Asymptotic stability analysis for transition front solutions in Cahn–Hilliard systems

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Abstract

We consider the asymptotic behavior of perturbations of transition front solutions arising in Cahn–Hilliard systems on $\mathbb{R}$. Such equations arise naturally in the study of phase separation, and systems describe cases in which three or more phases are possible. When a Cahn–Hilliard system is linearized about a transition front solution, the linearized operator has an eigenvalue at 0 (due to shift invariance), which is not separated from essential spectrum. In cases such as this, nonlinear stability cannot be concluded from classical semigroup considerations and a more refined development is appropriate. Our main result asserts that spectral stability—a necessary condition for stability, defined in terms of an appropriate Evans function—implies nonlinear stability.

1 Introduction

We consider the stability of transition front solutions $\bar{u}(x)$, $\bar{u}(\pm\infty) = u_\pm$, $u_- \neq u_+$, for Cahn–Hilliard systems on $\mathbb{R}$,

$$u_t = \Big(M(u)(-\Gamma u_{xx} + f(u))_x\Big)_x, \tag{1.1}$$

where $u, f \in \mathbb{R}^m$, m an integer greater than or equal to 2 ($m+1$ phases are possible) and $M, \Gamma \in \mathbb{R}^{m\times m}$. A brief discussion of the history and physicality of this equation is given in [10], and reasonable (physical) choices for f, M, and Γ are also discussed. We omit such a discussion here, but state, for convenient reference, the assumptions of [10], which we will assume throughout this paper.

(H0) (Assumptions on Γ) Γ denotes a constant, symmetric, positive definite matrix.

(H1) (Assumptions on f) $f \in C^3(\mathbb{R}^m)$, and f has at least two zeros on $\mathbb{R}^m$. For convenience we denote this set

$$\mathcal{M} := \{u \in \mathbb{R}^m : f(u) = 0\}. \tag{1.2}$$

(H2) (Transition front existence and structure) There exists a transition front solution to (1.1) $\bar{u}(x)$, so that

$$-\Gamma \bar{u}_{xx} + f(\bar{u}) = 0, \tag{1.3}$$

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$. When (1.3) is written as a first order autonomous ODE system $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

(H3) (Assumptions on M and Γ) $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the front; i.e., there exists $\theta > 0$ so that for all $\xi \in \mathbb{R}^m$ and all $x \in \mathbb{R}$ we have

$$\xi^{tr} M(\bar{u}(x))\xi \geq \theta |\xi|^2.$$

(H4) (Symmetry and Endstate Assumptions) We assume the $m \times m$ Jacobian matrix $f'(\bar{u}(x))$ is symmetric for all $x \in \mathbb{R}$. Setting $B_\pm := f'(u_\pm)$ and $M_\pm := M(u_\pm)$, we assume $B_\pm$ and $M_\pm$ are both symmetric and positive definite. (Of course, $M_\pm$ is already positive definite from (H3).) In addition, we assume that for each of the matrices $M_\pm B_\pm$ and $\Gamma^{-1}B_\pm$, the spectrum is distinct except possibly for repeated eigenvalues that have an associated eigenspace with dimension equal to eigenvalue multiplicity. In the case of repeated eigenvalues, we assume additionally that the solutions μ of

$$\det\Big(- \mu^4 M_\pm \Gamma + \mu^2 M_\pm B_\pm - \lambda I \Big) = 0$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\lambda)$ can be written as $\mu(\lambda) = \sqrt{\lambda} h(\lambda)$, where h is analytic in λ for $|\lambda|$ sufficiently small.

Regarding (H1) we observe that for Cahn-Hilliard systems we can often write f as the gradient of an appropriate bulk free energy density F (i.e. $f(u) = F'(u)$), where F has $m+1$ local minima on $\mathbb{R}^m$. In this way, it's natural for f to have precisely $m+1$ zeros. Since F would appear in (1.1) with both a u and an x derivative, we can subtract from it any affine function without changing (1.1). It is often convenient to subtract a supporting hyperplane from F so that F is also 0 on $\mathcal{M}$.

Regarding (H4), we first observe that the symmetry condition on $f'(\bar{u}(x))$ is natural since $F''(u)$ is a Hessian matrix. Also, we note that we can ensure that our system satisfies the determinant condition by taking arbitrarily small perturbations of the matrices M and Γ. Since we expect stability to be insensitive to such perturbations, we view this assumption as purely for technical convenience. In particular, our estimates of Lemma 2.1 would take a more complicated form if we removed them. Generally speaking, (H0)-(H4) hold for physically relevant choices of Γ, M, and f; particular examples can be found in [10].

When the Cahn–Hilliard system (1.1) is linearized about a standing wave solution $\bar{u}(x)$, as described in (H2), the resulting linear equation is

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x, \tag{1.4}$$

where (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$ and $B(x) := f'(\bar{u}(x))$. Assumptions (H0)–(H4) imply the following (stated with some redundancy so that these assumptions can be referred to independently of (H0)-(H4)):

(C1) $B \in C^2(\mathbb{R})$; there exists a constant $\alpha_B > 0$ so that

$$\partial_x^j (B(x) - B_\pm) = \mathbf{O}(e^{-\alpha_B |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both positive definite matrices.

(C2) $M \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial_x^j (M(x) - M_\pm) = \mathbf{O}(e^{-\alpha_M |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; M is uniformly positive definite on $\mathbb{R}$; Γ denotes a constant, symmetric, positive definite matrix. We will set $\alpha = \min\{\alpha_B, \alpha_M\}$.

The eigenvalue problem associated with (1.4) has the form

$$L\phi := \Big(M(x)(-\Gamma\phi'' + B(x)\phi)'\Big)' = \lambda\phi. \tag{1.5}$$

In many cases it's possible to verify that the only non-negative eigenvalue for this equation is $\lambda = 0$ (see, for example, [1, 2, 16] and our companion spectral paper [10]), and so stability depends entirely on the nature of this neutral eigenvalue. In [10], we identify an appropriate stability condition for this leading eigenvalue. Briefly, this condition is constructed in terms of the asymptotically growing/decaying solutions of (1.5). For $|\lambda| > 0$ sufficiently small, and $\mathrm{Arg}\lambda \neq \pi$ (i.e., excluding negative real numbers), there are $2m$ linearly independent solutions of (1.5) that decay as $x \to -\infty$ and $2m$ linearly independent solutions of (1.5) that decay as $x \to +\infty$. Moreover, these functions can be constructed so that they are analytic in $\rho = \sqrt{\lambda}$. If we denote these functions $\{\phi_j^\pm(x;\rho)\}_{j=1}^{2m}$ and set $\Phi_j^\pm = (\phi_j^\pm, \phi_j^{\pm\prime}, \phi_j^{\pm\prime\prime}, \phi_j^{\pm\prime\prime\prime})^{\mathrm{tr}}$, the Evans function can be expressed as

$$D_a(\rho) = \det(\Phi_1^+, \dots, \Phi_{2m}^+, \Phi_1^-, \dots, \Phi_{2m}^-)\Big|_{x=0}. \tag{1.6}$$

In terms of this function the stability condition of [10] can be stated as follows:

Condition 1.1. *The set $\sigma(L)\backslash\{0\}$ lies entirely in the negative half-plane $Re\lambda < 0$, and*

$$\frac{d^{m+1}}{d\rho^{m+1}} D_a(\rho)\Big|_{\rho=0} \neq 0.$$

Remark 1.1. *As discussed in Section 3 of [10], our assumptions (H0)-(H4) ensure that the essential spectrum of L (defined here as any value that is neither in the point spectrum nor the resolvent set of L) is confined to the negative real axis* $(-\infty, 0]$. *(This follows immediately from our assumptions that* Γ*,* $B_\pm$*, and* $M_\pm$ *are all symmetric and positive definite.) In addition, our straightforward energy estimate in Section 4 shows that Condition 1.1 implies that aside from the leading eigenvalue* $\lambda = 0$ *the point spectrum of L is bounded to the left of a wedge with vertex on the negative real axis:*

$$\Gamma_\theta := \{\lambda : Re\ \lambda = -\theta_1 - \theta_2 |Im\ \lambda|\} \tag{1.7}$$

for some positive values θ_1*,* θ_2 *sufficiently small. If we make one additional natural assumption, that* $M(\bar{u}(x))$ *is symmetric for all* $x \in \mathbb{R}$*, we can ensure that the point spectrum of L is entirely real-valued. Finally, we verify in [10] that*

$$D_a(0) = D_a'(0) = \cdots = D_a^{(m)}(0) = 0$$

(see also the brief note in Section 2 of the current paper.)

Our main goal in the current analysis is to establish that Condition 1.1 is sufficient to guarantee nonlinear (phase-asymptotic) stability for the front $\bar{u}(x)$. We employ the pointwise Green's function approach of [6, 7, 19], along with the local tracking developed in [12].

Generally, if the initial value for (1.1) is taken as a small perturbation of $\bar{u}(x)$, the solution $u(t, x)$ will approach a shift of $\bar{u}(x)$ rather than the front itself (orbital stability). Following [12], we proceed by tracking this shift locally in time, our location denoted by $\delta(t)$, which is standard notation in the literature and should not be confused with a Dirac delta function. More precisely, we include this shift in our analysis by defining our perturbation $v(t, x)$ as

$$v(t, x) := u(t, x + \delta(t)) - \bar{u}(x). \tag{1.8}$$

At this point, $\delta(t)$ is yet undetermined, and indeed one of the most important aspects of our approach to this problem is that it allows us to make an effective choice of $\delta(t)$. Upon substitution of $u(t, x + \delta(t))$ into (1.1) we obtain the perturbation equation

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x + \bar{u}'(x)\dot{\delta}(t) + v_x\dot{\delta}(t) + Q_x, \tag{1.9}$$

where $Q = Q(x, v, v_x, v_{xxx})$ is at least C^2 in all its variables, and if

$$|v| + |v_x| + |v_{xxx}| \leq \tilde{C}$$

for some constant $\tilde{C}$, then there exists a constant C so that

$$|Q| \leq C\Big(|v||v_x| + e^{-\alpha|x|}|v|^2 + |v||v_{xxx}|\Big), \tag{1.10}$$

where α is described in (C1)-(C2) above. On one hand, this is a beneficial nonlinearity, because $|v_x|$ and $|v_{xxx}|$ will generally decay faster than $|v|$ as $|x|$ or t tends to ∞, and so each of these bounds is better than the standard nonlinearity $|v|^2$ encountered in the analysis of viscous conservation laws. On the other hand, for small values of t, derivatives of v generally blow up, and v_{xxx} is problematic in this regard. Our short time analysis of Section 7.1 is designed primarily to address this difficulty.

Let $G(t, x; y)$ denote the Green's function associated with the linear equation $v_t = Lv$, where L is as in (1.5), so that, in the standard distributional sense,

$$\begin{aligned} G_t &= LG \\ G(0, x; y) &= \delta_y(x) I, \end{aligned} \tag{1.11}$$

where I denotes an $m \times m$ identity matrix, and of course $\delta_y(x)$ is a standard Dirac delta function. Integrating (1.9), we find

$$\begin{aligned} v(t, x) = &\int_{-\infty}^{+\infty} G(t, x; y) v_0(y) dy + \delta(t) \bar{u}'(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s, x; y) \Big[\dot{\delta}(s) v(s, y) + Q(s, y) \Big] dy ds, \end{aligned} \tag{1.12}$$

where in deriving this equation we have (1) observed that since $\bar{u}'(x)$ is a stationary solution for $v_t = Lv$ we must have $e^{Lt} \bar{u}'(x) = \bar{u}'(x)$; (2) assumed our eventual choice of $\delta(t)$ has the natural property $\delta(0) = 0$; and (3) integrated the standard nonlinear integral by parts. To be clear, we do not assume at this stage that solutions of (1.12) are necessarily solutions of (1.9). Rather, our approach will be to work directly with (1.12) and use our estimates on G and v to establish the correspondence. We consider the condition $\delta(0) = 0$ to be natural, because $\delta(t)$ should capture the shift obtained as perturbation mass accumulates near the transition layer, and generally this accumulation will take some time.

We remark that (1.12) can be expressed in terms of the semigroup e^{Lt} as

$$v(t, x) = e^{Lt} v_0 + \delta(t) \bar{u}'(x) + \int_0^t e^{L(t-s)} \Big[\dot{\delta}(s) v(s, \cdot) + Q(s, \cdot) \Big]_y ds. \tag{1.13}$$

Using the resolvent kernel estimates we derive in Section 3.2, we can verify that $-L$ is sectorial on $L^p(\mathbb{R})$, $1 \le p < \infty$, and so generates an analytic semigroup. (See, for example, [13] or [15].) If we assume additional regularity on f and M we can ensure $-L$ is sectorial on $W^{k,p}$ spaces: precisely, if $f \in C^{3+k}$ and $M \in C^{1+k}$ then $-L$ will be sectorial on $W^{k,p}$. This serves to establish the spectral representation (inverse Laplace transform)

$$e^{Lt} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I - L)^{-1} d\lambda, \tag{1.14}$$

where Γ is a contour in the resolvent set of L, entirely to the right of $\sigma(L)$, so that $\arg \lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. (In fact, we can relax this last condition to the extent allowed by analyticity and Cauchy's Theorem.) If ϕ is in an appropriate Banach space, such as those listed above, then

$$(\lambda I - L)^{-1}\phi = \int_{-\infty}^{+\infty} G_\lambda(x,y)\phi(y)dy, \tag{1.15}$$

where G_λ denotes the resolvent kernel associated with L. We have, then,

$$e^{Lt}\phi = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\int_{-\infty}^{+\infty} G_\lambda(x,y)\phi(y)dyd\lambda. \tag{1.16}$$

Our estimates on G_λ, derived in Section 3, will verify that we can exchange the order of integration, and so we have

$$e^{Lt}\phi = \int_{-\infty}^{+\infty} G(t,x;y)\phi(y)dy,$$

where

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)d\lambda. \tag{1.17}$$

More directly, we can employ our estimates on $G_\lambda(x,y)$ to show that if G is defined as in (1.17) then (1.11) can be verified directly.

In our detailed analysis of G (carried out in Section 5) we proceed by decomposing G into two parts, an *excited* term E that does not decay as $t\to\infty$ (and is associated with the leading eigenvalue $\lambda = 0$), and a higher order term $\tilde{G}(t,x;y)$ that does decay as $t\to\infty$. This approach, following [8, 12, 17, 19] and others, will allow us to choose our shift $\delta(t)$. We will find that E can be written as $E(t,x;y) = \bar{u}'(x)e(t,y)$, and so we can express G as

$$G(t,x;y) = \bar{u}'(x)e(t;y) + \tilde{G}(t,x;y), \tag{1.18}$$

so that (1.12) becomes

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t\int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\ &+ \bar{u}'(x)\Big\{\delta(t) + \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy - \int_0^t\int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds\Big\}. \end{aligned} \tag{1.19}$$

Our goal will be to choose $\delta(t)$ in such a way that the entire expression multiplying $\bar{u}'(x)$ in (1.19) is annihilated. That is, we would like $\delta(t)$ to solve the integral equation

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy + \int_0^t\int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds. \tag{1.20}$$

In principle now, we would like to establish existence of v, along with a bound on asymmptotic behavior, by closing an iteration on (1.12). For such an argument we must be clear about which functions must be carried through the iteration and which can be analyzed after the iteration, using the obtained bounds. Of particular importance in this regard, $\delta(t)$ does not appear directly in (1.12), and so it suffices to couple (1.12) with an equation for $\dot{\delta}(t)$, rather than for $\delta(t)$ itself. (Of course, v depends on δ, and we will accomodate that dependence with our short-time theory; see Section 7.2.) Afterward, estimates on $\delta(t)$ can be obtained directly from (1.20). Also, the nonlinearity Q depends on v_x and v_{xxx} (in addition, of course, to dependence on x and v), and so we must either couple (1.12) with integral equations for these functions or obtain estimates on them in terms of the functions we do iterate. It's straightforward to show that v_{xxx} can be bounded in terms of x, v_x, and $\delta(t)$ for t bounded away from 0, and can easily be estimated for t near 0, and so our approach will be to iterate with the variables v, v_x, and $\dot{\delta}(t)$, and to obtain estimates on v_{xxx} and $\delta(t)$ after the iteration. (Though the connection between v_x and v_{xxx} will be used during the course of the iteration.) In this way, we will carry out an iteration on the $2m+1$ integral equations,

$$
\begin{aligned}
v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\
v_x(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\
\dot{\delta}(t) &= -\int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds.
\end{aligned}
\tag{1.21}
$$

Our first result regards estimates on $G(t,x;y)$ and its derivatives. In expressing this theorem, it's convenient to separate short and long time behavior by taking a function $\varrho(t)$ so that $\varrho \in C^\infty[0,\infty)$, with $\varrho(t) = 0$ for $0 \le t < T_1$ and $\varrho(t) = 1$ for $t \ge T_2$, where T_1 and T_2 are any constants so that $0 < T_1 < T_2$. Simply to be specific, we take

$$\varrho(t) := \varphi_\epsilon * \chi(t),$$

where φ is the standard mollifier

$$\varphi(t) := \begin{cases} Ce^{\frac{1}{t^2-1}} & |t| < 1 \\ 0 & |t| \ge 1, \end{cases}$$

(C chosen so that $\int_{\mathbb{R}} \varphi(t)dt = 1$), $\varphi_\epsilon(t) := \frac{1}{\epsilon}\varphi(\frac{t}{\epsilon})$, and $\chi(t)$ denotes a characteristic function on $[\frac{1}{2},\infty)$. Taking $\epsilon = \frac{1}{4}$, we obtain $T_1 = \frac{1}{4}$ and $T_2 = \frac{3}{4}$.

Theorem 1.1. *Suppose Conditions (C1)-(C2) hold, and also that spectral Condition 1.1 holds. Then given any time thresholds $T_1 > 0$ and $T_2 > 0$ there exist constants $\eta > 0$ (sufficiently small), and $C > 0$, $K > 0$, $M > 0$ (sufficiently large) so that the Green's function described in (1.11) can be bounded as follows: there exists a splitting*

$$G(t,x;y) = \bar{u}'(x)e(t;y) + \tilde{G}(t,x;y),$$

so that for $y<0$:

(I) (Excited terms)

(i) Main estimates:

$$
\begin{aligned}
e(t;y) &= \Big(\frac{2}{\sqrt{\pi}}\sum_{j=m+1}^{2m} c_j^-\tilde r_j^-(0)\int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2}dz + R_e(t;y)\Big)\varrho(t)\\
e_y(t;y) &= \Big(\sum_{j=m+1}^{2m}\frac{c_j^-\tilde r_j^-(0)}{\sqrt{\beta_{j-m}^-\pi t}}e^{-\frac{y^2}{4\beta_{j-m}^- t}} + \partial_y R_e(t;y)\Big)\varrho(t)
\end{aligned}
$$

where

$$
\begin{aligned}
|R_e(t,y)| &\le Ct^{-1/2}e^{-y^2/Mt}\\
|\partial_y R_e(t,y)| &\le C\Big(t^{-1}e^{-y^2/Mt} + t^{-1/2}e^{-y^2/Mt}e^{-\eta|y|}\Big).
\end{aligned}
$$

For brevity the (constant) values $\{\beta_j^-\}_{j=1}^m$ and $\{c_j^-\}_{j=m+1}^{2m}$, and the vectors $\{\tilde r_j^-(0)\}_{j=m+1}^{2m}$ are specified in a remark following the theorem statement.

(ii) Time derivatives:

$$
\begin{aligned}
\Big|e_t(t;y)\Big| &\le C(1+t)^{-1}e^{-\frac{y^2}{Mt}}\\
\Big|e_{yt}(t;y)\Big| &\le C(1+t)^{-3/2}e^{-\frac{y^2}{Mt}}.
\end{aligned}
$$

(II) For $|x-y|\le Kt$ and $t\ge T_1$

$$
\begin{aligned}
\Big|\tilde G(t,x;y)\Big| &\le Ct^{-1/2}e^{-\frac{(x-y)^2}{Mt}}\\
\Big|\tilde G_y(t,x;y)\Big| &\le Ct^{-1}e^{-\frac{(x-y)^2}{Mt}}\\
\Big|\tilde G_x(t,x;y)\Big| &\le C\Big[t^{-1/2}e^{-\eta|x|} + t^{-1}\Big]e^{-\frac{(x-y)^2}{Mt}}\\
\Big|\tilde G_{xy}(t,x;y)\Big| &\le C\Big[t^{-1}e^{-\eta|x|}e^{-\frac{y^2}{Mt}} + t^{-1}e^{-\eta|x-y|} + t^{-3/2}e^{-\frac{(x-y)^2}{Mt}}\Big].
\end{aligned}
$$

(III) For $|x-y|\ge Kt$ or $0<t<T_2$

$$
\Big|\partial^\alpha\tilde G(t,x;y)\Big| \le C\Big[t^{-\frac{1+|\alpha|}{4}}e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}} + e^{-\eta(|x|+t)}e^{-\frac{y^2}{Mt}}\Big]
$$

where α is a standard multiindex in x and y with $|\alpha|\le 3$. In all cases symmetric estimates hold for $y>0$.

Remark 1.2. *Using the notation of (C1)-(C2) we can, up to a choice of scaling, specify the values* $\{\beta_j^{\pm}\}_{j=1}^m$ *and* $\{\tilde{r}_{m+j}^{-}(0)\}_{j=1}^m$ *by the relation*

$$\tilde{r}_{m+j}^{\pm}(0)M_{\pm}B_{\pm} = \beta_j^{\pm}\tilde{r}_{m+j}^{\pm}(0).$$

The values $\{c_j\}_{j=m+1}^{2m}$ *can be specified as*

$$c_j = h_{(2m)j}^{-}\tilde{c}_j^{-}(0),$$

where the $\{\tilde{c}_j^{-}\}_{j=m+1}^{2m}$ *are described in Lemma 3.5, while the values* $\{h_{(2m)j}^{-}\}_{j=m+1}^{2m}$ *are described in Lemma 3.9 Part (iv). Although we give these precise specifications to be complete, our analysis only requires the existence of such constants.*

The estimates on $\tilde{G}$ *could be expressed in a more detailed form, similar to the expressions for* $e(t;y)$*, but our analysis won't require that much precision, and we have chosen to omit it. See [3, 8] for more precise statements in the scalar case.*

Remark 1.3. *We will use the observation that by taking* $T_2 > T_1$ *we can ensure there is a region in the case* $|x-y| \le Kt$ *for which estimates (II) and (III) both hold.*

In Section 7 we show that the estimates of Theorem 1.1 are sufficient to close an iteration on the system (1.21). In this way, we establish the following theorem, which is the main result of our analysis.

Theorem 1.2. *Suppose* $\bar{u}(x)$ *is a transition front solution to (1.1) as described in (H2), and suppose (H0)-(H4) hold, as well as Condition 1.1. Then for Hölder continuous initial conditions* $u(0,x) \in C^{\gamma}(\mathbb{R})$*,* $0 < \gamma < 1$*, with*

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

for some $\epsilon > 0$ *sufficiently small, there exists a solution* $u(t,x)$ *of (1.1)*

$$u \in C^{1+\frac{\gamma}{4},4+\gamma}((0,\infty)\times\mathbb{R}) \cap C^{\frac{\gamma}{4},\gamma}([0,\infty)\times\mathbb{R})$$

and a shift function $\delta \in C^{1+\frac{\gamma}{4}}[0,\infty)$ *so that*

$$\lim_{t\to 0^+}\delta(t) = 0; \quad \lim_{t\to\infty}\delta(t) = \delta_\infty \in \mathbb{R},$$

for which the following estimates hold: there exist constants $C > 0$ *and* $L > 0$ *(sufficiently large) and a constant* $\tilde{\eta} > 0$ *(sufficiently small) so that*

$$\begin{aligned}
|u(t,x+\delta(t)) - \bar{u}(x)| &\le C\epsilon\Big[(1+t)^{-1/2}e^{-\frac{x^2}{Lt}} + (1+|x|+\sqrt{t})^{-3/2}\Big]\\
|u_x(t,x+\delta(t)) - \bar{u}'(x)| &\le C\epsilon t^{-1/4}\Big[(1+t)^{-3/4}e^{-\frac{x^2}{Lt}}\\
&\qquad + (1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2} + (1+t)^{-1/4}e^{-\tilde{\eta}|x|}e^{-\frac{x^2}{Lt}}\Big]\\
|\delta(t) - \delta_\infty| &\le C\epsilon(1+t)^{-1/4}\\
|\dot{\delta}(t)| &\le C\epsilon(1+t)^{-1}.
\end{aligned}$$

Remark 1.4. *We've chosen to state our theorem with initial decay* $(1+|x|)^{-3/2}$*, but a similar statement can be obtained for* $(1+|x|)^{-r}$*, any* $r > 1$*. Along these lines, we verify in [11] that under the same assumptions of Theorem 1.2 excepting the initial rate condition, stability in* $W^{1,p}(\mathbb{R})$ *follows so long as* $\|v_0\|_{L^1(\mathbb{R})} + \|v_0\|_{L^\infty(\mathbb{R})} \leq \epsilon$*.*

Plan of the paper. In Section 2 we review necessary ODE estimates from [10], and we use these estimates in Section 3 to obtain bounds on the resolvent kernel $G_\lambda(x, y)$. In Section 4 we carry out a brief energy estimate that restricts the point specturm of L (aside from $\lambda = 0$) to the left of wedge Γ_θ, and in Section 5 we combine the observations of Sections 3-5 to prove Theorem 1.1. In Section 6 we obtain estimates on the linear and nonlinear integrals in (1.21). Finally, in Section 7 we prove Theorem 1.2.

2 Preliminary ODE Estimates

If we formally take a Laplace transform of the Green's function equation (1.11), transforming t to λ, and we denote the Laplace transform of $G(t, x; y)$ by $G_\lambda(x; y)$ (as has become common in the pointwise semigroup literature), we obtain the ODE System

$$LG_\lambda - \lambda G_\lambda = -\delta_y(x) I, \tag{2.1}$$

where L is as defined in (1.5). We will construct $G_\lambda(x; y)$ from the solutions of (1.5) that decay at either $-\infty$ or $+\infty$ (or both). This analysis has been carried out in detail in [10], and we only summarize the main results here.

First, to set notation, we specify the (necessarily positive) eigenvalues of $\Gamma^{-1}B_\pm$ and $M_\pm B_\pm$

$$\begin{aligned} \sigma(\Gamma^{-1}B_\pm) &:= \{\nu_j^\pm\}_{j=1}^m \\ \sigma(M_\pm B_\pm) &:= \{\beta_j^\pm\}_{j=1}^m \end{aligned} \tag{2.2}$$

ordered so that $j < k \Rightarrow \nu_j^\pm \leq \nu_k^\pm$, and likewise for the $\beta_j^\pm$. According to our assumption (H4) each of these sets of eigenvalues corresponds with a collection of eigenvectors that spans $\mathbb{R}^m$.

Asymptotically, (1.5) has the form

$$-M_\pm \Gamma \phi'''' + M_\pm B_\pm \phi'' = \lambda \phi, \tag{2.3}$$

and we naturally look for solutions of the form $\phi = e^{\mu^\pm x} r^\pm$, for which we find

$$\Big(- M_\pm \Gamma (\mu^\pm)^4 + M_\pm B_\pm (\mu^\pm)^2 - \lambda I \Big) r^\pm = 0. \tag{2.4}$$

In [10] the authors show that the growth and decay rates $\mu^\pm(\lambda)$ are as follows: for $j =$

$1, 2, \ldots, m$

$$\begin{aligned}
\mu_j^\pm(\lambda) &= -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|)\\
\mu_{m+j}^\pm(\lambda) &= -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2})\\
\mu_{2m+j}^\pm(\lambda) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2})\\
\mu_{3m+j}^\pm(\lambda) &= \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|),
\end{aligned} \tag{2.5}$$

where the index convention has been chosen so that $j < k \Rightarrow \mu_j^\pm \le \mu_k^\pm$. Moreover, the *fast* rates $\{\mu_j^\pm\}_{j=1}^m$ and $\{\mu_j^\pm\}_{j=3m+1}^{4m}$ are analytic in λ, while the *slow* rates $\{\mu_j^\pm\}_{j=m+1}^{3m}$ are analytic as functions of $\sqrt{\lambda}$. (Our entire analysis will be restricted to the case $\mathrm{Arg}(\lambda) \neq \pi$.)

