Asymptotic L^p stability for transition fronts in Cahn–Hilliard systems

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Abstract

We consider the asymptotic behavior of perturbations of transition front solutions arising in Cahn–Hilliard systems on $\mathbb{R}$. Such equations arise naturally in the study of phase separation processes, and systems describe cases in which three or more phases are possible. When a Cahn–Hilliard system is linearized about a transition front solution, the linearized operator has an eigenvalue at 0 (due to shift invariance), which is not separated from essential spectrum. In cases such as this, nonlinear stability cannot be concluded from classical semigroup considerations and a more refined development is appropriate. Our main result asserts that if initial perturbations are small in $L^1 \cap L^\infty$ then spectral stability—a necessary condition for stability, defined in terms of an appropriate Evans function—implies asymptotic nonlinear stability in L^p for all $1 < p \leq \infty$.

1 Introduction

We consider the asymptotic L^p stability of transition front solutions $\bar{u}(x)$, $\bar{u}(\pm\infty) = u_\pm$, $u_- \neq u_+$, for Cahn–Hilliard systems on $\mathbb{R}$,

$$u_t = \Big(M(u)(-\Gamma u_{xx} + f(u))_x \Big)_x, \tag{1.1}$$

where $u, f \in \mathbb{R}^m$, m an integer greater than or equal to 2 ($m+1$ phases are possible) and $M, \Gamma \in \mathbb{R}^{m\times m}$. A brief discussion of the history and physicality of this equation is given in [10], and reasonable (physical) choices for f, M, and Γ are also discussed. We omit such a discussion here, but state, for convenient reference, the assumptions of [10], which we will assume throughout this paper.

(H0) (Assumptions on Γ) Γ denotes a constant, symmetric, positive definite matrix.

(H1) (Assumptions on f) $f \in C^3(\mathbb{R}^m)$, and f has at least two zeros on $\mathbb{R}^m$. For convenience we denote this set

$$\mathcal{M} := \{u \in \mathbb{R}^m : f(u) = 0\}. \tag{1.2}$$

(H2) (Transition front existence and structure) There exists a transition front solution to (1.1) $\bar{u}(x)$, so that

$$-\Gamma \bar{u}_{xx} + f(\bar{u}) = 0, \tag{1.3}$$

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$. When (1.3) is written as a first order autonomous ODE system $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

(H3) (Assumptions on M and Γ) $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the wave; i.e., there exists $\theta > 0$ so that for all $\xi \in \mathbb{R}^m$ and all $x \in \mathbb{R}$ we have

$$\xi^{tr} M(\bar{u}(x))\xi \geq \theta |\xi|^2.$$

(H4) (Symmetry and Endstate Assumptions) We assume the $m \times m$ Jacobian matrix $f'(\bar{u}(x))$ is symmetric for all $x \in \mathbb{R}$. Setting $B_\pm := f'(u_\pm)$ and $M_\pm := M(u_\pm)$, we assume $B_\pm$ and $M_\pm$ are both symmetric and positive definite. (Of course, $M_\pm$ is already positive definite from (H3).) In addition, we assume that for each of the matrices $M_\pm B_\pm$ and $\Gamma^{-1} B_\pm$, the spectrum is distinct except possibly for repeated eigenvalues that have an associated eigenspace with dimension equal to eigenvalue multiplicity. In the case of repeated eigenvalues, we assume additionally that the solutions μ of

$$\det\Big(-\mu^4 M_\pm \Gamma + \mu^2 M_\pm B_\pm - \lambda I \Big) = 0$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\lambda)$ can be written as $\mu(\lambda) = \sqrt{\lambda} h(\lambda)$, where h is analytic in λ for $|\lambda|$ sufficiently small.

Regarding (H1) we observe that for Cahn-Hilliard systems we can often write f as the gradient of an appropriate bulk free energy density F (i.e. $f(u) = F'(u)$), where F has $m+1$ local minima on $\mathbb{R}^m$. In this way, it's natural for f to have precisely $m+1$ zeros. Since F would appear in (1.1) with both a u and an x derivative, we can subtract from it any affine function without changing (1.1). It is often convenient to subtract a supporting hyperplane from F so that F is also 0 on $\mathcal{M}$.

Regarding (H4), we first observe that the symmetry condition on $f'(\bar{u}(x))$ is natural since $F''(u)$ is a Hessian matrix. Also, we note that we can ensure that our system satisfies the determinant condition by taking arbitrarily small perturbations of the matrices M and Γ. Since we expect stability to be insensitive to such perturbations, we view this assumption as purely for technical convenience. Generally speaking, (H0)-(H4) hold for physically relevant choices of Γ, M, and f; particular examples can be found in [10].

When the Cahn–Hilliard system (1.1) is linearized about a transition front $\bar{u}(x)$, as described in (H2), the resulting linear equation is

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x, \tag{1.4}$$

where (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$ and $B(x) := f'(\bar{u}(x))$. Assumptions (H0)–(H4) imply the following (stated with some redundancy so that these assumptions can be referred to independently of (H0)-(H4)):
(C1) $B \in C^2(\mathbb{R})$; there exists a constant $\alpha_B > 0$ so that

$$\partial_x^j (B(x) - B_\pm) = \mathbf{O}(e^{-\alpha_B |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both positive definite matrices.
(C2) $M \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial_x^j (M(x) - M_\pm) = \mathbf{O}(e^{-\alpha_M |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; M is uniformly positive definite on $\mathbb{R}$; Γ denotes a constant, symmetric, positive definite matrix. We will set $\alpha = \min\{\alpha_B, \alpha_M\}$.

