\theta)$. As discussed in the Morse index calculation for $\theta = 0$, the eigenvalues of $\tilde{W}(P;0,0)$ reside at -1 and (approximately) $-.0391 - .9992i$. As θ increases, the eigenvalue at -1 rotates monotonically clockwise to $.0391 + .9992$, while the eigenvalue at $-.0391 - .9992i$ rotates monotonically counterclockwise to $+1$. (We know at the outset that the arrangment of eigenvalues for $\theta = \pi$ must be the negative of the arrangement at $\theta = 0$, but the paths these eigenvalues take must be determined.) We conclude that for all $\theta > 0$

$$\text{Mas}(\ell_1(P,\cdot), \ell_2; [0,\theta]_{\lambda=0}) = -2,$$

and so by Theorem 1.2

$$\text{Mor}(\mathcal{H}_\theta) = \text{Mor}(\mathcal{H}_0) + 2 = 4.$$

We conclude that for any $\theta \in (0,\pi]$, H_θ has two eigenvalues below $\lambda_0 = 0$.

The full dispersion relation. The full dispersion relation for negative values of λ is shown in Figure 6.

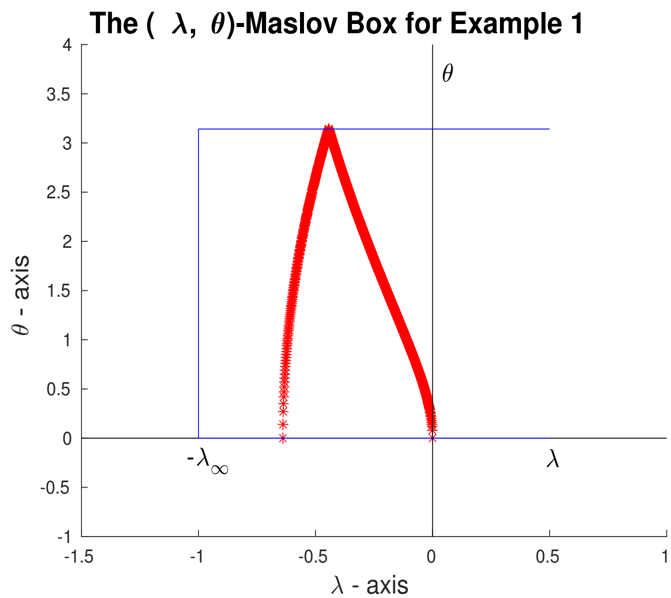

Figure 6: The (λ,θ)-Maslov Box for Example 1.

Plot of lowest eigenvalues. As discussed in [5], it can be useful when studying periodic waves to determine the dependence of the lower bound on spectrum as a function of wave amplitude. For such calculations, our counting framework can be used to ensure than an identified eigenvalue is indeed the lowest value in the spectrum. For example, we know from our calculations above that for $u_* = .5$, H_0 has precisely one eigenvalue below $\lambda_0 = 0$, and we find by calculation that H_0 has an eigenvalue at approximately $-.6394$. This must be the lowest eigenvalue of H_0. Proceeding in this way over a range of amplitudes $0 \leq u_* \leq 1$ we arrive at the relationship depicted in Figure 7.

Remark 6.1. *As suggested by Figure 6, the lowest eigenvalue of H_0 turns out for this example to be the lowest eigenvalue of H. I.e., for any $\theta \in (0,\pi]$ the lowest eigenvalue of H_θ is greater than the lowest eigenvalue of H_0.*

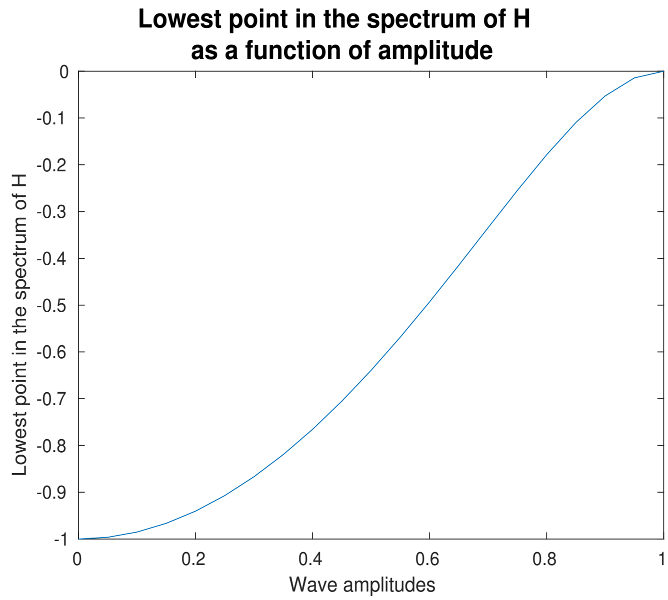

Figure 7: Lower limits for the spectrum of H for Example 1.