Our labeling for the eigenvectors $\{r_j^\pm\}$ will coincide with that of the $\mu_j^\pm$ so that for $j = 1, 2, \ldots, 4m$

$$\Big(- M_\pm \Gamma (\mu_j^\pm)^4 + M_\pm B_\pm (\mu_j^\pm)^2 - \lambda I \Big) r_j^\pm = 0. \tag{2.6}$$

Finally, it will be convenient to clarify the direct connection between $\{r_j^\pm(0)\}_{j=1}^{4m}$ and the eigenvalues $\{\nu_j^\pm\}_{j=1}^m$ and $\{\beta_j^\pm\}_{j=1}^m$. For $j = 1, 2, \ldots, m$

$$\begin{aligned}
\Gamma^{-1} B_\pm r_j^\pm(0) &= \nu_{m+1-j}^\pm r_j^\pm(0)\\
M_\pm B_\pm r_{m+j}^\pm(0) &= \beta_j^\pm r_{m+j}^\pm(0)\\
M_\pm B_\pm r_{2m+j}^\pm(0) &= \beta_{m+1-j}^\pm r_{2m+j}^\pm(0)\\
\Gamma^{-1} B_\pm r_{3m+j}^\pm(0) &= \nu_j^\pm r_{3m+j}^\pm(0).
\end{aligned} \tag{2.7}$$

Moreover, for $|\lambda|$ sufficiently small the eigenvectors $\{r_j^\pm(\lambda)\}_{j=1}^m$ and $\{r_j^\pm(\lambda)\}_{j=3m+1}^{4m}$ are analytic in λ, while the eigenvectors $\{r_j^\pm(\lambda)\}_{j=m+1}^{3m}$ are analytic as functions of $\sqrt{\lambda}$.

The following lemma is taken directly from [10].

Lemma 2.1. *Under Conditions (C1)–(C2), and for $|\lambda|$ sufficiently small, with $Arg\lambda \neq \pi$, there exists a value $\eta > 0$ for which we have the following estimates on a choice of linearly independent solutions of the eigenvalue problem (1.5).*

(I) For $x \le 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \ldots, 2m$

$$\partial_x^k \phi_j^-(x;\lambda) = e^{\mu_{2m+j}^-(\lambda)x} \Big(\mu_{2m+j}^-(\lambda)^k r_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x|}) \Big);$$

(ii) For $j = 1, \ldots, m$

$$\partial_x^k \psi_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x} \Big(\mu_j^-(\lambda)^k r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|}) \Big);$$

(iii) For $j = m+1, \dots, 2m$

$$\partial_x^k \psi_j^-(x;\lambda) = \frac{1}{\mu_j^-(\lambda)}\Big(\mu_j^-(\lambda)^k e^{\mu_j^-(\lambda)x} - (-\mu_j^-(\lambda))^k e^{-\mu_j^-(\lambda)x}\Big) r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|});$$

(II) For $x \geq 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \dots, 2m$

$$\partial_x^k \phi_j^+(x;\lambda) = e^{\mu_j^+(\lambda)x}\Big(\mu_j^+(\lambda)^k r_j^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) For $j = 1, \dots, m$

$$\partial_x^k \psi_j^+(x;\lambda) = \frac{1}{\mu_{2m+j}^+(\lambda)}\Big(\mu_{2m+j}^+(\lambda)^k e^{\mu_{2m+j}^+(\lambda)x} - (-\mu_{2m+j}^+(\lambda))^k e^{-\mu_{2m+j}^+(\lambda)x}\Big) r_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|});$$

(iii) For $j = m+1, \dots, 2m$

$$\partial_x^k \psi_j^+(x;\lambda) = e^{\mu_{2m+j}^+(\lambda)x}\Big(\mu_{2m+j}^+(\lambda)^k r_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).$$

Remark 2.1. *The cases (Ii), (Iii), (IIi), and (IIiii) can be established by standard methods such as those of Proposition 3.1 in [19]. Cases (Iiii) and (IIii) are established in [10]. The fast decay modes are $\{\phi_j^-\}_{j=m+1}^{2m}$ and $\{\phi_j^+\}_{j=1}^{m}$. Likewise, the slow decay modes are $\{\phi_j^-\}_{j=1}^{m}$ and $\{\phi_j^+\}_{j=m+1}^{2m}$.*

It will be convenient for our later calculations to briefly review the argument from [10] establishing that

$$D_a(0) = D_a'(0) = \cdots = D_a^m(0) = 0.$$

(The condition $D_a^{m+1}(0) \neq 0$ is much more difficult to verify, and we will not make any attempt to do so here. See [10].) First, we recall that $D_a(\rho)$ can be characterized as follows,

$$D_a(\rho) = W(\underbrace{\phi_1^+, \dots, \phi_m^+}_{\text{fast}}, \overbrace{\phi_{m+1}^+, \dots, \phi_{2m}^+}^{\text{slow}}, \overbrace{\phi_1^-, \dots, \phi_m^-}^{\text{slow}}, \underbrace{\phi_{m+1}^-, \dots, \phi_{2m}^-}_{\text{fast}}).$$

By a choice of our bases, we can take

$$\phi_1^+(x;0) = \bar{u}'(x) = \phi_{2m}^-(x;0). \tag{2.8}$$

To be clear, we observe that this choice typically requires a relabeling of the estimates in Lemma 2.1.

It's clear from (2.8) that if both ϕ_1^+ and ϕ_{2m}^- appear undifferentiated in a term $W(\dots)$ then that term will be 0 by linear dependence. Likewise, if only a single ρ-derivative appears on either of these modes the term will be 0 by analyticity. This means that we can only possibly have a non-zero term if at least one of these modes is differentiated twice with respect to ρ. At the same time, we observe that the twice-integrable form of our eigenvalue problem (1.5) ensures that the m terms $\phi_j^{\pm'''}$ can be eliminated from any term, fast or slow, that is undifferentiated. Since W is the determinant of a $4m \times 4m$ matrix this means we can only obtain a non-zero term if at least m terms are differentiated. If a single derivative appears on any fast mode then the corresponding W will be 0, so the lowest order derivative of $D_a(\rho)$ that can be non-zero at $\rho = 0$ is obtained by putting two ρ-derivatives on either ϕ_1^+ or ϕ_{2m}^- and one ρ-derivative on $m-1$ of the $2m$ slow modes. For a detailed discussion of these considerations, especially in the case $m = 2$, see [10].

3 Construction of the Resolvent Kernel

We turn now to the first stages in our construction of $G(t, x; y)$, beginning with (2.1). By construction, $G_\lambda(x; y)$ must decay to 0 as $x \to \pm\infty$ (for each fixed y), and so we expect that G_λ can be expressed as a linear combination of the asymptotically decaying solutions described in Lemma 2.1. The expansion coefficients will naturally depend on y, and in particular will be solutions of the transpose eigenvalue problem

$$\tilde{L}z := -\Big((z_x M)_x \Gamma\Big)_{xx} + \Big(z_x M\Big)_x B(x) = \lambda z, \tag{3.1}$$

where z denotes a $1 \times m$ row vector. (We will be more precise about this below.)

In our calculations below we will take advantage of the relationship between $G_\lambda(x; y)$ and the Green's function associated with (3.1). In particular, we have the following lemma, which is the analogue in the current setting to Lemma 4.3 of [19].

Lemma 3.1. *Suppose there exists a function $G_\lambda(x; y)$ that satisfies (2.1) and for each fixed $y \in \mathbb{R}$ decays to 0 as $x \to \pm\infty$. If $H_\lambda(x; y)$ likewise satisfies*

$$\tilde{L}H_\lambda - \lambda H_\lambda = -\delta_y(x) I.$$

and decays to 0 as $x \to \pm\infty$ for each fixed y, then $H_\lambda(x; y) = G_\lambda(y; x)$.

Proof of Lemma 3.1. Precisely as in the proof of Lemma 4.3 of [19], we compute directly.

For any $x_0, y_0 \in \mathbb{R}$, we have

$$\begin{aligned}
G_\lambda(x_0; y_0) &= \int_{-\infty}^{+\infty} \Big[\delta_{x_0}(x) I\Big] G_\lambda(x; y_0) dx \\
&= \int_{-\infty}^{+\infty} \Big[\Big((H_{\lambda_x} M)_x \Gamma\Big)_{xx} - \Big(H_{\lambda_x} M\Big)_x B(x) + \lambda H_\lambda\Big](x; x_0) G_\lambda(x; y_0) dx \\
&= \int_{-\infty}^{+\infty} H_\lambda(x, x_0) \Big[\Big(M(x)(\Gamma G_{\lambda_{xx}} - B(x) G_\lambda)_x\Big)_x + \lambda G_\lambda\Big](x; y_0) dx \\
&= \int_{-\infty}^{+\infty} H_\lambda(x; x_0) \Big[\delta_{y_0}(x)\Big] dx \\
&= H_\lambda(y_0; x_0),
\end{aligned}$$

where of course we have integrated by parts to obtain the third equality. □

In order for $G_\lambda(x; y)$ to solve (2.1), it must be continuous in all derivatives, including mixed partials, up to and including order 2, and it must have jumps in at least some of its order 3 derivatives. In order to efficiently describe this behavior, we will adopt the jump notation $[\cdot]$, so that, for example,

$$[G_\lambda](y) := \lim_{x \to y^+} G_\lambda(x; y) - \lim_{x \to y^-} G_\lambda(x; y). \tag{3.2}$$

It will also be notationally convenient for certain calculations to set

$$G_\lambda^{\pm}(x; y) := \lim_{z \to x^{\pm}} G_\lambda(z; y), \tag{3.3}$$

so that

$$[G_\lambda](y) = G_\lambda^+(y; y) - G_\lambda^-(y; y). \tag{3.4}$$

Finally, we will denote by $\mathcal{G}_\lambda$ the $4m \times 4m$ matrix

$$\mathcal{G}_\lambda(x; y) = \begin{pmatrix} G_\lambda & G_{\lambda_y} & G_{\lambda_{yy}} & G_{\lambda_{yyy}} \\ G_{\lambda_x} & G_{\lambda_{yx}} & G_{\lambda_{yyx}} & G_{\lambda_{yyyx}} \\ G_{\lambda_{xx}} & G_{\lambda_{yxx}} & G_{\lambda_{yyxx}} & G_{\lambda_{yyyxx}} \\ G_{\lambda_{xxx}} & G_{\lambda_{yxxx}} & G_{\lambda_{yyxxx}} & G_{\lambda_{yyyxxx}} \end{pmatrix}. \tag{3.5}$$

In the calculations that follow we will ease notation by setting

$$\partial^{i,j} G_\lambda := \frac{\partial^{i+j} G_\lambda}{\partial x^i \partial y^j}.$$

Lemma 3.2. *Suppose there exists a function $G_\lambda(x, y)$ that satisfies (2.1) and for each fixed $y \in \mathbb{R}$ decays to 0 as $x \to \pm\infty$. Then*

$$[\mathcal{G}_\lambda](y) = \begin{pmatrix} 0 & 0 & 0 & -\Gamma^{-1} M^{-1} \\ 0 & 0 & \Gamma^{-1} M^{-1} & 2\Gamma^{-1} \frac{dM^{-1}}{dy} \\ 0 & -\Gamma^{-1} M^{-1} & -\Gamma^{-1} \frac{dM^{-1}}{dy} & -\Gamma^{-1} B \Gamma^{-1} M^{-1} - \Gamma^{-1} \frac{d^2 M^{-1}}{dy} \\ \Gamma^{-1} M^{-1} & 0 & \Gamma^{-1} B \Gamma^{-1} M^{-1} & -\Gamma^{-1} B' \Gamma^{-1} M^{-1} + 2\Gamma^{-1} B \Gamma^{-1} \frac{dM^{-1}}{dy} \end{pmatrix},$$

where the matrices M and B, as well as their derivatives, are evaluated at y.

Moreover, we can invert $[\mathcal{G}_\lambda]$ to obtain

$$[\mathcal{G}_\lambda]^{-1} = \begin{pmatrix} -MB' & -MB & 0 & M\Gamma \\ MB - M''\Gamma & M'\Gamma & -M\Gamma & 0 \\ -2M'\Gamma & M\Gamma & 0 & 0 \\ -M\Gamma & 0 & 0 & 0 \end{pmatrix}.$$

Remark 3.1. *In putting Lemma 3.2 in its stated form, we have made liberal use of the standard identities*

$$\frac{dM}{dy} = -M\frac{dM^{-1}}{dy}M$$
$$\frac{dM^{-1}}{dy} = -M^{-1}\frac{dM}{dy}M^{-1}$$
$$M\frac{d^2M}{dy^2}M = 2M'M^{-1}M' - M''.$$

For notational brevity we will often suppress the dependence of B and M on y.

Proof of Lemma 3.2. First, G_λ and all of its partial derivatives are continuous up to and including order two, so the jumps for these functions must be 0.

Third order derivatives $[\partial^{3,0}G_\lambda]$, $[\partial^{2,1}G_\lambda]$, $[\partial^{1,2}G_\lambda]$, $[\partial^{0,3}G_\lambda]$. In order for the fourth x-derivative of G_λ to correspond with a Dirac delta function, we expect the third derivative to have a step jump with appropriate amplitude. That is, the distributional relationship

$$-M(x)\Gamma\partial^{4,0}G_\lambda = -\delta_y(x)I$$

suggests the jump relationship

$$[\partial^{3,0}G_\lambda](y) = \Gamma^{-1}M(y)^{-1}. \tag{3.6}$$

This completes the first column of $[\mathcal{G}_\lambda]$.

Next, we observe from Lemma 3.1 that for each fixed $x \in \mathbb{R}$ the function of y $G_\lambda(x,\cdot)$ satisfies

$$-\Big((G_{\lambda_y}M)_y\Gamma\Big)_{yy} + \Big(G_{\lambda_y}M\Big)_y B(y) - \lambda G_\lambda = -\delta_x(y)I.$$

In this case, we must have a jump in the third y-derivative of $G_\lambda(x,\cdot)$ of the form

$$[\partial^{0,3}G_\lambda](y)M\Gamma = -I \Rightarrow [\partial^{0,3}G_\lambda](y) = -\Gamma^{-1}M^{-1}.$$

For the purposes of keeping signs straight, we recall that $[\cdot]$ denotes a jump as x varies and the distributional relation $\partial^{0,4}G_\lambda M\Gamma = \delta_x(y)I$ suggests a jump up as y crosses x (increasing) and a jump down as x crosses y (increasing).

In order to compute the third order mixed partials, we differentiate our difference expressions. To begin, using (3.4) we have

$$0 = \frac{d}{dy}[G_\lambda](y) = G^+_{\lambda_x}(y,y) + G^+_{\lambda_y}(y,y) - G^-_{\lambda_x}(y,y) - G^-_{\lambda_y}(y,y) = [G_{\lambda_x}] + [G_{\lambda_y}].$$

Similarly,

$$\frac{d^3}{dy^3}[G_\lambda](y) = [\partial^{3,0}G_\lambda] + 3[\partial^{2,1}G_\lambda] + 3[\partial^{1,2}G_\lambda] + [\partial^{0,3}G_\lambda]. \tag{3.7}$$

Using our expressions for $[\partial^{3,0}G_\lambda]$ and $[\partial^{0,3}G_\lambda]$ we see that

$$[\partial^{2,1}G_\lambda] = -[\partial^{1,2}G_\lambda]. \tag{3.8}$$

Alternatively, we can begin with $[G_{\lambda_x}](y)$ and compute

$$0 = \frac{d^2}{dy^2}[G_{\lambda_x}](y) = [\partial^{3,0}G_\lambda] + 2[\partial^{2,1}G_\lambda] + [\partial^{1,2}G_\lambda]. \tag{3.9}$$

Combining (3.8) and (3.9) with our expression for $[\partial^{3,0}G_\lambda]$ we find

$$\begin{aligned}[\partial^{2,1}G_\lambda] &= -\Gamma^{-1}M^{-1}\\ [\partial^{1,2}G_\lambda] &= +\Gamma^{-1}M^{-1}.\end{aligned}$$

Fourth order derivatives $[\partial^{3,1}G_\lambda]$, $[\partial^{2,2}G_\lambda]$, $[\partial^{1,3}G_\lambda]$. In order to compute jumps in the fourth order derivatives, we'll first compute $[\partial^{4,0}G_\lambda]$ and $[\partial^{0,4}G_\lambda]$. To this end, we begin with

$$-\Big(M(x)\Gamma G^\pm_{\lambda_{xxx}}\Big)_x + \Big(M(x)(B(x)G^\pm_\lambda)_x\Big)_x - \lambda G^\pm_\lambda = 0. \tag{3.10}$$

Subtracting the G^-_λ equation from the G^+_λ equation we find

$$-M(y)\Gamma[\partial^{4,0}G_\lambda](y) - M'(y)\Gamma[\partial^{3,0}G_\lambda](y) = 0,$$

so that

$$[\partial^{4,0}G_\lambda](y) = -\Gamma^{-1}M(y)^{-1}M'(y)\Gamma\Gamma^{-1}M(y)^{-1} = \Gamma^{-1}\frac{dM^{-1}}{dy}.$$

Likewise

$$-\Big((G^\pm_{\lambda_y}M)_y\Gamma\Big)_{yy} + \Big(G^\pm_{\lambda_y}M\Big)_y B(y) - \lambda G^\pm_\lambda = 0. \tag{3.11}$$

We can compute jump values from (3.11) by subtracting the equation for G^-_λ from the equation for G^+_λ. We find

$$-[\partial^{0,4}G_\lambda](y)M\Gamma - 3[\partial^{0,3}G_\lambda](y)M'\Gamma = 0,$$

so that

$$[\partial^{0,4}G_\lambda](y) = 3\Gamma^{-1}M(y)^{-1}M'(y)M(y)^{-1} = -3\Gamma^{-1}\frac{dM^{-1}}{dy}.$$

We can now use the relation

$$\frac{d}{dy}[\partial^{3,0}G_\lambda](y) = [\partial^{4,0}G_\lambda](y) + [\partial^{3,1}G_\lambda](y),$$

to see that $[\partial^{3,1}G_\lambda](y) = 0$, and the relation

$$\frac{d}{dy}[\partial^{0,3}G_\lambda](y) = [\partial^{3,1}G_\lambda](y) + [\partial^{0,4}G_\lambda](y),$$

to see that $[\partial^{1,3}G_\lambda](y) = 2\Gamma^{-1}\frac{dM^{-1}}{dy}$. In order to compute $[\partial^{2,2}G_\lambda](y)$, we use the relation

$$\frac{d}{dy}[\partial^{2,1}G_\lambda](y) = [\partial^{3,1}G_\lambda](y) + [\partial^{2,2}G_\lambda](y),$$

along with our previously derived expressions for $[\partial^{2,1}G_\lambda](y)$ and $[\partial^{3,1}G_\lambda](y)$ to find

$$[\partial^{2,2}G_\lambda](y) = -\Gamma^{-1}\frac{dM^{-1}}{dy}.$$

Fifth order derivatives $[\partial^{3,2}G_\lambda]$, $[\partial^{2,3}G_\lambda]$. For these we begin by differentiating (3.10) with respect to x and subtracting the result we obtain for G_λ^- from the result we obtain for G_λ^+. We find

$$[\partial^{5,0}G_\lambda](y) = \Gamma^{-1}B(y)\Gamma^{-1}M(y)^{-1} + \Gamma^{-1}\frac{d^2M^{-1}}{dy^2}.$$

Likewise, working with (3.11) we find

$$[\partial^{0,5}G_\lambda](y) = -6\Gamma^{-1}\frac{d^2M^{-1}}{dy^2} - \Gamma^{-1}B(y)\Gamma^{-1}M(y)^{-1}.$$

We now subtract

$$\frac{d}{dy}[\partial^{4,0}G_\lambda](y) = [\partial^{5,0}G_\lambda](y) + [\partial^{4,1}G_\lambda](y)$$

from

$$\frac{d}{dy}[\partial^{3,1}G_\lambda](y) = [\partial^{4,1}G_\lambda](y) + [\partial^{3,2}G_\lambda](y),$$

to find

$$[\partial^{3,2}G_\lambda](y) = \frac{d}{dy}[\partial^{3,1}G_\lambda](y) - \frac{d}{dy}[\partial^{4,0}G_\lambda](y) + [\partial^{5,0}G_\lambda](y) = \Gamma^{-1}B(y)\Gamma^{-1}M(y)^{-1}.$$

Likewise, we subtract

$$\frac{d}{dy}[\partial^{0,4}G_\lambda](y) = [\partial^{1,4}G_\lambda](y) + [\partial^{0,5}G_\lambda](y)$$

from

$$\frac{d}{dy}[\partial^{1,3}G_\lambda](y) = [\partial^{2,3}G_\lambda](y) + [\partial^{1,4}G_\lambda](y),$$

to find

$$\begin{aligned}
[\partial^{2,3}G_\lambda](y) &= \frac{d}{dy}[\partial^{1,3}G_\lambda](y) - \frac{d}{dy}[\partial^{0,4}G_\lambda](y) + [\partial^{0,5}G_\lambda](y) \\
&= -\Gamma^{-1}\frac{d^2M^{-1}}{dy^2} - \Gamma^{-1}B(y)\Gamma^{-1}M(y)^{-1}.
\end{aligned}$$

Sixth order derivative $[\partial^{3,3}G_\lambda]$. Differentiating (3.11) twice, we find

$$[\partial^{0,6}G_\lambda](y) = -10\Gamma^{-1}\frac{d^3M^{-1}}{dy^3} - 5\Gamma^{-1}B(y)\Gamma^{-1}\frac{dM^{-1}}{dy} - 2\Gamma^{-1}B'(y)\Gamma^{-1}M(y)^{-1}.$$

We now combine the relations

$$\begin{aligned}
\frac{d}{dy}[\partial^{2,3}G_\lambda](y) &= [\partial^{3,3}G_\lambda](y) + [\partial^{2,4}G_\lambda](y) \\
\frac{d}{dy}[\partial^{1,4}G_\lambda](y) &= [\partial^{2,4}G_\lambda](y) + [\partial^{1,5}G_\lambda](y) \\
\frac{d}{dy}[\partial^{0,5}G_\lambda](y) &= [\partial^{1,5}G_\lambda](y) + [\partial^{0,6}G_\lambda](y),
\end{aligned}$$

to obtain

$$\begin{aligned}
[\partial^{3,3}G_\lambda](y) &= \frac{d}{dy}[\partial^{2,3}G_\lambda](y) - \frac{d}{dy}[\partial^{1,4}G_\lambda](y) + \frac{d}{dy}[\partial^{0,5}G_\lambda](y) - [\partial^{0,6}G_\lambda](y) \\
&= 2\Gamma^{-1}B(y)\Gamma^{-1}\frac{dM^{-1}}{dy} - \Gamma^{-1}B'(y)\Gamma^{-1}M(y)^{-1}.
\end{aligned}$$

Finally, we can verify that $[\mathcal{G}]^{-1}$ is the correct inverse by direct matrix multiplication. □

Lemma 3.3. *For each $\lambda \in \mathbb{C}$, if $z(\cdot;\lambda) \in C^4(\mathbb{R})$, and $\tilde{L}$ is defined as in (3.1), then $\tilde{L}z = \lambda z$ if and only if the entwining*

$$\begin{pmatrix} z & z' & z'' & z''' \end{pmatrix} [\mathcal{G}]^{-1} \begin{pmatrix} w \\ w' \\ w'' \\ w''' \end{pmatrix}$$

is constant (in x) for all $w(\cdot;\lambda) \in C^4(\mathbb{R})$ that satisfy $Lw = \lambda w$, where L is as defined in (1.5).

Proof. For notational convenience, we set $Z = (z, z', z'', z''')$ and $W = (w, w', w'', w''')^{tr}$. Computing directly, we find that

$$\frac{d}{dx}\Big(Z[\mathcal{G}]^{-1}W\Big) = z\Big((M\Gamma w''')' - (M(Bw)')'\Big) + z'\Big(M'B - M'''\Gamma\Big)w$$
$$+ z''\Big(MB - 3M''\Gamma\Big)w - 3z'''M'\Gamma w - z''''M\Gamma w$$
$$= \Big(-(z'M)'''\Gamma + (z'M)'B - \lambda z\Big)w = 0,$$

where we give the intermediate step as a convenient bookkeeping arrangement for the straightforward but tedious calculation, and where we have used (3.1) in obtaining the final inequality.

Since z and w can both be scaled by any constant (in x) we can clearly scale away λ dependence. $\square$

We now require a lemma, similar to Lemma 2.1, characterizing asymptotic behavior of solutions to $\tilde{L}z = \lambda z$. We will refer to these as *dual* solutions and typically we will distinguish them with a tilde.

Lemma 3.4. *Under Conditions (C1)–(C2), and for $|\lambda|$ sufficiently small, with $Arg\lambda \neq \pi$, there exists a value $\eta > 0$ for which we have the following estimates on a choice of linearly independent solutions of the eigenvalue problem $\tilde{L}z = \lambda z$.*

(I) For $x \leq 0$ and $k = 0, 1, 2, 3$ we have:

(i) (Slow growth) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\varphi}_j^-(x;\lambda) = e^{-\mu_{2m+j}^-(\lambda)x}\Big((-\mu_{2m+j}^-(\lambda))^k \tilde{r}_{2m+j}^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) (Fast growth) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\varphi}_j^-(x;\lambda) = e^{-\mu_{2m+j}^-(\lambda)x}\Big((-\mu_{2m+j}^-(\lambda))^k \tilde{r}_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(iii) (Fast decay) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\zeta}_j^-(x;\lambda) = e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(iv) (Slow decay) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\zeta}_j^-(x;\lambda) = e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big);$$

(II) For $x \geq 0$ and $k = 0, 1, 2, 3$ we have:

(i) (Fast growth) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\varphi}_j^+(x;\lambda) = e^{-\mu_j^+(\lambda)x}\Big((-\mu_j^+(\lambda))^k \tilde{r}_j^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) (Slow growth) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\varphi}_j^+(x;\lambda) = e^{-\mu_j^+(\lambda)x}\Big((-\mu_j^+(\lambda))^k \tilde{r}_j^+(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big);$$

(iii) (Slow decay) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\zeta}_j^+(x;\lambda) = e^{-\mu_{2m+j}^+(\lambda)x}\Big((-\mu_{2m+j}^+(\lambda))^k \tilde{r}_{2m+j}^+(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big).$$

(iv) (Fast decay) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\zeta}_j^+(x;\lambda) = e^{-\mu_{2m+j}^+(\lambda)x}\Big((-\mu_{2m+j}^+(\lambda))^k \tilde{r}_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).$$

Here, the $\tilde{r}_j^\pm$ satisfy

$$\tilde{r}_j^\pm\Big(-(\mu_j^\pm)^4 M_\pm\Gamma + (\mu_j^\pm)^2 M_\pm B_\pm - \lambda I\Big) = 0.$$

Notes on the proof. Lemma 3.4 can be proven by standard asymptotic techniques, as referenced following the statement of Lemma 2.1. The important point here is that for the slow modes the asymptotic error term goes to 0 (like $\sqrt{\lambda}$) as $\lambda \to 0$. This is a direct and straightforward consequence of the fact that for $\lambda = 0$ constants are solutions of the dual equation $\tilde{L}z = 0$. By contrast, this is not the case for $Lw = 0$. □

In practice, we will find it convenient to scale our bases of dual solutions according to an entwinement relationship with our basic ODE modes. To be precise, let $\{\phi_j^-\}_{j=1}^{2m}$ denote the decay modes of (1.5) at $-\infty$, and let $\{\bar{\psi}_j^-\}_{j=1}^{2m}$ denote the growth modes of (1.5) at $-\infty$ *obtained prior to the difference forms of Lemma 2.1* . That is, the fast $\bar{\psi}_j^-$ are precisely the same as the ψ_j^-, while the slow $\bar{\psi}_j^-$ satisfy the estimates ($j = m+1, \dots, 2m$)

$$\partial_x^k \bar{\psi}_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x}\Big(\mu_j^-(\lambda)^k r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).$$

In terms of $\bar{\psi}_j^-$, the difference growth mode ψ_j^- is defined as

$$\psi_j^- = \frac{1}{\mu_j^-}\Big(\bar{\psi}_j^- - \phi_{2m+1-j}^-\Big), \tag{3.12}$$

for $j = m+1, \dots, 2m$.