The eigenvalue problem associated with (1.4) has the form

$$L\phi := \Big(M(x)(-\Gamma\phi'' + B(x)\phi)'\Big)' = \lambda\phi. \tag{1.5}$$

In many cases it's possible to verify that the only non-negative eigenvalue for this equation is $\lambda = 0$ (see, for example, [1, 2, 15] and our companion spectral paper [10]), and so stability depends entirely on the nature of this neutral eigenvalue. In [10], we identify an appropriate stability condition for this leading eigenvalue. Briefly, this condition is constructed in terms of the asymptotically growing/decaying solutions of (1.5). For $|\lambda| > 0$ sufficiently small, and $\mathrm{Arg}\lambda \neq \pi$ (i.e., excluding negative real numbers), there are $2m$ linearly independent solutions of (1.5) that decay as $x \to -\infty$ and $2m$ linearly independent solutions of (1.5) that decay as $x \to +\infty$. Moreover, these functions can be constructed so that they are analytic in $\rho = \sqrt{\lambda}$. If we denote these functions $\{\phi_j^\pm(x;\rho)\}_{j=1}^{2m}$ and set $\Phi_j^\pm = (\phi_j^\pm, \phi_j^{\pm\prime}, \phi_j^{\pm\prime\prime}, \phi_j^{\pm\prime\prime\prime})^{\mathrm{tr}}$, the Evans function can be expressed as

$$D_a(\rho) = \det(\Phi_1^+, \dots, \Phi_{2m}^+, \Phi_1^-, \dots, \Phi_{2m}^-)\Big|_{x=0}. \tag{1.6}$$

In terms of this function the stability condition of [10] can be stated as follows:

Condition 1.1. *The set $\sigma(L)\backslash\{0\}$ lies entirely in the negative half-plane $Re\lambda < 0$, and*

$$\frac{d^{m+1}}{d\rho^{m+1}} D_a(\rho)\Big|_{\rho=0} \neq 0.$$

Remark 1.1. *As discussed in Section 3 of [10], our assumptions (H0)-(H4) ensure that the essential spectrum of L (defined here as any value that is neither in the point spectrum nor the resolvent set of L) is confined to the negative real axis* $(-\infty, 0]$. *(This follows immediately from our assumptions that* Γ, $B_\pm$, *and* $M_\pm$ *are all symmetric and positive definite.) In addition, it is shown in [11] by a straightforward energy estimate that Condition 1.1 implies that aside from the leading eigenvalue* $\lambda = 0$ *the point spectrum of L is bounded to the left of a wedge with vertex on the negative real axis:*

$$\Gamma_\theta := \{\lambda : Re\ \lambda = -\theta_1 - \theta_2 |Im\ \lambda|\} \tag{1.7}$$

for some positive values θ_1, θ_2 *sufficiently small. If we make one additional natural assumption, that* $M(\bar{u}(x))$ *is symmetric for all* $x \in \mathbb{R}$, *we can ensure that the point spectrum of L is entirely real-valued. Finally, we verify in [10] that*

$$D_a(0) = D_a'(0) = \cdots = D_a^{(m)}(0) = 0.$$

Our main goal in the current analysis is to establish that Condition 1.1 is sufficient to guarantee asymptotic stability for the wave $\bar{u}(x)$ in L^p spaces, $1 < p \leq \infty$ (more precisely $L^1 \cap L^\infty \to L^p$ phase-asymptotic stability). We employ the pointwise Green's function approach of [6, 7, 17], along with the local tracking developed in [12, 16] and the L^p framework of [14, 16].

Generally, if the initial value for (1.1) is taken as a small perturbation of $\bar{u}(x)$, the solution $u(t, x)$ will approach a shift of $\bar{u}(x)$ rather than the wave itself (orbital stability). Following [12], we proceed by tracking this shift locally in time, our location denoted by $\delta(t)$, which is standard notation in the literature and should not be confused with a Dirac delta function. More precisely, we include this shift in our analysis by defining our perturbation $v(t, x)$ as

$$v(t, x) := u(t, x + \delta(t)) - \bar{u}(x). \tag{1.8}$$

At this point, $\delta(t)$ is yet undetermined, and indeed one of the most important aspects of our approach to this problem is that it allows us to make an effective choice of $\delta(t)$. Upon substitution of $u(t, x + \delta(t))$ into (1.1) we obtain the perturbation equation

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x \Big)_x + \bar{u}'(x)\dot{\delta}(t) + v_x \dot{\delta}(t) + Q_x, \tag{1.9}$$

where $Q = Q(x, v, v_x, v_{xxx})$ is at least C^2 in all its variables, and if

$$|v| + |v_x| + |v_{xxx}| \leq \tilde{C}$$

for some constant $\tilde{C}$, then there exists a constant C so that

$$|Q| \leq C\Big(|v||v_x| + e^{-\alpha|x|}|v|^2 + |v||v_{xxx}| \Big), \tag{1.10}$$

where α is described in (C1)-(C2) above. On one hand, this is a beneficial nonlinearity, because $|v_x|$ and $|v_{xxx}|$ will generally decay faster than $|v|$ as $|x|$ or t tends to ∞, and so each of these bounds is better than the standard nonlinearity $|v|^2$ encountered in the analysis of viscous conservation laws (see, e.g., [17]). On the other hand, for small values of t, derivatives of v generally blow up, and v_{xxx} is problematic in this regard. Our short time analysis of Section 4 is designed primarily to address this difficulty.

Let $G(t,x;y)$ denote the Green's function associated with the linear equation $v_t = Lv$, where L is as in (1.5), so that, in the standard distributional sense,

$$\begin{aligned} G_t &= LG \\ G(0,x;y) &= \delta_y(x)I, \end{aligned} \tag{1.11}$$

where I denotes an $m \times m$ identity matrix, and of course $\delta_y(x)$ is a standard Dirac delta function. Integrating (1.9), we find

$$\begin{aligned} v(t,x) = &\int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}'(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds, \end{aligned} \tag{1.12}$$

where in deriving this equation we have (1) observed that since $\bar{u}'(x)$ is a stationary solution for $v_t = Lv$ we must have $e^{Lt}\bar{u}'(x) = \bar{u}'(x)$; (2) assumed our eventual choice of $\delta(t)$ has the natural property $\delta(0) = 0$; and (3) integrated the standard nonlinear integral by parts. To be clear, we do not assume at this stage that solutions of (1.12) are necessarily solutions of (1.9). Rather, our approach will be to work directly with (1.12) and use our estimates on G and v to establish the correspondence. We consider the condition $\delta(0) = 0$ to be natural, because $\delta(t)$ should capture the shift obtained as perturbation mass accumulates near the transition layer, and generally this accumulation will take some time.

