The Maslov index and spectral counts for linear Hamiltonian systems on $[0, 1]$

Peter Howard, Soyeun Jung, and Bongsuk Kwon

October 6, 2017

Abstract

Working with general linear Hamiltonian systems on $[0, 1]$, and with a wide range of self-adjoint boundary conditions, including both separated and coupled, we develop a general framework for relating the Maslov index to spectral counts. Our approach is illustrated with applications to Schrödinger systems on $\mathbb{R}$ with periodic coefficients, and to Euler-Bernoulli systems in the same context.

1 Introduction

For values λ in some interval $I \subset \mathbb{R}$, we consider linear Hamiltonian systems

$$J_{2n} y' = \mathbb{B}(x; \lambda) y; \quad x \in [0, 1], \quad y(x) \in \mathbb{R}^{2n}, \quad n \in \{1, 2, \dots\}, \tag{1.1}$$

where J_{2n} denotes the standard symplectic matrix

$$J_{2n} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

and we assume throughout that $\mathbb{B} \in C([0, 1] \times I : \mathbb{R}^{2n \times 2n})$ is symmetric. Moreover, we assume $\mathbb{B}$ is differentiable in λ and that $\mathbb{B}_\lambda \in C([0, 1] \times I : \mathbb{R}^{2n \times 2n})$. We consider two types of self-adjoint boundary conditions, *separated* and *generalized*.

(BC1). We consider separated self-adjoint boundary conditions

$$\alpha y(0) = 0; \quad \beta y(1) = 0,$$

where we assume

$$\alpha \in \mathbb{R}^{n \times 2n}, \quad \operatorname{rank} \alpha = n, \quad \alpha J_{2n} \alpha^t = 0;$$
$$\beta \in \mathbb{R}^{n \times 2n}, \quad \operatorname{rank} \beta = n, \quad \beta J_{2n} \beta^t = 0.$$

(BC2). We consider general self-adjoint boundary conditions

$$\Theta \begin{pmatrix} y(0) \\ y(1) \end{pmatrix} = 0; \quad \Theta \in \mathbb{R}^{2n \times 4n}, \quad \operatorname{rank} \Theta = 2n, \quad \Theta \mathcal{J}_{4n} \Theta^t = 0,$$

where

$$\mathcal{J}_{4n} := \begin{pmatrix} -J_{2n} & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

Remark 1.1. *We will see from a short calculation in Section 4 that the form of $\mathcal{J}_{4n}$ is a natural consequence of (1.1) and our convention for specifying boundary conditions in the form $\Theta\binom{y(0)}{y(1)} = 0$. We will also verify in Section 4 that the boundary conditions (BC1) can always be expressed in the form (BC2), while the converse is not true. (This is why we describe (BC2) as general rather than coupled.) We analyze both boundary conditions, because the approach taken with (BC2) is not generally as efficient as the approach taken with (BC1). Finally, we note that in [14], the authors show that for Sturm-Liouville systems, problems with general boundary conditions corresponding with (BC2) can be replaced by problems with separated self-adjoint boundary conditions that have precisely the same eigenvalues, including multiplicities. Indeed, the approach of [14] immediately extends to our setting, providing an alternative way to tackle (BC2). We do not pursue this approach further in the current paper.*

In some cases it will be natural to assume that the matrix Θ varies with a parameter ξ in some interval $\mathcal{I} \subset \mathbb{R}$. This leads to a third family of boundary conditions, which we specify as follows.

(BC2)$_\xi$. In (BC2), suppose the matrix Θ depends continuously on a variable $\xi \in \mathcal{I} \subset \mathbb{R}$. In particular, assume that $\Theta \in C(\mathcal{I}; \mathbb{R}^{2n \times 4n})$ and that for each $\xi \in \mathcal{I}$ the matrix $\Theta(\xi)$ satisfies the assumptions of (BC2).

One canonical setting in which equations of form (1.1) arise is from second order eigenvalue problems such as

$$-\phi'' + V(x)\phi = \lambda\phi, \tag{1.2}$$

where $\phi(x) \in \mathbb{R}^n$ and $V \in C([0,1]; \mathbb{R}^{n \times n})$ is symmetric for all $x \in [0,1]$. In this case, it's natural to express separated self-adjoint boundary conditions in the form

$$\alpha_1\phi(0) + \alpha_2\phi'(0) = 0$$
$$\beta_1\phi(1) + \beta_2\phi'(1) = 0,$$

with $\alpha = (\alpha_1 \; \alpha_2)$ and $\beta = (\beta_1 \; \beta_2)$ assumed to satisfy the conditions of (BC1). The form (1.1) is then obtained by setting $y_1 = \phi$ and $y_2 = \phi'$, and for $y = \binom{y_1}{y_2}$ expressing the resulting system as $J_{2n}y' = \mathbb{B}(x; \lambda)y$. Equations of form (1.2) have, of course, been studied extensively, and the introduction of symplectic methods such as those employed here goes back to early work of Maslov and Arnol'd ([1, 2, 28]). More recently, (1.2) has been analyzed in [20], using symplectic methods along with a projective formulation of (BC1) due to Berkolaiko and Kuchment [5, 26].

In order to accommodate the possibility of coupled boundary conditions such as periodic conditions, we have introduced the formulation (BC2). While our analysis will be for this general condition, our direct motivation for (BC2) stems from its role in the spectral analysis of periodic solutions to certain evolutionary PDE (discussed more in Section 4.4.1).

General linear Hamiltonian systems of form (1.1) comprise a broad class of equations that enjoy properties similar to those of (1.2) (see, for example, the text [29] for an overview). Such equations are a natural setting for symplectic methods, and in particular several recent analyses have centered on the Maslov index (for example, [7, 8, 9, 10, 16]). One of our goals in developing the general framework with (1.1)-(BC1) and (1.1)-(BC2) is to facilitate

2

the extension of results that are fairly standard in the Sturm-Liouville setting to the more general class of self-adjoint ODE. As examples of the systems we have in mind, we discuss two applications in Section 4, the first to Schrödinger systems (Section 4.4.1), and the second to Euler-Bernoulli systems (Section 4.4.2). In both cases, we consider θ-periodic boundary conditions, and a reformulation that sets the problem in the form (1.1)-(BC2)$_\xi$.

Our main results will involve relationships between the number of values $\lambda \in I$ for which (1.1) has a solution (with a specified choice of boundary conditions) and related Maslov indices for the equation. In the case of boundary conditions (BC1) we are particularly motivated by the development of [20] (for Schrödinger operators on $[0,1]$), while for boundary conditions (BC2) we are motivated by [17, 24] (again for Schrödinger operators). This correspondence between spectral counts and the Maslov index has its origins in the work of Arnol'd and Maslov noted above. Recently, the theory has been applied in a wide range of settings, including ODE on bounded domains [17, 20, 23, 24, 30, 31], ODE on unbounded domains [6, 7, 8, 9, 10, 19, 21, 22, 25], and PDE on bounded domains [11, 13, 27].

Our goal for the remainder of this introduction is to provide an informal development of the Maslov index in the current context, and to state our main results. A more systematic development of the Maslov index is provided in Section 2, and a thorough discussion, targeted toward the current setting, can be found in [18]. For a broader view of the Maslov index, we refer the reader to [12, 15, 34].

As a starting point, we define what we will mean by a *Lagrangian subspace* of $\mathbb{R}^{2n}$.

Definition 1.1. *We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and*

$$(J_{2n}u, v)_{\mathbb{R}^{2n}} = 0,$$

for all $u, v \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$. We sometimes adopt standard notation for symplectic forms, $\omega(u, v) = (J_{2n}u, v)_{\mathbb{R}^{2n}}$. In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.

Any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often write $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote paths of Lagrangian subspaces $\ell_i : \mathbb{I} \to \Lambda(n)$, for some parameter interval $\mathbb{I}$. The Maslov index associated with these paths, which we will denote $\mathrm{Mas}(\ell_1, \ell_2; \mathbb{I})$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (Precise definitions of what we mean in this context by *multiplicity* and *direction* will be given in Section 2.) In some cases, the Lagrangian subspaces will be defined along some path in the (α, β)-plane

$$\Gamma = \{(\alpha(t), \beta(t)) : t \in \mathbb{I}\},$$

and when it is convenient we will use the notation $\mathrm{Mas}(\ell_1, \ell_2; \Gamma)$.

Although there are certainly cases in which the Maslov index can be computed analytically, our point of view is that in most applications it will be computed numerically. In

particular, the general character of our theorems involves starting with a quantity that is relatively difficult to compute numerically, and expressing it in terms of one or more quantities that are relatively easy to compute numerically. Such calculations can be made via the associated frames, so the computational difficulty associated with the Maslov index is determined by the computational difficulty associated with computing the frames.

For (1.1) with boundary conditions (BC1) we begin by specifying an evolving Lagrangian subspace $\ell_1(x; \lambda)$ as follows. Let $\mathbf{X}_1$ denote a matrix solution of the initial value problem

$$
\begin{aligned}
J_{2n}\mathbf{X}_1' &= \mathbb{B}(x; \lambda)\mathbf{X}_1 \\
\mathbf{X}_1(0; \lambda) &= J_{2n}\alpha^t.
\end{aligned}
\tag{1.3}
$$

We will show in Section 3 that for all $(x, \lambda) \in [0, 1] \times I$, the matrix $\mathbf{X}_1(x; \lambda)$ is the frame for a Lagrangian subspace. In addition, we will check that the matrix $\mathbf{X}_2 = J_{2n}\beta^t$ is the frame for a (fixed) Lagrangian subspace, which we denote ℓ_2.

Suppose that for some value $\lambda \in I$ the equation (1.1) with boundary conditions (BC1) admits one or more linearly independent solutions. We denote the subspace spanned by these solutions by $\mathbb{E}(\lambda)$, noting that $\dim \mathbb{E}(\lambda) \leq 2n$. Given any two values $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, we will see that under the monotonicity assumption of Theorem 1.1 (see below), the *spectral count*

$$
\mathcal{N}([\lambda_1, \lambda_2)) := \sum_{\lambda \in [\lambda_1, \lambda_2)} \dim \mathbb{E}(\lambda),
\tag{1.4}
$$

is well-defined.

We will establish the following theorem.

Theorem 1.1. *For equation (1.1) with boundary conditions (BC1), suppose $\mathbb{B} \in C([0, 1] \times I : \mathbb{R}^{2n \times 2n})$ is symmetric, and that $\mathbb{B}$ is differentiable in λ, with $\mathbb{B}_\lambda \in C([0, 1] \times I : \mathbb{R}^{2n \times 2n})$. Fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$. If the matrix*

$$
\int_0^1 \mathbf{X}_1(x; \lambda)^t \mathbb{B}_\lambda(x; \lambda)\mathbf{X}_1(x; \lambda)dx
$$

is positive definite for all $\lambda \in [\lambda_1, \lambda_2]$, then

$$
\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell_1(\cdot; \lambda_2), \ell_2; [0, 1]) + \operatorname{Mas}(\ell_1(\cdot; \lambda_1), \ell_2; [0, 1]).
$$

Our notation $\operatorname{Mas}(\ell_1(\cdot; \lambda_2), \ell_2; [0, 1])$ denotes the Maslov index for Lagrangian subspaces $\ell_1(\cdot, \lambda_2)$ and ℓ_2, as x varies from 0 to 1, and similarly for $\operatorname{Mas}(\ell_1(\cdot; \lambda_1), \ell_2; [0, 1])$, except that λ_2 is replaced by λ_1. We emphasize that the strength of Theorem 1.1 lies in the numerical computability of these Maslov indices, which can both be computed by evolving forward solutions to appropriate initial value problems.

Remark 1.2. *Fixing $\lambda_0 \in I$, it's clear that if $\mathbb{B}_\lambda(x; \lambda_0)$ is positive definite for all $x \in [0, 1]$, then the matrix*

$$
\int_0^1 \mathbf{X}_1(x; \lambda_0)^t \mathbb{B}_\lambda(x; \lambda_0)\mathbf{X}_1(x; \lambda_0)dx
$$

will be positive definite. Moreover, we will see in our applications that this matrix is often positive definite in cases for which $\mathbb{B}_\lambda(x; \lambda_0)$ *is only non-negative for all* $x \in [0, 1]$. *Finally, we note that during the proof of Theorem 1.1, we will obtain the relation*

$$\int_0^1 \mathbf{X}_1(x; \lambda_0)^t \mathbb{B}_\lambda(x; \lambda_0)\mathbf{X}_1(x; \lambda_0)dx = \mathbf{X}_1(1; \lambda_0)^t J_{2n}(\partial_\lambda \mathbf{X}_1)(1; \lambda_0),$$

and so theorem 1.1 could equivalently be stated in terms of this last matrix expression.