For the $\{\phi_j^-\}_{j=1}^{2m}$ we define the extended vectors

$$\Phi_j^- = \begin{pmatrix} \phi_j^- \\ \phi_j^{-\prime} \\ \phi_j^{-\prime\prime} \\ \phi_j^{-\prime\prime\prime} \end{pmatrix},$$

and we similarly define Ψ_j^- and $\bar{\Psi}_j^-$. In addition it will be convenient to define the matrices

$$\begin{aligned} \Phi^\pm &= \begin{pmatrix} \Phi_1^\pm & \Phi_2^\pm & \dots & \Phi_{2m}^\pm \end{pmatrix} \\ \tilde{\Psi}^\pm &= \begin{pmatrix} \tilde{\Psi}_1^\pm \\ \tilde{\Psi}_2^\pm \\ \vdots \\ \tilde{\Psi}_{2m}^\pm \end{pmatrix}. \end{aligned} \tag{3.13}$$

Now scale the associated duals $\{\tilde{\Phi}_j^-\}_{j=1}^{2m}$ and $\{\tilde{\Psi}_j^-\}_{j=1}^{2m}$ as follows: for $j \in \{1, 2, \dots, 2m\}$

$$\begin{aligned} \tilde{\Phi}_j^-[\mathcal{G}_\lambda]^{-1}\Phi_k^- &= \delta_k^j; \quad k = 1, 2, \dots, 2m \\ \tilde{\Phi}_j^-[\mathcal{G}_\lambda]^{-1}\Psi_k^- &= 0; \quad k = 1, 2, \dots, 2m \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \tilde{\Psi}_j^-[\mathcal{G}_\lambda]^{-1}\Phi_k^- &= 0; \quad k = 1, 2, \dots, 2m \\ \tilde{\Psi}_j^-[\mathcal{G}_\lambda]^{-1}\Psi_k^- &= \delta_j^k; \quad k = 1, 2, \dots, 2m, \end{aligned} \tag{3.15}$$

or more briefly

$$\begin{pmatrix} \tilde{\Phi}^- \\ \tilde{\Psi}^- \end{pmatrix} [\mathcal{G}_\lambda]^{-1} \left(\Phi^-, \Psi^-\right) = I.$$

We now state a lemma specifying asymptotic behavior for the choice of dual bases we'll use for the analysis.

Lemma 3.5. *Under Conditions (C1)–(C2), and for $|\lambda|$ sufficiently small, with $Arg\lambda \neq \pi$, there exists a value $\eta > 0$ for which we have the following estimates on a choice of linearly independent solutions of the eigenvalue problem $\tilde{L}z = \lambda z$.*

(I) For $x \leq 0$ and $k = 0, 1, 2, 3$ we have:

(i) (Slow growth) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\phi}_j^-(x;\lambda) = \tilde{c}_j^-(\lambda)\Big((-\mu_{2m+j}^-)^k e^{-\mu_{2m+j}^-(\lambda)x} - (\mu_{2m+j}^-)^k e^{\mu_{2m+j}^-(\lambda)x}\Big)\tilde{r}_{2m+j}^- + \mathbf{O}(e^{-\eta|x|})$$

(ii) (Fast growth) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\phi}_j^-(x;\lambda) = e^{-\mu_{2m+j}^-(\lambda)x}\Big((-\mu_{2m+j}^-(\lambda))^k \tilde{r}_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(iii) (Fast decay) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\psi}_j^-(x;\lambda) = e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(iv) (Slow decay) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\psi}_j^-(x;\lambda) = \tilde{c}_j^-(\lambda) e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big);$$

(II) For $x \geq 0$ and $k = 0, 1, 2, 3$ we have:

(i) (Fast growth) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\phi}_j^+(x;\lambda) = e^{-\mu_j^+(\lambda)x}\Big((-\mu_j^+(\lambda))^k \tilde{r}_j^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);$$

(ii) (Slow growth) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\varphi}_j^+(x;\lambda) = \tilde{c}_j^+(\lambda)\Big((-\mu_j^+)^k e^{-\mu_j^+(\lambda)x} - (\mu_j^+)^k e^{\mu_j^+(\lambda)x}\Big)\tilde{r}_j^+ + \mathbf{O}(e^{-\eta|x|})$$

(iii) (Slow decay) For $j = 1, \dots, m$

$$\partial_x^k \tilde{\psi}_j^+(x;\lambda) = \tilde{c}_j^+(\lambda) e^{-\mu_{2m+j}^+(\lambda)x}\Big((-\mu_{2m+j}^+(\lambda))^k \tilde{r}_{2m+j}^+(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big).$$

(iv) (Fast decay) For $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\psi}_j^+(x;\lambda) = e^{-\mu_{2m+j}^+(\lambda)x}\Big((-\mu_{2m+j}^+(\lambda))^k \tilde{r}_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).$$

Here, the $\tilde{r}_j^\pm$ satisfy

$$\tilde{r}_j^\pm\Big(-(\mu_j^\pm)^4 M_\pm \Gamma + (\mu_j^\pm)^2 M_\pm B_\pm - \lambda I\Big) = 0.$$

Finally, there exists a constant $C > 0$ sufficiently large so that for $j = 1, 2, \dots, m$ $|\tilde{c}_j^-(\lambda)| \leq C|\lambda|^{-1/2}$ and $|\tilde{c}_j^+(\lambda)| \leq C$, while for $j = m+1, m+2, \dots, 2m$ $|\tilde{c}_j^-(\lambda)| \leq C$ and $|\tilde{c}_j^+(\lambda)| \leq C|\lambda|^{-1/2}$.

Proof. Since the considerations are similar for Cases (I) and (II), we will restrict our discussion to Case (I). First, let $\{\bar{\Psi}_j^-\}_{j=1}^{2m}$ be as discussed just above (3.13). We will work initially with the duals for $\{\Phi_j^-\}_{j=1}^{2m}$ and $\{\bar{\Psi}_j^-\}_{j=1}^{2m}$. That is, let $\{\bar{\tilde{\Phi}}_j^-\}_{j=1}^{2m}$ and $\{\bar{\tilde{\Psi}}_j^-\}_{j=1}^{2m}$ satisfy: for $j \in \{1, 2, \dots, 2m\}$

$$\begin{aligned} \bar{\tilde{\Phi}}_j^-[\mathcal{G}_\lambda]^{-1}\Phi_k^- &= \delta_k^j; \quad k = 1, 2, \dots, 2m \\ \bar{\tilde{\Phi}}_j^-[\mathcal{G}_\lambda]^{-1}\bar{\Psi}_k^- &= 0; \quad k = 1, 2, \dots, 2m \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\tilde{\bar{\Psi}}_j^-[\mathcal{G}_\lambda]^{-1}\Phi_k^- &= 0; \quad k = 1, 2, \dots, 2m \\ \tilde{\bar{\Psi}}_j^-[\mathcal{G}_\lambda]^{-1}\bar{\Psi}_k^- &= \delta_j^k; \quad k = 1, 2, \dots, 2m,\end{aligned} \tag{3.17}$$

or briefly

$$\begin{pmatrix} \tilde{\bar{\Phi}}^- \\ \tilde{\bar{\Psi}}^- \end{pmatrix} [\mathcal{G}_\lambda]^{-1} \left(\Phi^-, \bar{\Psi}^-\right) = I.$$

We grant that this notation is unwieldy, but it does not carry beyond this proof.

It's clear from our entwinement relations, and the form of $[\mathcal{G}_\lambda]^{-1}$ that an entwinement can only be non-zero if the fundamental rate of the dual variable is the negative of the fundamental rate of the ODE variable. In this way, the entwined duals $\{\tilde{\bar{\Phi}}_j^-\}_{j=1}^{2m}$, $\{\tilde{\bar{\Psi}}_j^-\}_{j=1}^{2m}$ must respectively be multiples of the (extended vector) forms from Lemma 3.4. For the slow decay duals $\{\tilde{\bar{\Psi}}_j^-\}_{j=m+1}^{2m}$ this means we have

$$\partial_x^k \tilde{\bar{\psi}}_j^-(x;\lambda) = \tilde{\bar{c}}_j^-(\lambda) e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big)$$

for some constant $\tilde{\bar{c}}_j^-(\lambda)$.

In order to understand the nature of this constant, we recall that by definition

$$\tilde{\bar{\Psi}}_j^-[\mathcal{G}]^{-1}\bar{\Psi}_j^- = 1,$$

and if we take a limit as $x \to -\infty$ in this last relation we obtain

$$\tilde{\bar{c}}_j^-(\lambda)\Big(- 2\mu_j^- \tilde{r}_j^- M_- B_- r_j^- + 3(\mu_j^-)^3 \tilde{r}_j^- M_- \Gamma \tilde{r}_j^- \Big) = 1,$$

and we see that $\tilde{\bar{c}}_j^-(\lambda)$ scales like $1/\mu_j^-$; i.e., like $\lambda^{-1/2}$. (Here, $M_- B_- r_j^-(0) = \beta_{j-m}^- r_j^-(0)$, and by our standard scaling $\tilde{r}_j^-(0) \cdot r_j^-(0) = 1$.)

Likewise, for $j = 1, 2, \dots, m$

$$\begin{aligned}\partial_x^k \tilde{\bar{\phi}}_j^-(x;\lambda) &= \tilde{\bar{c}}_j^-(\lambda) e^{-\mu_{2m+j}^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_{2m+j}^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x|})\Big) \\ \partial_x^k \tilde{\bar{\psi}}_j^-(x;\lambda) &= \tilde{\bar{c}}_j^-(\lambda) e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big),\end{aligned}$$

where $|\tilde{\bar{c}}_j^-(\lambda)| \le C|\lambda|^{-1/2}$ and $|\tilde{\bar{c}}_j^-(\lambda)| \le C$, and finally for $j = m+1, \dots, 2m$

$$\partial_x^k \tilde{\bar{\phi}}_j^-(x;\lambda) = \tilde{\bar{c}}_j^-(\lambda) e^{-\mu_j^-(\lambda)x}\Big((-\mu_j^-(\lambda))^k \tilde{r}_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big),$$

where $|\tilde{\bar{c}}_j^-(\lambda)| \le C$.

We now define the duals of Lemma 3.5 in terms of $\{\tilde{\bar{\Phi}}_j^-\}_{j=1}^{2m}$ and $\{\tilde{\bar{\Psi}}_j^-\}_{j=1}^{2m}$ as follows: for $j = 1, 2, \dots, m$

$$\begin{aligned}\tilde{\Phi}_j^- &= \Big(\tilde{\bar{\Phi}}_j^- + \tilde{\bar{\Psi}}_{2m+1-j}\Big) \\ \tilde{\Psi}_j^- &= \tilde{\bar{\Psi}}_j^-,\end{aligned}$$

while for $j = m+1, \dots, 2m$

$$\tilde{\Phi}_j^- = \bar{\tilde{\Phi}}_j^-$$
$$\tilde{\Psi}_j^- = \mu_j^- \bar{\tilde{\Psi}}_j^-.$$

We can verify by direct calculation, using (3.16) and (3.17) that these satisfy (3.14) and (3.15). The claimed estimates follow from the estimates on $\{\bar{\tilde{\Phi}}_j^-\}_{j=1}^{2m}$ and $\{\bar{\tilde{\Psi}}_j^-\}_{j=1}^{2m}$.

Since these calculations are straightforward, we consider only the case in which an addition dual entwines with a subtraction mode. That is, for $j = 1, \dots, m$ and $k = m+1, \dots, 2m$ we compute

$$\begin{aligned}
\tilde{\Phi}_j^- [\mathcal{G}_\lambda]^{-1} \Psi_k^- &= (\bar{\tilde{\Phi}}_j + \bar{\tilde{\Psi}}_{2m+1-j})[\mathcal{G}_\lambda]^{-1} \frac{1}{\mu_k^-}(\bar{\Psi}_k^- - \Phi_{2m+1-k}^-) \\
&= \frac{1}{\mu_k}\Big[\bar{\tilde{\Phi}}_j [\mathcal{G}_\lambda]^{-1} \bar{\Psi}_k^- - \bar{\tilde{\Phi}}_j [\mathcal{G}_\lambda]^{-1} \Phi_{2m+1-k}^- \\
&+ \bar{\tilde{\Psi}}_{2m+1-j}[\mathcal{G}_\lambda]^{-1} \bar{\Psi}_k^- - \bar{\tilde{\Psi}}_{2m+1-j}[\mathcal{G}_\lambda]^{-1} \Phi_{2m+1-k}^-\Big] \\
&= \frac{1}{\mu_k^-}\Big[0 - \delta_j^{2m+1-k} + \delta_k^{2m+1-j} - 0\Big] = 0.
\end{aligned}$$

The other cases are similar. □

We observe now that since $\mathcal{G}_\lambda(x;y)$ should decay as $x \to \pm\infty$ for each fixed $y \in \mathbb{R}$, we can express it as a linear combination (with coefficients depending on y) of the asymptotically decaying extended vector solutions $\Phi^\pm(x;\lambda)$; at the same time, by Lemma 3.1 we can express $\mathcal{G}_\lambda(x;y)$ as a linear combination (with coefficients depending on x) of the asymptotically decaying extended dual solutions $\{\tilde{\Psi}^\pm(y;\lambda)\}$. Combining these observations, we conclude that there must exists $2m \times 2m$ matrices $M^\pm(\lambda)$ so that

$$\mathcal{G}_\lambda(x;y) = \begin{cases} \Phi^+(x;\lambda) M^+(\lambda) \tilde{\Psi}^-(y;\lambda) & x > y \\ \Phi^-(x;\lambda) M^-(\lambda) \tilde{\Psi}^+(y;\lambda) & x < y, \end{cases} \tag{3.18}$$

or equivalently

$$\mathcal{G}_\lambda(x;y) = \begin{cases} \begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} \begin{pmatrix} \tilde{\Psi}^+(y;\lambda) \\ 0 \end{pmatrix} & x > y \\ \begin{pmatrix} 0 & -\Phi^-(x;\lambda) \end{pmatrix} \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\Psi}^-(y;\lambda) \end{pmatrix} & x < y, \end{cases} \tag{3.19}$$

By definition of our notation $[\mathcal{G}_\lambda](y)$ we have

$$[\mathcal{G}_\lambda](y) = \Phi^+(y;\lambda) M^+(\lambda) \tilde{\Psi}^-(y;\lambda) - \Phi^-(y;\lambda) M^-(\lambda) \tilde{\Psi}^+(y;\lambda),$$

which can be re-written in the form

$$[\mathcal{G}_\lambda](y) = \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix} \begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} \begin{pmatrix} \tilde{\Psi}^+(y;\lambda) \\ \tilde{\Psi}^-(y;\lambda) \end{pmatrix}.$$

Solving for the coefficient matrix, we find

$$\begin{pmatrix} M^+(\lambda) & 0 \\ 0 & -M^-(\lambda) \end{pmatrix} = \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y) \begin{pmatrix} \tilde{\Psi}^+(y;\lambda) \\ \tilde{\Psi}^-(y;\lambda) \end{pmatrix}^{-1}. \tag{3.20}$$

Following [19] we find it notationally convenient to define (suppressing λ dependence) the ODE solution maps

$$\begin{aligned} \mathcal{F}^{z\to x} &:= \begin{pmatrix} \Phi^+(x) & \Phi^-(x) \end{pmatrix} \begin{pmatrix} \Phi^+(z) & \Phi^-(z) \end{pmatrix}^{-1} \\ \tilde{\mathcal{F}}^{z\to y} &:= \begin{pmatrix} \tilde{\Psi}^-(z) \\ \tilde{\Psi}^+(z) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^-(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}, \end{aligned} \tag{3.21}$$

and the associated projective maps

$$\begin{aligned} \Pi_+(y) &:= \begin{pmatrix} \Phi^+(y) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y) & \Phi^-(y) \end{pmatrix}^{-1} \\ \Pi_-(y) &:= \begin{pmatrix} 0 & \Phi^-(y) \end{pmatrix} \begin{pmatrix} \Phi^+(y) & \Phi^-(y) \end{pmatrix}^{-1} \\ \tilde{\Pi}_+(y) &:= \begin{pmatrix} \tilde{\Psi}^-(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \tilde{\Psi}^+(y) \end{pmatrix} \\ \tilde{\Pi}_-(y) &:= \begin{pmatrix} \tilde{\Psi}^-(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^-(y) \\ 0 \end{pmatrix}. \end{aligned} \tag{3.22}$$

Clearly, we have the relations

$$\begin{aligned} \Pi_-(y) + \Pi_+(y) &= I \\ \tilde{\Pi}_-(y) + \tilde{\Pi}_+(y) &= I. \end{aligned} \tag{3.23}$$

We can now state three lemmas that will be useful in our analysis of $\mathcal{G}_\lambda$.

Lemma 3.6. *Let $\lambda \in \mathbb{C}$ and suppose that for (1.5), the solutions $\{\phi_j^-(x;\lambda)\}_{j=1}^{2m}$ decay as $x \to -\infty$, the solutions $\{\psi_j^-(x;\lambda)\}_{j=1}^{2m}$ grow as $x \to -\infty$, and that together these solutions comprise a full basis of solutions to (1.5), and likewise at $+\infty$ for $\{\phi_j^+(x;\lambda)\}_{j=1}^{2m}$ and $\{\psi_j^+(x;\lambda)\}_{j=1}^{2m}$. Define $\Phi^\pm$ and $\tilde{\Psi}^\pm$ as in (3.13), and let the definitions (3.21) and (3.22) hold. Then*

$$\mathcal{G}_\lambda(x;y) = \begin{cases} \mathcal{F}^{z\to x}\Pi_+(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_-(z)\tilde{\mathcal{F}}^{z\to y} & x > y \\ -\mathcal{F}^{z\to x}\Pi_-(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_+(z)\tilde{\mathcal{F}}^{z\to y} & x < y. \end{cases}$$

Proof. In both cases we verify the claim by direct computation. Since the cases are similar, we proceed only for $x < y$. First, it's clear from (3.19) and (3.20) that for $x < y$

$$\mathcal{G}_\lambda(x;y) = -\begin{pmatrix} 0 & \Phi^-(x;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(z;\lambda) & \Phi^-(z;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](z) \begin{pmatrix} \tilde{\Psi}^+(z;\lambda) \\ \tilde{\Psi}^-(z;\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \tilde{\Psi}^-(y;\lambda) \end{pmatrix},$$

where we have observed that the quantity obtained from (3.20) is independent of y and can be evaluated at any $z \in \mathbb{R}$.

Computing directly from the definitions, we find (suppressing dependence on λ for notational brevity)

$$\begin{aligned}
&-\mathcal{F}^{z\to x}\Pi_-(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_+(z)\tilde{\mathcal{F}}^{z\to y} \\
&= -\begin{pmatrix}\Phi^+(x) & \Phi^-(x)\end{pmatrix}\begin{pmatrix}\Phi^+(z) & \Phi^-(z)\end{pmatrix}^{-1}\begin{pmatrix}0 & \Phi^-(z)\end{pmatrix}\begin{pmatrix}\Phi^+(z) & \Phi^-(z)\end{pmatrix}^{-1} \\
&\times [\mathcal{G}_\lambda](z)\begin{pmatrix}\tilde{\Psi}^-(z)\\ \tilde{\Psi}^+(z)\end{pmatrix}^{-1}\begin{pmatrix}0\\ \tilde{\Psi}^+(z)\end{pmatrix}\begin{pmatrix}\tilde{\Psi}^-(z)\\ \tilde{\Psi}^+(z)\end{pmatrix}^{-1}\begin{pmatrix}\tilde{\Psi}^-(y)\\ \tilde{\Psi}^+(y)\end{pmatrix} \\
&= -\begin{pmatrix}\Phi^+(x) & \Phi^-(x)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\Phi^+(z) & \Phi^-(z)\end{pmatrix}^{-1} \\
&\times [\mathcal{G}_\lambda](z)\begin{pmatrix}\tilde{\Psi}^-(z)\\ \tilde{\Psi}^+(z)\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\tilde{\Psi}^-(y)\\ \tilde{\Psi}^+(y)\end{pmatrix} \\
&= -\begin{pmatrix}0 & \Phi^-(x)\end{pmatrix}\begin{pmatrix}\Phi^+(z) & \Phi^-(z)\end{pmatrix}^{-1}[\mathcal{G}_\lambda](z)\begin{pmatrix}\tilde{\Psi}^-(z)\\ \tilde{\Psi}^+(z)\end{pmatrix}^{-1}\begin{pmatrix}0\\ \tilde{\Psi}^+(y)\end{pmatrix}
\end{aligned}.$$

Comparing the last expression here with the observation at the start of the proof, we obtain the claim. □

Likewise, we can establish the following *vanishing pi lemma*.

Lemma 3.7. *Under the assumptions of Lemma 3.6, we have*

$$\begin{aligned}
\Pi_+(z)[\mathcal{G}_\lambda](z) &= \Pi_+(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_-(z) = [\mathcal{G}_\lambda](z)\tilde{\Pi}_-(z)\\
\Pi_-(z)[\mathcal{G}_\lambda](z) &= \Pi_-(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_+(z) = [\mathcal{G}_\lambda](z)\tilde{\Pi}_+(z).
\end{aligned} \tag{3.24}$$

Proof. First, we note that all quantities are evaluated at z, so there will be no confusion if we leave off this dependence. Also, the proof is similar for each of the claimed equalities, so we will only establish the choice $\Pi_-[\mathcal{G}_\lambda] = \Pi_-[\mathcal{G}_\lambda]\tilde{\Pi}_+$. To this end, we begin by observing that it's clear from our entwinement relations that

$$\begin{pmatrix}\tilde{\Psi}^-\\ \tilde{\Psi}^+\end{pmatrix}[\mathcal{G}_\lambda]^{-1}\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix} = \begin{pmatrix}\tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^+ & \tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^-\\ \tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^+ & \tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-\end{pmatrix} = \begin{pmatrix}\tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^+ & 0\\ 0 & \tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-\end{pmatrix},$$

so that

$$\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}[\mathcal{G}_\lambda]\begin{pmatrix}\tilde{\Psi}^-\\ \tilde{\Psi}^+\end{pmatrix}^{-1} = \begin{pmatrix}(\tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^+)^{-1} & 0\\ 0 & (\tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-)^{-1}\end{pmatrix}. \tag{3.25}$$

Using this expression, we compute

$$\begin{aligned}
\Pi_-[\mathcal{G}_\lambda] &= \begin{pmatrix}0 & \Phi^-\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}[\mathcal{G}_\lambda]\\
&= \begin{pmatrix}0 & \Phi^-\end{pmatrix}\begin{pmatrix}(\tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^+)^{-1} & 0\\ 0 & (\tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-)^{-1}\end{pmatrix}\begin{pmatrix}\tilde{\Psi}^-\\ \tilde{\Psi}^+\end{pmatrix} = \Phi^-(\tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-)^{-1}\tilde{\Psi}^+.
\end{aligned}$$

Likewise,

$$
\begin{aligned}
\Pi_-[\mathcal{G}_\lambda]\tilde{\Pi}_+ &= \begin{pmatrix} 0 & \Phi^- \end{pmatrix} \begin{pmatrix} \Phi^+ & \Phi^- \end{pmatrix}^{-1} [\mathcal{G}_\lambda] \begin{pmatrix} \tilde{\Psi}^- \\ \tilde{\Psi}^+ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \tilde{\Psi}^+ \end{pmatrix} \\
&= \begin{pmatrix} 0 & \Phi^- \end{pmatrix} \begin{pmatrix} (\tilde{\Psi}^-[\mathcal{G}_\lambda]^{-1}\Phi^+)^{-1} & 0 \\ 0 & (\tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\Psi}^+ \end{pmatrix} \\
&= \Phi^-(\tilde{\Psi}^+[\mathcal{G}_\lambda]^{-1}\Phi^-)^{-1}\tilde{\Psi}^+.
\end{aligned}
$$

□

Lemma 3.8. *Under the assumptions of Lemma 3.6, we have*

$$
\mathcal{G}_\lambda(x;y) = \begin{cases} \begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y) & x > y \\ -\begin{pmatrix} 0 & \Phi^-(x;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y) & x < y. \end{cases}
$$

Proof. Since the proof is similar for each case we proceed only for $x < y$. Using Lemmas 3.6 and 3.7, we compute

$$
\begin{aligned}
\mathcal{G}_\lambda(x,y) &= -\mathcal{F}^{z\to x}\Pi_-(z)[\mathcal{G}_\lambda](z)\tilde{\Pi}_+(z)\tilde{\mathcal{F}}^{z\to y} \\
&= -\mathcal{F}^{z\to x}\Pi_-(z)[\mathcal{G}_\lambda](z)\tilde{\mathcal{F}}^{z\to y} \\
&= -\begin{pmatrix} \Phi^+(x) & \Phi^-(x) \end{pmatrix} \begin{pmatrix} \Phi^+(z) & \Phi^-(z) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Phi^-(z) \end{pmatrix} \begin{pmatrix} \Phi^+(z) & \Phi^-(z) \end{pmatrix}^{-1} \\
&\times [\mathcal{G}_\lambda](z) \begin{pmatrix} \tilde{\Psi}^-(z) \\ \tilde{\Psi}^+(z) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^-(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}.
\end{aligned}
$$

Now, z is arbitrary here, so we can take $z = y$ to get

$$
\mathcal{G}_\lambda(x;y) = -\begin{pmatrix} \Phi^+(x) & \Phi^-(x) \end{pmatrix} \begin{pmatrix} \Phi^+(y) & \Phi^-(y) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Phi^-(y) \end{pmatrix} \begin{pmatrix} \Phi^+(y) & \Phi^-(y) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y).
$$

The claim is now clear from the relation

$$
\begin{pmatrix} \Phi^+(y) & \Phi^-(y) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Phi^-(y) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
$$

□

We will divide our analysis of $G_\lambda(x;y)$ into three cases: (A) $|\lambda| < r$, $r > 0$ sufficiently small; (B) $|\lambda| > R$, $R > 0$ sufficiently large; and (C) $r \le |\lambda| \le R$.

3.1 Small $|\lambda|$ estimates

In this section, we focus on the case $|\lambda| < r$ for $r > 0$ sufficiently small. Our notation $\mathbf{O}(\cdot)$ in this section will always describe behavior for $|\lambda| < r$. For example, we will write

$h(\lambda) = \mathbf{O}(|\lambda^{-1/2}|)$ if there exists a constant C so that $|h(\lambda)| \le C|\lambda^{-1/2}|$ for $|\lambda| < r$, $\lambda \neq 0$. As always, we assume $\text{Arg}\lambda \neq \pi$. If h depends additionally on x and y, then the notation $|h(\lambda)| \le C|\lambda^{-1/2}|$ refers to behavior uniform in x and y. Likewise, if we write $h = \mathbf{O}(e^{-\eta|x|})$, we mean there exists a constant C so that for all $|\lambda| < r$ and all y in its specified domain, we have $|h| \le Ce^{-\eta|x|}$.