Our approach will be to take advantage of the analysis of [11] in which G is decomposed into two parts, an *excited* term E that does not decay as $t \to \infty$ (and is associated with the leading eigenvalue $\lambda = 0$), and a higher order term $\tilde{G}(t,x;y)$ that does decay as $t \to \infty$. This approach, following [8, 12, 16, 17] and others, will allow us to choose our shift $\delta(t)$. We will find that E can be written as $E(t,x;y) = \bar{u}'(x)e(t,y)$, and so we can express G as

$$G(t,x;y) = \bar{u}'(x)e(t;y) + \tilde{G}(t,x;y), \tag{1.13}$$

so that (1.12) becomes

$$\begin{aligned} v(t,x) = &\int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\ &+ \bar{u}'(x)\Big\{\delta(t) + \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds\Big\}. \end{aligned} \tag{1.14}$$

Our goal will be to choose $\delta(t)$ in such a way that the entire expression multiplying $\bar{u}'(x)$ in (1.14) is annihilated. That is, we would like $\delta(t)$ to solve the integral equation

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds. \tag{1.15}$$

In principle now, we would like to establish existence of v, along with a bound on asymmptotic behavior, by closing an iteration on (1.12). For such an argument we must be clear about which functions must be carried through the iteration and which can be analyzed after the iteration, using the obtained bounds. Of particular importance in this regard, $\delta(t)$ does not appear directly in (1.12), and so it suffices to couple (1.12) with an equation for $\dot{\delta}(t)$, rather than for $\delta(t)$ itself. (Of course, v depends on δ, and this dependence is accomodated in the short-time analysis of [11].) Afterward, estimates on $\delta(t)$ can be obtained directly from (1.15). Also, the nonlinearity Q depends on v_x and v_{xxx} (in addition, of course, to dependence on x and v), and so we must either couple (1.12) with integral equations for these functions or obtain estimates on them in terms of the functions we do iterate. It's straightforward to show that v_{xxx} can be bounded in terms of x, v_x, and $\delta(t)$ for t bounded away from 0, and can easily be estimated for t near 0, and so our approach will be to iterate with the variables v, v_x, and $\dot{\delta}(t)$, and to obtain estimates on v_{xxx} and $\delta(t)$ after the iteration. (Though the connection between v_x and v_{xxx} will be used during the course of the iteration; our principal reference for this calculation is [13], though v_{xxx} does not appear there.) In this way, we will carry out an iteration on the $2m+1$ integral equations,

$$\begin{aligned}
v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\
v_x(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\
\dot{\delta}(t) &= -\int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds.
\end{aligned} \tag{1.16}$$

Our first result regards L^p estimates on $G(t,x;y)$ and its derivatives. We will prove this theorem in Section 2.

Theorem 1.1. *Suppose Conditions (C1)-(C2) hold, and also that spectral Condition 1.1 holds. Then given any time thresholds $T_1 > 0$ and $T_2 > 0$ there exists a constant $C > 0$ (depending on T_1 and T_2) so that the Green's function described in (1.11) can be bounded as follows: there exists a splitting*

$$G(t,x;y) = \bar{u}'(x)e(t;y) + \tilde{G}(t,x;y),$$

so that:

(I) For all $t \geq 0$

$$\|e_t(t;\cdot)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}; \qquad \|e_{ty}(t;\cdot)\|_{L^p} \leq C(1+t)^{-1-\frac{1}{2}(1-\frac{1}{p})},$$

and $e(t;y) \equiv 0$ for all $t \leq 1/4$.

(II) For $t \geq T_1$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}(t,\cdot;y)\|_{L^p_x} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}(t,x;\cdot)\|_{L^p_y} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}_y(t,\cdot;y)\|_{L^p_x} \leq Ct^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}_y(t,x;\cdot)\|_{L^p_y} \leq Ct^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}_x(t,\cdot;y)\|_{L^p_x} \leq Ct^{-\frac{1}{2}}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}_x(t,x;\cdot)\|_{L^p_y} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}_{xy}(t,\cdot;y)\|_{L^p_x} \leq Ct^{-1}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}_{xy}(t,x;\cdot)\|_{L^p_y} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}.$$

(III) For $0 < t < T_2$

$$\sup_{y\in\mathbb{R}} \|\partial^\alpha \tilde{G}(t,\cdot;y)\|_{L^p_x} \leq Ct^{-\frac{|\alpha|}{4}-\frac{1}{4}(1-\frac{1}{p})}$$

$$\sup_{x\in\mathbb{R}} \|\partial^\alpha \tilde{G}(t,x;\cdot)\|_{L^p_y} \leq Ct^{-\frac{|\alpha|}{4}-\frac{1}{4}(1-\frac{1}{p})}$$

where α is a standard multiindex in x and y and $|\alpha| \leq 3$;

Remark 1.2. *We will use the observation that by taking $T_2 > T_1$ we can ensure there is a region in the case $|x-y| \leq Kt$ for which estimates (II) and (III) both hold. Detailed expressions for $e(t;y)$ and $e_y(t;y)$ are given below in Theorem 2.1, taken from [11]. Here and below we only use a subscript on L^p if the expression under norm depends on both x and y; in all other cases, L^p will denote norm with respect to the spatial variable.*

In Section 4 we show that the estimates of Theorem 1.1 are sufficient to close an iteration on the system (1.16) in L^p norms. In this way, we establish the following theorem, which is the main result of our analysis.

Theorem 1.2. *Suppose $\bar{u}(x)$ is a transition front solution to (1.1) as described in (H2), and suppose (H0)-(H4) hold, as well as Condition 1.1. Then for Hölder continuous initial conditions $u(0,x) \in C^\gamma(\mathbb{R})$, $0 < \gamma < 1$, with*

$$\|u(0,x) - \bar{u}(x)\|_{L^\infty} + \|u(0,x) - \bar{u}(x)\|_{L^1} \leq \epsilon,$$

for some $\epsilon > 0$ sufficiently small, there exists a solution $u(t,x)$ of (1.1)

$$u \in C^{1+\frac{\gamma}{4},4+\gamma}((0,\infty)\times\mathbb{R}) \cap C^{\frac{\gamma}{4},\gamma}([0,\infty)\times\mathbb{R})$$

and a shift function $\delta \in C^{1+\frac{\gamma}{4}}[0,\infty)$ so that

$$\lim_{t\to 0^+} \delta(t) = 0; \quad \lim_{t\to\infty} \delta(t) = \delta_\infty \in \mathbb{R},$$

for which the following estimates hold: there exists a constant $C > 0$ so that for each $1 \le p \le \infty$

$$\|u(t, x + \delta(t)) - \bar{u}(x)\|_{L^p} \le C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$$
$$\|u_x(t, x + \delta(t)) - \bar{u}'(x)\|_{L^p} \le C\epsilon t^{-1/4}(1+t)^{-1/4}$$
$$|\delta(t) - \delta_\infty| \le C\epsilon(1+t)^{-1/4}$$
$$|\dot{\delta}(t)| \le C\epsilon(1+t)^{-1}.$$

Remark 1.3. *This is the $L^1 \cap L^\infty \to L^p$ analog to the pointwise theorem of [11] for which the authors assume*

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

which (with a slightly different value for ϵ) is a special case of the assumption made in Theorem 1.2.