6.2 Allen-Cahn Systems

As a toy system model, we consider Allen-Cahn systems of the form

$$\begin{aligned} u_t &= u_{xx} - F'(u) + \delta(u - v) \\ v_t &= v_{xx} - F'(v) - \delta(u - v), \end{aligned} \tag{6.7}$$

where $F(u)$ is specified in (6.2), and $\delta \in \mathbb{R}$ is a coupling constant.

System (6.7) has been contrived to have a stationary periodic vector solution

$$\begin{aligned} \bar{u}(x; u_*) &= u_* \operatorname{sn}(\frac{\sqrt{-2F(u_*)}}{u_*}x; k) \\ \bar{v}(x; u_*) &= u_* \operatorname{sn}(\frac{\sqrt{-2F(u_*)}}{u_*}x; k). \end{aligned}$$

The associated eigenvalue problem is

$$\begin{aligned} -\phi'' + F''(\bar{u})\phi - \delta\phi + \delta\psi &= \lambda\phi \\ -\psi'' + F''(\bar{u})\psi + \delta\phi - \delta\psi &= \lambda\psi. \end{aligned} \tag{6.8}$$

That is, we have our form (1.5) with

$$V(x) = \begin{pmatrix} -\delta + F''(\bar{u}) & \delta \\ \delta & -\delta + F''(\bar{u}) \end{pmatrix}.$$

Adding the equations in (6.8) we see that $\zeta = \phi + \psi$ solves

$$-\zeta'' + F''(\bar{u})\zeta = \lambda\zeta, \tag{6.9}$$

and subtracting the equations in (6.8) we see that $\eta = \phi - \psi$ solves

$$-\eta'' + F''(\bar{u})\eta = (\lambda + 2\delta)\eta. \tag{6.10}$$

We see from (6.9) that any eigenvalue from (6.5) in Example 1 will be an eigenvalue of (6.8) (with eigenfunction $(\phi, \psi) = (\zeta, \zeta)$), and we see from (6.10) that if $\lambda + 2\delta$ is an eigenvalue of (6.5) then λ will be an eigenvalue of (6.8) (with eigenfunction $(\phi, \psi) = (\eta, -\eta)$). For example, since $-.6394$ is an eigenvalue for (6.5) we should find that for $\delta = 1$, -2.6394 is an eigenvalue of (6.8).

In order to carry out a numerical calculation, we will take the specific values $\delta = 1$, $\alpha = 1$, $\beta = 1$, and the amplitude $u_* = .5$. In this case

$$\|V\|_{L^\infty} = \max_{u \in [0, u_*], \pm} |F''(u) - \delta \pm |\delta||,$$

where we note the appearance of $\pm$ in the maximization. For $u_* = .5$, $F''(u)$ varies from -1 to $-\frac{1}{4}$, and for $\delta = 1$, $\|V\|_{L^\infty} = 3$.

Lower bound computation. We start by showing that $H := -\partial_x^2 + V(x)$ has $L^2(\mathbb{R})$ spectrum below $\lambda_0 = 0$. When $\theta = 0$, $\bar{u}_x$ is an eigenfunction associated with $\lambda = 0$, and consequently $\tilde{W}(P; 0, 0)$ will certainly have -1 as an eigenvalue, with multiplicity at least two. According to Theorem 1.3, we will be able to conclude that H has spectrum below $\lambda_0 = 0$ so long as $\Delta\tilde{n}(P; 0, 0)) \neq 0$.