Lemma 3.9. *Suppose Conditions (C1)-(C2) hold, and also that spectral Condition 1.1 holds. Let $\mathcal{G}_\lambda$ be as defined in (3.5). There exists a value $r > 0$ sufficiently small so that for $|\lambda| < r$, with Arg $\lambda \neq \pi$ we have the following representation for $y < x < 0$:*

$$\mathcal{G}_\lambda(x;y) = \Phi^-(x;\lambda)E(\lambda)\tilde{\Psi}^-(y;\lambda) + \Psi^-(x;\lambda)\tilde{\Psi}^-(y;\lambda).$$

Here, $E(\lambda) \in \mathbb{R}^{2m\times 2m}$ with components $\{e_{ij}(\lambda)\}_{i,j=1}^{2m}$ can be characterized as follows: there exist real values $\{h_{ij}^-\}_{i,j=1}^{2m}$ so that

(i) For $i = 1, 2, \dots, 2m-1$ and $j = 1, 2, \dots, m$

$$e_{ij}(\lambda) = h_{ij}^- + \mathbf{O}(|\lambda^{1/2}|)$$

(ii) For $i = 1, 2, \dots, 2m-1$ and $j = m+1, m+2, \dots, 2m$

$$e_{ij}(\lambda) = h_{ij}^-\lambda^{-1/2} + \mathbf{O}(1)$$

(iii) For $i = 2m$ and $j = 1, 2, \dots, m$

$$e_{ij}(\lambda) = h_{ij}^-\lambda^{-1/2} + \mathbf{O}(1)$$

(iv) For $i = 2m$ and $j = m+1, m+2, \dots, 2m$

$$e_{ij}(\lambda) = h_{ij}^-\lambda^{-1} + \mathbf{O}(|\lambda^{-1/2}|).$$

Proof. First, for $y < x$ we have

$$\mathcal{G}_\lambda(x;y) = \Phi^+(x;\lambda)M^+(\lambda)\tilde{\Psi}^-(y;\lambda),$$

and we observe that for $x < 0$ we need to expand $\Phi^+(x;\lambda)$ in terms of $\Phi^-(x;\lambda)$ and $\Psi^-(x;\lambda)$. More precisely, there exist $2m \times 2m$ matrices $E(\lambda)$ and $F(\lambda)$ so that

$$\mathcal{G}_\lambda(x;y) = \Phi^-(x;\lambda)E(\lambda)\tilde{\Psi}^-(y;\lambda) + \Psi^-(x;\lambda)F(\lambda)\tilde{\Psi}^-(y;\lambda).$$

At the same time, according to Lemma 3.8, we have

$$\mathcal{G}_\lambda(x;y) = \begin{pmatrix}\Phi^+(x;\lambda) & 0\end{pmatrix}\begin{pmatrix}\Phi^+(y;\lambda) & \Phi^-(y;\lambda)\end{pmatrix}^{-1}[\mathcal{G}_\lambda](y).$$

If we equate these expressions for $\mathcal{G}_\lambda$, multiply each on the right by $[\mathcal{G}_\lambda]^{-1}\Psi^-$ and use the identity $\tilde{\Psi}^-[\mathcal{G}_\lambda]\Psi^- = I$, we find

$$\begin{pmatrix}\Phi^+(x;\lambda) & 0\end{pmatrix}\begin{pmatrix}\Phi^+(y;\lambda) & \Phi^-(y;\lambda)\end{pmatrix}^{-1}\Psi^-(y;\lambda) = \begin{pmatrix}\Phi^-(x;\lambda) & \Psi^-(x;\lambda)\end{pmatrix}\begin{pmatrix}E(\lambda)\\ F(\lambda)\end{pmatrix}.$$

Solving this system for the scattering coefficients $E(\lambda)$ and $F(\lambda)$, we find

$$\begin{pmatrix} E(\lambda) \\ F(\lambda) \end{pmatrix} = \begin{pmatrix} \Phi^-(x;\lambda) & \Psi^-(x;\lambda) \end{pmatrix}^{-1} \begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda).$$

Since the right-hand side is independent of x and y, we can take $y = x$. Recalling the definition of $\Pi_+(y)$ in (3.22) and using (3.23), we compute

$$\begin{aligned}
\begin{pmatrix} E(\lambda) \\ F(\lambda) \end{pmatrix} &= \begin{pmatrix} \Phi^-(y;\lambda) & \Psi^-(y;\lambda) \end{pmatrix}^{-1} \Pi_+(y)\Psi^-(y;\lambda) \\
&= \begin{pmatrix} \Phi^-(y;\lambda) & \Psi^-(y;\lambda) \end{pmatrix}^{-1} (I - \Pi_-(y))\Psi^-(y;\lambda) \\
&= \begin{pmatrix} \Phi^-(y;\lambda) & \Psi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda) \\
&- \begin{pmatrix} \Phi^-(y;\lambda) & \Psi^-(y;\lambda) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Phi^-(y;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda) \\
&= \begin{pmatrix} 0 \\ I \end{pmatrix} - \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda).
\end{aligned}$$

In this way, it's clear that $F(\lambda) = I$ and

$$E(\lambda) = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda). \tag{3.26}$$

We can characterize the components of $E(\lambda)$, denoted here $\{e_{ij}\}_{i,j=1}^{2m}$ by Cramer's Rule as

$$e_{ij}(\lambda) = -\frac{\det\left(\Phi_1^+, \ldots, \Phi_{2m}^+, \Phi_1^-, \ldots, \Psi_j^-, \ldots, \Phi_{2m}^-\right)}{\det\left(\Phi^+, \Phi^-\right)}, \tag{3.27}$$

where Ψ_j^- appears in the $(2m+i)$th slot. We focus our attention on the numerator, which for convenience in the following discussion we label

$$N_{ij} = \det\left(\Phi_1^+, \ldots, \Phi_{2m}^+, \Phi_1^-, \ldots, \Psi_j^-, \ldots, \Phi_{2m}^-\right).$$

The discussion here is based on the paragraph immediately following Remark 2.1.

Slow-fast. First, suppose $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m\}$ so that a slow decay mode is replaced by a fast growth mode. In this case N_{ij} vanishes at least to the same order as $D_a(\rho)$, and we can conclude $e_{ij}(\lambda) = h_{ij}^- + \mathbf{O}(|\lambda^{1/2}|)$.

Slow-slow. For the cases $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, 2m\}$ a slow decay mode is replaced by a slow growth mode. It's important to keep in mind at this point that we are working with the difference forms of our slow growth modes (see Part (iii) of Lemma 2.1), and so in particular

$$\psi_j^{-\prime}(x;\lambda) = \left(e^{\mu_j^-(\lambda)x} + e^{-\mu_j^-(\lambda)x}\right) r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|}), \tag{3.28}$$

so that

$$\lim_{x\to-\infty}\psi_j^{-\prime}(x;0)=2r_j^-(0).$$

In this way, if we set $\lambda=0$ in (1.5) for ψ_j^- and integrate once we obtain

$$M(x)\Big(-\Gamma\psi_j^{-\prime\prime\prime}(x;0)+B(x)\psi_j^-(x;0)\Big)'=2M_-B_-r_j^-(0).$$

In this way, we cannot eliminate $\psi_j^{-\prime\prime\prime}(x;0)$ with matrix row operations, and so $\frac{d^m N_{ij}}{d\rho^m}(0)$ may not be 0 in these cases. Since N_{ij} vanishes to a lower order than $D_a(\rho)$, we conclude $e_{ij}(\lambda)=h_{ij}^-\lambda^{-1/2}+\mathbf{O}(|\lambda|)$.

Fast-fast. For the cases $i\in\{m+1,\dots,2m-1\}$ and $j\in\{1,\dots,m\}$ a fast decay mode (though not ϕ_{2m}^-, which corresponds with $\bar{u}'$) is replaced by a fast growth mode. Since the fast growth modes have been chosen by scaling so that

$$-\Gamma\psi_j^{-\prime\prime}(x;0)+B(x)\psi_j^-(x;0)=0,$$

we find that in these cases the $N_{ij}(\rho)$ vanish to the same order in ρ as $D_a(\rho)$. We conclude $e_{ij}(\lambda)=h_{ij}^-+\mathbf{O}(|\lambda^{1/2}|)$.

Fast-slow. For the cases $i\in\{m+1,\dots,2m-1\}$ and $j=\{m+1,\dots,2m\}$, we replace a fast decay mode (though not ϕ_{2m}^-, which corresponds with $\bar{u}'$) with a slow growth mode. As discussed in the *slow-slow* case above, slow growth modes can reduce the order to which $N_{ij}(\rho)$ vanishes by one order, and so as in that case we conclude $e_{ij}(\lambda)=h_{ij}^-\lambda^{-1/2}+\mathbf{O}(|\lambda|)$.

Excited-fast. For the cases $i=2m$ and $j\in\{1,\dots,m\}$ the mode ϕ_{2m}^- (which corresponds with $\bar{u}'$) is replaced by a fast growth mode. In this case, the first non-zero ρ derivative occurs when a single ρ derivative appears on each of m different slow decay modes. In this way, $N_{(2m)j}(\rho)$ vanishes to order $m-1$, and we conclude $e_{ij}(\lambda)=h_{ij}^-\lambda^{-1/2}+\mathbf{O}(|\lambda|)$.

excited-slow. For the cases $i=2m$ and $j\in\{m+1,\dots,2m\}$ the mode ϕ_{2m}^- (which corresponds with $\bar{u}'$) is replaced by a slow growth mode. In this case the (un-differentiated) slow growth mode contributes a full column (no entries necessarily 0), and so for the determinant of a $4m\times 4m$ matrix we require only $m-1$ additional full columns, which can correspond with a ρ derivative on each of $m-1$ slow decay modes. In this way, $N_{(2m)j}(\rho)$ may only vanish up to order $m-2$ in ρ, and we conclude $e_{(2m)j}(\lambda)=h_{(2m)j}^-\lambda^{-1}+\mathbf{O}(|\lambda^{-1/2}|)$. □

Lemma 3.10. *Let the assumptions of Lemma 3.9 hold and consider the case $x<y<0$. There exists a value $r>0$ sufficiently small so that for $|\lambda|<r$, with Arg $\lambda\neq\pi$, we have the representation*

$$\mathcal{G}_\lambda(x;y)=-\Phi^-(x;\lambda)\tilde{\Phi}^-(y;\lambda)+\Phi^-(x;\lambda)E(\lambda)\tilde{\Psi}^-(y;\lambda),$$

where $E(\lambda)\in\mathbb{R}^{2m\times 2m}$ is precisely as described in Lemma 3.9.

Proof. First, for $y>x$ we have

$$\mathcal{G}_\lambda(x,y)=\Phi^-(x;\lambda)M^+(\lambda)\tilde{\Psi}^+(y;\lambda),$$

and we observe that for $y < 0$ we need to expand $\tilde{\Psi}^+(y;\lambda)$ in terms of $\tilde{\Phi}^-(y;\lambda)$ and $\tilde{\Psi}^-(y;\lambda)$. More precisely, there exist $2m \times 2m$ matrices $\tilde{E}(\lambda)$ and $\tilde{F}(\lambda)$ so that

$$\mathcal{G}_\lambda(x,y) = \Phi^-(x;\lambda)\tilde{E}(\lambda)\tilde{\Psi}^-(y;\lambda) + \Phi^-(x;\lambda)\tilde{F}(\lambda)\tilde{\Phi}^-(y;\lambda).$$

At the same time, according to Lemma 3.8, we have

$$\mathcal{G}_\lambda(x,y) = -\begin{pmatrix} 0 & \Phi^+(x;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y).$$

Equating these representations, we have

$$-\begin{pmatrix} 0 & \Phi^+(x;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y) = \Phi^-(x;\lambda)\Big\{\tilde{E}(\lambda)\tilde{\Psi}^-(y;\lambda) + \tilde{F}(\lambda)\tilde{\Phi}^-(y;\lambda)\Big\}. \tag{3.29}$$

We multiply this equality on the right by $[\mathcal{G}_\lambda]^{-1}(y)\Phi^-(y;\lambda)$ and use (3.14) and (3.15) to obtain

$$-\begin{pmatrix} 0 & \Phi^+(x;\lambda) \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Phi^-(y;\lambda) = \Phi^-(x;\lambda)\tilde{F}(\lambda),$$

and we multiply this new equality on the left by $\tilde{\Psi}^-(x;\lambda)[\mathcal{G}_\lambda]^{-1}(x)$ to obtain

$$\tilde{F}(\lambda) = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Phi^-(y;\lambda)[\mathcal{G}_\lambda]^{-1}(y) = -I.$$

Likewise, if we multiply (3.29) on the right by $[\mathcal{G}_\lambda]^{-1}(y)\Psi^-(y;\lambda)$ and on the left by $\tilde{\Psi}(x;\lambda)[\mathcal{G}_\lambda]^{-1}(x)$ we find

$$\tilde{E}(\lambda) = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} \Psi^-(y;\lambda),$$

and comparing with (3.26), we observe that $\tilde{E}(\lambda) = E(\lambda)$. □

Lemma 3.11. *Let the assumptions of Lemma 3.9 hold and consider the case $y < 0 < x$. There exists a value $r > 0$ sufficiently small so that for $|\lambda| < r$, with Arg $\lambda \neq \pi$, we have the representation*

$$\mathcal{G}_\lambda(x,y) = \Phi^-(x;\lambda)M^+(\lambda)\tilde{\Psi}^-(y;\lambda),$$

where $M^+(\lambda) \in \mathbb{R}^{2m\times 2m}$ with components $\{m_{ij}^+(\lambda)\}_{i,j=1}^{2m}$ characterized as follows: there exist real values $\{s_{ij}^+\}_{i,j=1}^{2m}$ so that

(i) For $i = 2, \dots, 2m$ and $j = 1, 2, \dots, m$

$$m_{ij}^+(\lambda) = s_{ij}^+ + \mathbf{O}(|\lambda^{1/2}|)$$

(ii) For $i = 2, \dots, 2m$ and $j = m+1, m+2, \dots, 2m$

$$m_{ij}^+(\lambda) = s_{ij}^+\lambda^{-1/2} + \mathbf{O}(1)$$

(iii) For $i = 1$ and $j = 1, 2, \dots, m$

$$m_{ij}^+(\lambda) = s_{ij}^+\lambda^{-1/2} + \mathbf{O}(1)$$

(iv) For $i = 1$ and $j = m+1, m+2, \dots, 2m$

$$m^+_{ij}(\lambda) = s^+_{ij}\lambda^{-1} + \mathbf{O}(|\lambda^{-1/2}|).$$

Moreover, $s^+_{1j} = h^-_{(2m)j}$ for all $j = 1, \dots, 2m$.

Proof. Using Lemma 3.8, we have

$$\Phi^-(x;\lambda)M^+(\lambda)\tilde{\Psi}^-(y;\lambda) = \begin{pmatrix}\Phi^+(x;\lambda) & 0\end{pmatrix}\begin{pmatrix}\Phi^+(y;\lambda) & \Phi^-(y;\lambda)\end{pmatrix}^{-1}[\mathcal{G}_\lambda](y).$$

We multiply both sides on the left by $\tilde{\Phi}^+(x;\lambda)[\mathcal{G}_\lambda]^{-1}(x)$ and on the right by $[\mathcal{G}_\lambda]^{-1}(y)\Psi^-(y;\lambda)$ to obtain

$$M^+(\lambda) = \begin{pmatrix}I & 0\end{pmatrix}\begin{pmatrix}\Phi^+(y;\lambda) & \Phi^-(y;\lambda)\end{pmatrix}^{-1}\Psi^-(y;\lambda).$$

According to Cramer's rule, the components m^+_{ij} can be written as

$$m^+_{ij}(\lambda) = \frac{\det\Big(\Phi^+_1, \dots, \Psi^-_j, \dots, \Phi^+_{2m}, \Phi^-_1, \dots, \Phi^-_{2m}\Big)}{\det\Big(\Phi^+, \Phi^-\Big)}, \tag{3.30}$$

where Ψ^-_j appears in the ith slot. Claims (i)-(iv) now follow almost precisely as in the proof of Lemma 3.9.

Finally, regarding s^+_{1j}, we observe that the associated numerator satisfies

$$\begin{aligned} N(0) &= W(\psi^-_j, \phi^+_2, \dots, \phi^+_{2m}, \phi^-_1, \dots, \bar{u}') = -W(\bar{u}', \phi^+_2, \dots, \phi^+_{2m}, \phi^-_1, \dots, \psi^-_j) \\ &= -W(\phi^+_1, \dots, \phi^+_{2m}, \phi^-_1, \dots, \psi^-_j), \end{aligned} \tag{3.31}$$

which is precisely the numerator in $e_{(2m)j}(\lambda)$, evaluated at $\lambda = 0$ (cf. (3.27)). □

In what follows, we will obtain estimates on $\mathcal{G}_\lambda(x,y)$ in three different regions: (i) $y < x < 0$; (ii) $x < y < 0$; and (iii) $y < 0 < x$. However, we find that the leading order terms (in λ) have the form $E_\lambda(x,y) = \bar{u}'(x)e_\lambda(y)$, where $e_\lambda(y)$ is case independent. We begin, then, with a lemma in which we define and characterize $E_\lambda(x,y)$ in all cases.

Lemma 3.12. *Let the assumptions of Lemma 3.9 hold and consider the case $y < 0$. Set*

$$e_\lambda(y) := \sum_{j=1}^{m} h^-_{(2m)j}\lambda^{-1/2}\tilde{\psi}^-_j(y;\lambda) + \sum_{j=m+1}^{2m} h^-_{(2m)j}\lambda^{-1}\tilde{\psi}^-_j(y;\lambda).$$

There exists a value $r > 0$ sufficiently small, and a constant $\eta > 0$ so that for $|\lambda| < r$, with $Arg\,\lambda \neq \pi$, the following estimates hold:

$$e_\lambda(y) = \frac{1}{\lambda}\sum_{j=m+1}^{2m} h^-_{(2m)j}e^{-\mu^-_j(\lambda)y}\tilde{c}^-_j(0)\tilde{r}^-_j(0) + \mathbf{O}(|\lambda^{-1/2}|)e^{-\mu^-_{2m}(\lambda)y}$$

$$e'_\lambda(y) = \frac{1}{\lambda}\sum_{j=m+1}^{2m} h^-_{(2m)j}e^{-\mu^-_j(\lambda)y}(-\mu^-_j(\lambda))\tilde{c}^-_j(0)\tilde{r}^-_j(0) + \mathbf{O}(1)e^{-\mu^-_{2m}(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(e^{-\eta|y|}).$$

Remark 3.2. *The fact that $e_\lambda(y)$ is the same for all three regions (i)-(iii) follows from the appearance of $E(\lambda)$ in both Lemma 3.9 and Lemma 3.10, and from the Moreover part of Lemma 3.11.*

Proof. For the fast $\tilde{\psi}_j^-$ (i.e., for $j = 1, \dots, m$), we have

$$\sum_{j=1}^{m} h^-_{(2m)j} \lambda^{-1/2} \tilde{\psi}_j^-(y;\lambda) = \sum_{j=1}^{m} h^-_{(2m)j} \lambda^{-1/2} \mathbf{O}(e^{-\eta|y|}) = \lambda^{-1/2}\mathbf{O}(e^{-\eta|y|}).$$

For the slow $\tilde{\psi}_j^-$ (i.e., for $j = m+1, \dots, 2m$), we have

$$\begin{aligned}\sum_{j=m+1}^{2m} h^-_{(2m)j} \lambda^{-1} \tilde{\psi}_j^-(y;\lambda) &= \sum_{j=m+1}^{2m} h^-_{(2m)j} \lambda^{-1} \tilde{c}_j^-(\lambda) e^{-\mu_j^-(\lambda)y} (\tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|y|})) \\ &= \sum_{j=m+1}^{2m} h^-_{(2m)j} \tilde{c}_j^-(0) \tilde{r}_j^-(0) \lambda^{-1} e^{-\mu_j^-(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|) e^{-\mu^-_{2m}(\lambda)y},\end{aligned}$$

where we have observed that $\mu^-_{2m}(\lambda)$ is the slow mode closest to 0 for small values of $|\lambda|$.

The derivative estimate follows similarly. □

Lemma 3.13. *Let the assumptions of Lemma 3.9 hold and consider the case $y < x < 0$. There exists a value $r > 0$ sufficiently small and a value $\eta > 0$ so that for $|\lambda| < r$, with $Arg\, \lambda \neq \pi$, we have the representation*

$$G_\lambda(x;y) = \bar{u}'(x) e_\lambda(y) + R_\lambda(x;y),$$

where $e_\lambda(y)$ is specified in Lemma 3.12 and $R_\lambda(x;y)$ satisfies the following estimates:

$$\begin{aligned}R_\lambda(x;y) = &\sum_{j=m+1}^{2m} \frac{\tilde{c}_j^-(\lambda)}{\mu_j^-(\lambda)} \Big(e^{\mu_j^-(\lambda)(x-y)} - e^{-\mu_j^-(\lambda)(x+y)}\Big) r_j^-(\lambda) \tilde{r}_j^-(\lambda) \\ &+ \mathbf{O}(|\lambda^{-1/2}|) e^{\mu^-_{2m+1}(\lambda)x} e^{-\mu^-_{2m}(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}).\end{aligned}$$

$$\begin{aligned}\partial_y R_\lambda(x;y) = -&\sum_{j=m+1}^{2m} \Big(e^{\mu_j^-(\lambda)(x-y)} - e^{-\mu_j^-(\lambda)(x+y)}\Big) r_j^-(\lambda) \tilde{c}_j^-(\lambda) \tilde{r}_j^-(\lambda) \\ &+ \mathbf{O}(1) e^{-\mu^-_{2m}(\lambda)(x+y)} + \mathbf{O}(e^{-\eta|x-y|}).\end{aligned}$$

$$\begin{aligned}\partial_x R_\lambda(x;y) = &\sum_{j=m+1}^{2m} \Big(e^{\mu_j^-(\lambda)(x-y)} + e^{-\mu_j^-(\lambda)(x+y)}\Big) r_j^-(\lambda) \tilde{c}_j^-(\lambda) \tilde{r}_j^-(\lambda) \\ &+ \mathbf{O}(1) e^{\mu^-_{2m}(\lambda)(x+y)} + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(e^{-\eta|x|}) e^{-\mu^-_{2m}(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}).\end{aligned}$$

$$\partial_{xy}R_\lambda(x;y) = -\sum_{j=m+1}^{2m}\Big(e^{\mu_j^-(\lambda)(x-y)} + e^{-\mu_j^-(\lambda)(x+y)}\Big)\mu_j^-(\lambda)r_j^-(\lambda)\tilde{c}_j^-(\lambda)\tilde{r}_j^-(\lambda)$$
$$+\mathbf{O}(e^{-\eta|x|})e^{-\mu_{2m}^-(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}) + \mathbf{O}(|\lambda^{1/2}|)e^{-\mu_{2m}^-(\lambda)(x+y)}.$$

Proof. First, we observe that for $y < x < 0$ Lemma 3.9 allows us to write

$$G_\lambda(x;y) = \sum_{i,j=1}^{2m} e_{ij}(\lambda)\phi_i^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda) + \sum_{i=1}^{2m}\psi_i^-(x;\lambda)\tilde{\psi}_i^-(y;\lambda),$$

and likewise, of course, expressions for x and y derivatives of $G_\lambda(x,y)$ can be obtained by placing x and y derivatives appropriately on the right-hand side.

As in the proof of Lemma 3.9, we will divide the analysis into cases.

Excited terms. In this case, the excited terms comprise the summands

$$\sum_{j=1}^{2m} e_{(2m)j}(\lambda)\phi_{2m}^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda),$$

where we recall our convention

$$\phi_{2m}^-(x;0) = \bar{u}'(x),$$

and by a perturbation argument that employs analyticity of $\phi_{2m}^-(x;\lambda)$,

$$\phi_{2m}^-(x;\lambda) = \bar{u}'(x) + \lambda\mathbf{O}(e^{-\eta|x|}),$$

for some $\eta > 0$. (Recall that our notation $\mathbf{O}(e^{-\eta|x|})$ indicates an estimate uniform for $|\lambda| < r$.)

For the fast $\tilde{\psi}_j^-$ (i.e., for $j = 1, \dots, m$), we have

$$\begin{aligned}
&\sum_{j=1}^{m} e_{(2m)j}(\lambda)\phi_{2m}^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda)\\
&\qquad= \sum_{j=1}^{m}(h_{(2m)j}^-\lambda^{-1/2} + \mathbf{O}(1))(\bar{u}'(x) + \lambda\mathbf{O}(e^{-\eta|x|}))\tilde{\psi}_j^-(y;\lambda)\\
&\qquad= \bar{u}'(x)\sum_{j=1}^{m} h_{(2m)j}^-\lambda^{-1/2}\tilde{\psi}_j^-(y;\lambda) + \mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|}).
\end{aligned}$$

The first of these will appear in $E_\lambda(x,y)$ while the second will appear in $R_\lambda(x,y)$.

For the slow $\tilde{\psi}_j^-$ (i.e., for $j = m+1, \ldots, 2m$), we have

$$
\begin{aligned}
&\sum_{j=m+1}^{2m} e_{(2m)j}(\lambda)\phi_{2m}^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda) \\
&\qquad = \sum_{j=m+1}^{2m} (h_{(2m)j}^-\lambda^{-1} + \mathbf{O}(|\lambda^{-1/2}|))(\bar{u}'(x) + \lambda\mathbf{O}(e^{-\eta|x|}))\tilde{\psi}_j^-(y;\lambda) \\
&\qquad = \bar{u}'(x)\sum_{j=m+1}^{2m} h_{(2m)j}^-\lambda^{-1}\tilde{\psi}_j^-(y;\lambda) \\
&\qquad + \sum_{j=m+1}^{2m} \lambda^{-1/2}\mathbf{O}(e^{-\eta|x|}))\tilde{c}_j^-(\lambda)e^{-\mu_j^-(\lambda)y}(\tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|y|})) \\
&\qquad = \bar{u}'(x)\sum_{j=m+1}^{2m} h_{(2m)j}^-\lambda^{-1}\tilde{\psi}_j^-(y;\lambda) + \lambda^{-1/2}\mathbf{O}(e^{-\eta|x|})e^{-\mu_{2m}^-(\lambda)y},
\end{aligned}
$$

where we have observed that $\mu_{2m}^-(\lambda)$ is the slow mode closest to 0 for small values of $|\lambda|$. Again, the first term will be taken as a summand in $E_\lambda(x,y)$ while the second will be a summand in $R_\lambda(x,y)$.

The remaining terms will all be incorporated into $R_\lambda(x,y)$.

Fast-fast terms. The fast-fast terms comprise the summands

$$
\begin{aligned}
&\sum_{i=m+1}^{2m-1}\sum_{j=1}^{m} e_{ij}(\lambda)\phi_i^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda) \\
&\qquad = \sum_{i=m+1}^{2m-1}\sum_{j=1}^{m}(h_{ij}^- + \mathbf{O}(|\lambda^{1/2}|))\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|}) = \mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|}).
\end{aligned}
\tag{3.32}
$$

Fast-slow terms. The fast-slow terms comprise the summands

$$
\begin{aligned}
&\sum_{i=m+1}^{2m-1}\sum_{j=m+1}^{2m} e_{ij}(\lambda)\phi_i^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda) \\
&\qquad = \sum_{i=m+1}^{2m-1}\sum_{j=m+1}^{2m}(h_{ij}^-\lambda^{-1/2} + \mathbf{O}(1))\mathbf{O}(e^{-\eta_j|x|})e^{-\mu_j^-(\lambda)y} = \lambda^{-1/2}\mathbf{O}(e^{-\eta|x|})e^{-\mu_{2m}^-(\lambda)y}.
\end{aligned}
\tag{3.33}
$$

Slow-fast terms. The slow-fast terms comprise the summands

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{m} e_{ij}(\lambda)\phi_i^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda) \\
&\qquad = \sum_{i=1}^{m}\sum_{j=1}^{m}(h_{ij}^- + \mathbf{O}(|\lambda^{1/2}|))e^{\mu_{2m+i}^-(\lambda)x}\mathbf{O}(e^{-\eta_j|y|}) = e^{\mu_{2m+1}^-(\lambda)x}\mathbf{O}(e^{-\eta|y|}).
\end{aligned}
\tag{3.34}
$$

Slow-slow terms. The slow-slow terms comprise the summands

$$\sum_{i=1}^{m}\sum_{j=1}^{m} e_{ij}(\lambda)\phi_i^-(x;\lambda)\tilde{\psi}_j^-(y;\lambda)$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m}(h_{ij}^-\lambda^{-1/2} + \mathbf{O}(1))e^{\mu_{2m+i}^-(\lambda)x}\mathbf{O}(1)e^{-\mu_{2m}^-(\lambda)y}\mathbf{O}(1) = \lambda^{-1/2}e^{\mu_{2m+1}^-(\lambda)(x+y)}\mathbf{O}(1). \tag{3.35}$$

Fast growth-decay terms. The fast growth-decay terms comprise the summands

$$\sum_{i=1}^{m}\psi_i^-(x;\lambda)\tilde{\psi}_i^-(y;\lambda) = \sum_{i=1}^{m} e^{\mu_i^-(\lambda)(x-y)}\mathbf{O}(1) = \mathbf{O}(e^{-\eta|x-y|}).$$

Slow growth-decay terms. The slow growth-decay terms comprise the summands

$$\sum_{i=m+1}^{2m}\psi_i^-(x;\lambda)\tilde{\psi}_i^-(y;\lambda) = \sum_{i=m+1}^{2m}\Big\{\Big(\frac{1}{\mu_j^-(\lambda)}(e^{\mu_j^-(\lambda)x} - e^{-\mu_j^-(\lambda)x})r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big)$$
$$\times\, \tilde{c}_j^-(\lambda)e^{-\mu_j^-(\lambda)y}(\tilde{r}_j^-(\lambda) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|y|}))\Big\}$$
$$= \sum_{i=m+1}^{2m}\frac{\tilde{c}_j^-(\lambda)}{\mu_j^-(\lambda)}(e^{\mu_j^-(\lambda)(x-y)} - e^{-\mu_j^-(\lambda)(x+y)})r_j^-(\lambda)\tilde{r}_j^-(\lambda)$$
$$+\, e^{\mu_{2m}^-(\lambda)(x-y)}\mathbf{O}(e^{-\eta|y|}) + e^{-\mu_{2m}^-(\lambda)y}\mathbf{O}(e^{-\eta|x|}) + \sqrt{\lambda}\mathbf{O}(e^{-\eta|x-y|}).$$

The claimed estimate consists of the larger of these terms. The remaining cases for this lemma are established in a very similar manner, with derivatives on the growth and decay modes as appropriate. ☐

Estimates on $G_\lambda(x;y)$ and its derivatives for the cases $x < y < 0$ and $y < 0 < x$ are proven similarly, and we state the results without proof in the next two lemmas.