2 Proof of Theorem 1.1

In this section we carry out a straightforward proof of Theorem 1.1. The proof is based on Theorem 1.2 from [11], for which we need to make one preliminary definition. We let $\varrho(t)$ denote a $C^\infty([0,\infty))$ function that is identically 0 for $t \le 1/4$ and identically 1 for $t \ge 3/4$. (In order to be definite, a precise choice is made in [11].) We now re-state Theorem 1.2 from [11].

Theorem 2.1. *Under the assumptions of Theorem 1.1, and given any time thresholds $T_1 > 0$ and $T_2 > 0$, there exist constants $\eta > 0$ (sufficiently small), and $C > 0$, $K > 0$, $M > 0$ (sufficiently large) so that the Green's function described in (1.11) can be bounded as follows: there exists a splitting*

$$G(t, x; y) = \bar{u}'(x)e(t; y) + \tilde{G}(t, x; y),$$

so that for $y < 0$:

(I) (Excited terms)

(i) Main estimates:

$$e(t; y) = \Big(\frac{2}{\sqrt{\pi}} \sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0) \int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2} dz + R_e(t; y)\Big)\varrho(t)$$
$$e_y(t; y) = \Big(\sum_{j=m+1}^{2m} \frac{c_j^- \tilde{r}_j^-(0)}{\sqrt{\beta_{j-m}^- \pi t}} e^{-\frac{y^2}{4\beta_{j-m}^- t}} + \partial_y R_e(t; y)\Big)\varrho(t)$$

where

$$|R_e(t, y)| \le Ct^{-1/2}e^{-y^2/Mt}$$
$$|\partial_y R_e(t, y)| \le C\Big(t^{-1}e^{-y^2/Mt} + t^{-1/2}e^{-y^2/Mt}e^{-\eta|y|}\Big).$$

For brevity the (constant) values $\{\beta_j^-\}_{j=1}^m$ *and* $\{c_j^-\}_{j=m+1}^{2m}$*, and the vectors* $\{\tilde{r}_j^-(0)\}_{j=m+1}^{2m}$ *are specified in a remark following the theorem statement.*

(ii) Time derivatives:

$$\Big|e_t(t;y)\Big| \le C(1+t)^{-1}e^{-\frac{y^2}{Mt}}$$
$$\Big|e_{yt}(t;y)\Big| \le C(1+t)^{-3/2}e^{-\frac{y^2}{Mt}}.$$

(II) For $|x-y| \le Kt$ *and* $t \ge T_1$

$$\Big|\tilde{G}(t,x;y)\Big| \le Ct^{-1/2}e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_y(t,x;y)\Big| \le Ct^{-1}e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_x(t,x;y)\Big| \le C\Big[t^{-1/2}e^{-\eta|x|} + t^{-1}\Big]e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_{xy}(t,x;y)\Big| \le C\Big[t^{-1}e^{-\eta|x|}e^{-\frac{y^2}{Mt}} + t^{-1}e^{-\eta|x-y|} + t^{-3/2}e^{-\frac{(x-y)^2}{Mt}}\Big].$$

(III) For $|x-y| \ge Kt$ *or* $0 < t < T_2$

$$\Big|\partial^\alpha \tilde{G}(t,x;y)\Big| \le C\Big[t^{-\frac{1+|\alpha|}{4}}e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}} + e^{-\eta(|x|+t)}e^{-\frac{y^2}{Mt}}\Big]$$

where α *is a standard multiindex in* x *and* y *with* $|\alpha| \le 3$*. In all cases symmetric estimates hold for* $y > 0$*.*

Remark 2.1. *Using the notation of (C1)-(C2) we can, up to a choice of scaling, specify the values* $\{\beta_j^-\}_{j=1}^m$ *and* $\{\tilde{r}_{m+j}^-(0)\}_{j=1}^m$ *by the relation*

$$\tilde{r}_{m+j}^-(0)M_-B_- = \beta_j^-\tilde{r}_{m+j}^-(0).$$

I.e., the β_j^- *are the (necessarily positive) eigenvalues of the asymptotic* $m \times m$ *matrix* M_-B_-*, and the* $\{\tilde{r}_{m+j}^-(0)\}_{j=1}^m$ *are the associated left eigenvectors (which span* $\mathbb{R}^m$ *by (H4)). For convenient reference we are adopting the notation of [11], where the* $\tilde{r}_{m+j}^-$ *are functions of* λ*, but fot he current discussion we only require the leading order. The values* $\{c_j\}_{j=m+1}^{2m}$ *can be specified as*

$$c_j = h_{(2m)j}^-\tilde{c}_j^-(0),$$

where the $\{\tilde{c}_j^-\}_{j=m+1}^{2m}$ *are described in Lemma 3.5 of [11], while the values* $\{h_{(2m)j}^-\}_{j=m+1}^{2m}$ *are described in Lemma 3.9 Part (iv) of the same reference. Although we give these precise specifications to be complete, our analysis only requires the existence of such constants.*

The estimates on $\tilde{G}$ *could be expressed in a more detailed form, similar to the expressions for* $e(t;y)$*, but our analysis won't require that much precision, and we have chosen to omit it. See [3, 8] for more precise statements in the scalar case.*

Using Theorem 2.1, the proof of Theorem 1.1 is straightforward, and we carry out details only for one example case. For $t \geq T_1$, we verify

$$\sup_{y\in\mathbb{R}} \|\tilde{G}(t,\cdot;y)\|_{L^p_x} \leq C t^{-\frac{1}{2}(1-\frac{1}{p})}.$$

Noting that we have different estimates on $\tilde{G}$ for $|x-y| \leq Kt$ and $|x-y| > Kt$, we write

$$\begin{aligned}
\|\tilde{G}(t,\cdot;y)\|^p_{L^p_x} &= \int_{|x-y|>Kt} |\tilde{G}(t,x;y)|^p dx + \int_{|x-y|\leq Kt} |\tilde{G}(t,x;y)|^p dx \\
&\leq C_1 \int_{|x-y|>Kt} \Big[t^{-\frac{p}{4}} e^{-\frac{p|x-y|^{4/3}}{Mt^{1/3}}} + e^{-\eta p(|x|+t)} e^{-\frac{py^2}{Mt}} \Big] dx \\
&+ C_2 \int_{|x-y|\leq Kt} t^{-p/2} e^{-\frac{p(x-y)^2}{Mt}} dx.
\end{aligned}$$

For the first of these three terms, we observe that since $|x-y| \geq Kt$ we have

$$e^{-\frac{p|x-y|^{4/3}}{Mt^{1/3}}} = e^{-\frac{p|x-y|^{4/3}}{2Mt^{1/3}}} e^{-\frac{p|x-y|^{4/3}}{2Mt^{1/3}}} \leq e^{-\frac{p|x-y|^{4/3}}{2Mt^{1/3}}} e^{-\frac{pK^{4/3}}{2M} t}.$$

In this way, we obtain exponential decay in t for both the first two terms. The claimed estimate now follows by direct integration of the third term.

