Hörmander's Index and Oscillation Theory

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Abstract

One of the obstacles that arises in the generalization of Sturm's oscillation theorem to the case of general linear Hamiltonian systems is the need to associate a sign with each crossing point, necessitating a signed count rather than a direct count of such points. It's known, however, that this difficulty does not arise for all system/boundary-condition combinations, and so it can be overcome in some cases by exchanging one boundary condition for another while also keeping track of any ancillary counts that accompany the exchange. The primary tool for making such an exchange is Hörmander's index, and in this analysis we develop a straightforward method for computing Hörmander's index, and employ our method to formulate oscillation-type theorems for linear Hamiltonian systems on both bounded and unbounded intervals.

1 Introduction

The notion of *oscillation theory* dates back to about 1836, when the French mathematician Jacques Sturm published a striking result asserting that for Sturm-Liouville equations (as they are now called) under certain conditions, the number of eigenvalues below a given eigenvalue λ can be counted as the number of roots for the eigenfunction $\phi(x;\lambda)$ associated to λ [42]. A natural generalization of Sturm's Oscillation Theorem to systems was carried out by H. C. Marston Morse in 1934 [36], and we set the context for the current analysis by briefly discussing a consequence of Morse's Index Theorem. For a Sturm-Liouville system with Dirichlet boundary conditions,

$$\begin{aligned} -(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi; \quad x \in (0,1), \quad \phi(x;\lambda) \in \mathbb{C}^n, \\ \phi(0) = 0, \quad \phi(1) = 0, \end{aligned} \tag{1.1}$$

suppose $P^{-1}, V, Q \in L^1((0,1), \mathbb{C}^{n\times n})$, with $P(x), V(x), Q(x)$ self-adjoint for a.e. $x \in (0,1)$, and that there exist constants $\theta_P, \theta_Q, C_V > 0$ so that

$$(P(x)v, v) \geq \theta_P |v|^2; \quad (Q(x)v, v) \geq \theta_Q |v|^2, \quad |(V(x)v, v)| \leq C_V |v|^2,$$

for a.e. $x \in (0,1)$ (for every $v \in \mathbb{C}^n$). (Here and throughout, our restriction to $(0,1)$ is taken for notational convenience and serves to indicate generic bounded intervals.) For some fixed $\lambda \in \mathbb{R}$, let $X(x;\lambda)$ denote an $n \times n$ matrix solution of (1.1), initialized with $X(0;\lambda) = 0$ and

$P(0)X'(0;\lambda) = I$. Then one consequence of the Morse Index Theorem is that the number of eigenvalues that (1.1) has below λ—for which we will use the notation $\mathcal{N}((-\infty,\lambda))$—can be computed as the sum

$$\mathcal{N}((-\infty,\lambda)) = \sum_{x\in(0,1)} \dim\ker X(x;\lambda). \tag{1.2}$$

(As is clear from the title of Morse's original work, the Morse Index Theorem was developed in a variational framework, and is typically stated in that context. The statement given here simply fits better with our development.)

Expressed this way, the Morse Index Theorem is a natural generalization of the Sturm Oscillation Theorem. However, if we try to express a Morse-type theorem for general equations and/or more general self-adjoint boundary conditions (not just Dirichlet), we find, among other complications, that the right-hand side of (1.2) must generally be replaced with a *signed* count of the dimensions of certain kernels. Early work along these lines with a specific emphasis on generalizing Sturm's theorems was carried out by Edwards [15] and Arnol'd [2], and more generally the literature in this direction has become quite vast (see, for example, [6, 8, 9, 11, 13, 24, 26, 34, 40] and the references therein). While the current analysis is closely related to such work, the emphasis here is limited to developing a framework that allows us to readily compute counts such as $\mathcal{N}((-\infty,\lambda))$ via unsigned sums of the dimensions of certain kernels. In particular, the primary application we have in mind is to the study of spectral stability of stationary and traveling waves arising as solutions to certain nonlinear PDE (see particularly Theorems 1.3 and 1.4), and in such cases the count we are interested in is $\mathcal{N}((-\infty,0))$ (with a sign convention placing unstable spectrum in the negative half-plane).

In order to place Sturm-Liouville systems in the context of linear Hamiltonian systems, we can set $y = \binom{y_1}{y_2}$, with $y_1(x) = \phi(x)$ and $y_2(x) = P(x)\phi'(x)$, and express (1.1) as a linear Hamiltonian system

$$Jy' = \mathbb{B}(x;\lambda)y; \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V(x) & 0 \\ 0 & P(x)^{-1} \end{pmatrix}. \tag{1.3}$$

More generally, the setting for the current analysis will be as follows: for values λ in some open interval $I \subset \mathbb{R}$ we consider general linear Hamiltonian systems

$$Jy' = \mathbb{B}(x;\lambda)y; \quad x \in (0,1), \quad y(x;\lambda) \in \mathbb{C}^{2n}, \tag{1.4}$$

where throughout most of the analysis we will make the following assumptions on $\mathbb{B}(x;\lambda)$:

(A) We assume $\mathbb{B}(\cdot;\lambda) \in L^1((0,1),\mathbb{C}^{2n\times 2n})$, with $\mathbb{B}(x;\lambda)$ self-adjoint for a.e. $x \in (0,1)$, and additionally that $\mathbb{B}$ is differentiable in λ, with $\mathbb{B}_\lambda(\cdot;\lambda) \in L^1((0,1),\mathbb{C}^{2n\times 2n})$ for each $\lambda \in I$.

In the case of separated self-adjoint boundary conditions, we can write

$$\alpha y(0) = 0; \quad \beta y(1) = 0, \tag{1.5}$$

where the boundary operator matrices α and β satisfy

$$\alpha \in \mathbb{C}^{n\times 2n},\ \operatorname{rank}\alpha = n,\ \alpha J\alpha^* = 0; \quad \beta \in \mathbb{C}^{n\times 2n},\ \operatorname{rank}\beta = n,\ \beta J\beta^* = 0. \tag{1.6}$$

Throughout the remainder of the analysis, whenever we refer to (1.5), we assume the specifications (1.6) hold.

We will say that $\lambda \in I$ is an eigenvalue of (1.4)-(1.5) if there exists an absolutely continuous function $y \in \mathrm{AC}([0,1], \mathbb{C}^{2n}) \backslash \{0\}$ that satisfies (1.4)-(1.5) (here, $AC(\cdot)$ denotes absolute continuity). In the usual way, we will refer to the dimension of the space of all such solutions to (1.4)-(1.5) as the geometric multiplicity of λ.

Our primary tool for this analysis will be the Maslov index, and as a starting point for a discussion of this object, we define what we will mean by a Lagrangian subspace of $\mathbb{C}^{2n}$.

Definition 1.1. *We say $\ell \subset \mathbb{C}^{2n}$ is a Lagrangian subspace of $\mathbb{C}^{2n}$ if ℓ has dimension n and*

$$(Ju, v)_{\mathbb{C}^{2n}} = 0, \tag{1.7}$$

for all $u, v \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{C}^{2n}}$ denotes the standard inner product on $\mathbb{C}^{2n}$. In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{C}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.

Any Lagrangian subspace of $\mathbb{C}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{C}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often coordinatize our frames as $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices. Following [16] (p. 274), we specify a metric on $\Lambda(n)$ in terms of appropriate orthogonal projections. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^* \mathbf{X}_i)^{-1} \mathbf{X}_i^*$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\| \cdot \|$ can denote any matrix norm. We will say that a path of Lagrangian subspaces $\ell : \mathcal{I} \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote continuous paths of Lagrangian subspaces $\ell_i : \mathcal{I} \to \Lambda(n)$, $i = 1, 2$, for some parameter interval $\mathcal{I}$. The Maslov index associated with these paths, which we will denote $\mathrm{Mas}(\ell_1, \ell_2; \mathcal{I})$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (In this setting, if we let t_* denote the point of intersection (often referred to as a *crossing* or *crossing point*; see, e.g., Definition 3.20 in [16]), then multiplicity corresponds with the dimension of the intersection $\ell_1(t_*) \cap \ell_2(t_*)$; a precise definition of what we mean in this context by *direction* will be given in Section 2.)

In order to formulate a standard spectral-counting theorem via the Maslov index, we let $\mathbf{X}_0(x; \lambda)$ denote a $2n \times n$ matrix solving

$$\begin{aligned} J\mathbf{X}_0' &= \mathbb{B}(x; \lambda)\mathbf{X}_0 \\ \mathbf{X}_0(0; \lambda) &= J\alpha^*. \end{aligned} \tag{1.8}$$

Under our assumptions **(A)** on $\mathbb{B}(x; \lambda)$, we can conclude that for each $\lambda \in I$, $\mathbf{X}_0(\cdot; \lambda) \in AC([0,1]; \mathbb{C}^{2n \times n})$. In addition, $\mathbf{X}_0 \in C([0,1] \times I; \mathbb{C}^{2n \times n})$, and $\mathbf{X}_0(x; \lambda)$ is differentiable in λ, with $\partial_\lambda \mathbf{X}_0 \in C([0,1] \times I; \mathbb{C}^{2n \times n})$. (See, for example, [43].) As shown in [21], for each pair $(x, \lambda) \in [0,1] \times I$, $\mathbf{X}_0(x; \lambda)$ is the frame for a Lagrangian subspace, which we will denote $\ell_0(x;\lambda)$. (In [21], the authors make slightly stronger assumptions on $\mathbb{B}(x;\lambda)$, but their proof carries over immediately into our setting.) In addition to $\ell_0(x;\lambda)$, we specify a fixed Lagrangian subspace ℓ_1 with frame $\mathbf{X}_1 = J\beta^*$.

The following *positivity* assumption will be have an important role in our analysis.

(B1) For any $\lambda \in I$ and any $x \in (0,1]$, the matrix

$$\int_0^x \mathbf{X}_0(\xi;\lambda)^* \mathbb{B}_\lambda(\xi;\lambda)\mathbf{X}_0(\xi;\lambda)d\xi \tag{1.9}$$

is positive definite. The assumption that **(B1)** holds for $x = 1$ (but not necessarily for other values of x) will be denoted $\mathbf{(B1)_1}$.

Suppose $\lambda \in I$ is an eigenvalue of (1.4)-(1.5), and denote by $\mathbb{E}(\lambda)$ the linear subspace of solutions of (1.4)-(1.5) corresponding to λ. Given any two values $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, it is shown in [21] that under positivity assumption $\mathbf{(B1)_1}$ the *spectral count*

$$\mathcal{N}([\lambda_1, \lambda_2)) := \sum_{\lambda \in [\lambda_1, \lambda_2)} \dim \mathbb{E}(\lambda), \tag{1.10}$$

is well-defined, and in our notation is precisely $\operatorname{Mas}(\ell_0(1;\cdot), \ell_1; [\lambda_1, \lambda_2))$. It's clear that $\mathcal{N}([\lambda_1, \lambda_2))$ is a count of the eigenvalues of (1.4)-(1.5) on $[\lambda_1, \lambda_2)$, counted with geometric multiplicity. In order to understand the nature of essential spectrum in this setting, and also the notion of algebraic multiplicity, it's useful to frame our discussion in terms of the operator pencil

$$\mathcal{L}(\lambda) = J\frac{d}{dx} - \mathbb{B}(x;\lambda),$$

specified on the domain (independent of λ)

$$\begin{aligned}\mathcal{D} := \{&y \in L^2((0,1), \mathbb{C}^{2n}) : y \in \operatorname{AC}([0,1], \mathbb{C}^{2n}), \\ &\mathcal{L}y \in L^2((0,1), \mathbb{C}^{2n}),\ \alpha y(0) = 0,\ \beta y(1) = 0\}.\end{aligned}$$

Using the methods of [43], we can readily verify that for each $\lambda \in I$, $\mathcal{L}(\lambda)$ is Fredholm and self-adjoint on $\mathcal{D}$, from which we can conclude that $\mathcal{L}$ has no essential spectrum on I. Moreover, under slightly stronger assumptions on $\mathbb{B}$ (in particular, $\mathbb{B}(\cdot;\lambda) \in L^2((0,1), \mathbb{C}^{2n\times 2n})$ for all $\lambda \in I$), we can verify that $\mathcal{L}$ has no Jordan chains of length greater than one, implying that the algebraic and geometric multiplicities of its eigenvalues agree. (See the appendix of [25] for further discussion, and also Section 1.2 of [24], in which the authors consider the same operator pencil under slightly stronger assumptions on $\mathbb{B}(x;\lambda)$.)

In [21], the authors establish the following theorem.