For the case of (1.1) with boundary conditions (BC2), we begin by defining a Lagrangian subspace in terms of the "trace" operator

$$\mathcal{T}_x y := \mathcal{M}\begin{pmatrix} y(0) \\ y(x) \end{pmatrix}, \tag{1.5}$$

where

$$\mathcal{M} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

Precisely, we will verify in Section 4 that the subspace

$$\ell_3(x; \lambda) := \{\mathcal{T}_x y : J_{2n}y' = \mathbb{B}(x; \lambda)y \text{ on } (0, 1)\} \tag{1.6}$$

is Lagrangian for all $(x, \lambda) \in [0, 1] \times I$.

In order to establish notation for the statement of our second theorem, we let $\Phi(x; \lambda)$ denote the $2n \times 2n$ fundamental matrix solution to

$$J_{2n}\Phi' = \mathbb{B}(x; \lambda)\Phi; \quad \Phi(0; \lambda) = I_{2n}. \tag{1.7}$$

If we introduce the notation

$$\Phi(x; \lambda) = \begin{pmatrix} \Phi_{11}(x; \lambda) & \Phi_{12}(x; \lambda) \\ \Phi_{21}(x; \lambda) & \Phi_{22}(x; \lambda) \end{pmatrix},$$

then we can express the frame for $\ell_3(x; \lambda)$ as

$$\mathbf{X}_3(x, \lambda) = \begin{pmatrix} X_3(x, \lambda) \\ Y_3(x, \lambda) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \Phi_{11}(x; \lambda) & \Phi_{12}(x; \lambda) \\ 0 & -I_n \\ \Phi_{21}(x; \lambda) & \Phi_{22}(x; \lambda) \end{pmatrix}. \tag{1.8}$$

In addition, we will verify in Section 4 that the matrix

$$\mathbf{X}_4 := \mathcal{M}\mathcal{J}_{4n}\Theta^t \tag{1.9}$$

is the frame for a Lagrangian subspace, which we will denote ℓ_4.

We will establish the following theorem.

5

Theorem 1.2. *For equation (1.1) with boundary conditions (BC2), suppose $\mathbb{B} \in C([0,1] \times I : \mathbb{R}^{2n \times 2n})$ is symmetric, and that $\mathbb{B}$ is differentiable in λ, with $\mathbb{B}_\lambda \in C([0,1] \times I : \mathbb{R}^{2n \times 2n})$. Fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$. If the matrix*

$$\int_0^1 \Phi(x;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda)dx$$

is positive definite for all $\lambda \in [\lambda_1, \lambda_2]$, then

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell_3(\cdot;\lambda_2), \ell_4; [0,1]) + \operatorname{Mas}(\ell_3(\cdot;\lambda_1), \ell_4; [0,1]).$$

Remark 1.3. *The considerations discussed in Remark 1.2 hold for the integral*

$$\int_0^1 \Phi(x;\lambda_0)^t \mathbb{B}_\lambda(x;\lambda_0)\Phi(x;\lambda_0)dx,$$

and we note that during the proof of Theorem 1.2, we will obtain the relation

$$\int_0^1 \Phi(x;\lambda_0)^t \mathbb{B}_\lambda(x;\lambda_0)\Phi(x;\lambda_0)dx = \Phi(1;\lambda_0)^t J_{2n}(\partial_\lambda \Phi)(1;\lambda_0).$$

Theorem 1.2 could equivalently be stated in terms of this last matrix expression.

Finally, for boundary conditions $(BC2)_\xi$ we consider the path of Lagrangian subspaces obtained by fixing $x = 1$ in our specification above of $\ell_3(x; \lambda)$. Rather than introducing new notation, we will express this path as $\ell_3(1; \cdot)$. The matrix Θ now depends on ξ, and we specify the evolving matrix $\mathbf{X}_4(\xi) := \mathcal{M}\mathcal{J}_{4n}\Theta(\xi)^t$, which is the frame for an evolving Lagrangian subspace $\ell_4(\xi)$. Again we do not introduce new notation to distinguish this from our previous use of ℓ_4; indeed, we can view the previous case as specialized to functions constant in ξ. In this case, $\mathcal{N}([\lambda_1, \lambda_2))$ will depend on the value of ξ, so we will accordingly expand the notation to $\mathcal{N}([\lambda_1, \lambda_2); \xi)$.

We obtain the following theorem.

Theorem 1.3. *For equation (1.1) with boundary conditions $(BC2)_\xi$, suppose $\mathbb{B} \in C([0,1] \times I : \mathbb{R}^{2n \times 2n})$ is symmetric, and that $\mathbb{B}$ is differentiable in λ, with $\mathbb{B}_\lambda \in C([0,1] \times I : \mathbb{R}^{2n \times 2n})$. Fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$. If the matrix*

$$\int_0^1 \Phi(x;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda)dx$$

is positive definite for all $\lambda \in [\lambda_1, \lambda_2]$, then

$$\begin{aligned}
\mathcal{N}([\lambda_1, \lambda_2); \xi_2) &- \mathcal{N}([\lambda_1, \lambda_2); \xi_1) \\
&= -\operatorname{Mas}(\ell_3(1;\lambda_2), \ell_4; [\xi_1, \xi_2]) + \operatorname{Mas}(\ell_3(1;\lambda_1), \ell_4; [\xi_1, \xi_2]).
\end{aligned}$$

For $\xi_1 \in \mathcal{I}$ fixed, we can use Theorem 1.2 to compute

$$\mathcal{N}([\lambda_1, \lambda_2); \xi_1) = -\operatorname{Mas}(\ell_3(\cdot;\lambda_2), \ell_4(\xi_1); [0,1]) + \operatorname{Mas}(\ell_3(\cdot;\lambda_1), \ell_4(\xi_1); [0,1]). \qquad (1.10)$$

We immediately conclude the following corollary.

6

Corollary 1.1. *Let the assumptions and notation of Theorem 1.3 hold, and fix any $\xi_1 \in \mathcal{I}$. Then for any $\xi_2 \in \mathcal{I}$, $\xi_2 > \xi_1$,*

$$\mathcal{N}([\lambda_1, \lambda_2); \xi_2) = -\operatorname{Mas}(\ell_3(1; \lambda_2), \ell_4; [\xi_1, \xi_2]) + \operatorname{Mas}(\ell_3(1; \lambda_1), \ell_4; [\xi_1, \xi_2])$$
$$- \operatorname{Mas}(\ell_3(\cdot; \lambda_2), \ell_4(\xi_1); [0, 1]) + \operatorname{Mas}(\ell_3(\cdot; \lambda_1), \ell_4(\xi_1); [0, 1]).$$

The importance of Corollary 1.1 is that in order to evaluate the right-hand side we only need to generate two (matrix) solutions of (1.1), one at λ_1 and the other at λ_2 (both at ξ_1).

The remainder of the article is organized as follows. In Section 2, we briefly review theory associated with the Maslov index, with an emphasis on the properties we will need in the remaining sections of the article. In Section 3, we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2, Theorem 1.3, and Corollary 1.1. Applications are provided at the ends of Sections 3 and 4.

2   The Maslov Index

In this section, we provide a short overview of the Maslov index in the current setting. Interested readers can find a more thorough discussion in [18] and the references found there.

Given any two Lagrangian subspaces ℓ_1 and ℓ_2, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we can define the complex $n \times n$ matrix

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{2.1}$$

As verified in [18], the matrices $(X_1 - iY_1)$ and $(X_2 + iY_2)$ are both invertible, and $\tilde{W}$ is unitary. We have the following theorem from [18].

Theorem 2.1. *Suppose $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ are Lagrangian subspaces, with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, and let $\tilde{W}$ be as defined in (2.1). Then*

$$\dim \ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2).$$

That is, the dimension of the eigenspace of $\tilde{W}$ associated with the eigenvalue -1 is precisely the dimension of the intersection of the Lagrangian subspaces ℓ_1 and ℓ_2.

Remark 2.1. *As noted in [18], Theorem 2.1 can be viewed as a convenient restatement of more general results associated with the Soriau map (see, e.g., Lemma 1.2 in [4] or Proposition 2.52 in [15]). The relationship between $\tilde{W}$ and the Souriau map is discussed at length in [18].*

Following [4, 15], we use Theorem 2.1, along with an approach to spectral flow introduced in [32], to define the Maslov index. Given a parameter interval $\mathbb{I} = [a, b]$, which can be normalized to $[0, 1]$, we consider maps $\ell : \mathbb{I} \to \Lambda(n)$, which will be expressed as $\ell(t)$. In order to specify a notion of continuity, we need to define a metric on $\Lambda(n)$, and following [15] (p. 274), we do this in terms of orthogonal projections onto elements $\ell \in \Lambda(n)$. Precisely, let

$\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^t \mathbf{X}_i)^{-1}\mathbf{X}_i^t$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that $\ell : \mathbb{I} \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Given two continuous maps $\ell_1(t), \ell_2(t)$ on $\mathbb{I}$, we denote by $\mathcal{L}(t)$ the path

$$\mathcal{L}(t) = (\ell_1(t), \ell_2(t)).$$

In what follows, we will define the Maslov index for the path $\mathcal{L}(t)$, which will be a count, including both multiplicity and direction, of the number of times the Lagrangian paths ℓ_1 and ℓ_2 intersect. In order to be clear about what we mean by multiplicity and direction, we observe that associated with any path $\mathcal{L}(t)$ we will have a path of unitary complex matrices as described in (2.1). We have already noted that the Lagrangian subspaces ℓ_1 and ℓ_2 intersect at a value $t_0 \in I$ if and only if $\tilde{W}(t_0)$ has -1 as an eigenvalue. (We refer to the value t_0 as a *conjugate point*.) In the event of such an intersection, we define the multiplicity of the intersection to be the multiplicity of -1 as an eigenvalue of $\tilde{W}$ (since $\tilde{W}$ is unitary the algebraic and geometric multiplicities are the same; we see from Theorem 2.1 that this multiplicity is precisely the dimension of the intersection $\ell_1(t_0) \cap \ell_2(t_0)$). When we talk about the direction of an intersection, we mean the direction the eigenvalues of $\tilde{W}$ are moving (as t varies) along the unit circle S^1 when they cross -1 (we take counterclockwise as the positive direction). We note that we will need to take care with what we mean by a crossing in the following sense: we must decide whether to increment the Maslov index upon arrival or upon departure. Indeed, there are several different approaches to defining the Maslov index (see, for example, [12, 34]), and they often disagree on this convention.

Following [4, 15, 32] (and in particular Definition 1.5 from [4]), we proceed by choosing a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $\mathbb{I} = [a, b]$, along with numbers $\epsilon_j \in (0, \pi)$ so that $\ker\left(\tilde{W}(t) - e^{i(\pi \pm \epsilon_j)}I\right) = \{0\}$ for $t_{j-1} \leq t \leq t_j$; that is, $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \leq t \leq t_j$ and $j = 1, \ldots, n$. Moreover, we notice that for each $j = 1, \ldots, n$ and any $t \in [t_{j-1}, t_j]$ there are only finitely many values $\theta \in [0, \epsilon_j)$ for which $e^{i(\pi+\theta)} \in \sigma(\tilde{W}(t))$.