□

3 Preliminary Estimates

In order to motivate the estimates established in this section, we recall from (1.16) the equation

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y) v_0(y) dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y) \Big[\dot{\delta}(s) v(s,y) + Q(s,y) \Big] dy ds.$$

We will take L^p norms of this equation, and so our analysis will require L^p norms of expressions such as

$$\int_{-\infty}^{+\infty} \tilde{G}(t,x;y) f(y) dy,$$

where f is in some appropriate L^p space.

We begin with a useful straightforward lemma.

Lemma 3.1. *Let $p \in [1,\infty]$ and suppose $K(t,x;y)$ is any function so that for any pair $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ $K(t,x;\cdot) \in L^1 \cap L^p$ and for any pair $(t,y) \in \mathbb{R}_+ \times \mathbb{R}$ $K(t,\cdot;y) \in L^1 \cap L^p$. Then given any function $f \in L^1 \cap L^p$, we have*

$$\Big\| \int_{-\infty}^{+\infty} K(t,x;y) f(y) dy \Big\|_{L^p} \leq \min \Big\{ \sup_{y\in\mathbb{R}} \|K\|_{L^p_x} \|f\|_{L^1}, \sup_{y\in\mathbb{R}} \|K\|^{\frac{1}{p}}_{L^1_x} \sup_{x\in\mathbb{R}} \|K\|^{\frac{1}{q}}_{L^1_y} \|f\|_{L^p} \Big\},$$

where q is the Hölder conjugate $q = p/(p-1)$.

Proof. For the first term in the minimum, we simply bring the L^p norm inside the integration (the triangle inequality or Minkowski's integral inequality; see [5]).

In order to establish the second inequality in the minimum we write

$$\begin{aligned}\Big\| \int_{-\infty}^{+\infty} K(t,x;y)f(y)dy \Big\|_{L^p} &\le \Big\| \int_{-\infty}^{+\infty} |K(t,x;y)|^{\frac{1}{p}} |f(y)| |K(t,x;y)|^{\frac{1}{q}} dy \Big\|_{L^p} \\ &\le \Big\| \Big(\int_{-\infty}^{+\infty} |K(t,x;y)| |f(y)|^p dy \Big)^{\frac{1}{p}} \Big(\int_{-\infty}^{+\infty} |K(t,x;y)| dy \Big)^{\frac{1}{q}} \Big\|_{L^p},\end{aligned}$$

where we have used Hölder's inequality. We bring the L^1 norm on K outside the L^p norm by taking supremum over x, giving an estimate by

$$\begin{aligned}&\sup_{x\in\mathbb{R}} \|K\|_{L^1_y}^{\frac{1}{q}} \Big(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |K| |f|^p dy dx \Big)^{\frac{1}{p}} \\ &\quad \le \sup_{x\in\mathbb{R}} \|K\|_{L^1_y}^{\frac{1}{q}} \sup_{y\in\mathbb{R}} \|K\|_{L^1_x}^{\frac{1}{p}} \|f\|_{L^p}.\end{aligned}$$

□

Theorem 3.1. *Let $e(t;y)$ and $\tilde{G}(t,x;y)$ denote any functions satisfying the estimates of Theorem 1.1, and suppose $f \in L^1 \cap L^p$ for some $p \in [1,\infty]$. Then there exists a constant $C > 0$ so that the following estimates hold:*

(I) For all $t \ge 0$

$$\Big| \int_{-\infty}^{+\infty} e_t(t;y) f(y) dy \Big| \le C(1+t)^{-\frac{1}{2}-\frac{1}{2p}} \|f\|_{L^p}$$

$$\Big| \int_{-\infty}^{+\infty} e_{ty}(t;y) f(y) dy \Big| \le C(1+t)^{-1-\frac{1}{2p}} \|f\|_{L^p}.$$

(II) For $t \ge T_1$

$$\Big\| \int_{-\infty}^{+\infty} \tilde{G}(t,x;y) f(y) dy \Big\|_{L^p} \le C \min \Big\{ t^{-\frac{1}{2}(1-\frac{1}{p})} \|f\|_{L^1}, \|f\|_{L^p} \Big\}$$

$$\Big\| \int_{-\infty}^{+\infty} \tilde{G}_y(t,x;y) f(y) dy \Big\|_{L^p} \le C \min \Big\{ t^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})} \|f\|_{L^1}, t^{-\frac{1}{2}} \|f\|_{L^p} \Big\}$$

$$\Big\| \int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y) f(y) dy \Big\|_{L^p} \le C \min \Big\{ t^{-\frac{1}{2}} \|f\|_{L^1}, t^{-\frac{1}{2p}} \|f\|_{L^p} \Big\}$$

$$\Big\| \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t,x;y) f(y) dy \Big\|_{L^p} \le C \min \Big\{ t^{-1} \|f\|_{L^1}, t^{-\frac{1}{2}-\frac{1}{2p}} \|f\|_{L^p} \Big\}$$

(III) For $t \le T_2$

$$\Big\| \int_{-\infty}^{+\infty} \partial^\alpha \tilde{G}(t,x;y) f(y) dy \Big\|_{L^p} \le C t^{-\frac{|\alpha|}{4}} \|f\|_{L^p},$$

where α denotes a standard multiindex in x and y, $|\alpha| \le 3$.

Comment on the proof. The proof of Theorem 3.1 is a straightforward combination of the estimates of Theorem 1.1 and Lemma 3.1. We omit the details.