Theorem 1.1. *For equation (1.4) with boundary conditions (1.5), let Assumptions* **(A)** *and* $\mathbf{(B1)_1}$ *hold. Then for any $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, we have*

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell_0(\cdot;\lambda_2), \ell_1; [0,1]) + \operatorname{Mas}(\ell_0(\cdot;\lambda_1), \ell_1; [0,1]).$$

Remark 1.1. *In Corollary 1.7 of [24], the authors formulate a stronger version of Theorem 1.1, in which the spectral count $\mathcal{N}([\lambda_1, \lambda_2))$ is replaced by the spectral flow for the operator* *pencil $\mathcal{L}(\lambda)$ on the interval $[\lambda_1, \lambda_2]$—i.e., a count of the number of eigenvalues of $\mathcal{L}(\lambda)$ that cross zero in the positive direction as λ increases from λ_1 to λ_2 minus the number of eigenvalues of $\mathcal{L}(\lambda)$ that cross zero in the negative direction as λ increases from λ_1 to λ_2 (see Definition 2.4 in [24] for a precise definition). An advantage of the spectral-flow formulation is that it does not require our positivity assumption* $\mathbf{(B1)_1}$*, and indeed in the case of overlapping assumptions on $\mathbb{B}(x;\lambda)$, our Theorem 1.1 results as a special case of Corollary 1.7 of [24] when positivity is assumed.*

The Maslov indices in Theorem 1.1 are signed counts of intersections between the paths $\ell_0(\cdot, \lambda_i)$ $(i = 1, 2)$ and the target space ℓ_1. For certain targets, however (depending on the structure of $\mathbb{B}(x;\lambda)$), the intersections will all have the same direction, and in such cases we again have a direct count of intersections as in (1.2). Hörmander's index provides a way to replace the target frame ℓ_1 with a frame for which the Maslov index gives a monotonic count.

In order to clarify the manner in which this monotonicity can be determined, we recall from [25] a straightforward method for determining the direction of a crossing (effectively, a convenient method for evaluating the crossing form of [38] in our general framework). If λ is fixed, and x_* is a crossing point, it means $\ell_0(x_*;\lambda) \cap \ell_1 \neq \{0\}$. Let $P(x_*)$ denote an orthogonal projection onto this intersection space. Then the direction of the crossing as x increases through x_* is determined by the matrix $-P(x_*)\mathbb{B}(x_*;\lambda)P(x_*)$ in the following sense: if $\dim(\ell_0(x_*;\lambda) \cap \ell_1) = m$, then $-P(x_*)\mathbb{B}(x_*;\lambda)P(x_*)$ can have up to m non-zero eigenvalues. Suppose all m of these eigenvalues are non-zero (we say the crossing point x^* is *non-degenerate* in this case), and let m_+ denote the number that are positive and let m_- denote the number that are negative. Then the Maslov index increments by an amount $m_+ - m_-$ as x increases through x_*. (We note that the matrix $-P(x_*)\mathbb{B}(x_*;\lambda)P(x_*)$ is effectively the same object as the matrix $\Gamma(t)$ specified in Corollary 3.1 of [34]).

As an important motivating case, for the Sturm-Liouville system (1.3), suppose the boundary conditions are general separated self-adjoint at $x = 0$ (i.e., $\alpha y(0) = 0$, with α satisfying (1.6)) and Dirichlet at $x = 1$ (i.e., $\beta = (I \;\; 0)$). In this case, the target Lagrangian subspace ℓ_1 is $\ell_1 = \ell_D = \operatorname{colspan}\binom{0}{I}$. If $x_* \in [0, 1]$ is a crossing point and $P(x_*;\lambda)$ denotes projection onto the space $\ell_0(x_*;\lambda) \cap \ell_D$, then we must have $\operatorname{ran} P(x_*;\lambda) \subset \ell_D$. The eigenvalues of $-P(x_*;\lambda)\mathbb{B}(x_*;\lambda)P(x_*;\lambda)$ are determined precisely by the restriction of the quadratic form

$$Q(v) = -(\mathbb{B}(x_*;\lambda)v, v) \tag{1.11}$$

to $v \in \ell_0(x_*;\lambda) \cap \ell_1 \subset \ell_D$. We see that any $v \in \ell_0(x_*;\lambda) \cap \ell_D$ can be expressed as $v = \binom{0}{v_2}$ for some $v_2 \in \mathbb{C}^n$, and consequently, for any $v \in (\ell_0(x_*;\lambda) \cap \ell_1)\backslash\{0\}$, we have

$$\begin{aligned} Q(v) = -(\mathbb{B}(x_*;\lambda)v, v) &= -\left(\begin{pmatrix} 0 \\ P(x_*)^{-1}v_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right)_{\mathbb{C}^{2n}} \\ &= -(P(x_*)^{-1}v_2, v_2)_{\mathbb{C}^n} < 0, \end{aligned}$$

where the final inequality follows from our assumption that $P(x)$ is positive definite for all $x \in [0, 1]$. We can conclude that in this case each crossing is negatively directed, and indeed it is well known that for Sturm-Liouville systems with Dirichlet boundary conditions all crossings are negatively directed (see, e.g., [2, 6, 15]). More generally, it's clear that if $\mathbb{B}(x;\lambda)$ is the self-adjoint matrix function associated with a general linear Hamiltonian system, and the boundary condition at $x = 1$ is characterized by a Lagrangian subspace ℓ_1, then the direction of crossings will be monotonic so long as the quadratic form (1.11), restricted to ℓ_1, is either positive definite or negative definite.

We contrast this situation with the case in which the target space is the Neumann space $\ell_N = \operatorname{colspan} \binom{0}{I}$. Proceeding as above we see that any $v \in \ell_0(x_*;\lambda) \cap \ell_N$ can be expressed as $v = \binom{v_1}{0}$ for some $v_1 \in \mathbb{C}^n$, and consequently, for any $v \in (\ell_0(x_*;\lambda) \cap \ell_N) \backslash \{0\}$, we have

$$Q(v) = -((\lambda Q(x_*) - V(x_*))v_1, v_1).$$

For sufficiently negative values of λ, the restriction of Q to $\ell_0(x_*;\lambda) \cap \ell_N$ will be positive definite, but otherwise it will not generally be either positive definite or negative definite.

Before generalizing these considerations, we state another Atkinson-type positivity condition:

(B2) We assume that for some $\lambda \in I$ and some $\ell_1 \in \Lambda(n)$, the restriction $\mathbb{B}(x;\lambda)|_{\ell_1}$ is non-negative for a.e. $x \in (0,1)$, and moreover that if $y(x;\lambda)$ is any non-trivial solution of $Jy' = \mathbb{B}(x;\lambda)y$ with $y(x;\lambda) \in \ell_1$ for all x in some interval $[a,b] \subset [0,1]$, $a < b$, then we have

$$\int_a^b (\mathbb{B}(x;\lambda)y(x;\lambda), y(x;\lambda))dx > 0.$$

We note that Condition **(B2)** can be satisfied in the vacuous case that there are no such non-trivial functions $y(x;\lambda)$.

In Section 2, we will prove the following lemma.

Lemma 1.1. *Let Assumptions* **(A)** *and* **(B1)** *hold, and suppose* **(B2)** *holds for some $\lambda \in I$ and some $\ell_1 \in \Lambda(n)$. If $\mathbf{X}_0(x;\lambda)$ solves (1.8), then*

$$\begin{aligned}\operatorname{Mas}(\ell_0(\cdot;\lambda),\ell_1;[0,1]) &= -\sum_{x\in[0,1)} \dim(\ell_0(x;\lambda)\cap\ell_1)\\ &= -\sum_{x\in[0,1)} \dim\ker(\mathbf{X}_0(x;\lambda)^* J\mathbf{X}_1).\end{aligned}$$

Remark 1.2. *Our statement of Lemma 1.1 utilizes Lemma 2.2 from [25], which asserts that if $\mathbf{X}_0$ and $\mathbf{X}_1$ respectively denote frames for Lagrangian subspaces of $\mathbb{C}^{2n}$, ℓ_0 and ℓ_1, then*

$$\dim(\ell_0 \cap \ell_1) = \dim\ker(\mathbf{X}_0^* J\mathbf{X}_1).$$

In the event that the quadratic form Q is indefinite, we can sometimes use Hörmander's index to exchange ℓ_1 for a target frame for which Q is negative definite. The development of a systematic approach to making such exchanges is precisely the goal of this analysis.

To understand how this works in principle, we fix any four Lagrangian subspaces ν, σ, $\tilde{\nu}$, and $\tilde{\sigma}$, and we let $\mathcal{P}(\nu,\sigma)$ denote the collection of continuous paths $\ell : [0,1] \to \Lambda(n)$ such that $\ell(0) = \nu$ and $\ell(1) = \sigma$. It's well known (and will also be verified in Section 3) that the difference

$$\operatorname{Mas}(\ell(\cdot),\tilde{\sigma};[0,1]) - \operatorname{Mas}(\ell(\cdot),\tilde{\nu};[0,1]),$$

is independent of $\ell \in \mathcal{P}(\nu, \sigma)$, and so this difference is an integer depending only on the fixed Lagrangian subspaces ν, σ, $\tilde{\nu}$, and $\tilde{\sigma}$. Following standard terminology and notation (e.g., Equation (2.9) in [14] and Definition 3.9 in [44]), we refer to this value as *Hörmander's index* and express it as

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \operatorname{Mas}(\ell(\cdot), \tilde{\sigma}; [0,1]) - \operatorname{Mas}(\ell(\cdot), \tilde{\nu}; [0,1]). \tag{1.12}$$

(In addition to [14, 44], the reader is referred to the recent article [6], in which the authors obtain a number of useful estimates on Hörmander's index.) In Section 3, we will discuss a straightforward method for evaluating Hörmander's index, and we will compare our approach to a well-known formula of Hörmander's.

Remark 1.3. *Our use of Greek letters to denote Lagrangian subspaces is somewhat standard in this context, following e.g., [14, 44]. In practice, we're introducing this convention to distinguish general formulas, which will be stated with Greek letters denoting Lagrangian subspaces, with particular implementations, for which specific Lagrangian subspaces will typically be designated with ℓ. The most commonly used Greek letters for Lagrangian subspaces seem to be λ and μ, but for the current analysis, we prefer to reserve these to denote spectral parameters.*

In the setting of Theorem 1.1, let $\tilde{\ell}_1$ denote any fixed Lagrangian subspace $\tilde{\ell}_1 \in \Lambda(n)$. Then

$$\operatorname{Mas}(\ell_0(\cdot;\lambda_2), \ell_1; [0,1]) = \operatorname{Mas}(\ell_0(\cdot;\lambda_2), \tilde{\ell}_1; [0,1]) + s(\tilde{\ell}_1, \ell_1; \ell_0(0;\lambda_2), \ell_0(1;\lambda_2)).$$

If we use Lemma 1.1, this allows us to formulate the following theorem.

Theorem 1.2. *For equation (1.4)-(1.5), let Assumptions* **(A)** *and* **(B1)** *hold, and suppose that additionally* **(B2)** *holds for some $\lambda \in I$ and some $\tilde{\ell}_1 \in \Lambda(n)$. Then*

$$\operatorname{Mas}(\ell_0(\cdot;\lambda), \ell_1; [0,1]) = -\sum_{x \in [0,1)} \dim \ker(\mathbf{X}_0(x;\lambda)^* J \tilde{\mathbf{X}}_1) + s(\tilde{\ell}_1, \ell_1; \ell_0(0;\lambda), \ell_0(1;\lambda)).$$

Theorem 1.2 is clearly in the spirit of (1.2), and more generally of the equality $(A) = (B)$ in Theorem 3.1 of [15], though the emphasis in Theorem 1.2 is on the exchange of targets that allows the right-hand side to be expressed as an unsigned sum of nullities similarly as in (1.2).

Our approach also allows us to formulate theorems addressing linear Hamiltonian systems on unbounded domains, and we illustrate the idea with an application to Schrödinger operators in this setting. Our primary reference for this case is [23], and following the development provided there, we consider eigenvalue problems

$$H\phi := -\phi'' + V(x)\phi = \lambda\phi; \quad \operatorname{dom}(H) = H^2(\mathbb{R}), \tag{1.13}$$

and also (for any $s \in \mathbb{R}$)

$$H_s\phi := -\phi'' + s\phi' + V(x)\phi = \lambda\phi; \quad \operatorname{dom}(H_s) = H^2(\mathbb{R}), \tag{1.14}$$

where $\lambda \in \mathbb{R}$, $\phi(x) \in \mathbb{R}^n$ and $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ is a real-valued symmetric matrix potential satisfying the following asymptotic condition:

(S) The limits $\lim_{x\to\pm\infty} V(x) = V_\pm$ exist, and for each $M \in \mathbb{R}$,

$$\int_{-M}^{\infty} (1+|x|)|V(x) - V_+|dx < \infty; \quad \int_{-\infty}^{M} (1+|x|)|V(x) - V_-|dx < \infty.$$

Let $\kappa_{\min}$ denote the minimum among all eigenvalues of the matrices $V_\pm$. In [23], the authors verify that under our assumptions on $V(x)$, and for $\lambda < \kappa_{\min}$, (1.13) has n linearly independent solutions that decay to zero as $x \to -\infty$ and n linearly independent solutions that decay to zero as $x \to +\infty$. In order to remain consistent with the indexing of [23], we denote the former $\{\phi^-_{n+j}(x;\lambda)\}_{j=1}^n$ and the latter $\{\phi^+_j(x;\lambda)\}_{j=1}^n$. In [23], the authors verify that if we create a frame $\mathbf{X}^-(x;\lambda) = \binom{X^-(x;\lambda)}{Y^-(x;\lambda)}$ by taking $\{\phi^-_{n+j}(x;\lambda)\}_{j=1}^n$ as the columns of $X^-(x;\lambda)$ and $\{\partial_x\phi^-_{n+j}(x;\lambda)\}_{j=1}^n$ as the respective columns of $Y^-(x;\lambda)$ then $\mathbf{X}^-(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^-(x;\lambda)$. Likewise, we can create a frame $\mathbf{X}^+(x;\lambda) = \binom{X^+(x;\lambda)}{Y^+(x;\lambda)}$ by taking $\{\phi^+_j(x;\lambda)\}_{j=1}^n$ as the columns of $X^+(x;\lambda)$ and $\{\partial_x\phi^+_j(x;\lambda)\}_{j=1}^n$ as the respective columns of $Y^+(x;\lambda)$. Then $\mathbf{X}^+(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^+(x;\lambda)$.