Lemma 3.14. *Let the assumptions of Lemma 3.9 hold and consider the case $x < y < 0$. There exists a value $r > 0$ sufficiently small and a value $\eta > 0$ so that for $|\lambda| < r$, with Arg $\lambda \neq \pi$, we have the representation*

$$G_\lambda(x;y) = \bar{u}'(x)e_\lambda(y) + R_\lambda(x;y),$$

where $e_\lambda(y)$ is specified in Lemma 3.12 and $R_\lambda(x,y)$ satisfies the following estimates:

$$R_\lambda(x;y) = \sum_{j=1}^{m}\frac{\tilde{c}_{2m+j}^-(\lambda)}{\mu_{2m+j}^-(\lambda)}\Big(e^{\mu_{2m+j}^-(\lambda)(x-y)} - e^{-\mu_{2m+j}^-(\lambda)(x+y)}\Big)r_{2m+j}^-(\lambda)\tilde{r}_{2m+j}^-(\lambda)$$
$$+\, \mathbf{O}(|\lambda^{-1/2}|)e^{-\mu_{2m}^-(\lambda)(x+y)} + \mathbf{O}(e^{-\eta|x-y|}).$$

$$\partial_y R_\lambda(x;y) = -\sum_{j=1}^{m}\Big(e^{\mu^-_{2m+j}(\lambda)(x-y)} + e^{-\mu^-_{2m+j}(\lambda)(x+y)}\Big) r^-_{2m+j}(\lambda)\tilde{c}^-_{2m+j}(\lambda)\tilde{r}^-_{2m+j}(\lambda)$$
$$+ \mathbf{O}(1)e^{-\mu^-_{2m}(\lambda)(x+y)} + \mathbf{O}(e^{-\eta|x-y|}).$$
$$\partial_x R_\lambda(x;y) = \sum_{j=1}^{m}\Big(e^{\mu^-_{2m+j}(\lambda)(x-y)} - e^{-\mu^-_{2m+j}(\lambda)(x+y)}\Big) r^-_{2m+j}(\lambda)\tilde{c}^-_{2m+j}(\lambda)\tilde{r}^-_{2m+j}(\lambda)$$
$$+ \mathbf{O}(1)e^{\mu^-_{2m}(\lambda)(x+y)} + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(e^{-\eta|x|})e^{-\mu^-_{2m}(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}).$$
$$\partial_{xy} R_\lambda(x;y) = -\sum_{j=1}^{m}\Big(e^{\mu^-_{2m+j}(\lambda)(x-y)} + e^{-\mu^-_{2m+j}(\lambda)(x+y)}\Big) \mu^-_{2m+j}(\lambda) r^-_{2m+j}(\lambda)\tilde{c}^-_{2m+j}(\lambda)\tilde{r}^-_{2m+j}(\lambda)$$
$$+ \mathbf{O}(e^{-\eta|x|})e^{-\mu^-_{2m}(\lambda)y} + \mathbf{O}(e^{-\eta|x-y|}) + \mathbf{O}(|\lambda^{1/2}|)e^{-\mu^-_{2m}(\lambda)(x+y)}.$$

Lemma 3.15. *Let the assumptions of Lemma 3.9 hold and consider the case $y < 0 < x$. There exists a value $r > 0$ sufficiently small and a value $\eta > 0$ so that for $|\lambda| < r$, with Arg $\lambda \neq \pi$, we have the representation*

$$G_\lambda(x;y) = \bar{u}'(x)e_\lambda(y) + R_\lambda(x;y),$$

where $e_\lambda(y)$ is specified in Lemma 3.12 and $R_\lambda(x,y)$ satisfies the following estimates:

$$R_\lambda(x;y) = \mathbf{O}(|\lambda^{-1/2}|)e^{\mu^+_{2m}(\lambda)x - \mu^-_{2m}(\lambda)y}.$$
$$\partial_y R_\lambda(x;y) = \mathbf{O}(1)e^{\mu^+_{2m}(\lambda)x - \mu^-_{2m}(\lambda)y}.$$
$$\partial_x R_\lambda(x;y) = \mathbf{O}(1)e^{\mu^+_{2m}(\lambda)x - \mu^-_{2m}(\lambda)y} + \mathbf{O}(|\lambda^{-1/2}|)\mathbf{O}(e^{-\eta|x|})e^{-\mu^-_{2m}(\lambda)y}.$$
$$\partial_{xy} R_\lambda(x;y) = \mathbf{O}(|\lambda^{1/2}|)e^{\mu^+_{2m}(\lambda)x - \mu^-_{2m}(\lambda)y} + \mathbf{O}(e^{-\eta|x|})e^{-\mu^-_{2m}(\lambda)y}.$$

3.2 Large $|\lambda|$ estimates

In this section, we focus on the case $|\lambda| > R$ for $R > 0$ sufficiently large. Our notation $\mathbf{O}(\cdot)$ in this section will always describe behavior for $|\lambda| > R$. For example, we will write $h(\lambda) = \mathbf{O}(|\lambda^{-1/2}|)$ if there exists a constant C so that $|h(\lambda)| \leq C|\lambda^{-1/2}|$ for $|\lambda| > R$. As always, we assume Arg$\lambda \neq \pi$. If h depends additionally on x and y, then the notation $|h(\lambda)| \leq C|\lambda^{-1/2}|$ refers to behavior uniform in x and y.

The scaling argument in this section follows [5, 19], and we employ Zumbrun's *tracking lemma* (see [18], Corollary 8.25).

For the eigenvalue problem

$$\Big(M(x)(-\Gamma\phi'' + B(x)\phi)'\Big)' = \lambda\phi \tag{3.36}$$

(i.e., for (1.5)), we take $|\lambda| > R$ and set $\bar{x} := |\lambda^{1/4}|x$ and $\bar{\phi}(\bar{x}) := \phi(x)$. We find, suppressing the dependence of $\bar{\phi}$ on $\bar{x}$,

$$-M(\frac{\bar{x}}{|\lambda^{1/4}|})\Gamma\bar{\phi}'''' = \bar{\lambda}\bar{\phi} + \mathbf{O}(|\lambda^{-1/4}|)\bar{\phi}''' + \mathbf{O}(|\lambda^{-1/2}|)\bar{\phi}'' + \mathbf{O}(|\lambda^{-3/4}|)\bar{\phi}' + \mathbf{O}(|\lambda^{-1}|)\bar{\phi},$$

where $\bar{\lambda} := \frac{\lambda}{|\lambda|}$.

Setting $\bar{M}(\bar{x}) := M(\bar{x}/|\lambda^{1/4}|)$, and computing directly, we find

$$\bar{\phi}'''' = -\Gamma^{-1}\bar{M}(\bar{x})^{-1}\bar{\lambda}\bar{\phi} + \mathbf{O}(|\lambda^{-1/4}|)\bar{\phi}''' + \mathbf{O}(|\lambda^{-1/2}|)\bar{\phi}'' + \mathbf{O}(|\lambda^{-3/4}|)\bar{\phi}' + \mathbf{O}(|\lambda^{-1}|)\bar{\phi}. \quad (3.37)$$

We express this as a first order system with $\bar{W}_j = \bar{\phi}^{(j-1)}$, $j = 1, 2, 3, 4$, so that

$$\bar{W}' = \bar{\mathbb{M}}(\bar{x};\lambda)\bar{W} + \bar{\mathbb{E}}(\bar{x};\lambda)\bar{W}, \quad (3.38)$$

where

$$\bar{\mathbb{M}}(\bar{x};\lambda) = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\bar{\lambda}\Gamma^{-1}\bar{M}^{-1} & 0 & 0 & 0 \end{pmatrix},$$

and

$$\bar{\mathbb{E}}(\bar{x};\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{O}(|\lambda^{-1}|) & \mathbf{O}(|\lambda^{-3/4}|) & \mathbf{O}(|\lambda^{-1/2}|) & \mathbf{O}(|\lambda^{-1/4}|) \end{pmatrix}.$$

Our goal now is to verify that we can apply the Tracking Lemma from Appendix A4 of [18] in this case.

If we let $\{\bar{m}_j\}_{j=1}^n$ denote the eigenvalues of $\Gamma^{-1}\bar{M}(\bar{x})^{-1}$, so that (recalling $\bar{m}_j(\bar{x}) \geq m_0 > 0$ for all $j \in \{1, 2, \dots, n\}$, by our assumption that $M(x)$ is uniformly positive definite)

$$\bar{\mu}_j^4 = -\frac{\bar{\lambda}}{\bar{m}_j} \Rightarrow \bar{\mu}_j(\bar{x};\lambda) = (-\bar{\lambda})^{1/4}\frac{1}{\sqrt[4]{\bar{m}_j(\bar{x})}},$$

where our convention is that $(\cdot)^{1/4}$ is multi-valued, while $\sqrt[4]{\bar{m}_j(\bar{x})}$ is not.

In order to apply the Tracking Lemma, we need to work on a contour $\Omega \subset \mathbb{C}$ such that the eigenvalues of $\bar{\mathbb{M}}(\bar{x};\lambda)$ can be separated into two spectral groups. More precisely, if (briefly following the notation of [18] for convenient reference) we let $\{\alpha_k(\bar{x};\lambda)\}_{k=1}^{4n}$ denote the eigenvalues of $\bar{\mathbb{M}}(\bar{x};\lambda)$, ordered so that $i \leq j \Rightarrow \mathrm{Re}\alpha_i \leq \mathrm{Re}\alpha_j$, then there exist $\underline{\alpha}(\bar{x};\lambda)$ and $\bar{\alpha}(\bar{x};\lambda)$ so that

$$\mathrm{Re}\ \alpha_1(\bar{x};\lambda), \dots, \mathrm{Re}\ \alpha_l(\bar{x};\lambda) < \mathrm{Re}\ \underline{\alpha}(\bar{x};\lambda) < \mathrm{Re}\ \bar{\alpha}(\bar{x};\lambda) \leq \mathrm{Re}\ \alpha_{l+1}(\bar{x};\lambda), \dots, \mathrm{Re}\ \alpha_{4n}(\bar{x};\lambda), \quad (3.39)$$

with the uniformity condition ((8.177) of [18])

$$\text{Re } \bar{\alpha}(\bar{x};\lambda) - \text{Re } \underline{\alpha}(\bar{x};\lambda) \geq 2\eta > 0.$$

(See (8.176) of [18], and also (4.178) of the same reference; in [18] the α_j appear without real parts, but only because the inequality is taken as an ordering by real parts.) In our case,

$$\text{Re } \underline{\alpha}(\bar{x};\lambda) < 0 < \text{Re } \bar{\alpha}(\bar{x};\lambda), \tag{3.40}$$

and the uniformity condition will be clear along our contour from our assumption that $M(\bar{u}(x))$ is uniformly positive definite.

Set

$$\Omega_\theta := \{\lambda : \text{Re}\lambda \geq -\theta_1 - \theta_2|\text{Im}\lambda|\}.$$

For $R > 0$ sufficiently large, and $|\lambda| > R$, we will work along the boundary of

$$\Omega = \Omega_\theta - B(0,R). \tag{3.41}$$

If we write $\bar{\lambda} = e^{i\bar{\theta}}$, then the fourth roots of $-\bar{\lambda}$ must have one of the four forms

$$\frac{1}{\sqrt{2}}(\cos\frac{\bar{\theta}}{4} \pm \sin\frac{\bar{\theta}}{4}); \quad \frac{1}{\sqrt{2}}(-\cos\frac{\bar{\theta}}{4} \pm \sin\frac{\bar{\theta}}{4}).$$

Given any $\epsilon > 0$, we can take $\theta_1 > 0$ and $\theta_2 > 0$ sufficiently small to ensure that $|\bar{\theta}| \leq \frac{\pi}{2} + \epsilon$. For such values of θ, $|\cos(\theta/4)| \geq |\sin(\theta/4)|$, and so the real parts of the roots have have fixed signs. We conclude that (3.39) holds in Ω.

Using the Tracking Lemma, we can conclude that if $\bar{W}^+(\bar{x};\lambda)$ denotes a solution of (3.38) that decays as $\bar{x} \to +\infty$ and $\bar{W}^-(\bar{x};\lambda)$ denotes a solution of (3.38) that decays as $\bar{x} \to -\infty$, then there exist constants $\bar{m}_1 > 0$, $\bar{m}_2 > 0$, and $\bar{C} > 0$, independent of $\lambda \in \Omega_\theta$, so that

$$\begin{aligned} \Big|\frac{\bar{W}^+(\bar{x};\lambda)}{\bar{W}^+(\bar{y};\lambda)}\Big| &\leq Ce^{-\bar{m}_1|\bar{x}-\bar{y}|} \\ \Big|\frac{\bar{W}^-(\bar{x};\lambda)}{\bar{W}^-(\bar{y};\lambda)}\Big| &\geq C^{-1}e^{+\bar{m}_2|\bar{x}-\bar{y}|}, \end{aligned} \tag{3.42}$$

for $\bar{x} \geq \bar{y}$ and likewise

$$\begin{aligned} \Big|\frac{\bar{W}^+(\bar{x};\lambda)}{\bar{W}^+(\bar{y};\lambda)}\Big| &\geq C^{-1}e^{-\bar{m}_2|\bar{x}-\bar{y}|} \\ \Big|\frac{\bar{W}^-(\bar{x};\lambda)}{\bar{W}^-(\bar{y};\lambda)}\Big| &\geq Ce^{+\bar{m}_1|\bar{x}-\bar{y}|}, \end{aligned} \tag{3.43}$$

for $\bar{x} \leq \bar{y}$.

Returning now to original coordinates, if $\phi^+(x;\lambda)$ is any solution of (3.36) that decays as $x \to +\infty$, then there is a corresponding $\bar{W}^+(\bar{x};\lambda)$ so that

$$\bar{W}^+(\bar{x};\lambda) = \Big(\phi^+(x;\lambda) \quad \tfrac{1}{|\lambda^{1/4}|}\phi^{+\prime}(x;\lambda) \quad \tfrac{1}{|\lambda^{1/2}|}\phi^{+\prime\prime}(x;\lambda) \quad \tfrac{1}{|\lambda^{3/4}|}\phi^{+\prime\prime\prime}(x;\lambda)\Big)^{tr}.$$

We now set (suppressing λ dependence for notational brevity)

$$\bar{\Phi}^+(\bar{y}) := \begin{pmatrix} \bar{\phi}^+(\bar{y}) \\ \bar{\phi}^{+\prime}(\bar{y}) \\ \bar{\phi}^{+\prime\prime}(\bar{y}) \\ \bar{\phi}^{+\prime\prime}(\bar{y}) \end{pmatrix}; \quad I_{|\lambda^{1/4}|} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & |\lambda^{1/4}|I & 0 & 0 \\ 0 & 0 & |\lambda^{1/2}|I & 0 \\ 0 & 0 & 0 & |\lambda^{3/4}|I \end{pmatrix}. \tag{3.44}$$

We can express the relationship between $\Phi^{\pm}(x;\lambda)$ and $\bar{\Phi}^{\pm}(x;\lambda)$ as

$$\begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix} = I_{|\lambda^{1/4}|} \begin{pmatrix} \bar{\Phi}^+(\bar{y};\lambda) & \bar{\Phi}^-(\bar{y};\lambda) \end{pmatrix}. \tag{3.45}$$

For $x > y$, we recall from Lemma 3.8 the representation

$$\mathcal{G}_\lambda(x,y) = \begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} [\mathcal{G}_\lambda](y). \tag{3.46}$$

According to (3.45), we can write

$$\begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} = I_{|\lambda^{1/4}|} \begin{pmatrix} \bar{\Phi}^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \bar{\Phi}^+(y;\lambda) & \bar{\Phi}^-(y;\lambda) \end{pmatrix}^{-1} I_{|\lambda^{1/4}|},$$

where by (3.42) and (3.43), and the observation that for $|\lambda| > R$ (with R sufficiently large), the Evans function has a uniformly bounded condition number (denote here κ), we have

$$\begin{aligned} \left| \begin{pmatrix} \bar{\Phi}^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \bar{\Phi}^+(y;\lambda) & \bar{\Phi}^-(y;\lambda) \end{pmatrix}^{-1} \right| &\le \kappa \frac{\left| \begin{pmatrix} \bar{\Phi}^+(x;\lambda) & 0 \end{pmatrix} \right|}{\left| \begin{pmatrix} \bar{\Phi}^+(y;\lambda) & \bar{\Phi}^-(y;\lambda) \end{pmatrix} \right|} \\ &\le Ce^{-m_1|\bar{x}-\bar{y}|}. \end{aligned}$$

In this way, we see that

$$\begin{pmatrix} \Phi^+(x;\lambda) & 0 \end{pmatrix} \begin{pmatrix} \Phi^+(y;\lambda) & \Phi^-(y;\lambda) \end{pmatrix}^{-1} = I_{|\lambda^{1/4}|} \mathbf{O}(e^{-m_1|\lambda^{1/4}||x-y|}) I_{|\lambda^{1/4}|}.$$

Using (3.46), and the expression for $[\mathcal{G}_\lambda](y)$ in Lemma 3.2, we obtain estimates summarized in the following lemma. We note that the case $x < y$ is almost identical.

Lemma 3.16. *Suppose $G_\lambda(x,y)$ is a solution of (2.1) in the natural (distributional sense), the assumptions (C1) and (C2) hold for the operator L, along with Condition 1.1, and let Ω be as in (3.41) for some R taken sufficiently large. Then there exist constants $m > 0$ (sufficiently small) and $C > 0$ (sufficiently large) so that for all $\lambda \in \Omega$ and any multi-index α in x and y with $|\alpha| \le 3$, we have*

$$\left| \partial^\alpha G_\lambda(x;y) \right| \le C|\lambda|^{\frac{|\alpha|-3}{4}} e^{-m|\lambda^{1/4}||x-y|}.$$

Remark 3.3. *Clearly, this estimate holds for certain multi-indices $|\alpha| > 3$ for which the derivatives are distributed with respect to both x and y so that differentiation with respect to either individual variable or third order or less. We will not require this observation.*

We have now obtained estimates on $G_\lambda(x,y)$ for $|\lambda| < r$, some $r > 0$ sufficiently small (Lemmas 3.13, 3.14, and 3.15) and for $|\lambda| > R$, some $R > 0$ sufficiently large (Lemma 3.16). Finally, we observe in the statement of the next lemma that for $r \le |\lambda| \le R$, we have simple boundedness.

Lemma 3.17. *Suppose $G_\lambda(x,y)$ is a solution of (2.1) in the natural (distributional sense), the assumptions (C1) and (C2) hold for the operator L, along with Condition 1.1, and $c \le |\lambda| \le C$, for any constants $0 < c < C$. Then there exists a constant K sufficiently large so that for any multi-index α in x and y with $|\alpha| \le 3$, and for λ to the right of Γ_θ (Γ_θ as in (1.7)), we have*

$$\Big|\partial^\alpha G_\lambda(x;y)\Big| \le K.$$

4 Energy Estimate

Throughout our contour-integral analysis, carried out in the next section, we will use the observation that Condition 1.1, along with assumptions (H0)-(H4), imply that, aside from the leading eigenvalue $\lambda = 0$, the point spectrum of L lies entirely to the left of Γ_θ. We verify that observation in this section with a straightforward energy estimate.

We begin by observing that since the left-hand side of our eigenvalue problem (1.5) is a derivative, and since eigenfunctions by definition for our problem must decay at both $\pm\infty$ we must have that if $\phi(x;\lambda)$ is an eigenfunction of L for any $\lambda \ne 0$ then

$$\int_{-\infty}^{+\infty} \phi(x;\lambda)dx = 0.$$

This justifies our setting

$$\varphi(x;\lambda) := \int_{-\infty}^{x} \phi(y;\lambda)dy,$$

and we observe that φ solves the integrated equation

$$(-\Gamma\varphi_{xxx} + B(x)\varphi_x)_x = \lambda M(x)^{-1}\varphi. \tag{4.1}$$

We multiply both sides of (4.1) by the complex conjugate of φ and integrate the result by parts to obtain

$$-\langle\varphi_{xx}, \Gamma\varphi_{xx}\rangle - \langle\varphi_x, B(x)\varphi_x\rangle = \lambda\langle\varphi, M(x)^{-1}\varphi\rangle, \tag{4.2}$$

where $\langle\cdot,\cdot\rangle$ denotes complex L^2 inner product. We see immediately that if Γ, $B(x)$, and $M(x)$ are all symmetric for all values of x then the point spectrum of L will be entirely real-valued. In this case, it is also straightforward to see that there exists some value $c \in \mathbb{R}$ so that the point spectrum of L is bounded to the left of c.

In order to relax this symmetry assumption on $M(x)$ (while leaving the symmetry of Γ and $B(x)$ in place), we consider complex eigenvalues $\lambda = \alpha + i\beta$ with associated eigenfunctions $\varphi = u + iv$. Taking real and imaginary parts of (4.2) we obtain two equations

$$\begin{aligned}
&-\langle u_{xx}, \Gamma u_{xx}\rangle - \langle v_{xx}, \Gamma v_{xx}\rangle - \langle u_x, B(x)u_x\rangle - \langle v_x, B(x)v_x\rangle \\
&\quad = \alpha\Big(\langle u, M(x)^{-1}u\rangle + \langle v, M(x)^{-1}v\rangle\Big) - \beta\Big(\langle u, M(x)^{-1}v\rangle - \langle v, M(x)^{-1}u\rangle - \Big); \\
&\quad \langle v_x, B(x)u_x\rangle - \langle u_x, B(x)v_x\rangle = \\
&\quad \alpha\Big(\langle u, M(x)^{-1}v\rangle - \langle v, M(x)^{-1}u\rangle\Big) + \beta\Big(\langle u, M(x)^{-1}u\rangle + \langle v, M(x)^{-1}v\rangle\Big).
\end{aligned} \tag{4.3}$$

For the second equation in (4.3) (from imaginary parts), if $B(x)$ is symetric we have

$$0 = \alpha\Big(\langle u, M(x)^{-1}v\rangle - \langle v, M(x)^{-1}u\rangle\Big) + \beta\Big(\langle u, M(x)^{-1}u\rangle + \langle v, M(x)^{-1}v\rangle\Big). \tag{4.4}$$

By our assumption that $M(x)$ is uniformly positive definite (and bounded by regularity and boundedness of $\bar{u}(x)$) we have the inequalities

$$\begin{aligned}
\Big|\langle u, M(x)^{-1}v\rangle - \langle v, M(x)^{-1}u\rangle\Big| &\leq C(\|u\|_{L^2}^2 + \|v\|_{L^2}^2) \\
\Big|\langle u, M(x)^{-1}u\rangle + \langle v, M(x)^{-1}v\rangle\Big| &\geq \gamma(\|u\|_{L^2}^2 + \|v\|_{L^2}^2),
\end{aligned}$$

for some constants C and γ. For $\beta > 0$ we obtain from (4.4) the inequality

$$0 \geq -|\alpha|C(\|u\|_{L^2}^2 + \|v\|_{L^2}^2) + \beta\gamma(\|u\|_{L^2}^2 + \|v\|_{L^2}^2),$$

from which we conclude that eigenvalues in the positive imaginary half-plane are restricted to the region

$$\beta \leq \frac{C|\alpha|}{\gamma}.$$

Proceeding similarly for $\beta < 0$ we find that eigenvalues in the negative imaginary half-plane are restricted to the region

$$\beta \geq -\frac{C|\alpha|}{\gamma}.$$

Combining these observations, we see that there can be no eigenvalues in the pair of wedges $|\beta| \geq (C/\gamma)|\alpha|$. Denote this pair of wedges $\mathcal{W}$.

Now, if we assume Condition 1.1 then the point spectrum of L must lie entirely to the left of $\mathcal{W}$. In addition, by analyticity of $D_a(\rho)$ there is a point eigenvalue at $\lambda = 0$ and a ball $B(0, r)$ in the complex plane that contains no point eigenvalues other than $\lambda = 0$. We can choose θ_1 and θ_2 to ensure that $\Gamma_\theta \subset B(0, r) \cap \mathcal{W}$. This establishes our claim that the point spectrum of L, excepting $\lambda = 0$, lies entirely to the left of Γ_θ.

5 Green's Function Estimates

In this section we derive our estimates on the Green's function $G(t, x; y)$. We will employ our estimates on $G_\lambda(x, y)$ and the Laplace inversion formula

$$G(t, x; y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(x; y) d\lambda, \tag{5.1}$$

where Γ denotes a contour that passes through the point at ∞ and encloses the poles of $G_\lambda(x, y)$. (See our more detailed discussion in the introduction.)

It's important to recognize at the outset that our estimates on $G_\lambda(x, y)$ have different forms for different values of $|\lambda|$ (small, medium, and large). In light of this, we will let Γ_r denote the portion of Γ so that $|\lambda| < r$, Γ_m the portion of Γ so that $r \le |\lambda| \le R$, and Γ_R the portion of the contour so that $|\lambda| > R$. (We will use additional notation below when we turn to the selection of our contours for various cases.) Clearly,

$$\Gamma = \Gamma_r \cup \Gamma_m \cup \Gamma_R.$$

Our choice of contour will depend on the values x, y, and t. Since our contours must all pass through the point at ∞, we know Γ will always include a portion Γ_R, but Γ_r and Γ_m can both be the empty set in cases. Since the choice of contours depends on the values of x, y, and t, we must organize our analysis according to these cases. For some $\epsilon > 0$, which will be chosen during the analysis, and for some $K > 0$, also chosen during the analysis, we consider the three cases: (1) $|x - y| > Kt$; (2) $\epsilon t \le |x - y| \le Kt$, and (3) $|x - y| < \epsilon t$.

Since the analysis consists of evaluating numerous integrals of the basic form (5.1), though only over partial contours, so that the individual results are not $G(t, x; y)$, it will be convenient to employ the notation

$$\mathcal{I}(\Gamma) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(x; y) d\lambda. \tag{5.2}$$

Likewise, if Γ is parametrized by k we will use the notation $\mathcal{I}(\Gamma|J)$ to denote the restriction of Γ to $k \in J$.

5.1 The Case $|x - y| > Kt$

The case $|x - y| > Kt$ is certainly the most straightforward, because for this case we can proceed entirely along contours for which $|\lambda| > R$. In light of this, we can work entirely with the estimates of Lemma 3.16, and we can follow closely the analysis on pp. 822-824 of [19]. The first summand in the estimate stated in Part III of Theorem 1.1 is precisely the estimate we obtain from the analysis just described. The second summand is a correction we take for convenience, and it arises in the following way. Our estimates from Part I of Theorem 1.1 are derived for the case $|x - y| \le Kt$ with $t \ge 1$ (Section 5.2 below), and so for the current case $|x-y| > Kt$, we can only continue to use the same expressions for $e(t;y)$ if we correct them with $\tilde{G}$. Details on the relevant calculation are given at the end of this section. Regarding the case $0 < t < 1$, we observe that the expressions in Part (Ii) of Theorem 1.1 can be regarded as errors in Part III. For example, for $0 < t < 1$ and $y < 0$ we have

$$\Big| \bar{u}'(x) \frac{2}{\sqrt{\pi}} \sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0) \int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2} dz \Big| \le C e^{-\alpha|x|} e^{-\frac{y^2}{Mt}}.$$

Of course, for $0 < t < 1$ we can write this with exponential decay in t as well, and this clearly gives a result that can be subsumed into Part III of Theorem 1.1.

We set

$$\tilde{R} = \frac{|x-y|^{4/3}}{\tilde{L} t^{4/3}}, \tag{5.3}$$

for a constant $\tilde{L}$ (to be specified during the analysis) so that $\tilde{R} > R$. Consider in addition the wedge contour Γ_θ (as defined in (1.7)), which we can write in the parametric form

$$\lambda_\theta(k) = -\theta_1 - \theta_2|k| + ik, \tag{5.4}$$

for $k \in (-\infty, \infty)$, with clearly $|d\lambda_\theta| = \sqrt{(1+\theta_2^2)}dk$ and $|\lambda_\theta|^2 = (\theta_1 + \theta_2|k|)^2 + k^2$.

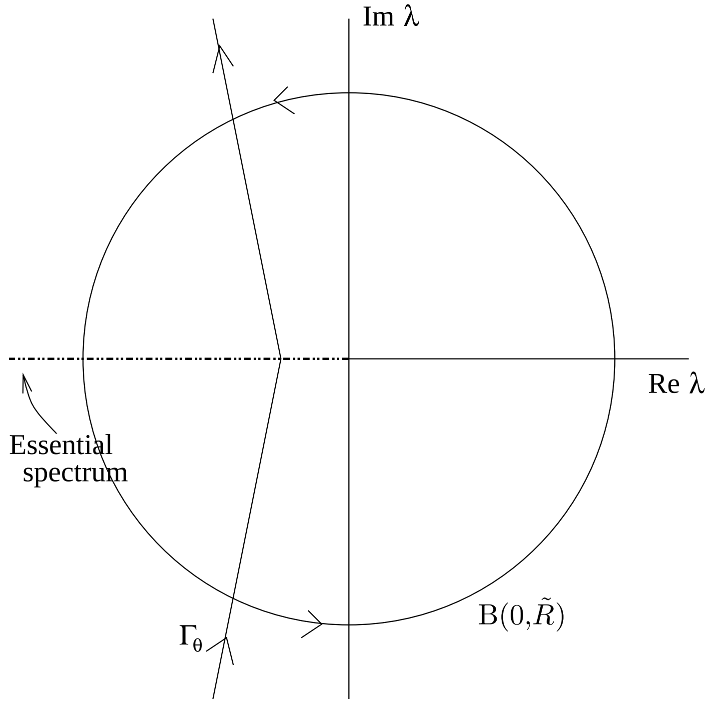

Figure 1: Contours for the case $|\lambda| > R$.