4 Short Time Theory

It will be useful to begin this section by defining the function we will ultimately show is bounded. We set

$$\zeta(t) := \sup_{\substack{0\le s\le t \\ 1\le p\le\infty}} \Big\{ \|v(s,\cdot)\|_{L^p}(1+s)^{\frac{1}{2}(1-\frac{1}{p})} + \|v_x(s,\cdot)\|_{L^p} s^{1/4}(1+s)^{1/4} + |\dot{\delta}(s)|(1+s) \Big\}. \tag{4.1}$$

The following inequalities are an immediate consequence for all $s \in [0,t]$:

$$\begin{aligned} \|v(s,\cdot)\|_{L^p} &\le \zeta(t)(1+s)^{-\frac{1}{2}(1-\frac{1}{p})} \\ \|v_x(s,\cdot)\|_{L^p} &\le \zeta(t) s^{-1/4}(1+s)^{-1/4} \\ |\dot{\delta}(s)| &\le \zeta(t)(1+s)^{-1}. \end{aligned} \tag{4.2}$$

In developing our short-time theory, our primary concern is the term $v_{xxx}(t,x)$, which appears in the nonlinearity Q. In order to control this term, we first note that our perturbation equation for v can be expressed as

$$\begin{aligned} v_t = &-M(\bar{u}+v)\Gamma v_{xxxx} - M'(\bar{u}+v)(\bar{u}_x+v_x)\Gamma v_{xxx} \\ &+ (M(\bar{u}+v)(\tilde{A}v)_x)_x + \dot{\delta}(t)v_x + \dot{\delta}(t)\bar{u}_x, \end{aligned} \tag{4.3}$$

where we have set

$$\tilde{A}(t,x) := \int_0^1 Df(\bar{u}+\gamma v)d\gamma.$$

As verified in [11] and the more detailed reference [9], we can view (4.3) as a linear equation in v. (Briefly, our point of view, following [9, 17], is that existence of a solution $v \in C^{\frac{\gamma}{4},\gamma}([0,T]\times\mathbb{R})$ is known—as established in [9]—and so expressions such as $M(\bar{u}+v)$ can be regarded as given coefficients for a linear problem.) Let $G^v(t,x;\tau,\xi)$ denote the Green's function associated with the homogeneous part of (4.3) (i.e., the equation with $\dot{\delta}(t)\bar{u}_x$ omitted), so that

$$v(t,x) = \int_{-\infty}^{+\infty} G^v(t,x;0,\xi)v_0(\xi)d\xi + \int_0^t \int_{-\infty}^{+\infty} G^v(t,x;\tau,\xi)\dot{\delta}(\tau)\bar{u}'(\xi)d\xi ds. \tag{4.4}$$

We are closely following our references [4, 9] here, and for ease of comparison we adopt their notation. Upon differentiating (4.4) three times with respect to x, we obtain

$$v_{xxx}(t,x) = \int_{-\infty}^{+\infty} G^v_{xxx}(t,x;0,\xi)v_0(\xi)d\xi + \int_0^t \int_{-\infty}^{+\infty} G^v_{xxx}(t,x;\tau,\xi)\dot{\delta}(\tau)\bar{u}'(\xi)d\xi ds. \tag{4.5}$$

Fix any time $T_0 > 0$ and consider times $0 < t \leq T_0$. Following Friedman [4], we obtain the estimate

$$\left|G^v_{xxx}(t,x;\tau,\xi)\right| \leq C(t-\tau)^{-1}e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}}.$$

According to our definition of $\zeta(t)$, we see upon letting $t \to 0$ and using $\dot{\delta}(0) = 0$ as well as the monotonicity of ζ that

$$\|v(0,\cdot)\|_{L^p} \leq \zeta(0) \leq \zeta(t). \tag{4.6}$$

Accordingly, we can estimate $v_{xxx}(t,x)$ as follows:

$$\begin{aligned}\|v_{xxx}(t,\cdot)\|_{L^p} \leq C_1 \Big\| \int_{-\infty}^{+\infty} t^{-1} e^{-\frac{(x-\xi)^{4/3}}{Mt^{1/3}}} |v_0(\xi)| d\xi \Big\|_{L^p} \\ + C_2 \int_0^t \Big\| \int_{-\infty}^{+\infty} (t-\tau)^{-1} e^{-\frac{(x-\xi)^{4/3}}{M(t-\tau)^{1/3}}} |\dot{\delta}(\tau)| |\bar{u}'(\xi)| d\xi \Big\|_{L^p} d\tau \\ \leq C\zeta(t) t^{-3/4},\end{aligned}$$

where we've used Lemma 3.1.

Since t is bounded, we can write this expression with more decay in t simply by increasing the size of C. In particular, we are justified in writing

$$\|v_{xxx}(t,\cdot)\|_{L^p} \leq \tilde{C}\zeta(t) t^{-3/4}(1+t)^{1/4}. \tag{4.7}$$

Next, we need to verify (4.7) for $t > T_0$. In this case we want to verify that for large time v_{xxx} inherits the increased decay rate of v_x, and so our goal will be to bound v_{xxx} in terms of v_x (rather than v, as in our bounded-time calculation). Formally differentiating (4.3) with respect to x, and setting $w = v_x$, we obtain

$$\begin{aligned}w_t = -M(\bar{u}+v)\Gamma w_{xxxx} - M'(\bar{u}+v)_x \Gamma w_{xxx} - (M'(\bar{u}_x + v_x)\Gamma w_{xx})_x \\ + (M(\bar{u}+v)(\tilde{A}v)_x)_{xx} + \dot{\delta}(t) w_x + \dot{\delta}(t)\bar{u}_{xx}.\end{aligned} \tag{4.8}$$

Using now our short-time theory for v, we see that (4.8) can be solved in Friedman's framework with two source terms:

$$q(t,x)v(t,x) + \dot{\delta}(t)\bar{u}_{xx}(x).$$

Here, $q(t,x)$ has several terms, but we need only recognize that each of these is multiplied by some derivative of $\bar{u}(x)$, so that $|q(t,x)| \leq Ce^{-\alpha|x|}$, uniformly in t.

Let $G^w(t,x;\tau,\xi)$ denote the Green's function associated with the homogeneous part of (4.8), so that for any fixed $\tau \geq 0$

$$\begin{aligned}w(t,x) = \int_{-\infty}^{+\infty} G^w(t,x;\tau,\xi) w(\tau,\xi) d\xi \\ + \int_\tau^t \int_{-\infty}^{+\infty} G^w(t,x;s,\xi) \Big[q(s,\xi)v(s,\xi) + \dot{\delta}(s)\bar{u}_{\xi\xi}(\xi)\Big] d\xi ds.\end{aligned} \tag{4.9}$$

Now, differentiating (4.9) twice with respect to x, and recalling $w = v_x$, we find

$$\begin{aligned} v_{xxx}(t,x) &= \int_{-\infty}^{+\infty} G^w_{xx}(t,x;\tau,\xi) v_\xi(\tau,\xi) d\xi \\ &\quad + \int_\tau^t \int_{-\infty}^{+\infty} G^w_{xx}(t,x;s,\xi) \Big[q(s,\xi)v(s,\xi) + \dot{\delta}(s)\bar{u}_{\xi\xi}(\xi) \Big] d\xi ds \\ &= I_1 + I_2. \end{aligned} \tag{4.10}$$

In what follows, we fix the increment $t - \tau =: T$ as a sufficiently small value, but let τ (and so t) grow. In particular, we take $T < T_0$ so that for $t > T_0$ we have $t > T$ so that $\tau > 0$.