We will show in Section 5.3 that the methods developed here can be used to prove the following theorems addressing the Morse index for the operators H and H_s respectively.

Theorem 1.3. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose* **(S)** *holds. Then for any $\lambda < \kappa_{\min}$,*

$$\mathrm{Mor}(H;\lambda) = \sum_{x\in\mathbb{R}} \dim\ker X^-(x;\lambda),$$

and likewise

$$\mathrm{Mor}(H;\lambda) = \sum_{x\in\mathbb{R}} \dim\ker X^+(x;\lambda).$$

Here, $\mathrm{Mor}(H;\lambda)$ denotes the number of eigenvalues that H has strictly below λ, counted with geometric multiplicity. For the final theorem, we let $\mathbf{X}^-_s(x;\lambda)$ and $\mathbf{X}^+_s(x;\lambda)$ be similar to $\mathbf{X}^\pm(x;\lambda)$ above, except specified in terms of asymptotically decaying solutions to (1.14) (see [23] for details).

Theorem 1.4. *Let $V \in C(\mathbb{R}; \mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose* **(S)** *holds. Then for any $\lambda < \kappa_{\min}$,*

$$\mathrm{Mor}(H_s;\lambda) = \sum_{x\in\mathbb{R}} \dim\ker X^-_s(x;\lambda),$$

and likewise

$$\mathrm{Mor}(H_s;\lambda) = \sum_{x\in\mathbb{R}} \dim\ker X^+_s(x;\lambda).$$

The precise results we take from [23] are stated as Theorems 5.1 and 5.2 in Section 5.3. In both cases, the Maslov index is computed as the intersection number of a path of Lagrangian subspaces with a fixed target, and Theorems 1.3 and 1.4 follow upon using Theorem 1.2 to replace the original target with the Dirichlet subspace. We emphasize at this point that the primary goal of the current analysis is to provide a general framework for readily making such a change of targets, and that the applications we have in mind are to the development of results such as Theorems 1.3 and 1.4 for a fairly general class of linear Hamiltonian systems. Results along these lines have been used to study the spectral stability of nonlinear waves arising in certain evolutionary PDE (see, e.g., [3, 4, 8, 29, 30, 31]), and the current approach extends the range of equations that can be analyzed in this way. We note particularly that the inclusion of s in (1.14) (making H_s non-self-adjoint) allows us to handle traveling waves in this setting.

Alternative approaches to those taken in Theorems 5.1 and 5.2 have been based on computing Maslov indices for appropriate pairs of evolving Lagrangian subspaces (i.e., with no fixed target). In [24], the authors evolve one path of Lagrangian subspaces forward from $-\infty$ and another backward from $+\infty$, and the associated spectral flow is captured where the two meet at $x = 0$ (see Theorem 1 in [24], which is formulated for a much more general class of linear Hamiltonian systems than those arising from Schrödinger systems, and stated in terms of the spectral flow of the operator pencil $\mathcal{L}(\lambda)$, rather than our $\mathrm{Mor}(H; \lambda)$). In [18, 19, 28], the authors use *renormalized oscillation theory*, in which the Maslov index is computed for a pair of Lagrangian paths, with one specified at some value λ_1 and the other specified at $\lambda_2 > \lambda_1$, leading to a count of the number of eigenvalues the operator has on (λ_1, λ_2). This latter method has the advantage of providing a naturally monotonic flow as x increases from $-\infty$ to $+\infty$ and being applicable in a wider range of cases than the approaches of [23, 24] (perhaps most notably, in the renormalized oscillation setting, there's no requirement on the existence of asymptotic endstates).

The remainder of the paper is organized as follows. In Section 2, we provide some background on the Maslov index, along with results that will be needed in the sequel, and in Section 3, we describe our approach to evaluating Hörmander's index. In Section 4, we compare our approach to computing Hörmander's index with analogous computations using Hörmander's formula, and in Section 5, we implement our framework in a variety of contexts, including the proofs of Theorems 1.3 and 1.4.

2 The Maslov Index on $\mathbb{C}^{2n}$

Approaches to computing the Maslov index for pairs of Lagrangian subspaces have been developed, for example, in [10, 22, 34]. Our approach is taken from [25], which adapts the development of [22] from the setting of Lagrangian subspaces of $\mathbb{R}^{2n}$ to the setting of Lagrangian subspaces of $\mathbb{C}^{2n}$.

2.1 Informal Definition of the Maslov Index

Given any pair of Lagrangian subspaces ℓ_1 and ℓ_2 with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we consider the matrix

$$\tilde{W} := -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{2.1}$$

In [25], the authors establish: (1) the inverses appearing in (2.1) exist; (2) $\tilde{W}$ is independent of the particular choice of frames $\mathbf{X}_1$ and $\mathbf{X}_2$, so long as these are indeed frames for ℓ_1 and ℓ_2; (3) $\tilde{W}$ is unitary; and (4) the identity

$$\dim(\ell_1 \cap \ell_2) = \dim(\ker(\tilde{W} + I)). \tag{2.2}$$

As context for (2.1), we note that in the case that $\mathbf{X}_1 = \binom{0}{I}$ and $\mathbf{X}_2 = \binom{I}{S}$, with S an $n \times n$ self-adjoint matrix, $\tilde{W}$ reduces to precisely the unitary matrix $(I - iS)(I + iS)^{-1}$ specified in equation (7) of [1]. The matrix (2.1) allows us to readily detect intersections between arbitrary pairs of Lagrangian subspaces. We also mention that up to sign conventions, $\tilde{W}$ is the same matrix as the matrix V^*U specified in Proposition 2 of [40].

Given two continuous paths of Lagrangian subspaces $\ell_i : [0,1] \to \Lambda(n)$, $i = 1, 2$, with respective frames $\mathbf{X}_i : [0,1] \to \mathbb{C}^{2n \times n}$, relation (2.2) allows us to compute the Maslov index $\operatorname{Mas}(\ell_1, \ell_2; [0,1])$ as a spectral flow through -1 for the path of matrices

$$\tilde{W}(t) := -(X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_2(t) - iY_2(t))(X_2(t) + iY_2(t))^{-1}. \tag{2.3}$$

In [25], the authors provide a rigorous definition of the Maslov index based on the spectral flow developed in [37]. Here, rather, we give only an intuitive discussion. As a starting point, if $-1 \in \sigma(\tilde{W}(t_*))$ for some $t_* \in [0,1]$, then t_* is a crossing point, and its multiplicity is taken to be $\dim(\ell_1(t_*) \cap \ell_2(t_*))$, which by virtue of (2.2) is equivalent to its multiplicity as an eigenvalue of $\tilde{W}(t_*)$. We compute the Maslov index $\operatorname{Mas}(\ell_1, \ell_2; [0,1])$ by allowing t to increase from 0 to 1 and incrementing the index whenever an eigenvalue crosses -1 in the counterclockwise direction, while decrementing the index whenever an eigenvalue crosses -1 in the clockwise direction. These increments/decrements are counted with multiplicity, so for example, if a pair of eigenvalues crosses -1 together in the counterclockwise direction, then a net amount of $+2$ is added to the index. Regarding behavior at the endpoints, if an eigenvalue of $\tilde{W}$ rotates away from -1 in the clockwise direction as t increases from 0, then the Maslov index decrements (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates away from -1 in the counterclockwise direction as t increases from 0, then the Maslov index does not change. Likewise, if an eigenvalue of $\tilde{W}$ rotates into -1 in the counterclockwise direction as t increases to 1, then the Maslov index increments (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates into -1 in the clockwise direction as t increases to 1, then the Maslov index does not change. Finally, it's possible that an eigenvalue of $\tilde{W}$ will arrive at -1 for $t = t_*$ and stay (i.e., remain at -1 for some interval $[t_*, t_* + \delta]$ with $\delta > 0$). In these cases, the Maslov index only increments/decrements upon arrival or departure, and the increments/decrements are determined as for the endpoints (departures determined as with $t = 0$, arrivals determined as with $t = 1$).

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(\mathcal{I})$ the collection of all paths $\mathcal{L}(\cdot) = (\ell_1(\cdot), \ell_2(\cdot))$, where $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t, s) \in \mathcal{I} \times [0, 1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties.

(P1) (Path Additivity) If $\mathcal{L} \in \mathcal{P}(\mathcal{I})$ and $a, b, c \in \mathcal{I}$, with $a < b < c$, then

$$\operatorname{Mas}(\mathcal{L}; [a, c]) = \operatorname{Mas}(\mathcal{L}; [a, b]) + \operatorname{Mas}(\mathcal{L}; [b, c]).$$

(P2) (Fixed Endpoints Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\operatorname{Mas}(\mathcal{L}; [a, b]) = \operatorname{Mas}(\mathcal{M}; [a, b]).$$

Straightforward proofs of these properties appear in [22] for Lagrangian subspaces of $\mathbb{R}^{2n}$, and proofs in the current setting of Lagrangian subspaces of $\mathbb{C}^{2n}$ are essentially identical.

2.2 Direction of Rotation

As noted previously, the direction we associate with a crossing point is determined by the direction in which eigenvalues of $\tilde{W}$ rotate through -1 (counterclockwise is positive, while clockwise is negative). In this subsection, we review the framework developed in [25] for analyzing this direction. Our starting point is the following lemma from [25].

Lemma 2.1. *Suppose $\ell_1, \ell_2 : I \to \Lambda(n)$ denote paths of Lagrangian subspaces of $\mathbb{C}^{2n}$ with absolutely continuous frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$ (respectively). If there exists $\delta > 0$ so that the matrices*

$$-\mathbf{X}_1(t)^* J \mathbf{X}_1'(t) = X_1(t)^* Y_1'(t) - Y_1(t)^* X_1'(t)$$

and (noting the sign change)

$$\mathbf{X}_2(t)^* J \mathbf{X}_2'(t) = -(X_2(t)^* Y_2'(t) - Y_2(t)^* X_2'(t))$$

are both a.e.-non-negative in $(t_0 - \delta, t_0 + \delta)$, and at least one is a.e.-positive definite in $(t_0 - \delta, t_0 + \delta)$ then the eigenvalues of $\tilde{W}(t)$ rotate in the counterclockwise direction as t increases through t_0. Likewise, if both of these matrices are a.e.-non-positive, and at least one is a.e.-negative definite, then the eigenvalues of $\tilde{W}(t)$ rotate in the clockwise direction as t increases through t_0.

Remark 2.1. *The corresponding statement Lemma 4.2 in [22] is stated in the slightly more restrictive case in which the frames are continuously differentiable.*

Using Lemma 2.1, we can readily verify that as λ increases along any subinterval of I, the eigenvalues of $\tilde{W}(x; \lambda)$ will rotate (strictly) monotonically in the clockwise direction.

Lemma 2.2. *Let Assumptions* **(A)** *and* **(B1)** *hold, and suppose α and β are as described in (1.6). If $\mathbf{X}_0(x;\lambda)$ is the matrix solution defined in (1.8) and $\mathbf{X}_1 = J\beta^*$, then for any $x \in (0,1]$, the following holds: as λ increases along any subinterval $[\lambda_1, \lambda_2] \subset I$, $\lambda_1 < \lambda_2$, the n eigenvalues of*

$$\tilde{W}(x;\lambda) = -(X_0(x;\lambda) + iY_0(x;\lambda))(X_0(x;\lambda) - iY_0(x;\lambda))^{-1}(X_1 - iY_1)(X_1 + iY_1)^{-1}$$

all rotate strictly monotonically in the clockwise direction.