The spectrum of $\tilde{W}(P; 0, 0)$ is depicited in Figure 8, from which we see that

$$\Delta\tilde{n}(P; 0, 0) = \tilde{n}_+(P; 0, 0) - \tilde{n}_-(P; 0, 0) = -8.$$

We emphasize that we know that -1 is an eigenvalue for $\tilde{W}(P; 0, 0)$ with multiplicity at least two, and so there is no issue with numerical error for that value. The other eigenvalues of $\tilde{W}(P; 0, 0)$ are located at approximately (moving counterclockwise from -1) $-.6984 - .7157i$, $-.0307 - .9995i$, and $.3243 - .9460i$, and while there is certainly numerical error associated with these values, the errors will not be enough to move any of the values into the upper semi-arc (though we do not prove this assertion here). We conclude from Theorem 1.3 that H_π has at least two eigenvalues below $\lambda = 0$.

Morse index for $\theta = 0$. We next use Theorem 1.1 to compute the number of eigenvalues below $\lambda_0 = 0$ for $\theta = 0$. This requires the numerical calculation of $\text{Mas}(\ell_1, \ell_2(0); [0, P]_{\lambda=0})$, and the flow runs as follows: the eight eigenvalues of $\tilde{W}(0; 0, 0)$ all reside at -1, and as x evolves from 0 to P they rotate monotonically clockwise, with four (i.e., two multiplicity-2 eigenvalues) crossing -1. Keeping in mind that the eight eigenvalues departing -1 in the clockwise direction decrement the Maslov index by 8, we conclude that

$$\text{Mas}(\ell_1, \ell_2(0); [0, P]_{\lambda=0}) = -12.$$

According then to Theorem 1.1 we have

$$\text{Mor}(\mathcal{H}_0) = 12 - 4 = 8.$$

We conclude that $\mathcal{H}_0$ has eight (i.e., four multiplicity-2) eigenvalues below $\lambda_0 = 0$, and H_0 has precisely four eigenvalues below $\lambda_0 = 0$. I.e. $\text{Mor}(H_0; 0) = 4$.

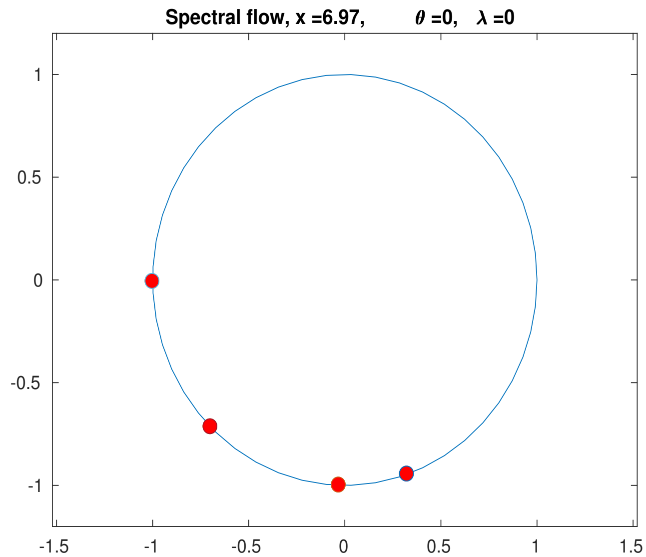

Figure 8: Eigenvalues of $\tilde{W}(P;0,0)$ for $\delta = 1$.

Similarly as in Section 6.1, we note that during the proof of Corollary 1.1, we have seen that the number of eigenvalues of $\tilde{W}(x;0,0)$ rotating away from -1 in the clockwise direction as x increases from 0 is equal to the number of negative eigenvalues of the matrix

$$-\mathbb{B}(0;0) = \begin{pmatrix} V(0)\otimes I_2 & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

In the current example, we have $n = 2$, $\lambda_0 = 0$, and

$$V(0) = \begin{pmatrix} -1+F''(0) & 1 \\ 1 & -1+F''(0) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues -3 and -1. We conclude that the matrix $-\mathbb{B}(0;0)$ has eight negative eigenvalues, and these correspond with the eight eigenvalues in our example that rotate from -1 in the clockwise direction as x increases from 0.

The full (λ, x)-Maslov box. For purposes of illustration, we depict the full (λ, x)-Maslov box in Figure 9. We find by this calculation that the eigenvalues reside at approximately $-.6394$, -1.6251, -2.0001, and -2.6394 (indicated by crossings along the top shelf in Figure 9). As in Figure 5, the entire bottom shelf is conjugate, and this is not depicted.