Fix some $j \in \{1, 2, \ldots, n\}$ and consider the value

$$k(t, \epsilon_j) := \sum_{0 \leq \theta < \epsilon_j} \dim \ker\left(\tilde{W}(t) - e^{i(\pi+\theta)}I\right). \tag{2.2}$$

for $t_{j-1} \leq t \leq t_j$. This is precisely the sum, along with multiplicity, of the number of eigenvalues of $\tilde{W}(t)$ that lie on the arc

$$A_j := \{e^{it} : t \in [\pi, \pi + \epsilon_j)\}.$$

(See Figure 1.) The stipulation that $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \leq t \leq t_j$ asserts that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $t_{j-1} \leq t \leq t_j$. In this way, we see that $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that enter A_j in the counterclockwise direction (i.e., through
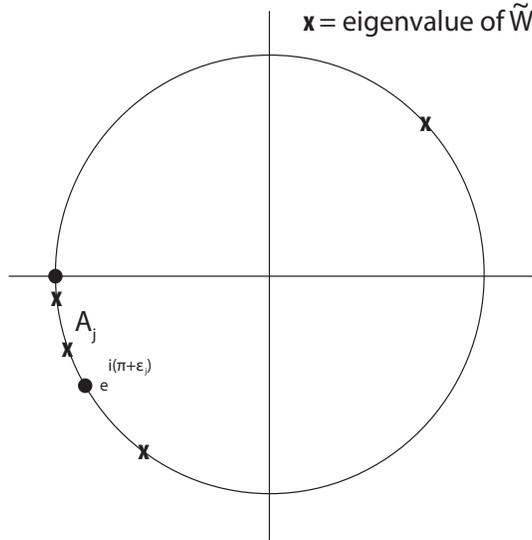
Figure 1: The arc A_j.

-1) minus the number that leave in the clockwise direction (again, through -1) during the interval $[t_{j-1}, t_j]$.

In dealing with the catenation of paths, it's particularly important to understand the difference $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ if an eigenvalue resides at -1 at either $t = t_{j-1}$ or $t = t_j$ (i.e., if an eigenvalue begins or ends at a crossing). If an eigenvalue moving in the counterclockwise direction arrives at -1 at $t = t_j$, then we increment the difference forward, while if the eigenvalue arrives at -1 from the clockwise direction we do not (because it was already in A_j prior to arrival). On the other hand, suppose an eigenvalue resides at -1 at $t = t_{j-1}$ and moves in the counterclockwise direction. The eigenvalue remains in A_j, and so we do not increment the difference. However, if the eigenvalue leaves in the clockwise direction then we decrement the difference. In summary, the difference increments forward upon arrivals in the counterclockwise direction, but not upon arrivals in the clockwise direction, and it decrements upon departures in the clockwise direction, but not upon departures in the counterclockwise direction.

We are now ready to define the Maslov index.

Definition 2.1. *Let $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathbb{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. The Maslov index $\mathrm{Mas}(\mathcal{L}; \mathbb{I})$ is defined by*

$$\mathrm{Mas}(\mathcal{L}; \mathbb{I}) = \sum_{j=1}^{n} (k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)). \tag{2.3}$$

Remark 2.2. *As we did in the introduction, we will typically refer explicitly to the individual paths with the notation $\mathrm{Mas}(\ell_1, \ell_2; \mathbb{I})$.*

Remark 2.3. *As discussed in [4], the Maslov index does not depend on the choices of $\{t_j\}_{j=0}^{n}$ and $\{\epsilon_j\}_{j=1}^{n}$, so long as they follow the specifications above.*

9

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(\mathbb{I})$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathbb{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathbb{I})$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t, s) \in \mathbb{I} \times [0, 1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties (see, for example, [18] in the current setting, or Theorem 3.6 in [15] for a more general result).

(P1) (Path Additivity) If $a < b < c$ then

$$\mathrm{Mas}(\mathcal{L}; [a, c]) = \mathrm{Mas}(\mathcal{L}; [a, b]) + \mathrm{Mas}(\mathcal{L}; [b, c]).$$

(P2) (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathbb{I})$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\mathrm{Mas}(\mathcal{L}; [a, b]) = \mathrm{Mas}(\mathcal{M}; [a, b]).$$

In practice, we work primarily with frames for Lagrangian subspaces, and we can use the following standard condition to verify that a given frame $\mathbf{X}$ is indeed the frame for a Lagrangian subspace (see, for example, [18, 34]).

Proposition 2.1. *A $2n \times n$ matrix $\mathbf{X}$ is a frame for a Lagrangian subspace if and only if the columns of $\mathbf{X}$ are linearly independent, and additionally*

$$\mathbf{X}^t J_{2n} \mathbf{X} = 0. \tag{2.4}$$

We refer to this relation as the Lagrangian property for frames.

3 Separated Boundary Conditions

For (1.1) with boundary conditions (BC1) we will work with the Lagrangian subspaces $\ell_1(x; \lambda)$ and ℓ_2, specified in the introduction (see (1.3) and the surrounding discussion). As a start, we verify that for all $(x, \lambda) \in [0, 1] \times I$, the matrix $\mathbf{X}_1(x; \lambda)$ specified in (1.3) is the frame for a Lagrangian subspace. First, it's clear from our assumptions on α and standard ODE theory that the columns of $\mathbf{X}_1(x; \lambda)$ are linearly independent. According to Proposition 2.1, it only remains to show that

$$\mathbf{X}_1(x; \lambda)^t J_{2n} \mathbf{X}_1(x; \lambda) = 0$$

for all $(x, \lambda) \in [0, 1] \times I$. Fix any $\lambda \in I$. First, for $x = 0$,

$$\mathbf{X}_1(0; \lambda)^t J_{2n} \mathbf{X}_1(0; \lambda) = \alpha J_{2n}^t J_{2n}^2 \alpha^t = \alpha J_{2n} \alpha^t = 0,$$

where the final equality is a condition of (BC1). For $x > 0$ temporarily set

$$\mathcal{A}(x; \lambda) = \mathbf{X}_1(x; \lambda)^t J_{2n} \mathbf{X}_1(x; \lambda).$$

We compute

$$\partial_x \mathcal{A}(x;\lambda) = \partial_x \mathbf{X}_1(x;\lambda)^t J_{2n} \mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t J_{2n} \partial_x \mathbf{X}_1(x;\lambda)$$
$$= -(J_{2n}\partial_x \mathbf{X}_1(x;\lambda))^t \mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t J_{2n}\partial_x \mathbf{X}_1(x;\lambda)$$
$$= -\mathbf{X}_1(x;\lambda)^t \mathbb{B}(x;\lambda)\mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t \mathbb{B}(x;\lambda)\mathbf{X}_1(x;\lambda) = 0.$$

Since $\mathcal{A}(0;\lambda) = 0$, we can conclude that $\mathcal{A}(x;\lambda) = 0$ for all $x \in [0,1]$. The same argument is true for every $\lambda \in I$, so we obtain the claim.

Turning to ℓ_2, the associated frame is $\mathbf{X}_2 = J_{2n}\beta^t$, and so similarly as for $\mathbf{X}_1(0;\lambda)$

$$\mathbf{X}_2^t J_{2n} \mathbf{X}_2 = \beta J_{2n}^t J_{2n}^2 \beta^t = \beta J_{2n} \beta^t = 0,$$

where the final equality is a condition of (BC1).

Using the notation

$$\mathbf{X}_1(x;\lambda) = \begin{pmatrix} X_1(x;\lambda) \\ Y_1(x;\lambda) \end{pmatrix}; \quad \mathbf{X}_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix},$$

we set

$$\tilde{W}(x;\lambda) = -(X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}.$$

3.1 The Maslov Box

Fix any values $\lambda_1, \lambda_2 \in I$ with $\lambda_1 < \lambda_2$, and consider the path in the (x,λ)-plane described as follows: (1) fix $x = 0$ and let λ run from λ_1 to λ_2 (the *bottom shelf*, which we will sometimes denote $[\lambda_1, \lambda_2]_{x=0}$); (2) fix $\lambda = \lambda_2$ and let x run from 0 to 1 (the *right shelf*, $[0,1]_{\lambda=\lambda_2}$); (3) fix $x = 1$ and let λ run from λ_2 to λ_1 (the *top shelf*, $[\lambda_2, \lambda_1]_{x=1}$); and (4) fix $\lambda = \lambda_1$ and let x run from 1 to 0 (the *left shelf*, $[1,0]_{\lambda=\lambda_1}$). We denote by Γ the simple closed curve obtained by following each of these paths precisely once. (See Figure 2.)

By catenation of paths, we have

$$\text{Mas}(\ell_1, \ell_2; \Gamma) = \text{Mas}(\ell_1(0;\cdot), \ell_2; [\lambda_1, \lambda_2]) + \text{Mas}(\ell_1(\cdot;\lambda_2), \ell_2; [0,1])$$
$$- \text{Mas}(\ell_1(1;\cdot), \ell_2; [\lambda_1, \lambda_2]) - \text{Mas}(\ell_1(\cdot;\lambda_1), \ell_2; [0,1]),$$

and by homotopy invariance $\text{Mas}(\ell_1, \ell_2; \Gamma) = 0$. Here, the latter two summands could alternatively be expressed with addition, but the parameter would then decrease rather than increase. The convention for this article will be to express every Maslov index in terms of an increasing parameter. The relation is simply that if $\mathbb{I}$ denotes an interval traversed in one direction, and $-\mathbb{I}$ denotes the same interval traversed in the opposite direction, then

$$\text{Mas}(\ell_1, \ell_2; \mathbb{I}) = -\text{Mas}(\ell_1, \ell_2; -\mathbb{I}).$$

This is a trivial consequence of our definition, and indeed must be true for any Maslov index (see Property X of [12]).

For the bottom shelf, $\mathbf{X}_1(0;\lambda) = J\alpha^t$, and if we introduce the block notation $\alpha = (\alpha_1 \ \alpha_2)$ and $\beta = (\beta_1 \ \beta_2)$ we find

$$\mathbf{X}_1(0;\lambda) = \begin{pmatrix} -\alpha_2^t \\ \alpha_1^t \end{pmatrix}; \quad \mathbf{X}_2 = \begin{pmatrix} -\beta_2^t \\ \beta_1^t \end{pmatrix},$$
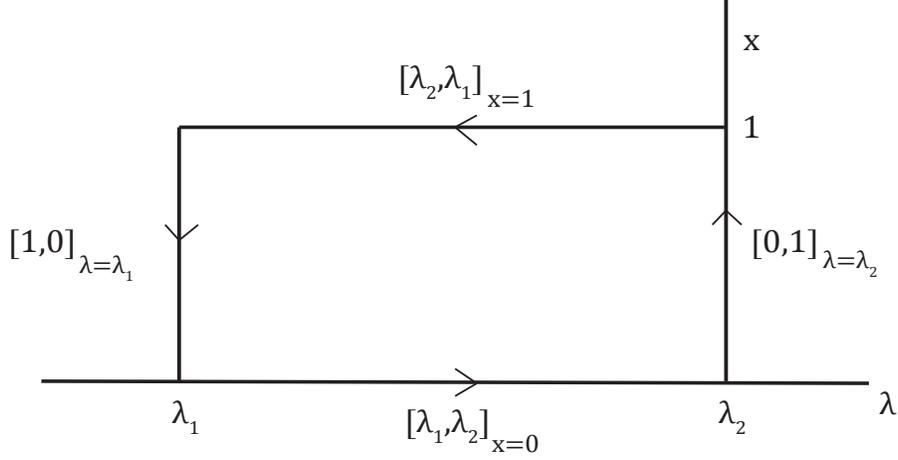
11

Figure 2: The (λ, x)-Maslov Box.

so that
$$\tilde{W}(0; \lambda) = -(-\alpha_2^t + i\alpha_1^t)(-\alpha_2^t - i\alpha_1^t)^{-1}(-\beta_2^t - i\beta_1^t)(-\beta_2^t + i\beta_1^t)^{-1}.$$

This is constant in λ, so trivially $\mathrm{Mas}(\ell_1(0; \cdot), \ell_2; [\lambda_1, \lambda_2]) = 0$.

For the left and right shelves we will generally compute $\mathbf{X}_1(x; \lambda_1)$ and $\mathbf{X}_1(x; \lambda_2)$ numerically, and generate the spectral flow from these calculations. The important point here is that $\mathbf{X}_1(x; \lambda_1)$ and $\mathbf{X}_1(x; \lambda_2)$ solve initial value problems which can typically be solved efficiently with a standard method such as Runge-Kutta.

For the top shelf, we will show that under certain circumstances the eigenvalues of $\tilde{W}(x; \lambda)$ rotate monotonically counterclockwise as λ decreases (with x fixed), and it will follow that

$$- \mathrm{Mas}(\ell_1(1; \cdot), \ell_2; [\lambda_1, \lambda_2]) = \mathcal{N}([\lambda_1, \lambda_2]),$$

where $\mathcal{N}([\lambda_1, \lambda_2])$ is described in (1.4). We note particularly that $\mathcal{N}([\lambda_1, \lambda_2])$ is well-defined, and that the key property we will used to show that it is well-defined is monotonicity.