Proof. According to Lemma 2.1, we only need to check the sign of the matrix

$$-\mathbf{X}_0(x;\lambda)^* J \partial_\lambda \mathbf{X}_0(x;\lambda).$$

For this, we compute

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbf{X}_0^*(x;\lambda) J \partial_\lambda \mathbf{X}_0(x;\lambda) &= (\mathbf{X}_0')^* J \partial_\lambda \mathbf{X}_0 + \mathbf{X}_0^* J \partial_\lambda \mathbf{X}_0' \\
&= -(\mathbf{X}_0')^* J^* \partial_\lambda \mathbf{X}_0 + \mathbf{X}_0^* \partial_\lambda J \mathbf{X}_0' \\
&= -\mathbf{X}_0^* \mathbb{B} \partial_\lambda \mathbf{X}_0 + \mathbf{X}_0^* \partial_\lambda (\mathbb{B}\mathbf{X}_0) = \mathbf{X}_0^* \mathbb{B}_\lambda \mathbf{X}_0.
\end{aligned}$$

Integrating on $[0,x]$, and noting that $\partial_\lambda \mathbf{X}_0(0;\lambda) = 0$, we see that

$$\mathbf{X}_0(x;\lambda)^* J \partial_\lambda \mathbf{X}_0(x;\lambda) = \int_0^x \mathbf{X}_0(y;\lambda)^* \mathbb{B}_\lambda(y;\lambda) \mathbf{X}_0(y;\lambda) dy.$$

Monotonicity follows immediately from our assumption **(B1)**. □

For monotonicity as the independent variable varies, we typically require additional information, starting with our next lemma. Although this lemma effectively just states the well-known fact that for regular crossings the direction is determined by the crossing form of [38], we state it in the current notation and for completeness provide a short proof. For a detailed discussion of the relation between such calculations in our notation and the development of [38], we refer the reader to Section 4.2 of [22]. In the event of degenerate crossings, the determination of direction becomes more complicated, and Lemma 2.3 does not address such cases. For an approach to accommodating degenerate crossings, we refer the reader to [17].

Lemma 2.3. *Suppose $\ell_1 : [a,b] \to \Lambda(n)$ denotes a path of Lagrangian subspaces of $\mathbb{C}^{2n}$ with absolutely continuous frames $\mathbf{X}_1 : [a,b] \to \mathbb{C}^{2n \times n}$, and suppose $\ell_2 \in \Lambda(n)$ is fixed. Let $t_* \in [a,b]$ be a crossing point for $\ell_1(\cdot)$ and ℓ_2 with multiplicity $\dim(\ell_1(t_*) \cap \ell_2) = m$, and let $\mathbb{P}_*$ denote projection onto $\ker(\mathbf{X}_2^* J \mathbf{X}_1(t_*))$. Fix $\delta_0 > 0$ sufficiently small so that t_* is the only crossing point for $\ell_1(\cdot)$ and ℓ_2 on $(t_* - \delta_0, t_* + \delta_0)$. If there exists $0 < \delta < \delta_0$ so that*

$$-\mathbb{P}_* \mathbf{X}_1(t_*) J \mathbf{X}_1'(t_*) \mathbb{P}_*$$

has m_- a.e.-negative eigenvalues on $(t_* - \delta, t_* + \delta) \cap [0,1]$ and m_+ a.e.-positive eigenvalues on $(t_* - \delta, t_* + \delta) \cap [0,1]$, and if in addition $m_- + m_+ = m$ (i.e., the crossing point t_* is non-degenerate), then the following hold:

(i) if $t_* \in (a,b)$,

$$\operatorname{Mas}(\ell_1(\cdot), \ell_2; [t_* - \delta, t_* + \delta]) = m_+ - m_-;$$

(ii) If $t_* = a$*, then*

$$\operatorname{Mas}(\ell_1(\cdot), \ell_2; [a, a+\delta]) = -m_-;$$

(iii) If $t_* = b$*, then*

$$\operatorname{Mas}(\ell_1(\cdot), \ell_2; [b-\delta, b]) = m_+.$$

Proof. We coordinatize $\mathbf{X}_1(t)$ in the usual way as $\mathbf{X}_1(t) = \binom{X_1(t)}{Y_1(t)}$. The lemma assumes that t_* is the only crossing point in $[t_* - \delta, t_* + \delta]$, so the Maslov indices stated in the three parts are all entirely determined by the direction of the rotation of the eigenvalues of

$$\tilde{W}(t) = -(X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}, \tag{2.4}$$

through -1 as t increases through t_*.

According to [25], the direction of rotation for the eigenvalues of $\tilde{W}(t)$ is determined by the restriction of the quadratic form

$$\tilde{Q}_1(w) := -2\Big(\mathbf{X}_1(t)^* J \mathbf{X}_1'(t)(X_1(t) + iY_1(t))^{-1} w, (X_1(t) + iY_1(t))^{-1} w\Big),$$

to the space $V_* := \ker(\tilde{W}(t_*) + I)$ in the following way: positive eigenvalues of $\tilde{Q}_1(w)$ correspond with rotation in the positive (counterclockwise) direction, while negative eigenvalues of $\tilde{Q}_1(w)$ correspond with rotation in the negative (clockwise) direction. In addition, according to Lemma 2.1 of [25],

$$\operatorname{ran}(X_1(t_*) + iY_1(t_*))^{-1}\Big|_{V_*} = \ker(\mathbf{X}_2^* J \mathbf{X}_1(t_*)).$$

Combining these observations, we see that the rotation of the eigenvalues of $\tilde{W}(t)$ through -1 as t increases through t_* is determined by the restriction of $-\mathbf{X}_1(t)^* J \mathbf{X}_1'(t)$ to

$$\ker(\mathbf{X}_2^* J \mathbf{X}_1(t_*))$$

for a.e. $t \in (t_* - \delta, t_* + \delta)$. (If $\mathbf{X}_1$ is differentiable at t_*, we can simply evaluate at t_*.) $\square$

In the current analysis, we have an evolving frame $\mathbf{X}_0(x;\lambda)$ specified by

$$J\mathbf{X}_0' = \mathbb{B}(x;\lambda)\mathbf{X}_0; \quad \mathbf{X}(0;\lambda) = J\alpha^*,$$

and a fixed target frame $\mathbf{X}_1 = J\beta^*$. The key point in adapting the results of [25] to the current setting is the observation that $\mathbf{X}_1$ can be viewed as an evolving frame solving

$$J\mathbf{X}_1' = \mathbb{B}_1(x;\lambda)\mathbf{X}_1; \quad \mathbf{X}_1(1;\lambda) = J\beta^*, \tag{2.5}$$

where $\mathbb{B}_1(x;\lambda) \equiv 0$.

We begin with the following lemma.

Lemma 2.4. *Let Assumptions* **(A)** *and* **(B1)** *hold, and assume that $\mathbb{P}_1 \mathbb{B}(x;\lambda)\mathbb{P}_1$ is non-negative for a.e. $x \in (0,1)$, where $\mathbb{P}_1$ denotes projection onto the Lagrangian subspace ℓ_1 with frame $\mathbf{X}_1 = J\beta^*$. For λ fixed and $\tilde{W}(x;\lambda)$ as in Lemma 2.2, let $x_* \in [0,1]$ be a crossing point. If $x_* \in (0,1]$, then no eigenvalue of $\tilde{W}(\cdot;\lambda)$ can arrive at -1 moving in the counterclockwise direction as x increases to x_*. If $x_* = 0$, then no eigenvalue of $\tilde{W}(\cdot;\lambda)$ can rotate away from -1 moving in the counterclockwise direction as x increases from 0.*

Proof. If we take the convention described in (2.5), then we are in precisely the setting of Claim 4.1 (Part (1)) of [25], except with the direction of rotation reversed. The statement follows immediately. □

Lemma 2.4 asserts that under its assumptions, no crossing point can have positive direction (with x increasing). We also need to check that the flow cannot get stuck at a crossing point. For this, we can adapt the framework from Section 3.1 of [25], beginning with the following claim (adapted from Claim 3.4 in [25]).

Lemma 2.5. *Let Assumption* **(A)** *hold, and assume that $\mathbb{P}_1 \mathbb{B}(x;\lambda)\mathbb{P}_1$ is non-negative for a.e. $x \in (0,1)$, where $\mathbb{P}_1$ denotes projection onto the Lagrangian subspace ℓ_1 with frame $\mathbf{X}_1 = J\beta^*$. With $\tilde{W}(x;\lambda)$ specified as in Lemma 2.2, assume that for some interval $[a,b] \subset [0,1]$, $a < b$, and some $m \in \{1,2,\dots,n\}$,*

$$\dim \ker(\tilde{W}(x;\lambda) + I) = m$$

for all $x \in [a,b]$. Then there exist functions $v, w \in \mathrm{AC}([a,b],\mathbb{C}^n)$ so that

$$\mathbf{X}_0(x;\lambda)v(x) = \mathbf{X}_1 w(x)$$

for all $x \in [a,b]$. Moreover,

$$\mathbf{X}_0(x;\lambda)v'(x) = \mathbf{X}_1 w'(x),$$

for a.e. $x \in (a,b)$.

Proof. If we again view $\mathbf{X}_1$ as described in (2.5), we arrive in the framework of Claim 3.4 from [25]. The only difference is that in Claim 3.4 from [25] the authors assume $\mathbb{B}_1(x;\lambda) - \mathbb{B}(x;\lambda)$ is semi-definite (non-negative in that case) for a.e. $x \in (a,b)$, and in the current setting we assume this difference is semi-definite (non-positive in our case) *when restricted to* ℓ_1. (We emphasize that the type of semi-definiteness is not the issue, but rather the restriction.) The proof from the referenced claim can be used precisely as given in [25], with the following change.

In [25], the authors observe that since $\mathbb{B}_1(x;\lambda) - \mathbb{B}(x;\lambda)$ is non-negative, they have that for a.e. $x \in (a,b)$

$$\Big((\mathbb{B}_1(x;\lambda) - \mathbb{B}(x;\lambda))y_0, y_0\Big) = 0 \quad \iff \quad y_0 \in \ker(\mathbb{B}_1(x;\lambda) - \mathbb{B}(x;\lambda)).$$

In the current setting, the vectors y_0 must belong to ℓ_1, so the difference $(\mathbb{B}_1(x;\lambda) - \mathbb{B}(x;\lambda))$ can be replaced by $\mathbb{P}_1 \mathbb{B}(x;\lambda)\mathbb{P}_1$ (using $\mathbb{B}_1 \equiv 0$ and changing signs). With this change, our Lemma 2.5 is established by the proof of Claim 3.4 from [25]. □

Lemma 2.6. *Let Assumption* **(A)** *hold, and assume that $\mathbb{P}_1 \mathbb{B}(x;\lambda)\mathbb{P}_1$ is non-negative for a.e. $x \in (0,1)$, where $\mathbb{P}_1$ denotes projection onto the Lagrangian subspace ℓ_1 with frame $\mathbf{X}_1 = J\beta^*$. If there exists an interval $[a,b] \subset [0,1]$, $a < b$, and $m \in \{1,2,\dots,n\}$ so that $\dim(\ell_0(x;\lambda) \cap \ell_1) = m$ for all $x \in [a,b]$, then there exists an interval $[c,d] \subset [a,b]$, $c < d$, and a constant vector $v_0 \in \mathbb{C}^n$ so that*

$$\mathbf{X}_0(x;\lambda)v_0 \in \ell_0(x;\lambda) \cap \ell_1$$

for all $x \in [c,d]$. Moreover, it follows that we must have $\mathbb{B}(x;\lambda)\mathbf{X}_0(x;\lambda)v_0 = 0$ for a.e. $x \in (c,d)$.

Proof. The proof is identical to that of Claim 3.5 in [25], using here Lemma 2.5 where Claim 3.4 is used in [25]. □

We are now in a position to prove Lemma 1.1.

Proof of Lemma 1.1. Under the assumptions of Lemma 1.1, we know from Lemma 2.4 that in the computation of $\operatorname{Mas}(\ell_0(\cdot;\lambda), \ell_1; [0,1])$, no crossing points can be associated with the positive direction. In particular, with the possible exception of $x = 0$, each crossing point must correspond with one or more eigenvalues of $\tilde{W}(x;\lambda)$ arriving at -1 from the clockwise direction. (If $x = 0$ is crossing, the associated eigenvalue(s) of $\tilde{W}(x;\lambda)$ residing at -1 cannot rotate away from -1 in the counterclockwise direction). In order to ensure that each such crossing point corresponds with a negative contribution to the Maslov index (according to multiplicity), we need to be sure that no eigenvalue of $\tilde{W}(x;\lambda)$ can reside at -1 on an interval $[a,b] \subset [0,1]$, $a < b$.

Suppose to the contrary that $\dim\ker(\tilde{W}(x;\lambda) + I) \neq 0$ for all $x \in [a,b]$ for some interval $[a,b] \subset [0,1]$, $a < b$. According to Claim 3.3 of [25], we can conclude that there exists a subinterval $[c,d] \subset [a,b]$, $c < d$, and an integer $m \in \{1,2,\dots,n\}$, so that in fact $\dim\ker(\tilde{W}(x;\lambda) + I) = m$ for all $x \in [c,d]$. We can now conclude from Lemma 2.6 that there exists a further subinterval $[\tilde{c},\tilde{d}] \subset [c,d]$, $\tilde{c} < \tilde{d}$, and a constant vector $v_0 \in \mathbb{C}^n$ so that $\mathbf{X}_0(x;\lambda)v_0 \in \ell_1$ for all $x \in [\tilde{c},\tilde{d}]$, and additionally

$$\mathbb{B}(x;\lambda)\mathbf{X}_0(x;\lambda)v_0 = 0$$

for a.e. $x \in (\tilde{c},\tilde{d})$. This contradicts our positivity assumption **(B2)**, so we can conclude that there can be no such interval $[a,b]$, $a < b$, so that $\dim\ker(\tilde{W}(x;\lambda) + I) \neq 0$ for all $x \in [a,b]$. This completes the proof. □

2.3 Spectral Curves

We see from the proof of Lemma 1.1 that, under its assumptions, as x increases, with λ fixed, the eigenvalues of $\tilde{W}(x;\lambda)$ can only rotate through -1 in the clockwise direction around S^1. Likewise, from Lemma 2.2, we see that as λ increases, with x fixed, the eigenvalues of $\tilde{W}(x;\lambda)$ rotate monotonically in the clockwise direction around S^1. By combining these two observations, we can establish the monotonicity of *spectral curves*, by which we mean certain connected subsets of the dispersion diagram for (1.4)–(1.5),

$$\mathcal{D} := \{(x,\lambda) \in [0,1] \times I : \dim\ker \mathbf{X}_0(x;\lambda)^* J\mathbf{X}_1 \neq 0\}.$$

(Examples appear in Figures 1, 3 and 4.)