We can write (from (4.10))

$$\begin{aligned} \|I_1\|_{L^p} &\le \Big\| \int_{-\infty}^{+\infty} |G^w_{xx}(t,x;\tau,\xi)| |v_\xi(\tau,\xi)| d\xi \Big\|_{L^p} \le C(t-\tau)^{-1/2} \|v_\xi(\tau,\cdot)\|_{L^p} \\ &\le C\zeta(t)(t-\tau)^{-1/2}\tau^{-1/4}(1+\tau)^{-1/4}. \end{aligned}$$

Likewise,

$$\begin{aligned} \|I_2\|_{L^p} &\le C \int_\tau^t (t-\tau)^{-1/2} \Big(\|q\|_{L^p}\|v\|_{L^\infty} + |\dot{\delta}(s)| \|\bar{u}''\|_{L^p} \Big) ds \\ &\le \tilde{C}\zeta(t) \int_\tau^t (t-\tau)^{-1/2} \Big(\|q\|_{L^p}(1+s)^{-1/2} + (1+s)^{-1} \|\bar{u}''\|_{L^p} \Big) ds \\ &\le \tilde{\tilde{C}}\zeta(t)(t-\tau)^{1/2}(1+\tau)^{-1/2}. \end{aligned}$$

Combining this observation with the case $0 < t < T_0$, we see that (4.7) holds for all $t > 0$. Finally, since $t = \tau + T$, with T small, we can find a (new) constant C so that

$$\|v_{xxx}(t,\cdot)\|_{L^p} \le C\zeta(t) t^{-3/4}(1+t)^{1/4}. \tag{4.11}$$

5 Proof of Theorem 1.2

We begin our proof of Theorem 1.2 by estimating the nonlinearity

$$\mathcal{N}(s,y) := \dot{\delta}(s)v(s,y) + Q(s,y) \tag{5.1}$$

in terms of ζ. We have (combining (1.10), (4.1), and (4.11))

$$\begin{aligned} \|\mathcal{N}(s,\cdot)\|_{L^p} &\le |\dot{\delta}(s)| \|v(s,\cdot)\|_{L^p} + C\Big[\| |v||v_x| \|_{L^p} + \|e^{-\alpha|\cdot|}|v|^2\|_{L^p} + \| |v||v_{xxx}| \|_{L^p} \Big] \\ &\le |\dot{\delta}(s)| \|v(s,\cdot)\|_{L^p} + C\Big[\|v\|_{L^\infty}\|v_x\|_{L^p} + \|v\|^2_{L^\infty}\|e^{-\alpha|\cdot|}\|_{L^p} + \|v\|_{L^\infty}\|v_{xxx}\|_{L^p} \Big] \\ &\le C\zeta(t)^2 s^{-3/4}(1+s)^{-1/4}. \end{aligned}$$

Lemma 5.1. *Under the assumptions of Theorem 1.2, and for $\zeta(t)$ as defined in (4.1), we have*

$$\|v(t,\cdot)\|_{L^p} \le C(\epsilon + \zeta(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$$
$$\|v_x(t,\cdot)\|_{L^p} \le C(\epsilon + \zeta(t)^2)t^{-1/4}(1+t)^{-1/4}$$
$$|\dot{\delta}(t)| \le C(\epsilon + \zeta(t)^2)(1+t)^{-1}.$$

Proof. For this calculation it will be convenient to take (referring to the statement of Theorem 1.1) $T_2 = 1$.

For $t \le 1$, we have

$$\begin{aligned}
\|v(t,\cdot)\|_{L^p} &\le \Big\| \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy \Big\|_{L^p} + \int_0^t \Big\| \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\mathcal{N}(s,y)dy \Big\|_{L^p} ds \\
&\le C_1\|v_0\|_{L^p} + C_2\zeta(t)^2 \int_0^t (t-s)^{-1/4}\|\mathcal{N}(s,\cdot)\|_{L^p} ds \\
&\le C_1\epsilon + \tilde{C}_2\zeta(t)^2 \int_0^t (t-s)^{-1/4}s^{-3/4}(1+s)^{-1/4} ds \\
&\le C(\epsilon + \zeta(t)^2).
\end{aligned}$$

Here, since t is bounded, we can (by taking a larger constant C) express this inequality as

$$\|v(t,\cdot)\|_{L^p} \le C(\epsilon + \zeta(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

Proceeding similarly for $\|v_x(t,\cdot)\|_{L^p}$ we find

$$\|v_x(t,\cdot)\|_{L^p} \le C(\epsilon + \zeta(t)^2)t^{-1/4},$$

and again since t is bounded we can express this with increased decay in t

$$\|v_x(t,\cdot)\|_{L^p} \le C(\epsilon + \zeta(t)^2)t^{-1/4}(1+t)^{-1/4}.$$

Likewise, we easily verify that

$$|\dot{\delta}(t)| \le C(\epsilon + \zeta(t)^2),$$

and for bounded time we can express this as

$$|\dot{\delta}(t)| \le C(\epsilon + \zeta(t)^2)(1+t)^{-1}.$$

For $t > 1$, we estimate $\|v(t,\cdot)\|_{L^p}$ as

$$\begin{aligned}
\|v(t,\cdot)\|_{L^p} \le & \Big\| \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy \Big\|_{L^p} \\
&+ \int_0^{t-1} \Big\| \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\mathcal{N}(s,y)dy \Big\|_{L^p} ds \\
&+ \int_{t-1}^{t} \Big\| \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\mathcal{N}(s,y)dy \Big\|_{L^p} ds.
\end{aligned} \tag{5.2}$$

Using Theorem 3.1, we estimate the integrals on the right hand side respectively by

$$\begin{aligned}C_1 t^{-\frac{1}{2}(1-\frac{1}{p})}\|v_0\|_{L^1} + C_2\zeta(t)^2 \int_0^{t-1} & (t-s)^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})} s^{-3/4}(1+s)^{-1/4} ds \\ & + C_3\zeta(t)^2 \int_{t-1}^{t} (t-s)^{-1/4} s^{-3/4}(1+s)^{-1/4} ds \\ & \le C(\epsilon + \zeta(t)^2) t^{-\frac{1}{2}(1-\frac{1}{p})}.\end{aligned}$$

Since $t \ge 1$ in this case, this is equivalent with the claimed estimate.