3.2   Monotonicity

From [18] we have that monotonicity of the eigenvalues of $\tilde{W}(x; \lambda)$ as λ varies is determined by the matrix $\mathbf{X}_1^t J_{2n} \partial_\lambda \mathbf{X}_1$ in the following sense: for $x \in (0, 1]$ fixed, if $\mathbf{X}_1(x; \lambda)^t J_{2n} \partial_\lambda \mathbf{X}_1(x; \lambda)$ is positive definite for all $\lambda \in I$, then the eigenvalues of $\tilde{W}(x; \lambda)$ will rotate monotonically counterclockwise as λ *decreases*. In order to get a sign for this matrix, we compute

$$\frac{\partial}{\partial x} \mathbf{X}_1^t J_{2n} \partial_\lambda \mathbf{X}_1 = (\mathbf{X}_1')^t J_{2n} \partial_\lambda \mathbf{X}_1 + \mathbf{X}_1^t J_{2n} \partial_\lambda \mathbf{X}_1' = -(\mathbf{X}_1')^t J_{2n}^t \partial_\lambda \mathbf{X}_1 + \mathbf{X}_1^t J_{2n} \partial_\lambda \mathbf{X}_1'$$

$$= -\mathbf{X}_1^t \mathbb{B}(x; \lambda) \partial_\lambda \mathbf{X}_1 + \mathbf{X}_1^t \partial_\lambda (\mathbb{B}(x; \lambda) \mathbf{X}_1) = \mathbf{X}_1^t \mathbb{B}_\lambda \mathbf{X}_1.$$

12

Integrating on $[0, x]$, and noting that $\partial_\lambda \mathbf{X}_1(0; \lambda) = 0$, we see that

$$\mathbf{X}_1(x; \lambda)^t J_{2n} \partial_\lambda \mathbf{X}_1(x; \lambda) = \int_0^x \mathbf{X}_1(y; \lambda)^t \mathbb{B}_\lambda(y; \lambda) \mathbf{X}_1(y; \lambda) dy.$$

If $\mathbb{B}_\lambda(y; \lambda)$ is positive definite for all $y \in [0, x]$, then monotonicity is clear. We will see in the examples that monotonicity can often be obtained even if $\mathbb{B}_\lambda(y; \lambda)$ is merely non-negative.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We have already seen that

$$\mathrm{Mas}(\ell_1(0; \cdot), \ell_2; [\lambda_1, \lambda_2]) = 0.$$

For $-\mathrm{Mas}(\ell_1(1; \cdot), \ell_2; [\lambda_1, \lambda_2])$, which we can view as the Maslov index computed as λ runs from λ_2 back to λ_1 (with $x = 1$), we note that each conjugate point will be a value $\lambda \in [\lambda_1, \lambda_2]$ so that (1.1)-(BC1) has a non-trivial solution, and moreover the dimension of the solution space associated with λ will correspond with $\dim(\ell_1(1; \lambda) \cap \ell_2)$. Two cases warrant careful consideration: a conjugate point at λ_1 and a conjugate point at λ_2. If λ_1 is a conjugate point then by monotonicity as λ decreases to λ_1 the eigenvalue(s) of $\tilde{W}(1; \lambda)$ corresponding with the crossing will be rotating in the counterclockwise direction, and will increment the Maslov index forward as they arrive at -1. On the other hand, if λ_2 is a conjugate point, as λ decreases from λ_2, the associated eigenvalue(s) of $\tilde{W}(1; \lambda)$ will rotate away from -1 in the counterclockwise direction, and the Maslov index will not record the intersection. We conclude that

$$-\mathrm{Mas}(\ell_1(1; \cdot), \ell_2; [\lambda_1, \lambda_2]) = \mathcal{N}([\lambda_1, \lambda_2]).$$

Note, in particular, that the left-hand side of this equality is well-defined, and so $\mathcal{N}([\lambda_1, \lambda_2])$ is well-defined. Effectively, monotonicity ensures that the sum defining $\mathcal{N}([\lambda_1, \lambda_2])$ in (1.4) is over a discrete set of values λ (because each eigenvalue of $\tilde{W}(x; \lambda)$ must complete a full loop of S^1 between crossings of -1, and continuity of $\tilde{W}(x; \lambda)$ on $[\lambda_1, \lambda_2]$ (for x fixed) ensures that there is a lower bound on the width of the intervals in I between these crossings).

The proof now follows immediately from the relation

$$0 = \mathrm{Mas}(\ell_1(0; \cdot), \ell_2; [\lambda_1, \lambda_2]) + \mathrm{Mas}(\ell_1(\cdot; \lambda_2), \ell_2; [0, 1])$$
$$- \mathrm{Mas}(\ell_1(1; \cdot), \ell_2; [\lambda_1, \lambda_2]) - \mathrm{Mas}(\ell_1(\cdot; \lambda_1), \ell_2; [0, 1]).$$

$\square$

3.3 Rotation in x

The eigenvalues of $\tilde{W}(x; \lambda)$ generally do not rotate monotonically as x increases, but we nonetheless remark here on how this rotation can be analyzed. According to Lemma 4.2 in [18], the rotation of the eigenvalues of $\tilde{W}(x; \lambda)$ as x varies is determined by the nature of

$$\Omega(x; \lambda) := \mathbf{X}_1(x; \lambda)^t J_{2n} \mathbf{X}_1'(x; \lambda),$$

where prime denotes differentiation with respect to x. Using (1.3) we see that

$$\Omega(x; \lambda) = \mathbf{X}_1(x; \lambda)^t \mathbb{B}(x; \lambda) \mathbf{X}_1(x; \lambda).$$

13

If this matrix is positive definite then as x increases the eigenvalues of $\tilde{W}(x;\lambda)$ will rotate monotonically clockwise.

One setting in which this relation becomes useful is when we would like to understand the rotation of an eigenvalue of $\tilde{W}(x;\lambda)$ for x near 0. This can be determined by the explicit matrix

$$\Omega(0;\lambda) = (J_{2n}\alpha^t)^t \mathbb{B}(0;\lambda) J_{2n}\alpha^t.$$

Moreover,

$$\begin{aligned}
\Omega'(x;\lambda) &= \mathbf{X}_1'(x;\lambda)^t \mathbb{B}(x;\lambda)\mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t \mathbb{B}'(x;\lambda)\mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t \mathbb{B}(x;\lambda)\mathbf{X}_1'(x;\lambda) \\
&= (-J\mathbb{B}(x;\lambda)\mathbf{X}_1)^t \mathbb{B}(x;\lambda)\mathbf{X}_1(x;\lambda) + \mathbf{X}_1(x;\lambda)^t \mathbb{B}'(x;\lambda)\mathbf{X}_1(x;\lambda) \\
&\quad + \mathbf{X}_1(x;\lambda)^t \mathbb{B}(x;\lambda)(-J\mathbb{B}(x;\lambda)\mathbf{X}_1) \\
&= \mathbf{X}_1(x;\lambda)^t \mathbb{B}'(x;\lambda)\mathbf{X}_1(x;\lambda),
\end{aligned}$$

so that

$$\Omega(x;\lambda) = (J_{2n}\alpha^t)^t \mathbb{B}(0;\lambda) J_{2n}\alpha^t + \int_0^x \mathbf{X}_1(y;\lambda)^t \mathbb{B}'(y;\lambda)\mathbf{X}_1(y;\lambda)dy.$$

3.4   Applications

In this section we briefly review how some common ODE boundary value problems can be expressed in the form (1.1)-(BC1). Our goal is not to specify minimal requirements on these equations, and we often assume more than is required for analysis.

3.4.1   Sturm-Liouville Systems

We consider Sturm-Liouville systems with separated self-adjoint boundary conditions

$$\begin{aligned}
L\phi &:= -(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi \\
&\alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) = 0 \\
&\beta_1\phi(1) + \beta_2 P(1)\phi'(1) = 0.
\end{aligned} \tag{3.1}$$

Here, $\phi(x) \in \mathbb{R}^n$, and our notational convention is to take $\alpha = (\alpha_1\ \alpha_2)$ and $\beta = (\beta_1\ \beta_2)$. We assume $P \in C^1([0,1];\mathbb{R}^{n\times n})$, $V,Q \in C([0,1];\mathbb{R}^{n\times n})$, and that all three matrices are symmetric. In addition, we assume that $P(x)$ is invertible for each $x \in [0,1]$, and that $Q(x)$ is positive definite for each $x \in [0,1]$. For the boundary conditions, we assume $\operatorname{rank}\alpha = n$, $\alpha J_{2n}\alpha^t = 0$ and likewise $\operatorname{rank}\beta = n$, $\beta J_{2n}\beta^t = 0$, which is equivalent to self-adjointness in this case. We note that in our motivating reference [20] the authors focus on Schrödinger operators, for which $P(x) = I_n$ and $Q(x) = I_n$.

For each $x \in [0,1]$, we define a new vector $y(x) \in \mathbb{R}^{2n}$ so that $y(x) = (y_1(x)\ y_2(x))^t$, with $y_1(x) = \phi(x)$ and $y_2(x) = P(x)\phi'(x)$. In this way, we express (3.1) in the form

$$\begin{aligned}
y' &= \mathbb{A}(x;\lambda)y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & P(x)^{-1} \\ V(x) - \lambda Q(x) & 0 \end{pmatrix}, \\
&\alpha_1 y_1(0) + \alpha_2 y_2(0) = 0 \\
&\beta_1 y_1(1) + \beta_2 y_2(1) = 0.
\end{aligned}$$

Upon multiplying both sides of this equation by J_{2n}, we obtain (1.1) with

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V(x) & 0 \\ 0 & P(x)^{-1} \end{pmatrix},$$

which satisfies our assumptions on $\mathbb{B}(x;\lambda)$.

In this case,

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} Q(x) & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\int_0^1 \mathbf{X}_1(y;\lambda)^t \mathbb{B}_\lambda(y;\lambda) \mathbf{X}_1(y;\lambda) dy = \int_0^1 X_1(y;\lambda)^t Q(y) X_1(y;\lambda) dy.$$

This matrix is clearly non-negative (since Q is positive definite), and moreover it cannot have 0 as an eigenvalue, because the associated eigenvector $v \in \mathbb{R}^n$ would necessarily satisfy $X_1(x;\lambda)v = 0$ for all $x \in [0,1]$, and this would contradict linear independence of the columns of $X_1(x;\lambda)$ (as solutions of (1.1)).