Suppose $(x_*, \lambda_*) \in [0,1] \times I$ is a crossing point for the flow of $\tilde{W}(x;\lambda)$. Due to monotonicity in λ, the eigenvalue(s) of $\tilde{W}(x_*;\lambda)$ that cross -1 as λ increases through λ_* must cross in the clockwise direction. Moreover, by differentiability of $\tilde{W}(x;\lambda)$ in λ, we can find $r > 0$ sufficiently small so that the rate of rotation is bounded below by some constant $\delta > 0$ for all $(x;\lambda) \in B(x_*, \lambda_*; r)$ (the closed ball centered at (x_*, λ_*) with radius r). (This follows from Theorem II.5.4. in [32]; the use of that theorem in the current setting is discussed in detail in Section 3 of [25].) We can now think of taking some small increment Δx and tracking the eigenvalues of $\tilde{W}(x;\lambda_*)$ as x increases from x_* to $x_* + \Delta x$. By monotonicity in x, the eigenvalues residing at -1 will all rotate away from -1 in the clockwise direction, and by choosing Δx sufficiently small, we can keep these eigenvalues as close as we like to -1. In particular, since the rate of rotation in λ is bounded below, we can take Δx sufficiently small so that there exists a small increment $\Delta\lambda$ so that as (x, λ) goes linearly from $(x_* + \Delta x, \lambda_*)$ to $(x_* + \Delta x, \lambda_* - \Delta\lambda)$ the eigenvalues of $\tilde{W}(x;\lambda)$ residing at -1 at (x_*, λ_*) have all rotated back through -1 in the counterclockwise direction. In this way, we see that the spectral curves associated with all of these crossing points decrease monotonically when viewed in the (λ, x)-plane. (See Figure 1; this monotonicity is also depicted on the right-hand side of Figure 3).

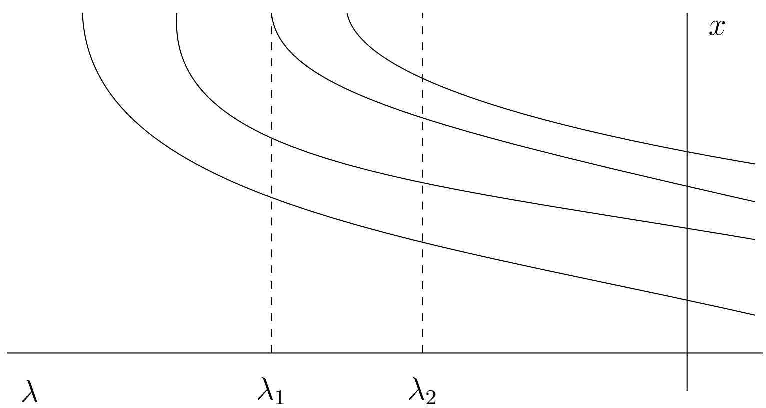

Figure 1: Monotonic spectral curves.

We see from this discussion that if a spectral curve contains the point (x_*, λ_*), then for any $\Delta x > 0$ sufficiently small, there will exist $\Delta\lambda > 0$, depending on Δx, so that the spectral curve will contain the point $(x_* + \Delta x, \lambda_l)$ for some $\lambda_l \in (\lambda_* - \Delta\lambda, \lambda_*)$, and likewise it will contain the point $(x_* - \Delta x, \lambda_r)$ for some $\lambda_r \in [\lambda_*, \lambda_* + \Delta\lambda]$. Moreover, given any $\epsilon > 0$, there exists $\delta > 0$ sufficiently small so that $\Delta x < \delta \implies \Delta\lambda < \epsilon$. In this way, we see that near x_* the crossing points $(x, \lambda(x))$ correspond with a well-defined function $\lambda(x)$ for all $x \in [x_* - \Delta x, x_* + \Delta x]$, and that $\lambda(x)$ is continuous in x.

For convenience, we can choose a distinct labeling of these curves so that the lower-most is always curve 1, the curve just above it curve 2 etc. In this way, even if the curves cross, we can express the i^{th} curve as

$$\mathcal{S}_i := \{(s_i(\lambda), \lambda) : \lambda \in I_i\},$$

for some continuous function s_i defined on some subinterval $I_i \subset I$.

3 Evaluating Hörmander's index

In this section, we introduce a method for computing Hörmander's index that is especially convenient for the applications we have in mind (see Section 5). As a starting point, we fix four Lagrangian subspaces $\nu, \sigma, \tilde{\nu}, \tilde{\sigma}$ and consider two continuous paths of Lagrangian subspaces $\ell, \tilde{\ell} : [0,1] \to \Lambda(n)$, with $\ell(0) = \nu$, $\ell(1) = \sigma$, $\tilde{\ell}(0) = \tilde{\nu}$, and $\tilde{\ell}(1) = \tilde{\sigma}$. If we let s denote the parameters for ℓ, and let $\tilde{s}$ denote the parameter for $\tilde{\ell}$, then we can compute the Maslov index associated with ℓ and $\tilde{\ell}$ along a *Maslov box*, which we describe as follows: fix $\tilde{s} = 0$ and let s increase from 0 to 1; fix $s = 1$ and let $\tilde{s}$ increase from 0 to 1; fix $\tilde{s} = 1$ and let s decrease from 1 to 0; fix $s = 0$ and let $\tilde{s}$ decrease from 1 to 0. (See Figure 2.)

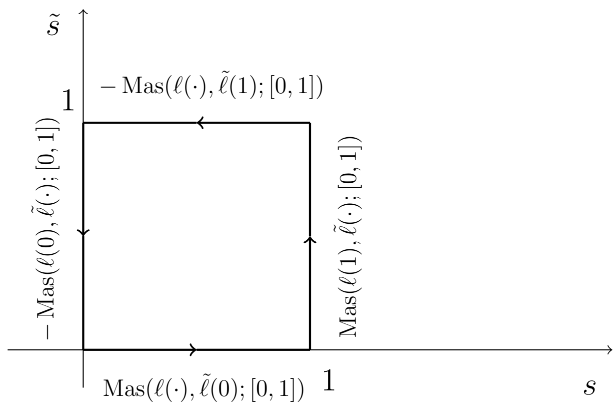

Figure 2: The Maslov Box in the $(s, \tilde{s})$-plane.

Using path additivity and homotopy invariance, we see that

$$\begin{aligned} &\operatorname{Mas}(\ell(\cdot), \tilde{\ell}(0); [0,1]) + \operatorname{Mas}(\ell(1), \tilde{\ell}(\cdot); [0,1]) \\ &\quad - \operatorname{Mas}(\ell(\cdot), \tilde{\ell}(1); [0,1]) - \operatorname{Mas}(\ell(0), \tilde{\ell}(\cdot); [0,1]) = 0. \end{aligned} \tag{3.1}$$

We see immediately that the difference

$$\operatorname{Mas}(\ell(\cdot), \tilde{\sigma}; [0,1]) - \operatorname{Mas}(\ell(\cdot), \tilde{\nu}; [0,1])$$

is independent of the path ℓ as long as ℓ has endpoints $\ell(0) = \nu$ and $\ell(1) = \sigma$, and likewise

$$\operatorname{Mas}(\sigma, \tilde{\ell}(\cdot); [0,1]) - \operatorname{Mas}(\nu, \tilde{\ell}(\cdot); [0,1])$$

is independent of the path $\tilde{\ell}$ as long as $\tilde{\ell}$ has endpoints $\tilde{\ell}(0) = \tilde{\nu}$ and $\tilde{\ell}(1) = \tilde{\sigma}$. This justifies our definition of the Hörmander index as

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) := \operatorname{Mas}(\ell(\cdot), \tilde{\sigma}; [0,1]) - \operatorname{Mas}(\ell(\cdot), \tilde{\nu}; [0,1]); \quad \ell(0) = \nu,\ \ell(1) = \sigma. \tag{3.2}$$

In addition, we see from (3.1) that

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \operatorname{Mas}(\sigma, \tilde{\ell}(\cdot); [0,1]) - \operatorname{Mas}(\nu, \tilde{\ell}(\cdot); [0,1]); \quad \tilde{\ell}(0) = \tilde{\nu},\ \tilde{\ell}(1) = \tilde{\sigma}. \tag{3.3}$$

3.1 Interpolation Paths

Formulas (3.2) and (3.3) are independent of the paths $\ell(\cdot)$ and $\tilde{\ell}(\cdot)$ (respectively), and this allows us to choose paths that are convenient to work with. Of particular interest here are interpolation paths, which we specify as follows.

Definition 3.1. *Let $\mathbf{X}_{\tilde{\nu}}$ and $\mathbf{X}_{\tilde{\sigma}}$ respectively denote frames for Lagrangian subspaces $\tilde{\nu}$ and $\tilde{\sigma}$, and assume*

$$\mathbf{X}_{\tilde{\nu}}^* J \mathbf{X}_{\tilde{\sigma}} + \mathbf{X}_{\tilde{\sigma}}^* J \mathbf{X}_{\tilde{\nu}} = 0, \tag{3.4}$$

and

$$\ker \Big(t\mathbf{X}_{\tilde{\nu}} + (1-t)\mathbf{X}_{\tilde{\sigma}} \Big) = \{0\},$$

for all $t \in [0,1]$. We define the interpolation frame by

$$\tilde{\mathbf{X}}(t) = t\mathbf{X}_{\tilde{\sigma}} + (1-t)\mathbf{X}_{\tilde{\nu}}. \tag{3.5}$$

Proposition 3.1. *Under the specifications of Definition 3.1, $\tilde{\mathbf{X}}(t)$ is the frame for a Lagrangian subspace of $\mathbb{C}^{2n}$ for all $t \in [0,1]$. In particular, if for each $t \in [0,1]$, $\tilde{\ell}(t)$ denotes the Lagrangian subspace associated with $\tilde{\mathbf{X}}(t)$, then $\tilde{\ell} : [0,1] \to \Lambda(n)$ is a continuous path of Lagrangian subspaces.*

Proof. Dimensionality is assumed in the definition, so we only need to check the Lagrangian property. We compute

$$\begin{aligned} \tilde{\mathbf{X}}(t)^* J \tilde{\mathbf{X}}(t) &= (t\mathbf{X}_{\tilde{\sigma}}^* + (1-t)\mathbf{X}_{\tilde{\nu}}^*)(tJ\mathbf{X}_{\tilde{\sigma}} + (1-t)J\mathbf{X}_{\tilde{\nu}}) \\ &= t(1-t)(\mathbf{X}_{\tilde{\sigma}}^* J \mathbf{X}_{\tilde{\nu}} + \mathbf{X}_{\tilde{\nu}}^* J \mathbf{X}_{\tilde{\sigma}}). \end{aligned}$$

from which it's clear that $\tilde{\mathbf{X}}(t)$ satisfies the Lagrangian property for all $t \in [0,1]$ if and only if (3.4) holds. $\square$

Interpolation paths constitute a natural tool in this context, and have been used, for example, in the proof of Theorem 3.5 from [38]. Nonetheless, the current analysis seems to be the first systematic use of such paths in the development of a general framework for computing Hörmander's index in applications.

It will be convenient to set some notation for the Maslov index obtained for an interpolation path $\tilde{\ell}(t)$ and a fixed target ν. Since the value of this index is entirely determined by the target ν and the frames $\mathbf{X}_{\tilde{\nu}}$ and $\mathbf{X}_{\tilde{\sigma}}$, we will denote by

$$\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) \tag{3.6}$$

the Maslov index $\operatorname{Mas}(\nu, \tilde{\ell}(\cdot); [0,1])$, where $\tilde{\ell} : [0,1] \to \Lambda(n)$ denotes the path of Lagrangian subspaces with frames (3.5). We emphasize that this value depends on the specific frames $\mathbf{X}_{\tilde{\nu}}$ and $\mathbf{X}_{\tilde{\sigma}}$, not just the spaces $\tilde{\nu}$ and $\tilde{\sigma}$, and as such is less fundamental than objects such as the quadratic form Q and triple index ι discussed below. Our rationale for introducing $\mathcal{I}$ is primarily computational convenience: as we'll see, it allows us to readily compute Hörmander's index in a wide variety of useful cases.