In this case, our contour Γ will follow Γ_θ until Γ_θ intersects $\partial B(0,\tilde{R})$, and then will remain on $\partial B(0,\tilde{R})$, moving in the counterclockwise direction, until it reaches the second interection of Γ_θ and $B(0,\tilde{R})$. Finally, Γ follows Γ_θ out to the point at ∞. For notational convenience, we will designate the portion of this contour along $\partial B(0,\tilde{R})$ by $\tilde{\Gamma}$. See Figure 1.

We begin with the analysis along $\tilde{\Gamma}$, parametrizing this contour as

$$\lambda(\omega) = \tilde{R}e^{i\omega}, \tag{5.5}$$

where θ_1 and θ_2 can be chosen small enough so that ω ranges from $-\omega_0$ to $+\omega_0$, with $\omega_0 > \frac{\pi}{2}$ as close to $\frac{\pi}{2}$ as we like.

Recalling from Lemma 3.16 the estimate

$$\Big|G_\lambda(x,y)\Big| \leq C|\lambda|^{-\frac{3}{4}}e^{-m|\lambda^{1/4}||x-y|},$$

we have (recalling (5.2))

$$\begin{aligned}\Big|\mathcal{I}(\tilde{\Gamma})\Big| &\leq \frac{C}{2\pi}\int_{-\omega_0}^{+\omega_0} e^{\tilde{R}t}\tilde{R}^{1/4}e^{-m\tilde{R}^{1/4}|x-y|}d\omega \\ &\leq \frac{C\omega_0}{\pi}\frac{|x-y|^{1/3}}{\tilde{L}^{1/4}t^{1/3}}e^{\frac{|x-y|^{4/3}}{\tilde{L}t^{1/3}}(1-m\tilde{L}^{3/4})}.\end{aligned}$$

By choosing $\tilde{L}$ large enough so that $m\tilde{L}^{3/4} > 1$, we obtain exponential decay, and we additionally use the estimate

$$z^\alpha e^{-\frac{z^{4/3}}{t^{1/3}}} \leq Ct^{\frac{\alpha}{4}}e^{-\frac{z^{4/3}}{2t^{1/3}}}, \tag{5.6}$$

to obtain the estimate (for the undifferentiated case) of Theorem 1.1 Part (III),

$$\Big|\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}G_\lambda(x,y)d\lambda\Big| \leq Ct^{-1/4}e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}},$$

for M sufficiently large.

For the portion of Γ along Γ_θ, let $\lambda_0 = \tilde{R}e^{i\omega_0}$ denote the value of λ where $\partial B(0,\tilde{R})$ intersects Γ_θ (in the second quadrant) so that

$$\mathrm{Re}\lambda - \mathrm{Re}\lambda_0 = -\frac{1}{\theta_2}(\mathrm{Im}\lambda - \mathrm{Im}\lambda_0).$$

We have, then,

$$\Big|\mathcal{I}(\bar{\Gamma}\cap B(0,R)^c)\Big| \leq C\int_{k_0}^{+\infty}\frac{1}{((\theta_1+\theta_2 k)^2+k^2)^{3/8}}e^{\mathrm{Re}\lambda_0 t}e^{\mathrm{Re}(\lambda-\lambda_0)t}e^{-m\tilde{R}^{1/4}|x-y|}dk,$$

where we have observed that along this contour $|\lambda| \geq R$. Here, we have the following relations:

$$\mathrm{Re}\lambda_0 t - m\tilde{R}^{1/4}|x-y| \leq \tilde{R}(\cos\omega_0)t - m\tilde{R}^{1/4}|x-y| = \frac{|x-y|^{4/3}}{\tilde{L}t^{1/3}}(\cos\omega_0 - m\tilde{L}^{3/4})$$
$$\mathrm{Re}(\lambda-\lambda_0)t = -\frac{1}{\theta_2}(k - \tilde{R}\sin\omega_0)t.$$

Combining these, we find

$$\mathrm{Re}\lambda_0 t + \mathrm{Re}(\lambda-\lambda_0)t - m_1\tilde{R}^{1/4}|x-y| \leq \frac{|x-y|^{4/3}}{\tilde{L}t^{1/3}}(\cos\omega_0 - m\tilde{L}^{3/4} + \frac{\sin\omega_0}{\theta_2}) - \frac{k}{\theta_2}t.$$

Clearly, we can choose $\tilde{L}$ sufficiently large to obtain the expected exponential rate of decay. That is, we have shown

$$\Big|\mathcal{I}(\bar{\Gamma}\cap B(0,\tilde{R})^c)\Big| \leq Ce^{\frac{|x-y|^{4/3}}{Mt^{1/3}}}\int_{k_0}^{\infty}\frac{1}{((\theta_1+\theta_2 k)^2+k^2)^{3/8}}e^{-\frac{k}{\theta_2}t}dk,$$

and we can use a straightforward scaling argument to show that this integral is bounded by $Ct^{-1/4}$ for some constant C. Our claimed derivative estimates follow from almost precisely the same calculation.

We now turn to the second summand in the estimate from Part (III) of Theorem 1.1. This term arises because our derivation of the estimates on $e(t;y)$ and its derivatives for Part I of the theorem are only valid for $|x-y| \leq Kt$, with $t \geq 1$, and so we must correct for these terms in Part III. We note, in particular, that this approach simplifies the analysis involved with choosing the local shift $\delta(t)$. For that choice, we require a precise definition of $e(t;y)$ for all $y \in \mathbb{R}$. We have already discussed the correction associated with the case $0 < t < 1$, so we focus here on the correction associated with $|x-y| > Kt$.

For the undifferentiated case, we have $\bar{u}'(x)e(t;y)$, where (as we will verify in Section 5.2.2)

$$e(t;y) = \frac{2}{\sqrt{\pi}}\sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0)\int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2}dz + R_e(t;y). \tag{5.7}$$

We need to determine an estimate on this term when $|x-y| > Kt$. First, we observe that in this case we must have either $|x|/2 > Kt$ or $|y|/2 > Kt$ (or both). If $|x|/2 > Kt$, we obtain exponential decay in t from $\bar{u}'(x)$. On the other hand, if $|y|/2 > Kt$, the integral in (5.7) is bounded by

$$\int_{-\infty}^{-\frac{K}{2\sqrt{4\beta_{j-m}^-}}\sqrt{t}} e^{-z^2}dz \leq \frac{2\sqrt{4\beta_{j-m}^-}}{K\sqrt{t}}\int_{-\infty}^{-\frac{K}{2\sqrt{4\beta_{j-m}^-}}\sqrt{t}}(-z)e^{-z^2}dz$$
$$= \frac{\sqrt{4\beta_{j-m}^-}}{K\sqrt{t}}e^{-\frac{K^2}{16\beta_{j-m}^-}t},$$

and again we obtain exponential decay in t. It's clear that the blow-up as $t \to 0$ in this last expression is an artifact of the approach and can be removed by considering $t \leq 1$ separately, as discussed above. Finally, we include the decay in y by observing

$$\int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^{-} t}}} e^{-z^2} dz \leq C e^{-\frac{y^2}{Mt}}.$$

This completes the proof of Theorem 1.1 Part (III) for the undifferentiated case $\alpha = 0$. Estimates on derivatives follow from similar calculations, and we omit those details.

5.2 The Case $|x-y| \leq Kt$

In this subsection we consider the case $|x-y| \leq Kt$, for which the estimates on $G_\lambda(x,y)$ will change along our contours.

5.2.1 Bounded Time Estimate

In this section we consider the case $t \leq T$ for any constant $T > 0$. (For specificity we often take $T = 1$.) We note at the outset that for T sufficiently small (which, in fact, is all we require) our estimate is a straightforward consequence of Friedmann's parabolic theory [4]. The calculations here are similar to those of the case $|x-y| > Kt$, and so we focus only on the different points.

For this case, we work entirely along integrals Γ for which the large $|\lambda|$ estimates of Lemma 3.16 apply. First, with R as in Lemma 3.16 we take any $\tilde{R} > R$, and then we proceed along the same contour taken in the case $|x-y| > Kt$ (see Figure 1).

Along $\partial B(0, \tilde{R})$,

$$\Big| \mathcal{I}(\partial B(0, \tilde{R}) | [-\omega_0, \omega_0]) \Big| \leq C \int_{-\omega_0}^{+\omega_0} \tilde{R}^{1/4} e^{-m|\tilde{R}^{1/4}||x-y|} \leq C_{\tilde{R}} e^{-m|\tilde{R}^{1/4}||x-y|}.$$

At this stage, we recall that we are now in the case $|x-y| < Kt$ so that

$$\frac{m\tilde{R}^{1/4}|x-y|^{4/3}}{t^{1/3}} < m\tilde{R}^{1/4} K^{1/3} |x-y|.$$

In this way,

$$e^{-m|\tilde{R}^{1/4}||x-y|} \leq e^{-\frac{m\tilde{R}^{1/4}|x-y|^{4/3}}{K^{1/3} t^{1/3}}}.$$

Similarly, we can proceed as in the case $|x-y| > Kt$ along Γ_θ, using again the observation that in the case $|x-y| < Kt$ exponential decay in $x-y$ is sufficient.

5.2.2 Leading Order Estimates

For the remaining cases we take $t \geq 1$. We will begin by analyzing the leading order terms, which arise from small $|\lambda|$ portions of our contour, and the excited terms from Lemma 3.12: $\bar{u}'(x)e_\lambda(y)$. First, we restrict to the case $|y| \leq \epsilon t$, where $\epsilon > 0$ will be chosen sufficiently small during the analysis.

The case $|\mathbf{y}| \leq \epsilon \mathbf{t}$. This is the determining case of the analysis and must be analyzed in considerable detail. For this single case, we have six fundamentally different calculations to consider, and so we divide the analysis into further subcases, which must be defined as we proceed. We postpone a full discussion of the contours to be taken until some further notation has been established, but for convenient reference we gather here our terminology: *Main term*, *extension correction*, *continuation correction 1*, *continuation correction 2*, *term higher order correction*, *sum higher order correction*.

Main term. Using the estimates of Lemma 3.12, we need to evaluate integrals of the form

$$\frac{1}{2\pi i}\int_{\Gamma_r}\frac{1}{\lambda}e^{\lambda t-\mu_j^-(\lambda)y}d\lambda, \tag{5.8}$$

where $j \in \{m+1, m+2, \ldots, 2m\}$, so that

$$\mu_j^-(\lambda) = -\sqrt{\frac{\lambda}{\beta_{j-m}^-}} + \mathbf{O}(|\lambda|^{3/2}).$$

In order to ease notation we will use β to denote a general eigenvalue β_{j-m}^-. In this way, we write

$$e^{-\mu_j^-(\lambda)y} = e^{\sqrt{\frac{\lambda}{\beta}}y+\mathbf{O}(|\lambda^{3/2}|)y} = e^{\sqrt{\frac{\lambda}{\beta}}y} + e^{\sqrt{\frac{\lambda}{\beta}}y}\Big(e^{\mathbf{O}(|\lambda^{3/2}|)y}-1\Big).$$

We focus first on integrals of the form

$$\frac{1}{2\pi i}\int_{\Gamma_r}\frac{1}{\lambda}e^{\lambda t+\sqrt{\frac{\lambda}{\beta}}y}d\lambda. \tag{5.9}$$

We proceed along the contour described parametrically by

$$\sqrt{\frac{\lambda}{\beta}} = -\frac{y}{2\beta t} + ik, \tag{5.10}$$

where we emphasize that we are in the case $y \leq 0$, so the real part is non-negative.

Computing directly, we find the following useful expressions:

$$\begin{aligned}
\lambda(k) &= \frac{y^2}{4\beta t^2} - ik\frac{y}{t} - \beta k^2 \\
|\lambda(k)| &= \frac{y^2}{4\beta t^2} + \beta k^2 \\
d\lambda &= 2i\sqrt{\beta}\sqrt{\lambda}dk \\
\lambda t + \sqrt{\frac{\lambda}{\beta}}y &= -\frac{y^2}{4\beta t} - \beta k^2 t.
\end{aligned}$$

Our requirement $|y| \leq \epsilon t$ clearly implies

$$|\lambda(k)| \leq \frac{\epsilon^2}{4\beta} + \beta k^2,$$

which ensures that this contour crosses the real axis in $B(0,r)$.

Substituting these expressions into (5.10) we obtain the integral

$$\frac{1}{\pi}e^{-\frac{y^2}{4\beta t}}\int_{-\infty}^{+\infty}\frac{e^{-\beta k^2 t}}{-\frac{y}{2\beta t}+ik}dk. \tag{5.11}$$

Multiplying by a complex conjugate, we obtain an integral of the form

$$\begin{aligned}
&\frac{1}{\pi}e^{-\frac{y^2}{4\beta t}}\int_{-\infty}^{+\infty}\frac{e^{-\beta k^2 t}}{\frac{y^2}{4\beta^2 t^2}+k^2}\Big(-\frac{y}{2\beta t}-ik\Big)dk \\
&= -\frac{2}{\pi}\frac{y}{2\beta t}e^{-\frac{y^2}{4\beta t}}\int_0^{+\infty}\frac{e^{-\beta k^2 t}}{\frac{y^2}{4\beta^2 t^2}+k^2}dk,
\end{aligned}$$

where the integrand has been separated into an odd summand (which integrates to 0) and an even summand.

We now have enough analysis in place to clarify our use of contours. For this discussion, refer to Figure 2. First, the contour described by (5.10) is depicted as the contour a-b-c along with the extension indicated by the dashed curves. We will denote this full contour $\hat{\Gamma}$. Let $-k_r$ denote the value of k for which Γ_r strikes Γ_θ (at point a) and notice by symmetry that $+k_r$ will then denote the value k for which Γ_r strikes Γ_θ (at point c). Our approach will be as follows: we will integrate the entirety of $\hat{\Gamma}$ (to obtain the *main* term) and then subtract off a residue term corresponding with the dashed lines in Figure 2 (the *extension correction*). (As will be clear below, the reason we carry out the full integration along $\hat{\Gamma}$ is because it provides, by way of an exact integration formula, a convenient expression for our leading order term.) We will then obtain estimates on the residue obtained from integration along $\Gamma_\theta\backslash\hat{\Gamma}$ (the *continuation correction*).

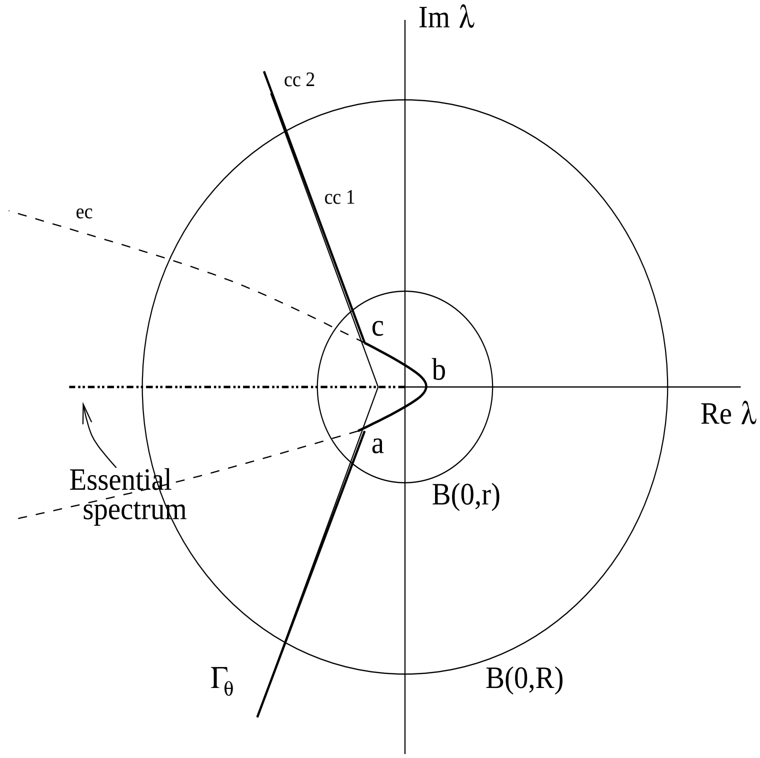

Figure 2: Contours for the leading order analysis.

We now employ the integral identity

$$\int_0^\infty \frac{e^{-b\zeta^2}}{a+\zeta^2}d\zeta = \frac{\pi}{2}\frac{e^{ab}}{\sqrt{a}}\Big(1-\mathrm{erf}(\sqrt{ab})\Big), \tag{5.12}$$

where a and b are positive constants and *erf* denotes the error function with scaling

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz. \tag{5.13}$$

Combining this identity with our previous observations, we find

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}e^{\lambda t+\sqrt{\frac{\lambda}{\beta}}y}d\lambda = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{y}{\sqrt{4\beta t}}} e^{-z^2}dz, \tag{5.14}$$

which can alternatively be expressed in terms of the complementary error function

$$\mathrm{erfc}(x) = 1-\mathrm{erf}(x). \tag{5.15}$$

We conclude a leading order estimate of the form

$$\frac{2}{\sqrt{\pi}}\sum_{j=1}^{2m} h^-_{(2m)j}\tilde{c}^-_j(0)\tilde{r}^-_j(0)\int_{-\infty}^{\frac{y}{\sqrt{4\beta^-_{j-m}t}}} e^{-z^2}dz. \tag{5.16}$$

This is precisely the leading order term in Part (Ii) of Theorem 1.1. (See Remark 1.2 regarding our notation for constants.)

While this analysis has led to a convenient final expression for our leading order term, we have ignored several terms, and at this point we need to make some *corrections*. Our terminology will be as follows: Corrections arising from extending contours beyond the applicability of our estimates on $G_\lambda(x;y)$ (such as along the dashed lines in Figure 2) will be termed *extension corrections*; corrections arising from omitted parts in Γ will be termed *continuation corrections*; corrections arising from Taylor expansions in λ on our exponents $\mu_j^\pm(\lambda)$ will be termed *term higher order corrections*; and finally corrections arising from higher order terms in the estimates on $G_\lambda(x;y)$ will be termed *sum higher order corrections.*

Extension Correction. The value of our contour integral along the dashed portion of our contour is clearly

$$-\frac{2}{\pi}\frac{y}{2\beta t}e^{-\frac{y^2}{4\beta t}}\int_{k_r}^{+\infty}\frac{e^{-\beta k^2 t}}{\frac{y^2}{4\beta^2t^2}+k^2}dk. \tag{5.17}$$

Since $k_r>0$ is fixed, this integral decays at exponential rate in t, and since we are in the case $|y|\le\epsilon t$, this means it decays at exponential rate in $|y|$ as well. We have then, after integration, an expression bounded by $Ct^{-1/2}e^{-c(|y|+t)}$ for positive constants c and C. Since $|y|^2/t\le\epsilon|y|$, we have

$$e^{-\delta|y|}\le e^{-c\frac{y^2}{\epsilon t}},$$

and so this term can be subsumed into residue terms of the form

$$Ct^{-1/2}e^{-\frac{y^2}{Mt}}. \tag{5.18}$$

Continuation Correction 1. We now turn to the residue obtained by integrating along Γ_θ from the points of its intersection with $\hat{\Gamma}$ out to the points of its intersection with $B(0,R)$. We can specify Γ_θ parametrically in (5.4). In this case, let $\pm\bar{k}_1$ denote the values of k at which Γ_θ strikes $\hat{\Gamma}$ and let $\pm\bar{k}_2$ denote the values of k at which Γ_θ strikes $B(0,R)$.

For $k \in [\bar{k}_1, \bar{k}_2]$, it will be sufficient to use simple boundedness of $G_\lambda(x;y)$. That is, we can estimate (5.1) by

$$\begin{aligned} C\int_{\bar{k}_1}^{\bar{k}_2} e^{\mathrm{Re}\lambda(k)t}|d\lambda(k)| &= C\int_{\bar{k}_1}^{\bar{k}_2} e^{-\theta_1 t-\theta_2 kt}\sqrt{\theta_2^2+1}dk \\ &\leq (\bar{k}_2-\bar{k}_1)e^{-\theta_1 t}, \end{aligned} \tag{5.19}$$

and this exponential decay in t provides an estimate smaller than (5.18) as before. Notice particularly that we have used the boundedness of our interval of integration to avoid the t^{-1} behavior we would naturally obtain upon integration. It is precisely this issue that requires our division of the continuation correction into two cases.

Continuation Correction 2. We now continue our contour Γ in the complement of $B(0,R)$, using now our estimate from Lemma 3.16. This analysis is similar to the second part of our analysis in Subsection 5.1—the integration along Γ_θ in the complement of $B(0,\tilde{R})$. More precisely, we need only use the inequality

$$\mathrm{Re}\Big(\lambda t - m_1 R^{1/4}|x-y|\Big) \leq \Big(-\theta_1 t - \theta_2|k|t - m_1 R^{1/4}|x-y|\Big),$$

from which we see that we have integrals of the form

$$e^{-\theta_1 t}\int_{\bar{k}_2}^{\infty}\frac{1}{((\theta_1+\theta_2 k)^2+k^2)^{3/8}}e^{-\frac{k}{\theta_2}t}dk \leq Ct^{-1/4}e^{-\theta_1 t},$$

where $\bar{k}_2$ is the same value described in our discussion of continuation correction 1.

In this case we have $|x-y| \leq Kt$, and so

$$K^2t^2 \geq |x-y|^2 \Rightarrow t \geq \frac{(x-y)^2}{K^2 t}. \tag{5.20}$$

This allows us to estimate this term with

$$Ct^{-1/4}e^{-\frac{\theta_1}{2}t}e^{-\frac{(x-y)^2}{Mt}},$$

for M sufficiently large, where our form is intended to demonstrate that we obtain fast decay in t for t large, but relatively low order blow-up for t small.

Term Higher Order Correction. We recall here that in the preceding calculations, we omitted a term of the form

$$e^{\sqrt{\frac{\lambda}{\beta}}y}\Big(e^{\mathbf{O}(|\lambda^{3/2}|)y}-1\Big).$$

If $\Big||\lambda|^{3/2}y\Big| \leq 1$, this is clearly bounded by

$$C|\lambda|^{3/2}|y|e^{\sqrt{\frac{\lambda}{\beta}}y},$$

while if $\Big||\lambda|^{3/2}y\Big| > 1$ we observe that for $|\lambda|$ sufficiently small

$$e^{\sqrt{\frac{\lambda}{\beta}}y+\mathbf{O}(|\lambda^{3/2}|)y} \leq e^{\sqrt{\frac{\lambda}{\beta'}}y},$$

for $\beta' > 0$ sufficiently large. Combining these observations, we can use the crude estimates

$$e^{\sqrt{\frac{\lambda}{\beta}}y}\Big(e^{\mathbf{O}(|\lambda^{3/2}|)y}-1\Big). \leq C'|\lambda|^{3/2}|y|e^{\sqrt{\frac{\lambda}{\beta'}}y} \leq C|\lambda|e^{\sqrt{\frac{\lambda}{\beta}}y}, \tag{5.21}$$

where $\beta > \beta'$.

These terms lead to integrals of the form

$$\int_{\Gamma_r} e^{\mathrm{Re}(\lambda t+\sqrt{\frac{\lambda}{\beta}}y)}d\lambda,$$

which can be analyzed as above with a resulting estimate by

$$C(1+t)^{-1}e^{-\frac{y^2}{Mt}}.$$

Sum Higher Order Correction. According to Lemma 3.12, the sum higher order correction for the leading order term is $\mathbf{O}(|\lambda^{-1/2}|)e^{-\mu_{2m}^{-}(\lambda)y}$, so that we must consider integrals of the form

$$\int_{\Gamma_r} e^{\lambda t}\mathbf{O}(|\lambda^{-1/2}|)e^{-\mu_{2m}^{-}(\lambda)y}d\lambda.$$

We can proceed as in the *Main Term* and *Term Higher Order Correction* arguments to obtain an estimate of the form

$$C(1+t)^{-1/2}e^{-\frac{y^2}{Mt}}.$$

The case $\epsilon \mathbf{t} \leq |\mathbf{y}| \leq \tilde{\mathbf{K}}\mathbf{t}$. We now turn to the case $\epsilon t \leq |y| \leq \tilde{K}t$, where we choose $\tilde{K}$ sufficiently large so that $|y| > \tilde{K}t$ implies $|x| > \eta t$ for some $\eta > 0$. (I.e., $\eta = \tilde{K} - K$, where we recall that we remain in the case $|x-y| \leq Kt$.) We note at the outset that since $|y| \geq \epsilon t$, decay in y gives decay in t, while the inequality $t \geq \frac{|y|}{\tilde{K}}$ ensures us that exponential t decay implies scaled decay of the kernel form $\exp(-|y|^2/t)$. In this way, we see that either exponential decay in $|y|$ or exponential decay in t will be sufficient to give a rate of decay faster than that of (5.18).

In this case we specify our contour as

$$\sqrt{\frac{\lambda}{\beta}} = -\frac{y}{Lt} + ik,$$

where L will be chosen sufficiently large during the analysis. In particular, we have

$$\lambda(k) = \beta\frac{(x-y)^2}{L^2t^2} + 2i\beta k\frac{x-y}{Lt} - \beta k^2,$$

and we choose L large enough so that $\mathrm{Re}\lambda(0) < r$, and so that this contour intersects Γ_θ inside $B(0,r)$.

Let $k_1 > 0$ denote the positive value of k for which this contour strikes Γ_θ. Proceeding similarly as in the case $|y| \leq \epsilon t$, we find

$$|\mathcal{I}(\Gamma|[-k,k])| \leq Ce^{\frac{\beta-L}{L^2}\frac{y^2}{t}} \int_{-k_1}^{k_1} \frac{1}{\sqrt{\frac{y^2}{L^2t^2} + k^2}} e^{-\beta k^2 t} dk.$$

The most important observation to make regarding expressions of this type is that since $|y| \geq \epsilon t$, we have exponential decay in t. Clearly, this term is smaller than (5.18).

In this case, there is no extension correction. Continuation corrections 1 and 2 can be carried out similarly as in the case $|y| \leq \epsilon t$, and the resulting exponential decay in t is sufficient to give a term smaller than (5.18). In this case, the term and sum higher order corrections clearly provide estimates smaller than those of the main term.

The case $|\mathbf{y}| \geq \tilde{\mathbf{K}}\mathbf{t}$. As discussed at the beginning of the previous case, for $|y| \geq \tilde{K}t$, we are guaranteed $|x| \geq \eta t$, so that $\bar{u}'(x)$ gives exponential decay in both $|x|$ and t. Since this is a subcase of $t \geq |x-y|/K$, this exponential decay in t gives scaled decay of the form

$$e^{-\delta_1 t} e^{-\frac{(x-y)^2}{Mt}},$$

for $\delta_1 > 0$ and M sufficiently large. To make this more precise, we can proceed along $\partial B(0,R)$ to the right of Γ_θ and follow Γ_θ out to the point at ∞.

This concludes our analysis of the leading order term $\bar{u}'(x)e(t;y)$.

5.2.3 Remainder Estimates

We refer to estimates on $G(t,x;y)$ arising from the terms denoted $R_\lambda(x;y)$ in Lemmas 3.13, 3.14, and 3.15 as *remainder estimates*. Although there are quite a few such terms, they can all be analyzed in a similar fashion, and indeed the analysis is similar to the one already carried out above for the leading order terms. Our exposition in this section will focus only on the most salient points. We begin with the determining case, $|x-y| \leq \epsilon t$, for some $\epsilon > 0$ sufficiently small.