3.4.2 Self-Adjoint Fourth Order Equations

We consider self-adjoint fourth order ODE with separated self-adjoint boundary conditions

$$\mathcal{L}\phi = (V_4(x)\phi'')'' - (V_2(x)\phi')' + V_0(x)\phi = \lambda Q(x)\phi$$
$$\alpha_1\phi(0) + \alpha_2\phi'(0) + \alpha_3 V_4(0)\phi''(0) + \alpha_4\Big((V_4\phi'')' - V_2\phi'\Big)\Big|_{x=0} = 0 \tag{3.2}$$
$$\beta_1\phi(1) + \beta_2\phi'(1) + \beta_3 V_4(1)\phi''(1) + \beta_4\Big((V_4\phi'')' - V_2\phi'\Big)\Big|_{x=1} = 0.$$

Here, $\phi(x) \in \mathbb{R}^n$, $Q, V_0 \in C([0,1]; \mathbb{R}^{n\times n})$, $V_2 \in C^1([0,1]; \mathbb{R}^{n\times n})$, $V_4 \in C^2([0,1]; \mathbb{R}^{n\times n})$. We assume each of the matrices $V_0(x), V_2(x), V_4(x)$, and $Q(x)$ is symmetric for each $x \in [0,1]$, that $V_4(x)$ is invertible for all $x \in [0,1]$, and that $Q(x)$ is positive definite for all $x \in [0,1]$. Also, α_j and β_j are $2n \times n$ matrices with real entries, and we set

$$\tilde{\alpha} := (\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) \in \mathbb{R}^{2n\times 4n}$$
$$\tilde{\beta} := (\beta_1\ \beta_2\ \beta_3\ \beta_4) \in \mathbb{R}^{2n\times 4n}. \tag{3.3}$$

(Our use of tildes will be clarified below when we make a slight adjustment to get the form (1.1)-(BC1).) We assume

$$\text{rank}\,\tilde{\alpha} = 2n; \quad \tilde{\alpha}\mathbb{J}_{4n}\tilde{\alpha}^t = 0$$
$$\text{rank}\,\tilde{\beta} = 2n; \quad \tilde{\beta}\mathbb{J}_{4n}\tilde{\beta}^t = 0,$$

with

$$\mathbb{J}_{4n} = \begin{pmatrix} 0 & J_{2n} \\ J_{2n} & 0 \end{pmatrix}.$$

It is straightforward to check that if we take the domain of $\mathcal{L}$ to be

$$\mathcal{D}(\mathcal{L}) := \{\phi \in H^4([0,1]) : \text{boundary conditions in (3.2) hold}\},$$

15

then $\mathcal{L}$ is self-adjoint. In order to express (3.2) as a first order system we set $y_1 = \phi$, $y_2 = V_4\phi''$, $y_3 = -(V_4\phi'')' + V_2\phi'$, and $y_4 = -\phi'$, so that

$$y' = \mathbb{A}(x;\lambda)y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & -V_2(x) \\ V_0(x) - \lambda Q(x) & 0 & 0 & 0 \\ 0 & -V_4(x)^{-1} & 0 & 0 \end{pmatrix},$$

and consequently

$$\mathbb{J}_{4n}y' = \mathbb{B}(x;\lambda)y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V_0(x) & 0 & 0 & 0 \\ 0 & V_4(x)^{-1} & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & -V_2(x) \end{pmatrix}.$$

If we set $\alpha = (\alpha_1 \ \alpha_3 \ -\alpha_4 \ -\alpha_2)$, and similarly for β, then we can express the boundary conditions as $\alpha y(0) = 0$ and $\beta y(1) = 0$. In this way, we obtain the form of (1.1) with boundary conditions (BC1).

Before checking the monotonicity assumption of Theorem 1.1, we remark on how this development goes if we start with the more conventional variables $z_1 = \phi$, $z_2 = \phi'$, $z_3 = V_4\phi''$, and $z_4 = (V_4\phi'')' - V_2\phi'$. In this case, we arrive at the system

$$z' = \tilde{\mathbb{A}}(x;\lambda)z; \quad \tilde{\mathbb{A}}(x;\lambda) = \begin{pmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & V_4(x)^{-1} & 0 \\ 0 & V_2(x) & 0 & I_n \\ \lambda Q(x) - V_0(x) & 0 & 0 & 0 \end{pmatrix},$$

and consequently

$$(-\mathbb{J}_{4n})z' = \tilde{\mathbb{B}}(x;\lambda)z; \quad \tilde{\mathbb{B}}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V_0(x) & 0 & 0 & 0 \\ 0 & -V_2(x) & 0 & -I_n \\ 0 & 0 & V_4(x)^{-1} & 0 \\ 0 & -I_n & 0 & 0 \end{pmatrix}.$$

We see that in some sense $-\mathbb{J}_{4n}$ is the natural skew-symmetric matrix for this problem (rather than $\mathbb{J}_{4n}$). Our choice of $-\mathbb{J}_{4n}$ as opposed to $\mathbb{J}_{4n}$ is merely a convention that is consistent with the variables we started with. If we proceeded with $\mathbb{J}_{4n}$ instead, it would lead to the variables $y_1 = -\phi$, $y_2 = -V_4\phi''$, $y_3 = V_2\phi' - (V_4\phi'')'$, and $y_4 = -\phi'$. We know (e.g., Theorem 8.5 in [3]) that there exists an invertible matrix M so that $M^t(-\mathbb{J}_{4n})M = \mathbb{J}_{4n}$. Setting $z = Mw$ we get

$$(-\mathbb{J}_{4n})Mw' = \tilde{\mathbb{B}}(x;\lambda)Mw,$$

so that

$$\mathbb{J}_{4n}w' = \mathbb{B}(x;\lambda)w; \quad \mathbb{B}(x;\lambda) = M^t\tilde{\mathbb{B}}(x;\lambda)M.$$

In our case,

$$M = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \end{pmatrix}; \quad M^t = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & -I_n & 0 & 0 \end{pmatrix},$$

16

which leads immediately to the change of variables we introduced originally.

In order to apply Theorem 1.1, we observe that

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} Q(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we introduce the notation

$$X_1 = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where X_1 is $2n \times 2n$ and each X_{ij} is $n \times n$, then

$$\mathbf{X}_1(x;\lambda)^t \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda) = X_{11}(x;\lambda)^t Q(x) X_{11}(x;\lambda),$$

and we can conclude monotonicity similarly as for the case of Sturm-Liouville operators.

3.4.3 Dirac Equations

Many of the examples we have in mind have the general form

$$J_{2n}y' = (\lambda Q(x) + V(x))y. \tag{3.4}$$

In this case, $\mathbb{B}(x;\lambda) = \lambda Q(x) + V(x)$, so that $\mathbb{B}_\lambda(x;\lambda) = Q(x)$, and the monotonicity criterion of Theorem 1.1 will be satisfied if $Q(x)$ is positive definite. (Though we have seen in our previous examples that we can have monotonicity in cases for which Q is only non-negative.) We note that in the event that $Q(x) = I_{2n}$, we refer to (3.4) as the Dirac equation.

3.4.4 Models with Convection

In this section, we consider equations of the form

$$J_{2n}\phi = \mathbb{B}(x;\lambda)\phi + \mathbb{S}\phi, \tag{3.5}$$

where $\mathbb{B}(x;\lambda)$ is precisely as described following (1.1) and

$$\mathbb{S} = \begin{pmatrix} 0 & sI_n \\ 0 & 0 \end{pmatrix},$$

for some real value $s \neq 0$. (We continue to use boundary conditions (BC1).)

Equations of this form arise naturally in the context of eigenvalue problems

$$-\psi'' - s\psi' + V(x)\psi = \lambda\psi,$$

where the term $-s\psi'$ would be associated with convection for the related PDE $u_t - su_x + F'(u) = u_{xx}$. If we set $\phi_1 = \psi$ and $\phi_2 = \psi'$ we arrive at the system

$$\phi' = \mathbb{A}(x;\lambda)\phi; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I_n \\ V(x) - \lambda I_n & -sI_n \end{pmatrix},$$

17

which we can express as

$$J_{2n}\phi' = \mathbb{B}(x;\lambda)\phi + \mathbb{S}\phi,$$

with

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} \lambda I_n - V(x) & 0 \\ 0 & I_n \end{pmatrix},$$

and $\mathbb{S}$ as above.

Now let ϕ satisfy the general equation (3.5) and set $\zeta(x;\lambda) = e^{\frac{s}{2}x}\phi(x;\lambda)$. Computing directly we find that

$$J_{2n}\zeta' = \tilde{\mathbb{B}}(x;\lambda)\zeta,$$

where

$$\tilde{\mathbb{B}}(x;\lambda) = \mathbb{B}(x;\lambda) + \begin{pmatrix} 0 & \frac{s}{2}I_n \\ \frac{s}{2}I_n & 0 \end{pmatrix},$$

and this equation satisfies our assumptions on (1.1) (with $\tilde{\mathbb{B}}$ replacing $\mathbb{B}$). The boundary conditions (BC1) remain unchanged.

4   General Self-Adjoint Boundary Conditions

In this section, we consider equation (1.1) with general self-adjoint boundary conditions (BC2). As a starting point, we check that equations with boundary conditions (BC1) can be formulated with boundary conditions (BC2). If we write

$$\Theta = \begin{pmatrix} \alpha & 0_{n \times 2n} \\ 0_{n \times 2n} & \beta \end{pmatrix},$$

then the condition $\Theta \begin{pmatrix} y(0) \\ y(1) \end{pmatrix} = 0$ is equivalent to the pair of conditions $\alpha y(0) = 0$ and $\beta y(1) = 0$. Moreover, if α and β each have rank n then Θ will have rank $2n$. Finally, by direct calculation,

$$\Theta \mathcal{J}_{4n} \Theta^t = \begin{pmatrix} -\alpha J_{2n}\alpha^t & 0_n \\ 0_n & \beta J_{2n}\beta^t \end{pmatrix} = 0.$$

Next, we observe that if we fix any $x \in (0,1]$ and take an $L^2([0,x])$ inner product on both sides of (1.1) with a second solution z then we obtain the relation

$$\langle J_{2n}y', z \rangle_{L^2([0,x])} - \langle y, J_{2n}z' \rangle_{L^2([0,x])} = (J_{2n}y(x), z(x))_{\mathbb{R}^{2n}} - (J_{2n}y(0), z(0))_{\mathbb{R}^{2n}}. \tag{4.1}$$

But since y and z both solve (1.1) (and keeping in mind that $\mathbb{B}(x;\lambda)$ is symmetric), the left-hand side is 0. This of course means that the right-hand side of (4.1) is 0, and we can express this observation as

$$\left(\mathcal{J}_{4n} \begin{pmatrix} y(0) \\ y(x) \end{pmatrix}, \begin{pmatrix} z(0) \\ z(x) \end{pmatrix} \right)_{\mathbb{R}^{4n}} = 0.$$

This suggests a Lagrangian subspace defined in terms of the skew-symmetric matrix $\mathcal{J}_{4n}$. As in Section 3.4.2, we note that there exists an invertible matrix $\mathcal{M}$ so that $\mathcal{M}^t J_{4n}\mathcal{M} = \mathcal{J}_{4n}$. If $\hat{\ell}$ denotes a Lagrangian subspace relative to $\mathcal{J}_{4n}$, then $\ell = \mathcal{M}(\hat{\ell})$ will be a Lagrangian

subspace relative to J_{4n}. I.e., given any $u, v \in \ell$, there exist $\hat{u}, \hat{v} \in \hat{\ell}$ so that $u = \mathcal{M}\hat{u}$, $v = \mathcal{M}\hat{v}$, and consequently

$$(J_{4n}u, v) = (J_{4n}\mathcal{M}\hat{u}, \mathcal{M}\hat{v}) = (\mathcal{M}^t J_{4n}\mathcal{M}\hat{u}, \hat{v}) = (\mathcal{J}_{4n}\hat{u}, \hat{v}) = 0.$$

In this case (i.e., for the matrices J_{4n} and $\mathcal{J}_{4n}$), we have

$$\mathcal{M} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix},$$

as introduced in Section 1.