We note that switching the roles of $\mathbf{X}_{\tilde{\nu}}$ and $\mathbf{X}_{\tilde{\sigma}}$ simply reverses the path, and so by homotopy invariance,

$$\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) = -\mathcal{I}(\nu; \mathbf{X}_{\tilde{\sigma}}, \mathbf{X}_{\tilde{\nu}}). \tag{3.7}$$

In addition, it will be clear from our development that if either $\nu \cap \tilde{\nu} = \{0\}$ or $\nu \cap \tilde{\sigma} = \{0\}$ we have the inequality

$$-n \leq \mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) \leq n. \tag{3.8}$$

(We will see that in these cases crossing points for the calculation of $\text{Mas}(\nu, \tilde{\ell}(\cdot); [0,1])$ are in bijective correspondence with eigenvalues of the generalized eigenvalue problem specified in (3.14).)

With this notation, we can express (3.3) as the useful relation

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) - \mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}). \tag{3.9}$$

A typical implementation of our framework will look as follows. Given some continuous path $\ell : [0,1] \to \Lambda(n)$, with $\ell(0) = \nu$ and $\ell(1) = \sigma$, and two fixed Lagrangian subspaces $\tilde{\nu}, \tilde{\sigma} \in \Lambda(n)$, we would like to relate the Maslov indices $\text{Mas}(\ell(\cdot), \tilde{\nu}; [0,1])$ and $\text{Mas}(\ell(\cdot), \tilde{\sigma}; [0,1])$. Our development allows us to do this by writing

$$\begin{aligned} &\text{Mas}(\ell(\cdot), \tilde{\sigma}; [0,1]) - \text{Mas}(\ell(\cdot), \tilde{\nu}; [0,1]) \\ &\quad = s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) - \mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}). \end{aligned}$$

Of course the efficacy of this approach is determined by the ease with which we can compute these interpolation values, and that's the topic we turn to next.

3.2 Computing $\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$

In practice, the evaluation of $\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ typically involves counting the number of positive, negative, and null eigenvalues of certain related matrices (counted with geometric multiplicity). For an $n \times n$ matrix A, we will respectively denote these counts $n_+(A)$, $n_-(A)$, and $n_0(A)$. Moreover, for a generalized eigenvalue problem

$$Av = \lambda Bv, \tag{3.10}$$

we will denote the corresponding counts $n_+(A,B)$, $n_-(A,B)$, and $n_0(A,B)$.

We fix three Lagrangian subspaces ν, $\tilde{\nu}$, and $\tilde{\sigma}$, along with an interpolation path $\tilde{\ell} : [0,1] \to \Lambda(n)$ with frame

$$\tilde{\mathbf{X}}(t) = t\mathbf{X}_{\tilde{\sigma}} + (1-t)\mathbf{X}_{\tilde{\nu}}.$$

Our goal is to compute

$$\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) = \text{Mas}(\nu; \tilde{\ell}(\cdot); [0,1]). \tag{3.11}$$

To fix notation, we will set

$$\begin{aligned} \nu &= \text{colspan}\, \mathbf{X}_{\nu} = \text{colspan} \begin{pmatrix} X_{\nu} \\ Y_{\nu} \end{pmatrix} \\ \tilde{\nu} &= \text{colspan}\, \mathbf{X}_{\tilde{\nu}} = \text{colspan} \begin{pmatrix} X_{\tilde{\nu}} \\ Y_{\tilde{\nu}} \end{pmatrix} \\ \tilde{\sigma} &= \text{colspan}\, \mathbf{X}_{\tilde{\sigma}} = \text{colspan} \begin{pmatrix} X_{\tilde{\sigma}} \\ Y_{\tilde{\sigma}} \end{pmatrix}. \end{aligned}$$

Interpolation frame properties. In order for $\tilde{\mathbf{X}}(t)$ to be an interpolation frame, we require that the conditions of Definition 3.1 hold. In particular, we assume

$$\mathbf{X}_{\tilde{\sigma}}^* J \mathbf{X}_{\tilde{\nu}} + \mathbf{X}_{\tilde{\nu}}^* J \mathbf{X}_{\tilde{\sigma}} = 0, \tag{3.12}$$

and

$$\ker \tilde{\mathbf{X}}(t) = \{0\}, \quad \text{for all } t \in [0,1]. \tag{3.13}$$

This latter condition can be expressed as

$$\text{for each } t \in [0,1], \quad (t\mathbf{X}_{\tilde{\sigma}} + (1-t)\mathbf{X}_{\tilde{\nu}})v = 0 \quad \Longrightarrow \quad v = 0.$$

If we multiply this last expression on the left by $\mathbf{X}_{\tilde{\nu}}^* J$, we see that this would be implied by

$$\mathbf{X}_{\tilde{\nu}}^* J \mathbf{X}_{\tilde{\sigma}} v = 0 \quad \Longrightarrow \quad v = 0, \quad \text{for all } t \in [0,1].$$

If $\tilde{\nu} \cap \tilde{\sigma} = \{0\}$, then $\mathbf{X}_{\tilde{\nu}}^* J \mathbf{X}_{\tilde{\sigma}}$ will be non-singular, so a sufficient (though not necessary) condition for the dimensionality condition is

$$\tilde{\nu} \cap \tilde{\sigma} = \{0\}.$$

Crossing points and directionality. In order to compute

$$\operatorname{Mas}(\nu; \tilde{\ell}(\cdot); [0,1]),$$

we proceed by identifying the crossing points for $\tilde{\ell}(\cdot)$ and ν and assigning a direction to each. First, a value $t_* \in [0,1]$ will be a crossing point for $\tilde{\ell}(\cdot)$ and ν if and only if there exists $v \in \mathbb{C}\backslash\{0\}$ so that

$$\mathbf{X}_{\nu}^* J \tilde{\mathbf{X}}(t_*) v = 0.$$

Rearranging this last expression, we obtain the generalized eigenvalue problem

$$\mathbf{X}_{\nu}^* J \mathbf{X}_{\tilde{\sigma}} v = -\frac{1-t_*}{t_*} \mathbf{X}_{\nu}^* J \mathbf{X}_{\tilde{\nu}} v.$$

We see that $t_* \in (0,1]$ is a crossing point for $\tilde{\ell}(\cdot)$ and ν of multiplicity m if and only if the generalized eigenvalue problem

$$\mathbf{X}_{\nu}^* J \mathbf{X}_{\tilde{\sigma}} v = \tau \mathbf{X}_{\nu}^* J \mathbf{X}_{\tilde{\nu}} v, \tag{3.14}$$

has a non-positive eigenvalue $\tau = -(1-t_*)/t_*$ with multiplicity m. We note that $t_* = 0$ will be a crossing point if and only if $\dim \ker \mathbf{X}_{\nu}^* J \mathbf{X}_{\tilde{\nu}} = m \neq 0$, and in this case, its multiplicity will be m.

For directionality, we know from Lemma 2.3 that the the direction(s) associated with a crossing point $t_* \in [0,1]$ will be determined by the eigenvalues of the restricted matrix

$$\tilde{\mathbf{X}}(t_*)^* J \tilde{\mathbf{X}}'(t_*)\Big|_{\ker \mathbf{X}_{\nu}^* J \tilde{\mathbf{X}}(t_*)}.$$

I.e., by the non-zero eigenvalues of

$$\mathbb{P}_* \tilde{\mathbf{X}}(t_*)^* J \tilde{\mathbf{X}}'(t_*) \mathbb{P}_*,$$

where $\mathbb{P}_*$ denotes projection onto $\ker(\mathbf{X}_\nu^* J\tilde{\mathbf{X}}(t_*))$.

We can readily compute

$$\begin{aligned}
\tilde{\mathbf{X}}(t_*)^* J\tilde{\mathbf{X}}'(t_*) &= (t\mathbf{X}_{\tilde{\sigma}}^* + (1-t)\mathbf{X}_{\tilde{\nu}}^*)(J\mathbf{X}_{\tilde{\sigma}} - J\mathbf{X}_{\tilde{\nu}}) \\
&= \mathbf{X}_{\tilde{\nu}}^* J\mathbf{X}_{\tilde{\sigma}} - t(\mathbf{X}_{\tilde{\sigma}}^* J\mathbf{X}_{\tilde{\nu}} + \mathbf{X}_{\tilde{\nu}}^* J\mathbf{X}_{\tilde{\sigma}}) \\
&= \mathbf{X}_{\tilde{\nu}}^* J\mathbf{X}_{\tilde{\sigma}},
\end{aligned}$$

where in obtaining the final equality we have used (3.12). In this way, we see that directionality will be determined by the eigenvalues of

$$\mathbf{X}_{\tilde{\nu}}^* J\mathbf{X}_{\tilde{\sigma}}\Big|_{\ker \mathbf{X}_\nu^* J\tilde{\mathbf{X}}(t_*)}. \tag{3.15}$$

In particular, suppose $t_* \in [0,1]$ is a crossing point with multiplicity m_*. Then the operator in (3.15) will have at most m_* non-zero eigenvalues, and each positive eigenvalue will correspond with an increase of the Maslov index according to multiplicity, while each negative eigenvalue will correspond with a decrease in the Maslov index according to multiplicity. In the event that $\mathbb{P}_*\mathbf{X}_{\tilde{\nu}} J\mathbf{X}_{\tilde{\sigma}}\mathbb{P}_*$ has fewer than m_* non-zero eigenvalues, directionality is not entirely determined; we will not address such cases in the current analysis. For $t_* = 0$, we are restricting to $\ker \mathbf{X}_\nu^* J\mathbf{X}_{\tilde{\nu}}$, while for any $t_* \in (0,1]$, we are restricting to the geometric eigenspace of $\tau_* = 1 - 1/t_*$ as an eigenvalue of the generalized eigenvalue problem (3.14).

At this point, the computation of $\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ has been reduced to matrix calculations. In many important cases, these calculations take particularly simple forms, and we turn next to such cases.

3.3 Exchanging the Dirichlet Plane

As discussed in the introduction, the Dirichlet Lagrangian plane enjoys a distinguished relationship with the evolution of Sturm-Liouville systems, and so constitutes an important special case.

In the framework of Section 3.2, suppose $\ell_{\tilde{\sigma}}$ is the Dirichlet plane, for which we use the natural frame $\mathbf{X}_{\tilde{\sigma}} = \binom{0}{I}$. In this case, condition (3.12) becomes

$$0 = \mathbf{X}_{\tilde{\sigma}}^* J\mathbf{X}_{\tilde{\nu}} + \mathbf{X}_{\tilde{\nu}}^* J\mathbf{X}_{\tilde{\sigma}} = X_{\tilde{\nu}} - X_{\tilde{\nu}}^*. \tag{3.16}$$

I.e., the interpolation frame $\tilde{\mathbf{X}}(t)$ will have the Lagrangian property for all $t \in [0,1]$ if and only if $X_{\tilde{\nu}}$ is self-adjoint. The following lemma will be useful for choosing Lagrangian frames for which the first coordinate matrix is self-adjoint.

Lemma 3.1. *Let $\mathbf{X} = \binom{X}{Y}$ be a frame for a Lagrangian subspace ℓ. Then the following hold.*

(i) If X is invertible, then the matrix $\hat{\mathbf{X}} := \binom{I}{YX^{-1}}$ is a frame for ℓ, and YX^{-1} is self-adjoint;

(ii) If Y is invertible, then the matrix $\hat{\mathbf{X}} := \binom{XY^{-1}}{I}$ is a frame for ℓ, and XY^{-1} is self-adjoint;

(ii) If there exists a self-adjoint matrix M so that $Y - MX$ is invertible (i.e., so that ℓ does not intersect the Lagrangian subspace with frame $\binom{I}{M}$), then the matrix $\hat{\mathbf{X}} := \binom{X(Y-MX)^{-1}}{Y(Y-MX)^{-1}}$ is a frame for ℓ, and $X(Y - MX)^{-1}$ is self-adjoint.

Proof. In each case, we find an invertible matrix A so that $\hat{\mathbf{X}} = \mathbf{X}A$, and then verify that $\mathbf{X}A$ has the stated properties. For (i), we take $A = X^{-1}$, which immediately gives the stated form of $\hat{\mathbf{X}}$. To see that YX^{-1} is self-adjoint, we observe from the Lagrangian property,

$$X^*Y = Y^*X \implies YX^{-1} = (X^*)^{-1}Y^*.$$

For (ii), we take $A = Y^{-1}$ and proceed similarly as with (i).