Morse index for $\theta \in (0,\pi]$. Having computed $\mathbf{X}_1(P;0)$, we can evaluate the Maslov index $\mathrm{Mas}(\ell_1(P,\cdot),\ell_2;[0,\theta]_{\lambda=0})$ for all $\theta \in (0,\pi]$ by evolving the matrix $\mathcal{R}_\theta$ and following the eigenvalues of $\tilde{W}(P;0,\theta)$. As discussed in the spectral count for $\theta = 0$, the eigenvalues of $\tilde{W}(P;0,0)$ reside at -1 and (approximately, moving counterclockwise from -1) $-.6984 - .7157i$, $-.0307 - .9995i$, and $.3243 - .9460i$. As θ increases, the eigenvalue at -1 rotates monotonically clockwise to $.0307+.9995i$, and one additional crossings occurs in the clockwise

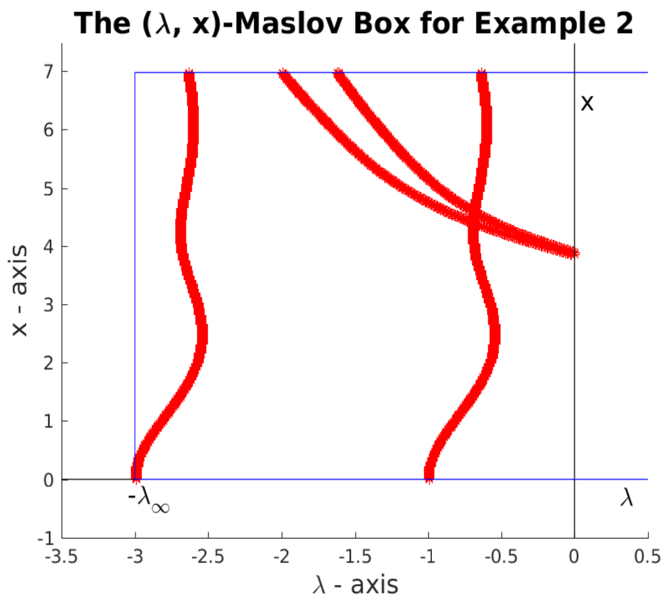

Figure 9: The (λ, x)-Maslov Box for Example 2.

direction at (approximately) $\theta = 1.31$. I.e.,

$$\operatorname{Mas}(\ell_1(P;\cdot), \ell_2; [0,\theta]_{\lambda=0}) = \begin{cases} -2 & 0 < \theta \le 1.31 \\ -4 & 1.31 < \theta \le \pi \end{cases}.$$

We conclude from Theorem 1.2 that

$$\operatorname{Mor}(\mathcal{H}_\theta) = \begin{cases} 8 & \theta = 0 \\ 10 & 0 < \theta \le 1.31 \\ 12 & 1.31 < \theta \le \pi \end{cases},$$

where the value 1.31 is approximate.

The full dispersion relation. The full dispersion relation for negative values of λ is shown in Figure 10.

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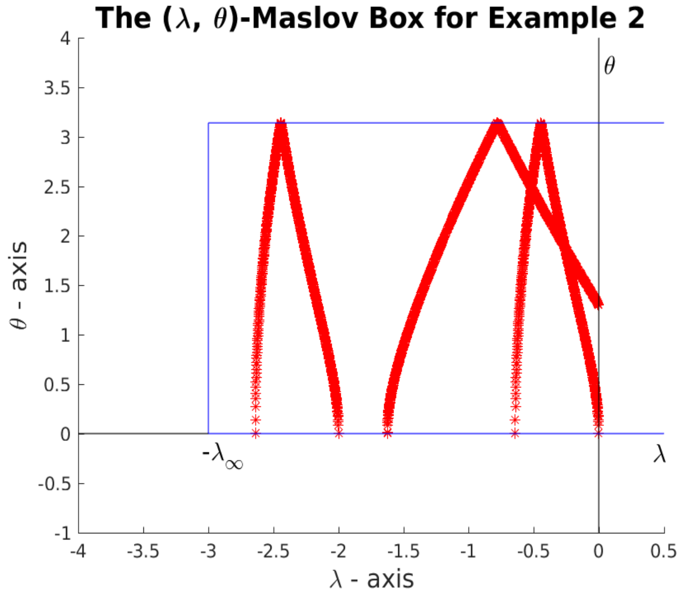

Figure 10: The (λ, θ)-Maslov Box for Example 2.

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