This suggests a Lagrangian subspace, which we define in terms of the trace-type operator

$$\mathcal{T}_x y := \mathcal{M} \begin{pmatrix} y(0) \\ y(x) \end{pmatrix}. \tag{4.2}$$

Precisely, we will verify below that the subspace

$$\ell_3(x; \lambda) = \{ \mathcal{T}_x y : J_{2n} y' = \mathbb{B}(x; \lambda) y \text{ on } (0, 1) \} \tag{4.3}$$

is Lagrangian for all $(x, \lambda) \in [0, 1] \times I$. (Our choice to designate this space as ℓ_3 is taken simply to distinguish it from the spaces ℓ_1 and ℓ_2 specified for separated boundary conditions; i.e., the two primary Lagrangian subspaces for the case of general self-adjoint boundary conditions will be denoted ℓ_3 and ℓ_4.) In order to construct a frame for this Lagrangian subspace, we follow the approach of [17] and let $\Phi(x; \lambda)$ denote the $2n \times 2n$ fundamental matrix solution to

$$J_{2n} \Phi' = \mathbb{B}(x; \lambda) \Phi; \quad \Phi(0; \lambda) = I_{2n}. \tag{4.4}$$

If we introduce the notation

$$\Phi(x; \lambda) = \begin{pmatrix} \Phi_{11}(x; \lambda) & \Phi_{12}(x; \lambda) \\ \Phi_{21}(x; \lambda) & \Phi_{22}(x; \lambda) \end{pmatrix},$$

then we can associate the collection of all vectors of the form $\begin{pmatrix} y(0) \\ y(x) \end{pmatrix}$ with the $4n \times 2n$ matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \\ \Phi_{11}(x; \lambda) & \Phi_{12}(x; \lambda) \\ \Phi_{21}(x; \lambda) & \Phi_{22}(x; \lambda) \end{pmatrix}.$$

Acting on this with $\mathcal{M}$, we find that our frame for $\ell_3(x; \lambda)$ is

$$\mathbf{X}_3(x; \lambda) = \begin{pmatrix} X_3(x; \lambda) \\ Y_3(x; \lambda) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \Phi_{11}(x; \lambda) & \Phi_{12}(x; \lambda) \\ 0 & -I_n \\ \Phi_{21}(x; \lambda) & \Phi_{22}(x; \lambda) \end{pmatrix}. \tag{4.5}$$

19

In order to verify that $\mathbf{X}_3(x;\lambda)$ is indeed the frame for a Lagrangian subspace, we use Proposition 2.1; i.e., we check the Lagrangian property

$$\mathbf{X}_3(x;\lambda)^t J_{4n} \mathbf{X}_3(x;\lambda) = 0,$$

for all $(x,\lambda) \in [0,1] \times I$. First, for $x = 0$ we have

$$\mathbf{X}_3(0;\lambda) = \begin{pmatrix} I_n & 0 \\ I_n & 0 \\ 0 & -I_n \\ 0 & I_n \end{pmatrix},$$

from which we compute directly to find $\mathbf{X}_3(0;\lambda)^t J_{4n} \mathbf{X}_3(0;\lambda) = 0$. For each fixed $\lambda \in I$, we temporarily set

$$\mathcal{A}(x) := \mathbf{X}_3(x,\lambda)^t J_{4n} \mathbf{X}_3(x,\lambda) = \Phi(x;\lambda)^t J_{2n} \Phi(x;\lambda) - J_{2n},$$

where dependence on the fixed value λ has been suppressed in $\mathcal{A}$, and the second equality follows from a straightforward calculation. We have, then,

$$\begin{aligned} \mathcal{A}'(x) &= \Phi'(x;\lambda)^t J_{2n} \Phi(x;\lambda) + \Phi(x;\lambda)^t J_{2n} \Phi'(x;\lambda) \\ &= -(J_{2n}\Phi'(x;\lambda))^t \Phi(x;\lambda) + \Phi(x;\lambda)^t J_{2n} \Phi'(x;\lambda) \\ &= -\Phi(x;\lambda)^t \mathbb{B}(x;\lambda)\Phi(x;\lambda) + \Phi(x;\lambda)^t \mathbb{B}(x;\lambda)\Phi(x;\lambda) = 0, \end{aligned}$$

where we have used $J_{2n}^t = -J_{2n}$ and the symmetry of $\mathbb{B}(x;\lambda)$. We conclude that $\mathcal{A}(x) = 0$ for all $x \in [0,1]$, and since this is true for any $\lambda \in I$ we conclude that $\ell_3(x;\lambda)$ is indeed Lagrangian for all $(x,\lambda) \in [0,1] \times I$.

Likewise, we specify a Lagrangian subspace associated with the boundary conditions (BC2). In this case, it is more convenient to specify the Lagrangian subspace directly from its frame. The boundary conditions (BC2) have been taken so that the matrix $\mathcal{J}_{4n}\Theta^t$ comprises $2n$ linearly independent columns that satisfy the boundary condition. It is immediate that the matrix $\mathcal{J}_{4n}\Theta^t$ is the frame for a Lagrangian subspace relative to $\mathcal{J}_{4n}$. I.e.,

$$(\mathcal{J}_{4n}\Theta^t)^t \mathcal{J}_{4n} \mathcal{J}_{4n}\Theta^t = \Theta \mathcal{J}_{4n}^t \mathcal{J}_{4n}^2 \Theta^t = \Theta \mathcal{J}_{4n}\Theta^t = 0.$$

Precisely as in the considerations leading up to our definition of $\ell_3(x;\lambda)$, this means that $\mathcal{M}\mathcal{J}_{4n}\Theta^t$ will be a frame of a Lagrangian subspace relative to J_{4n}. We let ℓ_4 denote the Lagrangian subspace with this frame. I.e., we set

$$X_4 = \mathcal{M}\mathcal{J}_{4n}\Theta^t.$$

4.1   The Maslov Box and Monotonicity

For any pair $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, we consider precisely the same Maslov Box as described in Section 3.1. (See Figure 2.) For the bottom shelf (i.e., for $[\lambda_1, \lambda_2]_{x=0}$) the Lagrangian frames $\mathbf{X}_3(0;\lambda)$ and $\mathbf{X}_4$ are both independent of λ, so we have

$$\mathrm{Mas}(\ell_3(0;\cdot), \ell_4; [\lambda_1, \lambda_2]) = 0.$$

20

For the top shelf (i.e., for $[\lambda_2, \lambda_1]_{x=1}$), and indeed for any intermediate horizontal shelf $[\lambda_2, \lambda_1]_{x=s}$, with $s \in (0, 1]$, we establish conditions under which the eigenvalues of $\tilde{W}(x, \lambda)$ rotate monotonically clockwise as λ increases. According to Lemma 4.2 of [18], monotonicity will be determined by the nature of

$$\Omega(x; \lambda) := \mathbf{X}_3(x; \lambda)^t J_{4n} \partial_\lambda \mathbf{X}_3(x; \lambda) = \Phi(x; \lambda)^t J_{2n} \partial_\lambda \Phi(x; \lambda),$$

where the second equality follows from a straightforward calculation. In particular, if $\Omega(x; \lambda)$ is positive definite at a conjugate point, then the associated eigenvalues of $\tilde{W}(x; \lambda)$ will rotate through -1 in the counterclockwise direction as λ decreases. Computing now almost exactly as in Section 3.2, we find

$$\Omega(x; \lambda) = \int_0^x \Phi(y; \lambda)^t \mathbb{B}_\lambda(y; \lambda) \Phi(y; \lambda) dy.$$

Similarly as for the top shelf in the proof of Theorem 1.1, we can conclude that

$$-\operatorname{Mas}(\ell_1(1; \cdot), \ell_2; [\lambda_1, \lambda_2]) = \mathcal{N}([\lambda_1, \lambda_2]).$$

Theorem 1.2 follows immediately by catenation of paths and homotopy invariance.

4.2   Rotation in x

As in the case of separated self-adjoint boundary conditions, the eigenvalues of $\tilde{W}(x; \lambda)$ don't generally rotate monotonically as x increases. Nonetheless, there are cases in which useful information can be obtained about such rotation. According to Lemma 4.2 of [18], the nature of this rotation will be determined by the matrix

$$\Omega(x; \lambda) = \mathbf{X}_3(x; \lambda)^t J_{4n} \mathbf{X}_3'(x; \lambda) = \Phi(x; \lambda)^t J_{2n} \Phi'(x; \lambda),$$

where prime denotes differentiation with respect to x. Precisely as with λ, if $\Omega(x; \lambda)$ is positive definite at a conjugate point, then the associated eigenvalues of $\tilde{W}(x; \lambda)$ will rotate through -1 in the counterclockwise direction as x decreases.

Using (1.7) we see that

$$\Omega(x; \lambda) = \Phi(x; \lambda)^t \mathbb{B}(x; \lambda) \Phi(x; \lambda).$$

We will find this expression useful in understanding rotation near $x = 0$, for which we have simply $\Omega(0; \lambda) = \mathbb{B}(0; \lambda)$. Similarly as in Section 3.3, we find that

$$\Omega(x; \lambda) = \mathbb{B}(0; \lambda) + \int_0^x \Phi(y; \lambda)^t \mathbb{B}'(y; \lambda) \Phi(y; \lambda) dy.$$

4.3   Parameter-Dependent Boundary Conditions

We now turn to the case in which Θ depends on a parameter $\xi \in \mathcal{I}$, for which we take boundary conditions $(BC2)_\xi$. As will be discussed in Section 4.4, our primary motivation

for this analysis is the role such eigenvalue problems play in spectral counts for operators obtained when certain nonlinear PDE are linearized about periodic stationary solutions.

For each fixed $\xi \in \mathcal{I}$, Theorem 1.2 holds with Θ replaced by $\Theta(\xi)$. In this section, we fix $x = 1$ and let ξ vary over $\mathcal{I}$. Rather than introducing new notation, we will let $\ell_3(1; \lambda)$ denote the path of Lagrangian subspaces obtained by fixing $x = 1$ for the Lagrangian subspaces $\ell_3(x; \lambda)$ from (4.3). I.e., we let $\ell_3(1; \lambda)$ denote the path of Lagrangian subspaces associated with frames

$$\mathbf{X}_3(1, \lambda) = \begin{pmatrix} I_n & 0 \\ \Phi_{11}(1; \lambda) & \Phi_{12}(1; \lambda) \\ 0 & -I_n \\ \Phi_{21}(1; \lambda) & \Phi_{22}(1; \lambda) \end{pmatrix}, \tag{4.6}$$

where $\Phi(x; \lambda)$ is the fundamental matrix described in (4.4), and we are using the notation just following (4.4). In addition, we expand the notation ℓ_4 to $\ell_4 = \ell_4(\xi)$, where $\ell_4(\xi)$ denotes the Lagrangian subspace with frame

$$\mathbf{X}_4(\xi) = \mathcal{M}\mathcal{J}_{4n}\Theta(\xi)^t.$$

Working with the Lagrangian paths $\ell_3(1; \lambda)$ and $\ell_4(\xi)$, we are in a position to consider a Maslov Box constructed as follows: fix values $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, and likewise $\xi_1, \xi_2 \in \mathcal{I}$, $\xi_1 < \xi_2$. For $\xi = \xi_1$, let λ run from λ_1 to λ_2 (the bottom shelf, $[\lambda_1, \lambda_2]_{\xi=\xi_1}$); for $\lambda = \lambda_2$, let ξ run from ξ_1 to ξ_2 (the right shelf, $[\xi_1, \xi_2]_{\lambda=\lambda_2}$); for $\xi = \xi_2$, let λ run from λ_2 to λ_1 (the top shelf, $[\lambda_2, \lambda_1]_{\xi=\xi_2}$); and for $\lambda = \lambda_1$ let ξ run from ξ_2 to ξ_1 (the left shelf, $[\xi_2, \xi_1]_{\lambda=\lambda_1}$). By path additivity and homotopy invariance we have

$$\mathrm{Mas}(\ell_3(1; \cdot), \ell_4(\xi_1); [\lambda_1, \lambda_2]) + \mathrm{Mas}(\ell_3(1; \lambda_2), \ell_4; [\xi_1, \xi_2])$$
$$- \mathrm{Mas}(\ell_3(1; \cdot), \ell_4(\xi_2); [\lambda_1, \lambda_2]) - \mathrm{Mas}(\ell_3(1; \lambda_1), \ell_4; [\xi_1, \xi_2]) = 0.$$

For any $\xi \in \mathcal{I}$, let $\mathcal{N}([\lambda_1, \lambda_2); \xi)$ denote the spectral count defined in (1.4) for $\Theta = \Theta(\xi)$. If the eigenvalues of $\tilde{W}(1, \lambda; \xi)$ rotate monotonically clockwise as λ increases then for $i = 1, 2$ we have

$$\mathrm{Mas}(\ell_3(1; \cdot), \ell_4(\xi_1); [\lambda_1, \lambda_2]) = -\mathcal{N}([\lambda_1, \lambda_2); \xi_1)$$
$$\mathrm{Mas}(\ell_3(1; \cdot), \ell_4(\xi_2); [\lambda_1, \lambda_2]) = -\mathcal{N}([\lambda_1, \lambda_2); \xi_2).$$

Precisely the same considerations that lead to Theorem 1.2 provide us with Theorem 1.3. Corollary 1.1 follows immediately, as discussed in the introduction.

4.4 Applications

In this section, we review two applications of our framework with general self-adjoint boundary conditions. Given appropriate boundary conditions, all of the examples from Sections 3.4.1, 3.4.2, and 3.4.3 can be adapted to the current setting, so our emphasis will be on the applications that motivated our interest in boundary conditions (BC2) and (BC2)$_\xi$. (The family of examples from Section 3.4.4 on models with convection cannot be adapted to the current setting, because the boundary conditions don't necessarily transform appropriately.)