Likewise,

$$\begin{aligned}\|v_x(t,\cdot)\|_{L^p} \le & \Big\| \int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y) v_0(y) dy \Big\|_{L^p} \\ & + \int_0^{t-1} \Big\| \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\mathcal{N}(s,y) dy \Big\|_{L^p} ds \\ & + \int_{t-1}^{t} \Big\| \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\mathcal{N}(s,y) dy \Big\|_{L^p} ds.\end{aligned} \tag{5.3}$$

Using Theorem 3.1, we estimate the integrals on the right hand side respectively by

$$\begin{aligned}C_1 t^{-\frac{1}{2}}\|v_0\|_{L^1} + C_2\zeta(t)^2 \int_0^{t-1} & (t-s)^{-1} s^{-3/4}(1+s)^{-1/4} ds \\ & + C_3\zeta(t)^2 \int_{t-1}^{t} (t-s)^{-1/2} s^{-3/4}(1+s)^{-1/4} ds \\ & \le C(\epsilon + \zeta(t)^2) t^{-\frac{1}{2}}.\end{aligned}$$

Since $t \ge 1$ in this case, this is equivalent with the claimed estimate.

Finally,

$$\begin{aligned}|\dot{\delta}(t)| \le & \Big| \int_{-\infty}^{+\infty} e_t(t;y) v_0(y) dy \Big| \\ & + \int_0^t \Big| \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\mathcal{N}(s,y) dy \Big| ds \\ & \le C_1(1+t)^{-1}\|v_0\|_{L^1} + C_2\zeta(t)^2 \int_0^t (1+(t-s))^{-3/2} s^{-3/4}(1+s)^{-1/4} ds \\ & \le C(\epsilon + \zeta(t)^2)(1+t)^{-1}.\end{aligned}$$

□

It's clear from Lemma 5.1 that

$$\|v(t,\cdot)\|_{L^p}(1+t)^{\frac{1}{2}(1-\frac{1}{p})} + \|v_x(t,\cdot)\|_{L^p} t^{1/4}(1+t)^{1/4} + |\dot{\delta}(t)|(1+t) \le 3C(\epsilon + \zeta(t)^2), \tag{5.4}$$

for all $t > 0$ such that the right-hand side is bounded. If we express this inequality with s replacing t and taking a supremum over both sides for $s \in [0, t]$, then by monotonicity of the right-hand side we conclude

$$\zeta(t) \le 3C(\epsilon + \zeta(t)^2).$$

As verified in [8] (see Claim 4.1 on p. 799), we can conclude from this last inequality that

$$\zeta(t) < 6C\epsilon,$$

for all $t \ge 0$. The estimates claimed in Theorem 1.2 are an immediate consequence of this last inequality. The existence follows by combining this estimate with the short-time theory of [9]. More precisely, by a standard continuation argument, we can verify that $v(t,x)$ exists so long as $\|v(t,\cdot)\|_{C^\gamma}$ remains bounded. But our bound

$$\|v_x(t,\cdot)\|_{L^p} \le 6C\epsilon t^{-1/4}(1+t)^{-1/4}$$

ensures (by Sobolev embedding) that $v(t,\cdot) \in C^\gamma(\mathbb{R})$ for any $0 < \gamma < 1$ and any $t > 0$. (The fact that $v(t,\cdot) \in C^\gamma(\mathbb{R})$ for $0 \le t \le T_0$, with T_0 sufficiently small is established in [9] by a direct contraction mapping argument.)

$\square$

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References

[1] N. D. Alikakos, S. I. Betelu, and X. Chen, *Explicit stationary solutions in multiple well dynamics and non-uniqueness of interfacial energy densities*, Euro. Jnl of Applied Mathematics **17** (2006) 525–556.

[2] N. D. Alikakos and G. Fusco, *On the connection problem for potentials with several global minima*, Indiana U. Math. J. **57** (2008) 1871–1906.

[3] J. Bricmont, A. Kupiainen, and J. Taskinen, *Stability of Cahn-Hilliard fronts*, Comm. Pure Appl. Math. **LII** (1999) 839-871.

[4] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[5] G. B. Folland, *Real analysis: modern techniques and their applications*, 2nd Ed., Wiley Interscience 1999.

[6] P. Howard, *Pointwise estimates on the Green's function for a scalar linear convection-diffusion equation*, J. Differential Equations **155** (1999) 327-367.

[7] P. Howard, *Pointwise Green's function approach to stability for scalar conservation laws*, Comm. Pure Appl. Math. **52** (1999) 1295-1313.

[8] P. Howard, *Asymptotic behavior near transition fronts for equations of generalized Cahn-Hilliard form*, Commun. Math. Phys. **269** (2007) 765–808.

[9] P. Howard, *Short-time existence theory toward stability for parabolic systems*, in preparation.

[10] P. Howard and B. Kwon, *Spectral analysis for transition front solutions in Cahn-Hilliard systems*, to appear in Discrete and Continuous Dynamical Systems A.

[11] P. Howard and B. Kwon, *Asymptotic stability analysis for transition wave solutions in Cahn-Hilliard systems*, Preprint 2011.

[12] P. Howard and K. Zumbrun, *Pointwise estimates and stability for dispersive-diffusive shock waves*, Arch. Rational Mech. Anal. **155** (2000) 85-169.

[13] P. Howard and K. Zumbrun, *Stability of undercompressive shock profiles*, J. Differential Eqns. **225** (2006) 308-360.

[14] M. Raoofi, *L^p asymptotic behavior of perturbed viscous shock profiles*, Hyperbolic Differ. Equ. **2** (2005) 595-644.

[15] V. Stefanopoulos, *Heteroclinic connections for multiple-well potentials: the anisotropic case*, Proc. Royal Soc. Edinburgh **138A** (2008) 1313-1330.

[16] K. Zumbrun, *Refined wave-tracking and nonlinear stability of viscous Lax shocks*, Methods Appl. Anal. **7** (2000) 747-768.

[17] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana U. Math. J. **47** (1998) 741–871. See also the errata for this paper: Indiana U. Math. J. **51** (2002) 1017–1021.