For (iii), we take $A = (Y - MX)^{-1}$, and we need to show that $X(Y - MX)^{-1}$ is self-adjoint. In order to do this, we compute

$$X^*(Y - MX) = X^*Y - X^*MX = Y^*X - X^*MX = (Y^* - X^*M)X.$$

If we now multiply on the right by $(Y - MX)^{-1}$ and on the left by $(Y^* - X^*M)$, we obtain the sought relation

$$(Y^* - X^*M)^{-1}X^* = X(Y - MX)^{-1}.$$

$\square$

For dimensionality, if $\mathbf{X}_{\tilde{\sigma}} = \binom{0}{I}$, then we have

$$\tilde{\mathbf{X}}(t) = \begin{pmatrix} (1-t)X_{\tilde{\nu}} \\ tI + (1-t)Y_{\tilde{\nu}} \end{pmatrix}.$$

We see that we only lose dimensionality if there exists $v \in \mathbb{C}^n \backslash \{0\}$ so that

$$\begin{aligned} X_{\tilde{\nu}}v &= 0 \\ Y_{\tilde{\nu}}v &= -\frac{t}{1-t}v, \end{aligned}$$

for some $t \in (0, 1)$. If we let $E_{(-\infty,0)}(Y_{\tilde{\nu}})$ denote the union of eigenspaces of $Y_{\tilde{\nu}}$ associated with negative eigenvalues, then the condition

$$\ker(X_{\tilde{\nu}}) \cap E_{(-\infty,0)}(Y_{\tilde{\nu}}) = \{0\} \tag{3.17}$$

implies $\dim \operatorname{colspan} \tilde{\mathbf{X}}(t) = n$ for all $t \in [0, 1]$.

In this setting, crossing points $t_* \in (0, 1]$ correspond with eigenvalues of the generalized eigenvalue problem

$$X_{\nu}^*v = \frac{1-t_*}{t_*}\mathbf{X}_{\nu}^*J\mathbf{X}_{\tilde{\nu}}v, \tag{3.18}$$

via the relation $\tau_* = -(1 - 1/t_*)$. The value $t_* = 0$ is a crossing if and only if $\ker(\mathbf{X}_{\nu}^*J\mathbf{X}_{\tilde{\nu}}) = m \neq 0$, and its multiplicity as a crossing point is m.

For any crossing point $t_* \in [0, 1]$, the direction of rotation associated with t_* is determined by

$$-X_{\tilde{\nu}}^*\Big|_{\ker \mathbf{X}_{\nu}^*J\tilde{\mathbf{X}}(t_*)}. \tag{3.19}$$

3.3.1 Useful Special Cases

In this section, we employ the preceding considerations to derive specific formulas for computing $\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ in two common and important special cases.

Dirichlet-Neumann Exchange. We consider the case

$$\mathbf{X}_{\tilde{\nu}} = \begin{pmatrix} I \\ 0 \end{pmatrix}; \quad \mathbf{X}_{\tilde{\sigma}} = \begin{pmatrix} 0 \\ I \end{pmatrix}; \quad \mathbf{X}_{\nu} = \begin{pmatrix} X_\nu \\ Y_\nu \end{pmatrix}. \tag{3.20}$$

In this case, the interpolation path of matrices is

$$\tilde{\mathbf{X}}(t) = \begin{pmatrix} (1-t)I \\ tI \end{pmatrix},$$

and for each $t \in [0,1]$, $\tilde{\mathbf{X}}(t)$ is clearly the frame for a Lagrangian subspace $\tilde{\ell}(t)$.

The equation for crossing points (3.18) is

$$X_\nu^* v = \frac{1-t_*}{t_*} Y_\nu^* v,$$

and the direction is determined by $-X_{\tilde{\nu}} = -I$. From the direction, we can conclude that all crossing points correspond with a crossing in the negative direction. Due to this direction, a crossing point at $t_* = 1$ will not contribute to the Maslov index, but a crossing point at $t_* = 0$ will contribute an amount equal to $\dim \ker Y_\nu^*$. I.e., in this case, we have

$$\mathcal{I}(\nu; \mathbf{X}_N, \mathbf{X}_D) = -n_+(X_\nu^*, Y_\nu^*) - n_0(Y_\nu^*).$$

Dirichlet-non-Dirichlet Exchange. If a Lagrangian subspace $\tilde{\nu}$ does not intersect the Dirichlet subspace, then we can choose a frame for it with the form $\mathbf{X}_{\tilde{\nu}} = \binom{I}{M_{\tilde{\nu}}}$, where $M_{\tilde{\nu}}$ is a self-adjoint matrix. We consider the case

$$\mathbf{X}_{\tilde{\nu}} = \begin{pmatrix} I \\ M_{\tilde{\nu}} \end{pmatrix}; \quad \mathbf{X}_{\tilde{\sigma}} = \begin{pmatrix} 0 \\ I \end{pmatrix}; \quad \mathbf{X}_{\nu} = \begin{pmatrix} I \\ M_\nu \end{pmatrix}. \tag{3.21}$$

First, $X_{\tilde{\nu}} = I$, so (3.16) is satisfied, as is (3.17), thus establishing that in this case $\tilde{\mathbf{X}}(t)$ remains Lagrangian for all $t \in [0,1]$. The eigenvalue problem for crossing points (3.18) can be expressed as

$$(M_{\tilde{\nu}} - M_\nu) v = -\frac{t_*}{1-t_*} v,$$

and directionality is determined by $-X_{\tilde{\nu}}^* = -I$. In this case, $t_* = 1$ cannot be a crossing, because $\ell_{\tilde{\sigma}} \cap \ell_\nu = \{0\}$. We see that each crossing point corresponds with a non-positive eigenvalue of $M_{\tilde{\nu}} - M_\nu$, and that each such crossing point decrements the Maslov index according to multiplicity. We can conclude that

$$\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_D) = -n_-(M_{\tilde{\nu}} - M_\nu) - n_0(M_{\tilde{\nu}} - M_\nu). \tag{3.22}$$

3.4 Exchanging the Neumann Plane

In the framework of Section 3.2, suppose $\ell_{\tilde{\sigma}}$ is the Neumann plane, for which we use the natural frame $\mathbf{X}_N = \binom{I}{0}$. I.e., we are addressing the computation of

$$\mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_N),$$

where $\nu \in \Lambda(n)$ is arbitrary, and we will make assumptions on $\mathbf{X}_{\tilde{\nu}}$ as we proceed.

The analysis in this case is analogous to that of Section 3.3, and we only summarize the results. First, the Lagrangian property holds if and only if $Y_{\tilde{\nu}}^* = Y_{\tilde{\nu}}$. For dimensionality, we let $E_{(-\infty,0)}(X_{\tilde{\nu}})$ denote the union of all eigenspaces of $X_{\tilde{\nu}}$ associated with negative eigenvalues. Then $\tilde{\mathbf{X}}(t)$ retains dimensionality for all $t \in [0,1]$ if and only if

$$\ker(Y_{\tilde{\nu}}) \cap E_{(-\infty,0)}(X_{\tilde{\nu}}) = \{0\}. \tag{3.23}$$

A value $t_* \in (0,1]$ is a crossing point if and only if there exists $v \in \mathbb{C}^n$ so that

$$Y_\nu^* v = -\frac{1-t_*}{t_*} \mathbf{X}_\nu^* J \mathbf{X}_{\tilde{\nu}} v, \tag{3.24}$$

and the multiplicity of t_* is precisely the geometric multiplicity of $\tau_* = -\frac{1-t_*}{t_*}$ as an eigenvalue of the generalized eigenvalue problem

$$Y_\nu^* v = \tau \mathbf{X}_\nu^* J \mathbf{X}_{\tilde{\nu}} v.$$

Likewise, $t_* = 0$ is a crossing point with multiplicity m if and only if

$$\dim \ker(\mathbf{X}_\nu^* J \mathbf{X}_{\tilde{\nu}}) = m.$$

Finally, the direction of rotation is determined by

$$Y_{\tilde{\nu}}^* \Big|_{\ker(\mathbf{X}_\nu^* J \tilde{\mathbf{X}}(t_*))}.$$

For $t_* \in (0,1]$, $\ker(\mathbf{X}_\nu^* J \tilde{\mathbf{X}}(t_*))$ is the geometric eigenspace associated with $\tau_* = -\frac{1-t_*}{t_*}$ as an eigenvalue of the generalized eigenvalue problem (3.24). For $t_* = 0$, $\ker(\mathbf{X}_\nu^* J \tilde{\mathbf{X}}(t_*)) = \ker(\mathbf{X}_\nu^* J \tilde{\mathbf{X}}_{\tilde{\nu}})$.

Special cases analogous to those taken for the Dirichlet case can be analyzed similarly.

4 Comparison with Hörmander's Formula

In this section, we briefly relate our evaluations of Hörmander's index to Hörmander's formula (equation (2.10) in [14], given below as (4.1)). In order to state Hörmander's formula, we need to introduce an associated bilinear form.

Definition 4.1. *Fix any $\tilde{\nu}, \tilde{\sigma} \in \Lambda(n)$ with $\tilde{\nu} \cap \tilde{\sigma} = \{0\}$. Then any n-dimensional linear subspace $\nu \subset \mathbb{C}^{2n}$ (i.e., ν not necessarily Lagrangian) with $\nu \cap \tilde{\sigma} = \{0\}$ can be expressed as*

$$\nu = \{u + Cu : u \in \tilde{\nu}\}$$

for some $2n \times 2n$ matrix C that maps $\tilde{\nu}$ to $\tilde{\sigma}$. We define a bilinear form

$$Q = Q(\tilde{\nu}, \tilde{\sigma}; \nu) : \tilde{\nu} \times \tilde{\nu} \to \mathbb{C}$$

by the relation

$$Q(u, v) := -(JCu, v),$$

for all $u, v \in \tilde{\nu}$.

Remark 4.1. *The negative sign in our specification of Q is an artifact of convention: while we're taking our symplectic form to be (Ju, v), our reference [14] uses (u, Jv), which simply has the opposite sign.*

Hörmander's Q-form is precisely the form defined in [14], and aside from a sign convention is also the same form specified in Section 3.1 of [44]. Suppose ν intersects neither $\tilde{\nu}$ nor $\tilde{\sigma}$ and likewise σ intersects neither $\tilde{\nu}$ nor $\tilde{\sigma}$. Then if $\tilde{\nu} \cap \tilde{\sigma} = \{0\}$, Hörmander's formula for $s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma)$ can be expressed as (equation (2.10) of [14])

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \frac{1}{2}\Big(\operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}; \nu) - \operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}; \sigma) \Big), \tag{4.1}$$

where $\operatorname{sgn}(\cdot)$ denotes the usual signature of a bilinear form (number of positive eigenvalues minus the number of negative eigenvalues).

In [27], the authors derive a convenient formula for $\operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}, \nu)$ in the case that $\tilde{\nu}$ is the Dirichlet plane and neither ν nor $\tilde{\sigma}$ intersects the Dirichlet plane. In this case, frames for ν and $\tilde{\sigma}$ can respectively be taken as

$$\mathbf{X}_\nu = \begin{pmatrix} I \\ M_\nu \end{pmatrix} \quad \text{and} \quad \mathbf{X}_{\tilde{\sigma}} = \begin{pmatrix} I \\ M_{\tilde{\sigma}} \end{pmatrix}, \tag{4.2}$$

where M_ν and $M_{\tilde{\sigma}}$ are self-adjoint matrices. Under the additional assumption that $\nu \cap \tilde{\sigma} = \{0\}$, Lemma 3.2 from [27] asserts that

$$\operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}, \nu) = \operatorname{sgn}(M_{\tilde{\sigma}} - M_\nu).$$

Proceeding similarly for $\operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}, \sigma)$ (assuming additionally that σ does not intersect the Dirichlet plane, and $\sigma \cap \tilde{\sigma} = \{0\}$), we can express Hörmander's index as

$$s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = \frac{1}{2}\Big(\operatorname{sgn}(M_{\tilde{\sigma}} - M_\nu) - \operatorname{sgn}(M_{\tilde{\sigma}} - M_\sigma) \Big). \tag{4.3}$$

In the current analysis, we can use (3.22) to express Hörmander's index under these assumptions as

$$\begin{aligned} s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) &= \mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) - \mathcal{I}(\nu; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}}) \\ &= -\mathcal{I}(\sigma; \mathbf{X}_{\tilde{\sigma}}, \mathbf{X}_{\tilde{\nu}}) + \mathcal{I}(\nu; \mathbf{X}_{\tilde{\sigma}}, \mathbf{X}_{\tilde{\nu}}) \\ &= n_-(M_{\tilde{\sigma}} - M_\sigma) + n_0(M_{\tilde{\sigma}} - M_\sigma) - n_-(M_{\tilde{\sigma}} - M_\nu) - n_0(M_{\tilde{\sigma}} - M_\nu). \end{aligned} \tag{4.4}$$

In order to accommodate the assumptions required for Hörmander's formula (though not required for the current analysis), we are taking $\nu \cap \tilde{\sigma} = \{0\}$ and $\sigma \cap \tilde{\sigma} = \{0\}$, and under these additional assumptions, we have

$$
\begin{aligned}
n_0(M_{\tilde{\sigma}} - M_\sigma) &= 0 \\
n_0(M_{\tilde{\sigma}} - M_\nu) &= 0.
\end{aligned}
$$

I.e., we obtain the relation

$$
s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) = n_-(M_{\tilde{\sigma}} - M_\sigma) - n_-(M_{\tilde{\sigma}} - M_\nu). \tag{4.5}
$$

We can write

$$
\operatorname{sgn}(M_{\tilde{\sigma}} - M_\nu) = n_+(M_{\tilde{\sigma}} - M_\nu) - n_-(M_{\tilde{\sigma}} - M_\nu),
$$

and similarly for $(M_{\tilde{\sigma}} - M_\sigma)$. Also, we notice that the assumption $\tilde{\sigma} \cap \nu = \{0\}$ implies $(M_{\tilde{\sigma}} - M_\nu)$ has no zero eigenvalues, and consequently

$$
n_-(M_{\tilde{\sigma}} - M_\nu) + n_+(M_{\tilde{\sigma}} - M_\nu) = n,
$$

and again similarly for $(M_{\tilde{\sigma}} - M_\sigma)$. Working from (4.3), we can compute

$$
\begin{aligned}
s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) &= \frac{1}{2}\Big(\operatorname{sgn}(M_{\tilde{\sigma}} - M_\nu) - \operatorname{sgn}(M_{\tilde{\sigma}} - M_\sigma)\Big) \\
&= \frac{1}{2}\Big(n - 2n_-(M_{\tilde{\sigma}} - M_\nu) - (n - 2n_-(M_{\tilde{\sigma}} - M_\sigma)\Big) \\
&= n_-(M_{\tilde{\sigma}} - M_\sigma) - n_-(M_{\tilde{\sigma}} - M_\nu),
\end{aligned}
$$

which is equivalent to (4.5).