4.4.1 Schrödinger Operators

Our interest in boundary conditions (BC2) and (BC2)$_\xi$ stems from our references [17, 24], in which the authors consider eigenvalue problems

$$
\begin{aligned}
H_\xi \phi &:= -\phi'' + V(x)\phi = \lambda\phi, \\
\phi(1) &= e^{i\xi}\phi(0), \\
\phi'(1) &= e^{i\xi}\phi'(0),
\end{aligned} \tag{4.7}
$$

where $\phi(x;\lambda) \in \mathbb{C}^n$, $V \in C([0,1]; \mathbb{R}^{n \times n})$ is a symmetric matrix-valued potential, and $\xi \in [-\pi, \pi)$. We take as our domain for H_ξ

$$
\mathcal{D}(H_\xi) = \{\phi \in H^2((0,1)) : (4.7) \text{ holds}\}, \tag{4.8}
$$

and note that with this choice of domain, H_ξ is self-adjoint. In particular, the spectrum of H_ξ is real-valued. If $\lambda \in \mathbb{R}$ is an eigenvalue of H_ξ then (by complex conjugate) λ will also be an eigenvalue of $H_{-\xi}$. In this way, we can focus on the interval $\xi \in [0, \pi]$.

Equations (4.7) arise naturally when a gradient reaction-diffusion system

$$
u_t + F'(u) = u_{xx}; \quad u \in \mathbb{R}^n, \, x \in \mathbb{R}, \, t \geq 0, \tag{4.9}
$$

is linearized about a stationary 1-periodic solution $\bar{u}(x)$. In this case, if we write $u = \bar{u} + v$ we obtain the perturbation equation

$$
v_t + F''(\bar{u})v = v_{xx} + \mathbf{O}(v^2), \tag{4.10}
$$

with associated eigenvalue problem

$$
H\phi := -\phi_{xx} + V(x)\phi = \lambda\phi; \quad V(x) = F''(\bar{u}(x)). \tag{4.11}
$$

By standard Floquet theory the $L^2(\mathbb{R})$ spectrum of H is purely continuous and corresponds with the union of λ so that (4.11) admits a bounded eigenfunction of the Bloch form

$$
\phi(x) = e^{i\xi x} w(x),
$$

for some $\xi \in \mathbb{R}$ and 1-periodic function $w(x)$. The periodicity of w allows us to write

$$
\phi(0) = w(0) = w(1) = e^{-i\xi}\phi(1),
$$

and proceeding similarly for ϕ' we find that the $L^2(\mathbb{R})$ spectrum of H corresponds with the union of λ that are eigenvalues of (4.7).

We can adapt (4.7) to the current setting by expressing ϕ in terms of its real and imaginary parts. Adopting the labeling convention of [24], we write

$$
\phi_k = u_{2k-1} + i u_{2k},
$$

for $k = 1, 2, \ldots, n$. When expressing the resulting system for $u = (u_1, u_2, \ldots, u_{2n})^t$, it is convenient to define the counterclockwise rotation matrix

$$
R_\xi := \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}. \tag{4.12}
$$

Also, for an $m \times n$ matrix $A = (a_{ij})_{i,j=1}^{m,n}$ and a $k \times l$ matrix $B = (b_{ij})_{i,j=1}^{k,l}$ we denote by $A \otimes B$ the *Kronecker product*, by which we mean the $mk \times nl$ matrix with ij block $a_{ij}B$. This allows us to express our equation for u as

$$
\begin{aligned}
\mathcal{H}_\xi u &:= -u'' + (V(x) \otimes I_2)u = \lambda u \\
u(1) &= (I_n \otimes R_\xi)u(0) \\
u'(1) &= (I_n \otimes R_\xi)u'(0).
\end{aligned} \tag{4.13}
$$

In [17], the authors verify that $\lambda \in \mathbb{R}$ is an eigenvalue of H_ξ with multiplicity m if and only if it is an eigenvalue of $\mathcal{H}_\xi$ with multiplicity $2m$.

We can now express (4.13) as a system in the usual way, with $y_1 = u$ and $y_2 = u'$. We find

$$
y' = \mathbb{A}(x; \lambda)y; \quad \mathbb{A}(x; \lambda) = \begin{pmatrix} 0 & I_{2n} \\ V(x) \otimes I_2 - \lambda I_{2n} & 0 \end{pmatrix},
$$
$$
y(1) = (I_{2n} \otimes R_\xi)y(0),
$$

where we note that the appearance of $I_n \otimes R_\xi$ has now been replaced by $I_{2n} \otimes R_\xi$. If we multiply the equation by J_{4n} we arrive at the expected form

$$
J_{4n}y' = \mathbb{B}(x; \lambda)y; \quad \mathbb{B}(x; \lambda) = \begin{pmatrix} \lambda I_{2n} - V(x) \otimes I_2 & 0 \\ 0 & I_{2n} \end{pmatrix}
$$
$$
\Theta(\xi)\begin{pmatrix} y(0) \\ y(1) \end{pmatrix} = 0; \quad \Theta(\xi) = \left(I_{2n} \otimes R_\xi, -I_{4n}\right). \tag{4.14}
$$

(The dimension $4n$ arose naturally in our development, and we simply note that this corresponds with replacing n by $2n$ in our general formulation.) In this case,

$$
\mathbb{B}_\lambda(x; \lambda) = \begin{pmatrix} I_{2n} & 0 \\ 0 & 0 \end{pmatrix},
$$

and we can conclude similarly as in Section 3.4.1 that the monotonicity condition of Theorem 1.1 is satisfied.

To check that $\Theta(\xi)$ satisfies the assumptions of $(BC2)_\xi$, we first note that it's clear from the appearance of I_{4n} that $\Theta(\xi)$ has rank $4n$ (rank $2n$ in the general formulation). We also compute

$$
\Theta(\xi)\mathcal{J}_{8n}\Theta(\xi)^t = -(I_{2n} \otimes R_\xi)J_{4n}(I_{2n} \otimes R_\xi)^t + J_{4n} = 0,
$$

where we have observed that J_{4n} commutes with $(I_{2n} \otimes R_\xi)^t$ and $(I_{2n} \otimes R_\xi)(I_{2n} \otimes R_\xi)^t = I_{4n}$.

These calculations serve to verify that (4.7) can be analyzed within the current framework. The frame for $\ell_3(x; \lambda)$ is precisely as defined in (1.8) with $\mathbb{B}(x; \lambda)$ as in (4.14), while for $\ell_4(\xi)$ we obtain

$$
\mathbf{X}_4(\xi) = \mathcal{M}_{8n}\mathcal{J}_{8n}\Theta(\xi)^t = \begin{pmatrix} 0 & (I_n \otimes R_\xi)^t \\ 0 & I_{2n} \\ (I_n \otimes R_\xi)^t & 0 \\ -I_{2n} & 0 \end{pmatrix}.
$$

For comparison with our reference [17], we observe that we can write

$$\mathbf{X}_4(\xi) = \begin{pmatrix} 0 & (I_n \otimes R_\xi)^t \\ 0 & I_{2n} \\ (I_n \otimes R_\xi)^t & 0 \\ -I_{2n} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I_{2n} \\ 0 & (I_n \otimes R_\xi) \\ I_{2n} & 0 \\ -(I_n \otimes R_\xi) & 0 \end{pmatrix} \begin{pmatrix} (I_n \otimes R_\xi)^t & 0 \\ 0 & (I_n \otimes R_\xi)^t \end{pmatrix},$$

from which it's clear that the frame $\mathbf{X}_4(\xi)$ given here corresponds with the same Lagrangian subspace as the corresponding frame in [17]. (The left-hand matrix in this final matrix product is precisely the frame used in [17].)

Fixing any $\xi \in [0, \pi]$, we can apply Theorem 1.2 to conclude that for any pair $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, we have

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell_3(\cdot; \lambda_2), \ell_4(\xi); [0, 1]) + \operatorname{Mas}(\ell_3(\cdot; \lambda_1), \ell_4(\xi); [0, 1]).$$

As discussed in [17] (and as can be verified by a straightforward energy argument), $\mathcal{H}_\xi$ will not have any eigenvalues below $-\|V\|_{L^\infty(\mathbb{R})}$ (for any $\xi \in \mathbb{R}$). If we choose any $\lambda = -\lambda_\infty < -\|V\|_{L^\infty(\mathbb{R})}$ and any $\lambda_0 > -\|V\|_{L^\infty(\mathbb{R})}$ then the total number of eigenvalues of $\mathcal{H}_\xi$ at or below λ_0 is precisely the negative of the Maslov index on the top shelf. Following [17] we denote this value

$$\operatorname{Mor}(\mathcal{H}_\xi; \lambda_0) = -\operatorname{Mas}(\ell_3(1; \cdot), \ell_4(\xi); [-\lambda_\infty, \lambda_0]).$$

I.e.,

$$\mathcal{N}([-\lambda_\infty, \lambda_0); \xi) = \mathcal{N}((-\infty, \lambda_0); \xi) =: \operatorname{Mor}(\mathcal{H}_\xi; \lambda_0).$$

It now follows from Theorem 1.2 that

$$\operatorname{Mor}(\mathcal{H}_\xi; \lambda_0) = -\operatorname{Mas}(\ell_3(\cdot; \lambda_0), \ell_4(\xi); [0, 1]) + \operatorname{Mas}(\ell_3(\cdot; -\lambda_\infty), \ell_4(\xi); [0, 1]),$$

where we have recalled that the Maslov index on the bottom shelf is 0.

In this case, we can take further advantage of the nature of (4.7) and compute the Maslov index $\operatorname{Mas}(\ell_3(\cdot; -\lambda_\infty), \ell_4(\xi); [1, 0])$ explicitly. First, it's shown in [17] that there can be no crossings along $[1, 0]_{\lambda = -\lambda_\infty}$, except possibly corresponding with an arrival or departure at $x = 0$. In this case (i.e., for $x = 0$) we have explicitly

$$\mathbf{X}_3(0; \lambda) = \begin{pmatrix} I_{2n} & 0 \\ I_{2n} & 0 \\ 0 & -I_{2n} \\ 0 & I_{2n} \end{pmatrix},$$

from which we find

$$\tilde{W}(0; \lambda, \xi) = -\begin{pmatrix} I_n \otimes R_\xi & 0 \\ 0 & (I_n \otimes R_\xi)^t \end{pmatrix}.$$

25

The eigenvalues of this matrix, which we denote μ, satisfy

$$\det\left\{\left(-(I_n \otimes R_\xi) - \mu I_{2n}\right)\left(-(I_n \otimes R_\xi)^t - \mu I_{2n}\right)\right\}$$
$$= \det\left\{I_{2n}(\mu^2 + 2\mu \cos\xi + 1)\right\},$$

where we have recalled the relation $(I_n \otimes R_\xi)^t(I_n \otimes R_\xi) = I_{2n}$ and observed the relation $(I_n \otimes R_\xi)^t + (I_n \otimes R_\xi) = 2(\cos\xi)I_{2n}$. We conclude that the $4n$ eigenvalues of $\tilde{W}(0; \lambda, \xi)$ are

$$\mu = -\cos\xi \pm \sqrt{\cos^2\xi - 1} = -\cos\xi \pm i\sin\xi,$$

each repeated $2n$ times.

We see that for $\xi = 0$ all $4n$ eigenvalues reside at -1, while for any $\xi \in (0, \pi]$ none of the eigenvalues reside at -1 (for $\xi = \pi$, all $4n$ eigenvalues reside at $+1$). It follows that for $\xi \in (0, \pi]$, no eigenvalue of $\tilde{W}(x; -\lambda_\infty)$ can arrive at -1 as $x \to 0^+$, and since no eigenvalue of $\tilde{W}(x; -\lambda_\infty)$ can be -1 for any $x \in (0, 1]$ (as observed above) we conclude that for any $\xi \in (0, \pi]$ we must have $\mathrm{Mas}(\ell_3(\cdot; -\lambda_\infty), \ell_4(\xi); [0, 1]) = 0$. For $\xi = 0$ we need to understand the rotation of the eigenvalues of $\tilde{W}(x; -\lambda_\infty)$ as x approaches 0, keeping in mind that we already know that none can reside at -1 for $x \in (0, 1]$. For this, we recall from Section 4.2 that the rotation at $x = 0$ will be determined by the nature of $\Omega(0; \lambda) = \mathbb{B}(0; \lambda)$. In this case

$$\mathbb{B}(0; \lambda) = \begin{pmatrix} \lambda I_{2n} - V(0) \otimes I_2 & 0 \\ 0 & I_{2n} \end{pmatrix}.$$

For $\lambda = -\lambda_\infty < 0$, with $|\lambda_\infty|$ sufficiently large, the matrix $\mathbb{B}(0; \lambda)$ has $2n$ negative eigenvalues and $2n$ positive eigenvalues. We can conclude from (the proof of) Lemma 3.11 in [20] that as x increases from 0, $2n$ eigenvalues will rotate from -1 in the clockwise direction and $2n$ eigenvalues will rotate from -1 in the counterclockwise direction. We conclude that in this case $\mathrm{Mas}(\ell_3(\cdot; -\lambda_\infty), \ell_4(0); [0, 1]) = -2n$ (corresponding with the $2n$ eigenvalues that arrive at -1 in the counterclockwise direction as x *decreases* to 0).