The case $|\mathbf{x}-\mathbf{y}| \le \epsilon\mathbf{t}$. In this case, we begin our contours in $B(0,r)$, where the estimates of Lemmas 3.13, 3.14, and 3.15 are valid. We will focus on the case $y < x < 0$ (Lemma 3.13), and in particular on the first remainder term in that case

$$\sum_{j=m+1}^{2m} \frac{\tilde{c}_j^-(\lambda)}{\mu_j^-(\lambda)} e^{\mu_j^-(\lambda)(x-y)} r_j^-(\lambda)\tilde{r}_j^-(\lambda),$$

where we recall for convenient reference that for $j \in \{m+1, m+2, \dots, 2m\}$,

$$\mu_j^-(\lambda) = -\sqrt{\frac{\lambda}{\beta_{j-m}^-}} + \mathbf{O}(|\lambda|^{3/2}).$$

We recall from Lemma 3.5 that $|\tilde{c}^-(\lambda)| \le C$ for $|\lambda|$ sufficiently small, and from our construction of the eigenvectors $r_j^-(\lambda)$ and $\tilde{r}_j^-(\lambda)$ that $|r_j^-(\lambda)| \le C$ and $|\tilde{r}_j^-(\lambda)| \le C$. In this way, we can focus on integrals of the form

$$\tilde{I} := \int_{\Gamma_r} e^{\mathrm{Re}(\lambda t - \sqrt{\frac{\lambda}{\beta_{j-m}^-}}(x-y))} |\lambda^{-1/2}||d\lambda|.$$

We will define our initial contour according to the relation

$$\sqrt{\frac{\lambda}{\beta_{j-m}^-}} = \frac{(x-y)}{2t\beta_{j-m}^-} + ik. \tag{5.22}$$

Since $|x-y| \le \epsilon t$, we can choose $\epsilon > 0$ sufficiently small, and the values θ_1 and θ_2 in Γ_θ sufficiently small so that this contour strikes Γ_θ inside $B(0,r)$. Let $k_1 > 0$ denote the the positive value of k at which this contour strikes Γ_θ. Proceeding as in our analysis of *Main Term* in Section 5.2.2, we find

$$\tilde{I} \le C t^{-1/2} e^{-\frac{(x-y)^2}{Mt}}.$$

As in our analysis in Section 5.2.2, the correction terms can all be subsumed into this estimate.

The case $\epsilon\mathbf{t} \le |\mathbf{x}-\mathbf{y}| \le \mathbf{K}\mathbf{t}$. The only change we require for this case is a replacement of our contour choice (5.22) with the choice

$$\sqrt{\frac{\lambda}{\beta_{j-m}^-}} = \frac{(x-y)}{Lt} + ik,$$

where as in the corresponding case for our leading order estimate we leave L free to be chosen sufficiently large. Proceeding similarly as in that corresponding case, we find an estimate of the form

$$C t^{-1/2} e^{-\eta t} e^{-\frac{(x-y)^2}{4Mt}},$$

for M sufficiently large.

6 Integral Estimates

In this section we state and prove two lemmas collecting estimates on the integrals that appear in our system of integral equations (1.21). The first lemma addresses integrals associated with linear signal propagation, and the second lemma addresses integrals associated with the nonlinear interactions.

Lemma 6.1. *Suppose $e(t;y)$, $\tilde{G}(t,x;y)$ and their relevant derivatives are bounded by the estimates claimed in Theorem 1.1, and v_0 satisfies the pointwise estimate*

$$|v_0(y)| \le (1+|y|)^{-3/2}.$$

Then there exist positive constants C, L and η so that the following estimates hold:

$$\begin{aligned}\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy &= \sum_{j=1}^{m} c_j^- \tilde{r}_j^+(0)\int_0^\infty v_0(y)dy + \sum_{j=m+1}^{2m} c_j^+ \tilde{r}_j^-(0)\int_{-\infty}^0 v_0(y)dy \\ &+ \mathbf{O}\Big((1+t)^{-1/4}\Big) \\ \Big|\int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy\Big| &\le C(1+t)^{-1}.\end{aligned}$$

and

$$\begin{aligned}\Big|\int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy\Big| &\le C\Big[(1+t)^{-1/2}e^{-\frac{x^2}{Lt}} + (1+|x|+t)^{-3/2}\Big] \\ \Big|\int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y)v_0(y)dy\Big| &\le Ct^{-1/4}\Big[(1+t)^{-3/4}e^{-\frac{x^2}{Lt}} + (1+t)^{-1/4}(1+|x|+t)^{-3/2} \\ &+ (1+t)^{-1/4}e^{-\eta|x|}e^{-\frac{x^2}{Lt}}\Big].\end{aligned}$$

Remark 6.1. *As will be clear from the proof, η can be taken as the value η from Theorem 1.1 and we can take $L = 4M$, with M as is Theorem 1.1. (We could use $L = \gamma M$ for any $\gamma > 1$.)*

Proof. For the first integral we recall from Theorem 1.1 (Ii) that for $y < 0$

$$e(t;y) = \Big(\frac{2}{\sqrt{\pi}}\sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0)\int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2}dz + R_e(t;y)\Big)\varrho(t),$$

where

$$|R_e(t;y)| \le C(1+t)^{-1/2}e^{-\frac{y^2}{Mt}}.$$

Focusing on the integration over $[0,\infty)$, and $t \geq 1$, we compute

$$\frac{2}{\sqrt{\pi}} \sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0) \int_{-\infty}^0 \int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2} dz v_0(y) dy + \int_{-\infty}^0 R_e(t,y) v_0(y) dy.$$

The second summand clearly decays with rate $(1+t)^{-1/2}$. For the first, we integrate by parts. Setting $V_0(y) = \int_{-\infty}^y v_0(z)dz$, we have

$$\frac{2}{\sqrt{\pi}} \sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0) \int_{-\infty}^0 e^{-z^2} dz V_0(0) - \frac{2}{\sqrt{\pi}} \sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0) \frac{1}{\sqrt{4\beta_{j-m}^- t}} \int_{-\infty}^0 e^{-\frac{y^2}{4\beta_{j-m}^- t}} V_0(y) dy.$$

The second of these last two summands decays at rate $(1+t)^{-1/4}$, while the first gives precisely the form stated in the lemma.

The second estimate follows immediately from our bound

$$\Big| e_t(t;y) \Big| \leq C(1+t)^{-1} e^{-\frac{y^2}{Mt}}$$

(Theorem 1.1 (Ii).) We note that since $\varrho(t) \equiv 1$ for $t \geq 1$ the estimate for $0 \leq t < 1$ can be subsumed into $\mathbf{O}((1+t)^{-1/4})$.

For the third estimate, we will proceed in the case $x < 0$. By symmetry, it will follow that it holds as well for $x > 0$. First, for $t \leq 1$ we have

$$|\tilde{G}(t,x;y)| \leq C\Big[t^{-1/4} e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} + e^{-\eta(|x|+t)} e^{-\frac{y^2}{Mt}}\Big] \tag{6.1}$$

for all $y \in \mathbb{R}$. Since t is bounded, we only require decay in $|x|$, and for the second summand in this estimate we clearly get exponential decay in $|x|$. For the first summand, we observe

$$\begin{aligned}
&\int_{-\infty}^{+\infty} t^{-1/4} e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} (1+|y|)^{-3/2} dy \\
&\quad = \int_{[\frac{3x}{2},\frac{x}{2}]} t^{-1/4} e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} (1+|y|)^{-3/2} dy + \int_{[\frac{3x}{2},\frac{x}{2}]^c} t^{-1/4} e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy \\
&\quad \leq C\Big[(1+|x|/2)^{-3/2} + e^{-\frac{x^{4/3}}{2^{7/3}M}}\Big].
\end{aligned}$$

(We recall in writing the interval $[\frac{3x}{2},\frac{x}{2}]$ that we are taking $x < 0$.) These final estimates decay at sufficient rate in $|x|$.

For $t \geq 1$, we divide the integration as

$$\int_{-\infty}^{+\infty} = \int_{|x-y|\geq Kt} + \int_{|x-y|<Kt}. \tag{6.2}$$

For $|x-y| \geq Kt$, we have integrals

$$\int_{|x-y|\geq Kt} \Big[t^{-1/4} e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} + e^{-\eta(|x|+t)} e^{-\frac{y^2}{Mt}}\Big] (1+|y|)^{-3/2} dy.$$

For the second summand, we clearly have exponential decay in both $|x|$ and t. For the first, we observe that for $|x-y| \geq Kt$ we have

$$e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} = e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} \leq e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} e^{-\frac{K^{4/3}}{2M} t}. \tag{6.3}$$

Using (6.3), we compute

$$\begin{aligned}
&\int_{|x-y|\geq Kt} t^{-1/4} e^{-\frac{(x-y)^{4/3}}{Mt^{1/3}}} (1+|y|)^{-3/2} dy \leq e^{-\frac{K^{4/3}}{2M} t} \int_{|x-y|\geq Kt} t^{-1/4} e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy \\
&\leq e^{-\frac{K^{4/3}}{2M} t} t^{-1/4} \int_{[\frac{3x}{2}, \frac{x}{2}]} \Big[e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy + \int_{[\frac{3x}{2}, \frac{x}{2}]^c} e^{-\frac{(x-y)^{4/3}}{2Mt^{1/3}}} (1+|y|)^{-3/2} dy \Big].
\end{aligned}$$

Respectively, the final two summands in this inequality are bounded by

$$C e^{-\frac{K^{4/3}}{2M} t} \Big[(1+|x|/2)^{-3/2} + e^{-\frac{x^{4/3}}{2^{10/3} M t^{1/3}}} \Big].$$

If $t \geq |x|$ we have exponential decay in both $|x|$ and t. If $t \leq |x|$ the first term gives an estimate with the form

$$(1+|x|/4+t/4)^{-3/2},$$

which is bounded by the claimed estimate, while for the second we again obtain exponential decay in both $|x|$ and t.

For integration over $|x-y| < Kt$, with $t > 1$, we proceed similarly as in the case $|x-y| \geq Kt$. We have

$$\begin{aligned}
&\int_{|x-y|<Kt} t^{-1/2} e^{-\frac{(x-y)^2}{Mt}} (1+|y|)^{-3/2} dy \\
&\qquad \leq \int_{[\frac{3x}{2}, \frac{x}{2}]} t^{-1/2} e^{-\frac{(x-y)^2}{Mt}} (1+|y|)^{-3/2} dy + \int_{[\frac{3x}{2}, \frac{x}{2}]^c} t^{-1/2} e^{-\frac{(x-y)^2}{Mt}} (1+|y|)^{-3/2} dy \\
&\qquad \leq C \Big[\min\Big\{ t^{-1/2}(1+|x|)^{-1/2}, (1+|x|)^{-3/2} \Big\} + t^{-1/2} e^{-\frac{x^2}{4Mt}} \Big].
\end{aligned}$$

For the second (kernel) estimate, we simply take $L = 4M$. For the first, we observe that if $|x| \geq l\sqrt{t}$ (any $l > 0$) we have an estimate of the form $(1+|x|/2+l\sqrt{t}/2)^{-3/2}$, while if $|x| \leq l\sqrt{t}$ we use the simply observation

$$\frac{x^2}{t} < l^2 \Rightarrow 1 < e e^{-\frac{x^2}{l^2 t}}$$

to verify

$$t^{-1/2}(1+|x|)^{-1/2} \le t^{-1/2}(1+|x|)^{-1/2}ee^{-\frac{x^2}{l^2t}}.$$

We take $L = l^2$.

The fourth case in Lemma 6.1 can be proved similarly as was the third case, and we omit the details. □

We now state and prove a lemma regarding the integrals in (1.21) associated with the nonlinear interactions.

Lemma 6.2. *Suppose $e(t;y)$, $\tilde{G}(t,x;y)$ and their relevant derivatives are bounded by the estimates claimed in Theorem 1.1, and suppose Υ is any function satisfying the pointwise estimates*

$$\Upsilon(s,y) \le \Big[\Upsilon_1(s,y) + \Upsilon_2(s,y) + \Upsilon_3(s,y)\Big],$$

where

$$\Upsilon_1(s,y) = s^{-3/4}(1+s)^{-3/4}e^{-\frac{y^2}{Ls}}$$
$$\Upsilon_2(s,y) = s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2}$$
$$\Upsilon_3(s,y) = s^{-3/4}(1+s)^{-1/4}e^{-\tilde{\eta}|y|}e^{-\frac{y^2}{Ls}},$$

for positive constants L, and $\tilde{\eta}$, where $L \ge 4M$ and $\tilde{\eta} \le \eta/2$, with M and η as in the statement of Theorem 1.1. Then there exists a positive constant C so that the following estimates hold.

$$\Big|\int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s,y)\Upsilon(s,y)dyds\Big| \le C(1+t)^{-1}$$

and

$$\Big|\int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Upsilon(s,y)dyds\Big| \le C\Big[(1+t)^{-1/2}e^{-\frac{x^2}{Lt}} + (1+|x|+\sqrt{t})^{-3/2}\Big]$$
$$\Big|\int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\Upsilon(s,y)dyds\Big| \le Ct^{-1/4}\Big[(1+t)^{-3/4}e^{-\frac{x^2}{Lt}}$$
$$+ (1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2} + t^{-1/4}e^{-\eta|x|}e^{-\frac{x^2}{Lt}}\Big].$$

Remark 6.2. *As noted in the introduction, our nonlinearity $\dot{\delta}(t)v + Q$ is smaller for large t than $|v|^2$, and so the integral estimates of Lemma 6.2 are easy to verify. In fact, we obtain rates faster than required by roughly $t^{-1/2}$ (rough because in some cases the scaling leads to a logarithm.*

Proof. First, according to Theorem 1.1 Part (Iii) we have the estimate

$$\Big|e_{yt}(t-s,y)\Big| \le C(1+(t-s))^{-3/2}e^{-\frac{y^2}{M(t-s)}}.$$

For Υ_1, we have integrals of the form

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{y^2}{Ls}} dyds \\
&\quad = \int_0^t \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} s^{-3/4}(1+s)^{-3/4} e^{-y^2 \frac{Mt+(L-M)s}{LMs(t-s)}} dyds \\
&\quad = \sqrt{\pi} \int_0^t (1+(t-s))^{-3/2} s^{-3/4}(1+s)^{-3/4} \sqrt{\frac{LMs(t-s)}{Mt+(L-M)s}} ds.
\end{aligned}
$$

It's useful to estimate integrals of this form over two intervals, $s \in [0, t/2]$ and $s \in [t/2, t]$. In this case, we obtain an estimate by

$$
\begin{aligned}
&\sqrt{\pi} \int_0^{t/2} (1+t/2)^{-3/2} s^{-3/4}(1+s)^{-3/4} \sqrt{\frac{LM(t/2)(t)}{Mt}} ds \\
&+\sqrt{\pi} \int_{t/2}^t (1+(t-s))^{-3/2} (t/2)^{-3/4}(1+t/2)^{-3/4} \sqrt{\frac{LMt(t/2)}{Mt+(L-M)(t/2)}} ds \\
&\quad \leq C(1+t)^{-1}.
\end{aligned}
$$

For Υ_2 we have integrals of the form

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} dyds \\
&= \int_0^{t/2} \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} dyds \\
&+ \int_{t/2}^t \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} ds \\
&\leq \int_0^{t/2} (1+t/2)^{-3/2} s^{-3/4}(1+s)^{-1/4}(1+\sqrt{s})^{-1/2} ds \\
&+ \int_{t/2}^t (1+(t-s))^{-3/2} (t/2)^{-3/4}(1+t/2)^{-1/4}(1+\sqrt{t/2})^{-1/2} ds \\
&\leq C(1+t)^{-5/4}.
\end{aligned}
$$

Likewise, for Υ_3 we have integrals of the form

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4} e^{-\tilde{\eta}|y|} e^{-\frac{y^2}{Ls}} dyds \\
&\quad = \int_0^{t/2} \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4} e^{-\tilde{\eta}|y|} e^{-\frac{y^2}{Ls}} dyds \\
&\quad + \int_{t/2}^{t} \int_{-\infty}^{+\infty} (1+(t-s))^{-3/2} e^{-\frac{y^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4} e^{-\tilde{\eta}|y|} e^{-\frac{y^2}{Ls}} dyds \\
&\quad \leq \int_0^{t/2} (1+t/2)^{-3/2} s^{-3/4}(1+s)^{-1/4} ds \\
&\quad + \int_{t/2}^{t} (1+(t-s))^{-3/2} (t/2)^{-3/4}(1+t/2)^{-1/4} ds \\
&\quad \leq C(1+t)^{-1}.
\end{aligned}$$

We turn now to the integrals involving $\tilde{G}_y$. We will carry out the analysis for $x<0$, observing by symmetry that we can conclude the same estimates for $x>0$. First, for $t-s \leq 3$ we have

$$\left|\tilde{G}_y(t-s,x;y)\right| \leq C\Big[(t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} + e^{-\eta(|x|+(t-s))} e^{-\frac{y^2}{M(t-s)}}\Big] \tag{6.4}$$

for all $y \in \mathbb{R}$. If $t \leq 3$ then certainly $t-s \leq 3$, and we have integrals of the form

$$\int_0^t \int_{-\infty}^{+\infty} \Big[(t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} + e^{-\eta(|x|+(t-s))} e^{-\frac{y^2}{M(t-s)}}\Big] \Upsilon(s,y) dyds. \tag{6.5}$$

For $t>3$ we will divide the integration into two intervals $[0,t-1]$ and $[t-1,t]$. Our choice of $t-1$ (rather than $t-3$) is taken simply to ensure $t/2 < t-1$, which is certainly true for $t>3$. We obtain integrals of the form

$$\int_{t-1}^t \int_{-\infty}^{+\infty} \Big[(t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} + e^{-\eta(|x|+(t-s))} e^{-\frac{y^2}{M(t-s)}}\Big] \Upsilon(s,y) dyds. \tag{6.6}$$

Since the analyses of (6.5) and (6.6) are almost identical, we focus here on the latter.

For the first estimate in (6.4) and integration against $\Upsilon_1(s,y)$, we have

$$\begin{aligned}
&\int_{t-1}^t \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{y^2}{Ls}} dyds \\
&\quad = \int_{t-1}^t \int_{[\frac{3x}{2},\frac{x}{2}]} (t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{y^2}{Ls}} dyds \\
&\quad + \int_{t-1}^t \int_{[\frac{3x}{2},\frac{x}{2}]^c} (t-s)^{-1/2} e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}} e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{y^2}{Ls}} dyds \\
&\quad \leq C\Big[t^{-3/2} e^{-\frac{x^2}{4Lt}} + e^{-\frac{x^{4/3}}{2^{7/3}M}}\Big].
\end{aligned} \tag{6.7}$$

For either estimate, if $t \geq |x|$ we have sufficient decay with rate $(1+|x|+t)^{-3/2}$, while if $t \leq |x|$ we have exponential decay in both $|x|$ and t.

Integrating the first summand in (6.4) against $\Upsilon_2(s,y)$ with a similar argument gives an estimate by

$$C\Big[t^{-1}(1+|x|/2+\sqrt{t})^{-3/2}+t^{-7/4}e^{-\frac{x^{4/3}}{2^{7/3}M}}\Big],$$

which is sufficient as for the case Υ_1. Likewise, integration of this same estimate against Υ_3 leads to an estimate by

$$C\Big[t^{-1}e^{-\frac{\tilde{\eta}}{2}|x|}+t^{-1}e^{-\frac{x^{4/3}}{2^{7/3}M}}\Big].$$

In this case, if $t \geq |x|^{3/2}$ we have sufficient algebraic decay, while for $t < |x|^{3/2}$ we have exponential decay in both $|x|$ and t.

For $t-s>1$, we divide our integration as

$$\int_0^{t-1} = \int_0^{t-1}\int_{|x-y|\geq K(t-s)} + \int_0^{t-1}\int_{|x-y|<K(t-s)}. \tag{6.8}$$

For $|x-y| \geq K(t-s)$ our estimate on $\tilde{G}_y$ is (6.4). For the first summand in (6.4), we observe that for $|x-y|\geq K(t-s)$ we have

$$e^{-\frac{(x-y)^{4/3}}{M(t-s)^{1/3}}} = e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}}e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}} \leq e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}}e^{-\frac{K^{4/3}}{2M}(t-s)}. \tag{6.9}$$

Integrating against Υ_1 we have

$$\begin{aligned}
&\int_0^{t-1}\int_{|x-y|\geq K(t-s)}(t-s)^{-1/2}e^{-\frac{K^{4/3}}{2M}(t-s)}e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}}s^{-3/4}(1+s)^{-3/4}e^{-\frac{y^2}{Ls}}dyds\\
&\quad\leq \int_0^{t-1}\int_{[\frac{3x}{2},\frac{x}{2}]}(t-s)^{-1/2}e^{-\frac{K^{4/3}}{2M}(t-s)}e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}}s^{-3/4}(1+s)^{-3/4}e^{-\frac{y^2}{Ls}}dyds\\
&\quad+\int_0^{t-1}\int_{[\frac{3x}{2},\frac{x}{2}]^c}(t-s)^{-1/2}e^{-\frac{K^{4/3}}{2M}(t-s)}e^{-\frac{(x-y)^{4/3}}{2M(t-s)^{1/3}}}s^{-3/4}(1+s)^{-3/4}e^{-\frac{y^2}{Ls}}dyds.
\end{aligned}$$

In both cases we integrate the fourth-order kernel, and we respectively obtain estimates by

$$\begin{aligned}
&C_1\int_0^{t-1}(t-s)^{-1/4}e^{-\frac{K^{4/3}}{2M}(t-s)}s^{-3/4}(1+s)^{-3/4}e^{-\frac{x^2}{4Lt}}ds\\
&\quad+\int_0^{t-1}(t-s)^{-1/4}e^{-\frac{K^{4/3}}{2M}(t-s)}e^{-\frac{x^{4/3}}{2^{10/3}Mt^{1/3}}}s^{-3/4}(1+s)^{-3/4}e^{-\frac{x^2}{4Ls}}ds.
\end{aligned}$$

In obtaining this inequality, we have observed that t is larger than both s and $t-s$ and so can replace them in the kernels. Also, for the second summand, we have further subdivided the fourth-order kernel. Finally, we obtain an estimate by

$$C(1+t)^{-3/2}\Big[e^{-\frac{x^2}{4Lt}}+e^{-\frac{x^{4/3}}{2^{10/3}Mt^{1/3}}}\Big].$$

If $|x| > t$ we have exponential decay in both $|x|$ and t, while if $|x| \le t$ we obtain sufficient algebraic decay with rate $(1+|x|+t)^{-3/2}$.

The integrations against Υ_2 and Υ_3 are similar. For the second summand in (6.4) we immediately have decay in $|x|$ at exponential rate, and decay in t is obtained as with the first summand in (6.4).

For $|x-y| \le K(t-s)$, beginning with integration against $\Upsilon_1(s,y)$, we have

$$\int_0^{t-1} \int_{|x-y|<K(t-s)} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{y^2}{Ls}} dyds.$$

We use the algebraic relationship

$$\frac{(x-y)^2}{M(t-s)} + \frac{y^2}{Ls} = \frac{Ls+M(t-s)}{LM(t-s)}\Big(y - \frac{xLs}{Ls+M(t-s)}\Big)^2 + \frac{x^2}{Ls+M(t-s)}, \tag{6.10}$$

and integrate over y to obtain an estimate by

$$\begin{aligned} &C\int_0^{t-1} (t-s)^{-1} \sqrt{\frac{LMs(t-s)}{Ls+M(t-s)}} s^{-3/4}(1+s)^{-3/4} e^{-\frac{x^2}{Ls+M(t-s)}} ds \\ &\le C_1(1+t)^{-1/2} \int_0^{t-1} (t-s)^{-1/2} s^{-1/4}(1+s)^{-3/4} e^{-\frac{x^2}{Lt}} ds, \end{aligned}$$

where we have obverved that with $L > M$ we must have $Mt \le Ls + M(t-s) \le Lt$. We proceed by subdividing the integration of s. We obtain an estimate by

$$\begin{aligned} &C_2 e^{-\frac{x^2}{Lt}} \int_0^{t/2} t^{-1} s^{-1/4}(1+s)^{-3/4} ds \\ &+ C_3 e^{-\frac{x^2}{Lt}} \int_{t/2}^{t-1} (t-s)^{-1/2} t^{-3/2}(1+t)^{-3/4} ds \\ &\le C_4(1+t)^{-1} \log t e^{-\frac{x^2}{Lt}}, \end{aligned}$$

which is smaller than the estimate we require.

For the integration against Υ_2, we have

$$\int_0^{t-1} \int_{|x-y|<K(t-s)} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} dyds.$$

First, focusing on decay in t, we obtain an esimate by

$$\begin{aligned} &\int_0^{t/2} (t-s)^{-1} s^{-3/4}(1+s)^{-1/4}(1+\sqrt{s})^{-1/2} ds \\ &+ \int_0^{t-1} (t-s)^{-1/2} s^{-3/4}(1+s)^{-1/4}(1+\sqrt{s})^{-3/2} ds \\ &\le Ct^{-1}, \end{aligned}$$

which is sufficient for $t \geq |x|^{3/2}$.

For $t < |x|^{3/2}$ (i.e., $|x| > t^{2/3}$), we focus on decay in $|x|$. We obtain an estimate by

$$\begin{aligned}
&\int_0^{t-1} \int_{[\frac{3x}{2},\frac{x}{2}]} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} dyds \\
&\quad + \int_0^{t-1} \int_{[\frac{3x}{2},\frac{x}{2}]^c} (t-s)^{-1} e^{-\frac{(x-y)^2}{M(t-s)}} s^{-3/4}(1+s)^{-1/4}(1+|y|+\sqrt{s})^{-3/2} dyds \\
&\quad \leq C_1 \int_0^{t-1} (t-s)^{-1/2} s^{-3/4}(1+s)^{-1/4}(1+|x|/2+\sqrt{s})^{-3/2} ds \\
&\quad + C_2 \int_0^{t-1} (t-s)^{-1} e^{-\frac{x^2}{4Mt}} s^{-3/4}(1+s)^{-1/4}(1+\sqrt{s})^{-1/2} \\
&\quad \leq C\Big[t^{-1/2}\log t(1+|x|)^{-3/2} + t^{-1}e^{-\frac{x^2}{4Mt}}\Big].
\end{aligned}$$

The first estimate is sufficient for $|x| \geq \sqrt{t}$ (and so for $|x| \geq t^{2/3}$), and the second is always sufficient.

Integration against Υ_3 is very similar to integration against Υ_1, using (6.10). The derivative estimate is obtained similarly.

□

7 Nonlinear Iteration

We now turn to the proof of Theorem 1.2, which proceeds by a combination of continuous induction on bounds obtained from the integral equations (1.21) and a short-time theory. The short-time theory is necessitated by the appearance of v_{xxx} in our nonlinearity Q, which would not be present for constant mobility M. The difficulty arises in integrals such as

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds,$$

for which differentiation three times with respect to x would appear to give a divergent nonlinear integral. Following [19], we use a short-time theory to bound v_{xxx} in terms of v, v_x, x, and $\dot{\delta}(t)$. In [19], which considers viscous conservation laws, $|v_x|$ is bounded in terms of $|x|$, $|v|$, and diffusion waves; see particularly Corollary 11.7 of that reference.

7.1 Short-time Theory

We begin the short-time theory by writing our general Cahn-Hilliard system (1.1) in the form

$$u_t = \Big(M(u)(-\Gamma u_{xxx} + f'(u)u_x)\Big)_x.$$

We consider the linear PDE obtained if we let $\tilde{u}(t,x)$ denote any given function (in an appropriate function class, defined below), and set

$$\begin{aligned}\tilde{M}(t,x) &:= M(\tilde{u}(t,x)) \\ \tilde{A}(t,x) &:= f'(\tilde{u}(t,x)).\end{aligned}$$

That is, we consider the linear PDE

$$u_t = \Big(\tilde{M}(t,x)(-\Gamma u_{xxx} + \tilde{A}(t,x)u_x)\Big)_x. \tag{7.1}$$

In [4] Friedman uses Levi's parametrix method to show that (7.1) can be solved for a classical solution $u \in C^{1,4}((\tau,T] \times \mathbb{R})$, for $T - \tau > 0$ sufficiently small, so long as $u(\tau,x)$ is continuous and doesn't grow too fast and (roughly) $\tilde{M}$, $\tilde{A}$, and the first x-derivatives of these functions are Hölder continuous. (This is Theorem 3 on p. 256 of [4]; by "doesn't grow too fast" we mean Friedman's condition (4.22), and the conditions on $\tilde{M}$ and $\tilde{A}$ can be deduced from Friedman's Conditions (A1) and (A2) on p. 251.) If, in addition to Friedman's assumptions, we take $u(0,x) \in C^{\gamma}(\mathbb{R})$ for some $0 < \gamma < 1$ then we obtain a solution $u \in C^{1+\frac{\gamma}{4},4+\gamma}((\tau,T] \times \mathbb{R})$.

In order to slightly relax these assumptions on $\tilde{M}$ and $\tilde{A}$, we follow the approach of [19] and consider the weak form of (7.1). Following Friedman's analysis, it is straightforward to show that the weak form of (7.1) can be solved for $u \in C^{\frac{\gamma}{4},3+\gamma}((\tau,T] \times \mathbb{R})$ so long as the following conditions hold on $\tilde{M}$ and $\tilde{A}$:

(A1) $\tilde{M}$ and $\tilde{A}$ are bounded continuous functions on $\Omega = [\tau,T] \times \mathbb{R}$, and $\tilde{M}$ is continuous in t uniformly with respect to (t,x) in Ω.