For convenient reference, we record these observations as a proposition.

Proposition 4.1. *Suppose $V \in C([0, 1]; \mathbb{R}^{n \times n})$ is a symmetric matrix-valued potential, $\mathcal{H}_\xi$ is as in (4.13) for some $\xi \in [0, \pi]$, and $\mathrm{Mas}(\ell_3(\cdot, \lambda_0), \ell_4(\xi); [0, 1])$ is defined as in this section. Then*

$$\mathrm{Mor}(\mathcal{H}_\xi; \lambda_0) = -\mathrm{Mas}(\ell_3(\cdot; \lambda_0), \ell_4(\xi); [0, 1]) - \begin{cases} 2n & \xi = 0 \\ 0 & \xi \in (0, \pi] \end{cases}.$$

For each fixed $\xi \in [0, \pi]$, Proposition 4.1 provides a computationally efficient way to determine the number of eigenvalues that $\mathcal{H}_\xi$ has at or below a fixed threshold λ_0. Since each eigenvalue of H_ξ with multiplicity m is an eigenvalue of $\mathcal{H}_\xi$ with multiplicity $2m$, we obtain a count of the number of eigenvalues that H_ξ has below λ_0.

Suppose that we have carried out this calculation for some particular value of ξ_0 so that $\mathrm{Mor}(\mathcal{H}_{\xi_0}; \lambda_0)$ is known. We ask the following question: can we find $\mathrm{Mor}(\mathcal{H}_\xi; \lambda_0)$ for each value $\xi \in (\xi_0, \xi]$ without recomputing solutions of (4.7)? Theorem 1.3 has been formulated with precisely this question in mind. In particular, it allows us to write (for $\xi_0 < \xi$)

$$\mathrm{Mor}(\mathcal{H}_\xi; \lambda_0) = \mathrm{Mor}(\mathcal{H}_{\xi_0}; \lambda_0) - \mathrm{Mas}(\ell_3(1; \lambda_0), \ell_4; [\xi_0, \xi]),$$

where we have observed that for λ_∞ sufficiently large

$$\mathrm{Mas}(\ell_3(1; -\lambda_\infty), \ell_4; [\xi_0, \xi]) = 0. \tag{4.15}$$

Detailed calculations are carried out numerically in [17] for operators H_ξ obtained when Allen-Cahn equations and systems are linearized about stationary periodic solutions.

4.4.2 Euler-Bernoulli Systems

We consider Euler-Bernoulli systems

$$(A(x)\phi'')'' = \lambda R(x)\phi, \tag{4.16}$$

where $A \in C^2(\mathbb{R}; \mathbb{R}^{n\times n})$, $R \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ and both are 1-periodic and symmetric. In addition, we assume $A(x)$ is invertible for all $x \in [0, 1]$, and that $R(x)$ is positive definite for all $x \in [0, 1]$. The case $n = 1$ has been analyzed in considerable detail by Papanicolaou (see [33] and the references therein).

Similarly as in the case of the Schrödinger operator, we know from standard Floquet theory that the $L^2(\mathbb{R})$ spectrum of (4.16) is purely continuous and corresponds with the union of λ so that (4.16) admits a bounded eigenfunction of the Bloch form

$$\phi(x) = e^{i\xi x}w(x),$$

for some $\xi \in \mathbb{R}$ and 1-periodic function $w(x)$. As discussed in the previous section, this leads to the boundary conditions

$$\phi^{(k)}(1) = e^{i\xi}\phi^{(k)}(0); \quad k = 0, 1, 2, 3.$$

Using again the convention $\phi_k = u_{2k-1} + iu_{2k}$, we obtain the system

$$\begin{aligned} ((A(x) \otimes I_2)u'')'' &= \lambda(R(x) \otimes I_2)u \\ u^{(k)}(1) &= (I_n \otimes R_\xi)u^{(k)}(0); \quad k = 0, 1, 2, 3. \end{aligned} \tag{4.17}$$

Aside from the boundary conditions, equation (4.17) is a special case of (3.2) with $V_4(x) = A(x) \otimes I_2$, $V_2(x) \equiv 0$, $V_0(x) \equiv 0$, and $Q(x) = R(x) \otimes I_2$.

Following our general framework for (3.2) we express (4.17) as a system by writing $y_1 = u$, $y_2 = (A(x) \otimes I_2)u''$, $y_3 = -((A(x) \otimes I_2)u'')'$, and $y_4 = -u'$ so that

$$y' = \mathbb{A}(x; \lambda)y; \quad \mathbb{A}(x; \lambda) = \begin{pmatrix} 0 & 0 & 0 & -I_{2n} \\ 0 & 0 & -I_{2n} & 0 \\ -\lambda(R(x) \otimes I_2) & 0 & 0 & 0 \\ 0 & -A(x)^{-1} \otimes I_2 & 0 & 0 \end{pmatrix},$$

$$y(1) = (I_{4n} \otimes R_\xi)y(0),$$

where we have observed that $(A(x) \otimes I_2)^{-1} = A(x)^{-1} \otimes I_2$. If we multiply our equation by J_{8n} we obtain our form (1.1)

$$J_{8n}y' = \mathbb{B}(x; \lambda)y; \quad \mathbb{B}(x; \lambda) = \begin{pmatrix} \lambda(R(x) \otimes I_2) & 0 & 0 & 0 \\ 0 & A(x)^{-1} \otimes I_2 & 0 & 0 \\ 0 & 0 & 0 & -I_{2n} \\ 0 & 0 & -I_{2n} & 0 \end{pmatrix},$$

$$\Theta(\xi)\begin{pmatrix} y(0) \\ y(1) \end{pmatrix} = 0; \quad \Theta(\xi) = \left(I_{4n} \otimes R_\xi, -I_{8n} \right).$$

27

We have already seen that $\Theta(\xi)$ satisfies the assumptions in $(BC2)_\xi$, so our general framework applies in this case.

References

[1] V. I. Arnold, *Characteristic class entering in quantization conditions,* Func. Anal. Appl. **1** (1967) 1 – 14.

[2] V. I. Arnold, *The Sturm theorems and symplectic geometry,* Func. Anal. Appl. **19** (1985) 1–10.

[3] M. Artin, *Algebra*, Prentice-Hall 1991.

[4] B. Booss-Bavnbek and K. Furutani, *The Maslov index: a functional analytical definition and the spectral flow formula,* Tokyo J. Math. **21** (1998), 1–34.

[5] G. Berkolaiko and P. Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs **186**, AMS 2013.

[6] M. Beck and S. Malham, *Computing the Maslov index for large systems*, Proc. Amer. Math. Soc. 143 (2015), no. 5, 2159–2173.

[7] F. Chardard, F. Dias and T. J. Bridges, *Computing the Maslov index of solitary waves. I. Hamiltonian systems on a four-dimensional phase space*, Phys. D **238** (2009) 1841 – 1867.

[8] F. Chardard, F. Dias and T. J. Bridges, *Computing the Maslov index of solitary waves, Part 2: Phase space with dimension greater than four.* Phys. D **240** (2011) 1334 – 1344.

[9] C-N Chen and X. Hu, *Maslov index for homoclinic orbits in Hamiltonian systems*, Ann. I. H. Poincaré – AN **24** (2007) 589 – 603.

[10] F. Chardard, *Stability of Solitary Waves*, Doctoral thesis, Centre de Mathematiques et de Leurs Applications, 2009. Advisor: T. J. Bridges.

[11] G. Cox, Christopher K. R. T. Jones, Y. Latushkin, A. Sukhtayenv, *The Morse and Maslov indices for multidimensional Schrödinger operators with matrix-valued potentials*, Trans. Amer. Math. Soc. 368 (2016), no. 11, 8145–8207.

[12] S. Cappell, R. Lee and E. Miller, *On the Maslov index*, Comm. Pure Appl. Math. **47** (1994), 121–186.

[13] J. Deng and C. Jones, *Multi-dimensional Morse Index Theorems and a symplectic view of elliptic boundary value problems,* Trans. Amer. Math. Soc. **363** (2011) 1487 – 1508.

[14] H. I. Dwyer and A. Zettl, *Computing eigenvalues of regular Sturm-Liouville problems*, Electronic J. Differential Equations **1994** (1994) 1–10.

[15] K. Furutani, *Fredholm-Lagrangian-Grassmannian and the Maslov index,* Journal of Geometry and Physics **51** (2004) 269 – 331.

[16] R. Fabbri, R. Johnson and C. Núñez, *Rotation number for non-autonomous linear Hamiltonian systems I: Basic properties,* Z. angew. Math. Phys. **54** (2003) 484 – 502.

[17] P. Howard and S. Jung, *Spectral analysis of θ-periodic Schrödinger operators and applications to periodic waves*, Preprint 2017.

[18] P. Howard, Y. Latushkin, and A. Sukhtayev, *The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$*, Journal of Mathematical Analysis and Applications **451** (2017) 794–821.

[19] P. Howard, Y. Latushkin, and A. Sukhtayev, *The Maslov and Morse indices for Schrödinger operators on $\mathbb{R}$*, to appear in Indiana U. Math. J.

[20] P. Howard and A. Sukhtayev, *The Maslov and Morse indices for Schrödinger operators on $[0, 1]$*, J. Differential Equations **260** (2016) 4499–4559.

[21] C. K. R. T. Jones, *Instability of standing waves for nonlinear Schrödinger-type equations*, Ergodic Theory Dynam. Systems **8** (1988) 119 – 138.

[22] C. K. R. T. Jones, *An instability mechanism for radially symmetric standing waves of a nonlinear Schrödinger equation*, J. Differential Equations **71** (1988) 34 – 62.

[23] C. K. R. T. Jones, Y. Latushkin, and R. Marangell, *The Morse and Maslov indices for matrix Hill's equations*, Proceedings of Symposia in Pure Mathematics **87** (2013) 205–233.

[24] C. K. R. T. Jones, Y. Latushkin, and S. Sukhtaiev, *Counting spectrum via the Maslov index for one dimensional θ-periodic Schrödinger operators*, Proceedings of the AMS **145** (2017) 363–377.

[25] C. K. R. T. Jones and R. Marangell, *The spectrum of travelling wave solutions to the Sine-Gordon equation*, Discrete and Cont. Dyn. Sys. **5** (2012) 925 – 937.

[26] P. Kuchment, *Quantum graphs: I. Some basic structures*, Waves in random media **14**.

[27] Y. Latushkin, A. Sukhtayev, and S. Sukhtaiev, *The Morse and Maslov indices for Schrödinger operators*, Preprint 2014, arXiv 1411.1656.

[28] V. P. Maslov, *Theory of perturbations and asymptotic methods*, Izdat. Moskov. Gos. Univ. Moscow, 1965. French tranlation Dunod, Paris, 1972.

[29] K. Meyer, G. Hall, and D. Offin, *Introduction to Hamiltonian dynamical systems and the N-body problem*, 2nd Ed., Springer 2009.

[30] D. Offin, *A spectral theorem for reversible second order equations with periodic coefficients*, Diff. Int. Eqns. **5** (1992) 615-629.

[31] D. Offin, *Variational structure of the zones of instability*, Diff. Int. Eqns. **14** (2001) 1111-1127.

[32] J. Phillips, Selfadjoint Fredholm operators and spectral flow, *Canad. Math. Bull.* **39** (1996), 460–467.

[33] V. G. Papanicolaou, *The inverse periodic spectral theory of the Euler-Bernoulli equation*, Dynamics of PDE **2** (2005) 127–148.

[34] J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993) 827 – 844.