(A2) $\tilde{M}$ and $\tilde{A}$ are Hölder continuous (exponent γ) in x, uniformly with respect to (t,x) in bounded subsets of Ω, and $\tilde{M}$ is Hölder continuous (expondent γ) in x uniformly with respect to (t,x) in Ω.

(For space considerations, we have omitted several details along these lines. For full details in the case of evolutionary PDE that are parabolic in the sense of Petrovskii—a class including Cahn-Hilliard systems—we refer the reader to the companion article [9].) In particular, $u(t,x)$ is constructed in terms of a Green's function for the weak form of (7.1). Let $\tilde{u} \in C^{\frac{\gamma}{4},\gamma}([\tau,T] \times \mathbb{R})$ so that $\tilde{M}$ and $\tilde{A}$ satisfy (A1)-(A2), and let $G^{\tilde{u}}(t,x;\tau,y)$ denote the Green's function for the weak form of (7.1). Define the map

$$\mathcal{T}\tilde{u} := \int_{-\infty}^{+\infty} G^{\tilde{u}}(t,x;\tau,y)u^{\tau}(y)dy.$$

In [9] we show that if $u^{\tau} \in C^{\gamma}$, then for $T - \tau > 0$ sufficiently small, $\mathcal{T}$ is a contraction on the space

$$\mathcal{S} := \Big\{u \in C^{\frac{\gamma}{4},\gamma}([\tau,T] \times \mathbb{R}) : u(\tau,x) = u^{\tau}(x), \|u\|_{C^{\frac{\gamma}{4},\gamma}} \leq K\Big\},$$

for some sufficiently large constant K. This ensures that we have a solution $u \in C^{\frac{\gamma}{4},\gamma}([\tau,T] \times \mathbb{R})$ to the weak form of (7.1). According to our linear theory, we conclude that in fact u is additionally in the space $C^{\frac{\gamma}{4},3+\gamma}((\tau,T]\times\mathbb{R})$. Using this choice of u in our definitions of $\tilde{M}$ and $\tilde{A}$ (i.e., taking $\tilde{u}=u$), we find that $\tilde{M}$ and $\tilde{A}$ satisfy Friedman's stronger conditions (on $[\sigma,T]\times\mathbb{R}$ for any $\tau<\sigma<T$), and we can solve the strong form of (7.1) for some $w\in C^{1+\frac{\gamma}{4},4+\gamma}((\tau,T]\times\mathbb{R})$. Finally, we show in [9] that w and u agree pointwise, so in fact $u\in C^{1+\frac{\gamma}{4},4+\gamma}((\tau,T]\times\mathbb{R})$.

We can express u in the form

$$u(t,x)=\int_{-\infty}^{+\infty}G^u(t,x;\tau,y)u^\tau(y)dy,$$

where G^u satisfies the following estimates established by Friedman:

$$\Big|\frac{\partial^l G^u(t,x;\tau,y)}{\partial x^l}\Big|\le C_l(t-\tau)^{-\frac{1+l}{4}}e^{-\frac{(x-y)^{4/3}}{M(t-\tau)^{1/3}}},\tag{7.2}$$

for $l\in\{0,1,\dots,4\}$. We also note that since the unique classical solution of (7.1) with $u^\tau(x)\equiv 1$ is $u(t,x)\equiv 1$ for all $(t,x)\in[\tau,T]\times\mathbb{R}$, we must have

$$\int_{-\infty}^{+\infty}G^u(t,x;\tau,y)dy=I,\tag{7.3}$$

where I denotes an $m\times m$ identity matrix.

We use these observations to obtain short-time estimates on $u(t,x)$. We have, taking $\tau=0$,

$$\begin{aligned}\frac{\partial^l u}{\partial x^l}&=\int_{-\infty}^{+\infty}\frac{\partial^l}{\partial x^l}G^u(t,x;0,y)u(0,y)dy\\&=\int_{-\infty}^{+\infty}\frac{\partial^l}{\partial x^l}G^u(t,x;0,y)u(0,x)dy+\int_{-\infty}^{+\infty}\frac{\partial^l}{\partial x^l}G^u(t,x;0,y)(u(0,x)-u(0,y))dy\\&=I_l+J_l,\end{aligned}$$

where according to (7.2) and (7.3) $I_0=u(0,x)$ while $I_l=0$ for $l=1,2,3,4$, and

$$|J_l|\le Ct^{-\frac{l}{4}+\frac{\gamma}{4}}.$$

We record these as the short-time estimates

$$\|u(t,\cdot)\|_{C^\gamma(\mathbb{R})}\le C\quad \|u(t,\cdot)\|_{C^l(\mathbb{R})}\le C_l t^{\frac{-l+\gamma}{4}},\tag{7.4}$$

$l=1,2,3,4$, for $t\in(0,T]$, with T sufficiently small. By continuous extension, we can continue u so long as $\|u(t,\cdot)\|_{C^\gamma(\mathbb{R})}$ remains bounded, obtaining the estimates

$$\|u(t,\cdot)\|_{C^\gamma(\mathbb{R})}\le C\quad \|u(t,\cdot)\|_{C^l(\mathbb{R})}\le C_l t^{\frac{-l+\gamma}{4}}+K_l,\tag{7.5}$$

$l=1,2,3,4$.

7.2 Short-time theory for the shift

In Section (7.1) we established a short-time theory for solutions u to (1.1), and we now continue a similar theory for our shift δ. We begin by observing that given any $\tilde{\delta} \in C^{1+\gamma}[0,T]$ our perturbation definition

$$v(t,x) = u(t, x + \delta(t)) - \bar{u}(x),$$

defines v (recall that, by assumption, our transition front $\bar{u}(x)$ is well-defined; certainly we don't require $\delta \in C^{1+\gamma}$ to make this definition, but this is the space we'll work in). We obtain from (7.5) the estimates

$$\|v(t,\cdot)\|_{C^l(\mathbb{R})} \le \tilde{C}_l t^{\frac{-l+\gamma}{4}} + \tilde{\tilde{C}}_l, \tag{7.6}$$

$l = 0, 1, \dots, 4$, so long as $\|u(t,\cdot)\|_{C^\gamma(\mathbb{R})}$ (or equivalently $\|v(t,\cdot)\|_{C^\gamma(\mathbb{R})}$) remains bounded.

Recalling from (1.21) that

$$\dot{\delta}(t) = -\int_{-\infty}^{+\infty} e_t(t;y) v_0(y) dy + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y) \Big[\dot{\delta}(s) v(s,y) + Q(s,y)\Big] dy ds,$$

we see that for any times τ, t so that $0 < \tau < t$ we have

$$\begin{aligned}
\dot{\delta}(t) = \dot{\delta}(\tau) &- \int_{-\infty}^{+\infty} \Big(e_t(t;y) - e_t(\tau;y)\Big) v_0(y) dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(e_{ty}(t-s;y) - e_{ty}(\tau-s;y)\Big) \Big[\dot{\delta}(s) v(s,y) + Q(s,y)\Big] dy ds \\
&+ \int_\tau^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y) \Big[\dot{\delta}(s) v(s,y) + Q(s,y)\Big] dy ds.
\end{aligned}$$

By our choice of $e(t;y)$ (so that $e(t;y) \equiv 0$ for $t \le 1/4$), we have that $\dot{\delta}(t) = 0$ for $t \le 1/4$. Let $\tau \ge 1/4$ denote any value so that $\dot{\delta} \in C^{\frac{\gamma}{4}}([0,\tau]$. In order to extend $\dot{\delta}$ to the interval $[0, \tau + T]$, for T sufficiently small, we define the nonlinear map $\mathcal{J}$

$$\begin{aligned}
\mathcal{J}\dot{\delta}(t) := \dot{\delta}(\tau) &- \int_{-\infty}^{+\infty} \Big(e_t(t;y) - e_t(\tau;y)\Big) v_0(y) dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(e_{ty}(t-s;y) - e_{ty}(\tau-s;y)\Big) \Big[\dot{\delta}(s) v(s,y) + Q(s,y)\Big] dy ds \\
&+ \int_\tau^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y) \Big[\dot{\delta}(s) v(s,y) + Q(s,y)\Big] dy,
\end{aligned}$$

along with the choice $\delta(t) = \int_0^t \dot{\delta}(s) ds$.

Taking now $\dot\delta(t)$ as a given fixed function on $[0,\tau]$, with $\dot\delta(\tau) =: \dot\delta^\tau$, we verify that for T sufficiently small, $\mathcal{J}$ is a contraction on the space

$$\mathcal{V} := \{\dot\delta \in C^{\frac{\gamma}{4}}[\tau, \tau+T] : \dot\delta(\tau) = \dot\delta^\tau, \|\dot\delta\|_{C^{\frac{\gamma}{4}}} \le K\}.$$

Here,

$$\|\dot\delta\|_{C^{\frac{\gamma}{4}}} := \|\dot\delta\|_{C([\tau,\tau+T])} + \sup_{\substack{t_1,t_2\in[\tau,\tau+T]\\ t_1\neq t_2}} \frac{|\dot\delta(t_1) - \dot\delta(t_2)|}{|t_1-t_2|^{\gamma/4}},$$

with

$$\|\dot\delta\|_{C([\tau,\tau+T])} := \sup_{s\in[\tau,\tau+T]} |\dot\delta(s)|.$$

We use inequalities (7.6), along with the L^1 inequalities,

$$\begin{aligned} \|e_{ty}(t;y)\|_{L^1(\mathbb{R})} &\le C(1+t)^{-1} \\ \|e_t(t_2,\cdot) - e_t(t_1,\cdot)\|_{L^1(\mathbb{R})} &\le C(1+t_1)^{-3/2}(t_2-t_1) \\ \|e_{ty}(t_2,\cdot) - e_{ty}(t_1,\cdot)\|_{L^1(\mathbb{R})} &\le C(1+t_1)^{-2}(t_2-t_1), \end{aligned} \tag{7.7}$$

and we recall

$$|Q| \le C(|v||v_x| + e^{-\eta|x|}|v|^2 + |v||v_{xxx}|).$$

First, in order to verify invariance of $\mathcal{J}$, we write

$$\begin{aligned} |\mathcal{J}\dot\delta(t)| &\le |\dot\delta(\tau)| + \|e_t(t,\cdot) - e_t(\tau,\cdot)\|_{L^1(\mathbb{R})}\|v_0\|_{C(\mathbb{R})} \\ &\quad + \int_0^\tau \|e_{ty}(t-s,\cdot) - e_{ty}(\tau-s,\cdot)\|_{L^1(\mathbb{R})}\Big[\|\dot\delta(s)v(s,\cdot)\|_{C(\mathbb{R})} + \|Q(s,\cdot)\|_{C(\mathbb{R})}\Big]ds \\ &\quad + \int_\tau^t \|e_{ty}(t-s,\cdot)\|_{L^1(\mathbb{R})}\Big[\|\dot\delta(s)v(s,\cdot)\|_{C(\mathbb{R})} + \|Q(s,\cdot)\|_{C(\mathbb{R})}\Big]ds \\ &= |\dot\delta(\tau)| + I_1 + I_2 + I_3. \end{aligned}$$

For this argument, we're assuming $\|u(t,\cdot)\|_{C^\gamma(\mathbb{R})}$ remains bounded on $[0,\tau+T]$, and so for $\dot\delta \in \mathcal{V}$ we have the nonlinearity estimate

$$\Big[\|\dot\delta(s)v(s,\cdot)\|_{C(\mathbb{R})} + \|Q(s,\cdot)\|_{C(\mathbb{R})}\Big] \le C_1 + C_2 s^{\frac{-3+\gamma}{4}}.$$

Using this, we easily verify

$$I_1 + I_2 + I_3 \le C(t-\tau).$$

By choosing K sufficiently large, and T sufficiently small (so that $t-\tau$ is small), we can ensure $\|\dot\delta\|_{C([\tau,\tau+T])} < K/2$.

Proceeding similarly, we find

$$|\mathcal{J}\dot\delta(t_1) - \mathcal{J}\dot\delta(t_2)| \le C|t_2 - t_1|,$$

which ensures $\mathcal{J}\dot{\delta} \in \mathcal{V}$. In fact, we have established Lipschitz continuity.

In order to establish that $\mathcal{J}$ is a contraction, we let $\dot{\delta}_1$ and $\dot{\delta}_2$ denote any two functions in $\mathcal{V}$ and write

$$\mathcal{J}\dot{\delta}_1(t)-\mathcal{J}\dot{\delta}_2(t) = \int_\tau^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\Big[\dot{\delta}_1(s)v_1(s,y)-\dot{\delta}_2(s)v_2(s,y)+Q_1(s,y)-Q_2(s,y)\Big]dyds.$$

Here, our notation is

$$v_j(s,y) := u(s,y+\delta_j(s)) - \bar{u}(y)$$
$$Q_j = Q(y, v_j, v_{j_y}, v_{j_{yyy}}).$$

According to our short-time regularity of v, we have

$$\begin{aligned}|v_1(s,y) - v_2(s,y)| &= |(u(s,y+\delta_1(s)) - \bar{u}(y)) - (u(s,y+\delta_2(s)) - \bar{u}(y))| \\ &= |u_y(s,y^*)(\delta_1(s) - \delta_2(s))| \le Cs^{\frac{-1+\gamma}{4}}|\delta_1(s) - \delta_2(s)|.\end{aligned}$$

Recalling that $\delta(t) := \int_0^t \dot{\delta}(\sigma)d\sigma$, and that $\dot{\delta}(t)$ is taken as given for $t \in [0,\tau]$, we have

$$\begin{aligned}|\delta_1(s) - \delta_2(s)| &= |\int_\tau^s \dot{\delta}_1(\sigma) - \dot{\delta}_2(\sigma)d\sigma| \le (s-\tau)\|\dot{\delta}_1 - \dot{\delta}_2\|_{C([\tau,s])} \\ &\le (s-\tau)^{1+\frac{\gamma}{4}}\|\dot{\delta}_1 - \dot{\delta}_2\|_{C^\gamma([\tau,s])}.\end{aligned}$$

Writing

$$\dot{\delta}_1(s)v_1(s,y) - \dot{\delta}_2(s)v_2(s,y) = (\dot{\delta}_1(s) - \dot{\delta}_2(s))v_1(s,y) + \dot{\delta}_2(s)(v_1(s,y) - v_2(s,y)),$$

we see that

$$\Big|\dot{\delta}_1(s)v_1(s,y) - \dot{\delta}_2(s)v_2(s,y)\Big| \le (C_1(\tau-s)^{\gamma/4} + C_1 s^{\frac{-1+\gamma}{4}}(\tau-s)^{1+\gamma/4})\|\dot{\delta}_1 - \dot{\delta}_2\|_{C^\gamma([\tau,\tau+T])}.$$

Likewise,

$$|Q_1(s,y) - Q_2(s,y)| \le (C_1 s^{\frac{-1+\gamma}{4}} + C_2 s^{\frac{-5+\gamma}{4}})(s-\tau)^{1+\gamma/4}\|\dot{\delta}_1 - \dot{\delta}_2\|_{C^\gamma([\tau,\tau+T])}.$$

Combining these observations, we find

$$\|\mathcal{J}\dot{\delta}_1 - \mathcal{J}\dot{\delta}_2\|_{C([\tau,\tau+T])} \le C(t-\tau)^{1+\gamma/4}\|\dot{\delta}_1 - \dot{\delta}_2\|_{C^\gamma([\tau,\tau+T])}.$$

Proceeding similarly, we verify

$$\|\mathcal{J}\dot{\delta}_1 - \mathcal{J}\dot{\delta}_2\|_{C^\gamma([\tau,\tau+T])} \le CT^{1-\gamma/4}\|\dot{\delta}_1 - \dot{\delta}_2\|_{C^\gamma([\tau,\tau+T])},$$

which verifies that $\mathcal{J}$ is a contraction on $\mathcal{V}$.

7.3 Estimation of v_{xxx}

For notational convenience in this calculation and the ones that follow, we'll set

$$\begin{aligned}
\psi_1(t,x) &:= (1+t)^{-1/2} e^{-\frac{x^2}{Lt}} \\
\psi_2(t,x) &:= (1+|x|+\sqrt{t})^{-3/2} \\
\psi_3(t,x) &:= t^{-1/4}(1+t)^{-3/4} e^{-\frac{x^2}{Lt}} \\
\psi_4(t,x) &:= t^{-1/4}(1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2} \\
\psi_5(t,x) &:= t^{-1/4}(1+t)^{-1/4} e^{-\eta|x|} e^{-\frac{x^2}{Lt}}.
\end{aligned}$$

With these definitions in place, we see that our goal in proving Theorem 1.2 can be expressed as

$$\begin{aligned}
|v(t,x)| &\le C\epsilon\Big(\psi_1(t,x) + \psi_2(t,x)\Big) \\
|v_x(t,x)| &\le C\epsilon\Big(\psi_3(t,x) + \psi_4(t,x) + \psi_5(t,x)\Big) \\
|\dot{\delta}(t)| &\le C\epsilon(1+t)^{-1}.
\end{aligned} \tag{7.8}$$

Throughout the nonlinear argument, we would like to replace v, v_x, and $\dot{\delta}$ with these estimates, but of course we must take care since these are the estimates that we are ultimately trying to establish. Following [14] (p. 38) (and many subsequent analyses based on [19]) we define

$$\zeta(t) := \sup_{\substack{s\in[0,t] \\ x\in\mathbb{R}}} \Big\{ \frac{|v(s,x)|}{\psi_1(s,x)+\psi_2(s,x)} + \frac{|v_x(s,x)|}{\psi_3(s,x)+\psi_4(s,x)+\psi_5(s,x)} + \dot{\delta}(s)(1+s) \Big\}. \tag{7.9}$$

Clearly, for all $(s,x) \in [0,t]\times\mathbb{R}$, we have

$$\begin{aligned}
|v(s,x)| &\le \zeta(t)(\psi_1(s,x)+\psi_2(s,x)) \\
|v_x(s,x)| &\le \zeta(t)(\psi_3(s,x)+\psi_4(s,x)+\psi_5(s,x)) \\
|\dot{\delta}(s)| &\le \zeta(t)(1+s)^{-1}.
\end{aligned} \tag{7.10}$$

In developing our short-time theory, our primary concern has been to obtain estimates on $v_{xxx}(t,x)$, so that we can avoid iterating an integral equation for this term. In order to do this, we first note that our perturbation equation for v can be expressed as

$$\begin{aligned}
v_t = &-M(\bar{u}+v)\Gamma v_{xxxx} - M'(\bar{u}+v)(\bar{u}_x+v_x)\Gamma v_{xxx} \\
&+ (M(\bar{u}+v)(\tilde{A}v)_x)_x + \dot{\delta}(t)v_x + \dot{\delta}(t)\bar{u}_x,
\end{aligned} \tag{7.11}$$

where we have set

$$\tilde{A}(t,x) := \int_0^1 Df(\bar{u}+\gamma v)d\gamma.$$

Let $G^v(t,x;\tau,\xi)$ denote the Green's function associated with the homogeneous part of (7.11) (i.e., the equation with $\dot{\delta}(t)\bar{u}_x$ omitted), so that

$$v(t,x) = \int_{-\infty}^{+\infty} G^v(t,x;0,\xi)v_0(\xi)d\xi + \int_0^t \int_{-\infty}^{+\infty} G^v(t,x;\tau,\xi)\dot{\delta}(\tau)\bar{u}'(\xi)d\xi ds. \tag{7.12}$$

Upon differentiating (7.12) three times with respect to x, we obtain

$$v_{xxx}(t,x) = \int_{-\infty}^{+\infty} G^v_{xxx}(t,x;0,\xi)v_0(\xi)d\xi + \int_0^t \int_{-\infty}^{+\infty} G^v_{xxx}(t,x;\tau,\xi)\dot{\delta}(\tau)\bar{u}'(\xi)d\xi ds. \tag{7.13}$$

Fix any time $T_0 > 0$ and consider times $0 < t \le T_0$. Following Friedman, we obtain the estimate

$$\left|G^v(t,x;\tau,\xi)\right| \le C(t-\tau)^{-1}e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}}.$$

According to our definition of $\zeta(t)$, we see upon letting $t \to 0$ and using the monotonicity of ζ that

$$|v(0,x)| \le (1+|x|)^{-3/2}\zeta(0) \le (1+|x|)^{-3/2}\zeta(t). \tag{7.14}$$

Accordingly, we can estimate $v_{xxx}(t,x)$ as follows:

$$\begin{aligned}
|v_{xxx}(t,x)| &\le C_1 \int_{-\infty}^{+\infty} t^{-1}e^{-\frac{(x-\xi)^{4/3}}{Mt^{1/3}}}|v_0(\xi)|d\xi \\
&\quad + C_2 \int_0^t \int_{-\infty}^{+\infty} (t-\tau)^{-1}e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}}|\dot{\delta}(\tau)||\bar{u}'(\xi)|d\xi d\tau \\
&\le C_1\zeta(t) \int_{-\infty}^{+\infty} t^{-1}e^{-\frac{(x-\xi)^{4/3}}{Mt^{1/3}}}(1+|\xi|)^{-3/2}d\xi \\
&\quad + C_2\zeta(t) \int_0^t \int_{-\infty}^{+\infty} (t-\tau)^{-1}e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}}(1+\tau)^{-1}|\bar{u}'(\xi)|d\xi d\tau.
\end{aligned}$$

Keeping in mind that $t-\tau \le T_0$, we find that

$$|v_{xxx}(t,x)| \le C\zeta(t)t^{-3/4}(1+|x|)^{-3/2}.$$

Since t is bounded, we can write this expression with more decay in t simply by increasing the size of C. In particular, we are justified in writing

$$\begin{aligned}
|v_{xxx}(t,x)| \le \tilde{C}\zeta(t)t^{-3/4}\Big[&(1+t)^{-1/4}e^{-\frac{x^2}{Lt}} + (1+t)^{1/4}(1+|x|+\sqrt{t})^{-3/2} \\
&+ (1+t)^{1/4}e^{-\tilde{\eta}|x|}e^{-\frac{x^2}{Lt}}\Big].
\end{aligned} \tag{7.15}$$

Next, we need to verify (7.15) for $t > T_0$. In this case we want to verify that for large time v_{xxx} inherits the increased decay rate of v_x, and so our goal will be to bound v_{xxx} in terms of v_x (rather than v, as in our bounded-time calculation). Formally differentiating (7.11) with respect to x, and setting $w = v_x$, we obtain

$$\begin{aligned} w_t = &-M(\bar{u}+v)\Gamma w_{xxxx} - M'(\bar{u}+v)_x \Gamma w_{xxx} - (M'(\bar{u}_x + v_x)\Gamma w_{xx})_x \\ &+ (M(\bar{u}+v)(\tilde{A}v)_x)_{xx} + \dot{\delta}(t) w_x + \dot{\delta}(t)\bar{u}_{xx}. \end{aligned} \tag{7.16}$$

Using now our short-time theory for v, we see that (7.16) can be solved in Friedman's framework with two source terms:

$$q(t,x)v(t,x) + \dot{\delta}(t)\bar{u}_{xx}(x).$$

Here, $q(t,x)$ has several terms, but we need only recognize that each of these is multiplied by some derivative of $\bar{u}(x)$, so that $|q(t,x)| \le Ce^{-\alpha|x|}$, uniformly in t.

Let $G^w(t,x;\tau,\xi)$ denote the Green's function associated with the homogeneous part of (7.16), so that for any fixed $\tau \ge 0$

$$\begin{aligned} w(t,x) = &\int_{-\infty}^{+\infty} G^w(t,x;\tau,\xi) w(\tau,\xi) d\xi \\ &+ \int_{\tau}^{t}\int_{-\infty}^{+\infty} G^w(t,x;s,\xi)\Big[q(s,\xi)v(s,\xi) + \dot{\delta}(s)\bar{u}_{\xi\xi}(\xi)\Big] d\xi ds. \end{aligned} \tag{7.17}$$

(Our analysis here is based on Friedman [4], and in places we have adopted his notation.) Now, differentiating (7.17) twice with respect to x, and recalling $w = v_x$, we find

$$\begin{aligned} v_{xxx}(t,x) = &\int_{-\infty}^{+\infty} G^w_{xx}(t,x;\tau,\xi) v_\xi(\tau,\xi) d\xi \\ &+ \int_{\tau}^{t}\int_{-\infty}^{+\infty} G^w_{xx}(t,x;s,\xi)\Big[q(s,\xi)v(s,\xi) + \dot{\delta}(s)\bar{u}_{\xi\xi}(\xi)\Big] d\xi ds \\ = &\, I_1 + I_2. \end{aligned} \tag{7.18}$$

In what follows, we fix the increment $t - \tau =: T$ as a sufficiently small value, but let τ (and so t) grow.

We can write (from (7.18))

$$|I_1| \le \int_{-\infty}^{+\infty} |G^w_{xx}(t,x;\tau,\xi)| \zeta(t)(\psi_3(\tau,\xi) + \psi_4(\tau,\xi) + \psi_5(\tau,\xi)) d\xi.$$

From Friedman [4] we have the estimate

$$|G^w_{xx}(t,x;\tau,\xi)| \le C(t-\tau)^{-3/4} e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}}.$$

Since $t-\tau$ is small, we have exponential decay in $|x-\xi|^{4/3}$, which of course provides exponential decay in $|x-\xi|$. Using this observation, we find

$$|I_1| \le C\zeta(t)(t-\tau)^{-1/2}\Big[\psi_3(\tau,x) + \psi_4(\tau,x) + \psi_5(\tau,x)\Big].$$

Likewise,

$$|I_2| \le \zeta(t) \int_\tau^t \int_{-\infty}^{+\infty} |G^w_{xx}(t,x;s,\xi)| \Big[e^{-\alpha|x|}(\psi_1(s,\xi) + \psi_2(s,\xi)) + e^{-\alpha|\xi|}(1+s)^{-1} \Big] d\xi ds$$
$$\le C\zeta(t)(t-\tau)^{1/2}\psi_5(\tau,x).$$

Since $t = \tau + T$, for T chosen sufficiently small, we can replace τ by t in these inequalities (increasing C). We conclude

$$|v_{xxx}(t,x)| \le C\zeta(t)(\psi_3(t,x) + \psi_4(t,x) + \psi_5(t,x)), \tag{7.19}$$

for $t > T_0$. Combining this observation with the case $0 < t < T_0$, we see that (7.15) holds for all $t > 0$.

7.4 Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2, which proceeds by continuous induction on estimates for the system of integral equations (1.21). First, by combining (7.19) with (7.10), we find

$$|\dot{\delta}(s)v(s,y)| + |Q(s,y)| \le C\zeta(t)^2\Upsilon(s,y), \tag{7.20}$$

where Υ is defined in the statement of Lemma 6.2. For the first integral in (1.21), and for

$$|v_0(y)| \le \epsilon(1+|y|)^{-3/2},$$

as in the statement of Theorem 1.2, we have (using Lemmas 6.1) and 6.2)

$$v(t,x) \le \int_{-\infty}^{+\infty} |\tilde{G}(t,x;y)||v_0(y)|dy$$
$$+ \int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(t-s,x;y)| \Big[|\dot{\delta}(s)v(s,y)| + |Q(s,y)| \Big] dyds$$
$$\le C\Big(\epsilon + \zeta(t)^2\Big)(\psi_1(t,x) + \psi_2(t,x)).$$

Likewise,

$$|v_x(t,x)| \le C\Big(\epsilon + \zeta(t)^2\Big)(\psi_3(t,x) + \psi_4(t,x) + \psi_5(t,x)),$$

and

$$|\dot{\delta}(t)| \le C\Big(\epsilon + \zeta(t)^2\Big)(1+t)^{-1}.$$

We conclude that there exists a constant $\tilde{C}$ so that

$$\Big\{ \frac{|v(t,x)|}{\psi_1(t,x) + \psi_2(t,x)} + \frac{|v_x(t,x)|}{\psi_3(t,x) + \psi_4(t,x) + \psi_5(t,x)} + \dot{\delta}(1+t) \Big\} \le \tilde{C}\Big(\epsilon + \zeta(t)^2\Big).$$

Since the right-hand side of this last inequality is non-decreasing in t, we must have

$$\zeta(t) \leq \tilde{C}\Big(\epsilon + \zeta(t)^2\Big).$$

As verified in [8] (see Claim 4.1 on p. 799), we can conclude from this last inequality that

$$\zeta(t) < 2\tilde{C}\epsilon.$$

Theorem 1.2 is an immediate consequence of this last inequality.

$\square$

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