It's clear from these calculations that despite the similarity between (4.1) and the first equality in (4.4), it's not the case that the value $\mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ is another way of expressing $\frac{1}{2} \operatorname{sgn} Q(\tilde{\nu}, \tilde{\sigma}; \nu)$. In fact, no such correspondence is possible, because $\mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ depends on the specific frames $\mathbf{X}_{\tilde{\nu}}$ and $\mathbf{X}_{\tilde{\sigma}}$, not just on $\tilde{\nu}$ and $\tilde{\sigma}$ (as with $Q(\tilde{\nu}, \tilde{\sigma}; \nu)$). In this way, $\mathcal{I}(\sigma; \mathbf{X}_{\tilde{\nu}}, \mathbf{X}_{\tilde{\sigma}})$ should rightly be viewed as a computational tool rather than an index.

We also note that while Hörmander's formula (4.1) requires the specified conditions $\nu \cap \tilde{\nu} = \nu \cap \tilde{\sigma} = \{0\}$ and $\sigma \cap \tilde{\nu} = \sigma \cap \tilde{\sigma} = \{0\}$, a corresponding formulation in [44] requires no such assumptions. Precisely, with $Q(\tilde{\nu}, \tilde{\sigma}; \nu)$ specified as above, the authors of [44] work with the *triple index*

$$
\iota(\tilde{\nu}, \tilde{\sigma}, \nu) := n_+ Q(\tilde{\nu}, \tilde{\sigma}; \nu) + \dim(\tilde{\nu} \cap \nu) - \dim(\tilde{\nu} \cap \tilde{\sigma} \cap \nu), \tag{4.6}
$$

and establish in their Theorem 1.1 that for any quadruple of Lagrangian subspaces $\tilde{\nu}$, $\tilde{\sigma}$ ν, σ (and no additional assumptions)

$$
\begin{aligned}
s(\tilde{\nu}, \tilde{\sigma}; \nu, \sigma) &= \iota(\tilde{\nu}, \tilde{\sigma}, \sigma) - \iota(\tilde{\nu}, \tilde{\sigma}, \nu) \\
&= \iota(\tilde{\nu}, \nu, \sigma) - \iota(\tilde{\sigma}, \nu, \sigma).
\end{aligned}
$$

(For simplicity of this brief discussion, (4.6) is actually an alternative formulation of the triple index, taken from Remark A.12 of [6]; for the original formulation, see equation (2.16) in [14], or equivalently Corollary 3.12 in [44]).

5 Applications

In this section, we illustrate our theory with three straightforward applications.

5.1 Schrödinger Systems on $[0,1]$

We consider a Schrödinger system with two equations,

$$\begin{aligned} -\phi'' + V(x)\phi &= \lambda\phi \\ \alpha_1\phi(0) + \alpha_2\phi'(0) &= 0 \\ \beta_1\phi(1) + \beta_2\phi'(1) &= 0, \end{aligned} \tag{5.1}$$

which can be expressed as a linear Hamiltonian system as in (1.3) with $Q(x) = I_2$ and $P(x) = I_2$. We will take

$$V(x) = \begin{pmatrix} 10x(1-x) & 25\sin(10x) \\ 25\sin(10x) & 10\cos(10x) \end{pmatrix},$$

and Neumann boundary conditions at both $x = 0$ and $x = 1$, $\alpha = (\alpha_1\ \alpha_2) = (0_2\ I_2)$ and $\beta = (\beta_1\ \beta_2) = (0_2\ I_2)$. (The specific matrix $V(x)$ has no particular significance, and is chosen merely to provide an example that clearly illustrates the nature and implementation of the results.)

We will first illustrate how Theorems 1.1 and 1.2 can be used to count the number of eigenvalues that (5.1) has between $\lambda_1 = -3$ and $\lambda_2 = 0$, and then we will discuss more generally the dynamics for this example.

To start, we have from Theorem 1.1 that

$$\mathcal{N}([\lambda_1, \lambda_2)) = -\operatorname{Mas}(\ell_0(\cdot;\lambda_2), \ell_1; [0,1]) + \operatorname{Mas}(\ell_0(\cdot;\lambda_1), \ell_1; [0,1]), \tag{5.2}$$

where $\ell_0(x;\lambda)$ denotes the path of Lagrangian subspaces associated with $\mathbf{X}_0(x;\lambda)$ specified in (1.8) and ℓ_1 denotes the Neumann Lagrangian subspace, for which we use the standard frame $\mathbf{X}_N = \binom{I_2}{0_2}$. We proceed by replacing ℓ_1 in both instances by the Dirichlet Lagrangian subspace ℓ_D, for which we use the standard frame $\mathbf{X}_D = \binom{0_2}{I_2}$. We recall from the introduction that for $\mathbb{B}(x;\lambda)$ specified from a Schrödinger system (and more generally from a Sturm-Liouville system), the restriction $\mathbb{B}(x;\lambda)_{\ell_D}$ is non-negative, and additionally Assumption **(B2)** holds. This allows us to apply Theorem 1.2, so that for each index on the right-hand side of (5.2), we can write

$$\operatorname{Mas}(\ell_0(\cdot;\lambda_i), \ell_D; [0,1]) - \operatorname{Mas}(\ell_0(\cdot;\lambda_i), \ell_1; [0,1]) = s(\ell_1, \ell_D; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i)). \tag{5.3}$$

Following the approach outlined in Section 3.1, we can compute $s(\ell_1, \ell_D; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i))$ with

$$s(\ell_1, \ell_D; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i)) = \mathcal{I}(\ell_0(1;\lambda_i); \mathbf{X}_1, \mathbf{X}_D) - \mathcal{I}(\ell_0(0;\lambda_i); \mathbf{X}_1, \mathbf{X}_D).$$

In this case, we have the frames

$$\mathbf{X}_0(0;\lambda_i) = \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix},\ i = 1, 2; \quad \mathbf{X}_1 = \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}; \quad \mathbf{X}_D = \begin{pmatrix} 0_2 \\ I_2 \end{pmatrix},$$

and we find by numerical computation that

$$\mathbf{X}_0(1;\lambda_1) = \begin{pmatrix} I_2 \\ M_0(1;\lambda_1) \end{pmatrix}; \quad M_0(1;\lambda_1) = \begin{pmatrix} 1.5181 & 1.1717 \\ 1.1717 & .7090 \end{pmatrix}$$
$$\mathbf{X}_0(1;\lambda_2) = \begin{pmatrix} I_2 \\ M_0(1;\lambda_2) \end{pmatrix}; \quad M_0(1;\lambda_2) = \begin{pmatrix} .6757 & 1.3237 \\ 1.3237 & -.1040 \end{pmatrix}.$$

In addition, the eigenvalues of $M_0(1;\lambda_1)$ were computed to be $-.1261$ and 2.3532, while the eigenvalues of $M_0(1;\lambda_2)$ were computed to be -1.0941 and 1.6658.

Using relations (3.22), we find that

$$\mathcal{I}(\ell_0(0;\lambda_i);\mathbf{X}_1,\mathbf{X}_D) = -n_-(0) - n_0(0) = -2,$$

for $i = 1, 2$, and likewise

$$\mathcal{I}(\ell_0(1;\lambda_i);\mathbf{X}_1,\mathbf{X}_D) = -n_-(-M_0(1;\lambda_i)) - n_0(-M_0(1;\lambda_i)) = -1,$$

for $i = 1, 2$. This allows us to compute

$$s(\ell_1,\ell_D;\ell_0(0;\lambda_i),\ell_0(1;\lambda_i)) = -1 - (-2) = 1,$$

for $i = 1, 2$. Using (5.3), we see that

$$\mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_1;[0,1]) = \mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_D;[0,1]) - 1,$$

for $i = 1, 2$. This relation clearly allows us to compute $\mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_1;[0,1])$ in terms of $\mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_D;[0,1])$. For this example, we find by numerical computation that

$$\mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_D;[0,1]) = -1,$$

for $i = 1, 2$, and we conclude $\mathrm{Mas}(\ell_0(\cdot;\lambda_i),\ell_1;[0,1]) = -2$ for $i = 1, 2$. Using (5.2), we can conclude that $\mathcal{N}([\lambda_1,\lambda_2)) = 0$.

In order to illustrate the spectral dynamics associated with these calculations, we depict in Figure 3 the Maslov box for the target space $\ell_1 = \ell_N$ (on the left), along with the Maslov box for the target space ℓ_D (on the right). In the latter case, the crossings are monotonic, and correspondingly the spectral curves are monotonic (see the discussion in of spectral curves in Section 2.3).

5.2 Fourth Order Equations

In this section, we consider a single fourth order equation

$$\phi'''' - (v_2(x)\phi')' + v_0(x)\phi = \lambda\phi \tag{5.4}$$

with separated self-adjoint boundary conditions

$$\begin{aligned} \alpha_1\phi(0) + \alpha_2\phi'(0) + \alpha_3\phi''(0) + \alpha_4(\phi'''(0) - v_2(0)\phi'(0)) = 0 \\ \beta_1\phi(1) + \beta_2\phi'(1) + \beta_3\phi''(1) + \beta_4(\phi'''(1) - v_2(1)\phi'(1)) = 0. \end{aligned} \tag{5.5}$$

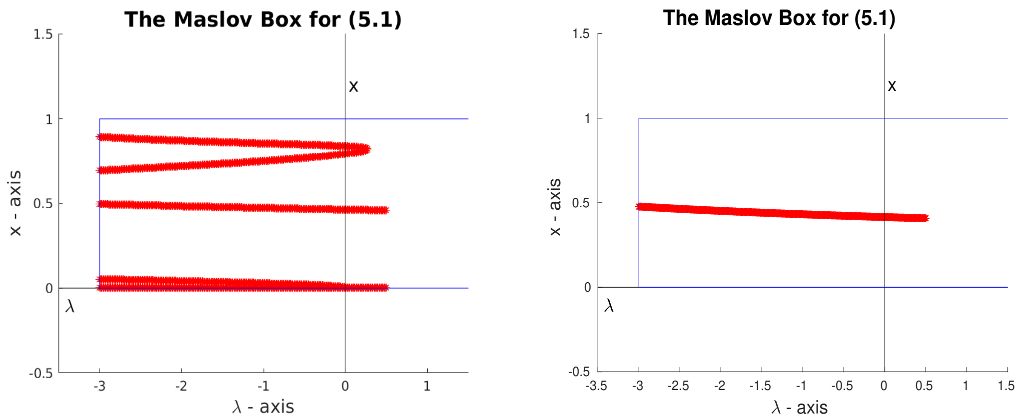

Figure 3: Spectral curves for (5.1): the Neumann target on the left; the Dirichlet target on the right.

Following the development of [21], we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \in \mathbb{C}^{2\times 4}$$
$$\tilde{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \in \mathbb{C}^{2\times 4},$$

and assume,

$$\operatorname{rank} \tilde{\alpha} = 2; \quad \tilde{\alpha}\mathbb{J}\tilde{\alpha}^* = 0$$
$$\operatorname{rank} \tilde{\beta} = 2; \quad \tilde{\beta}\mathbb{J}\tilde{\beta}^* = 0,$$

where

$$\mathbb{J} = \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix}.$$

Again following [21], we formulate (5.4) as a first-order system by writing $y = (y_1 \; y_2 \; y_3 \; y_4)^T$, with $y_1 = \phi$, $y_2 = \phi''$, $y_3 = -\phi''' + v_2\phi'$, and $y_4 = -\phi'$. With these choices, y solves the linear Hamiltonian system

$$Jy' = \mathbb{B}(x;\lambda)y; \quad \mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda - v_0(x) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -v_2(x) \end{pmatrix}. \tag{5.6}$$

If we set $\alpha = (\alpha_1 \; \alpha_3 \; -\alpha_4 \; -\alpha_2)$ and likewise $\beta = (\beta_1 \; \beta_3 \; -\beta_4 \; -\beta_2)$, then we can express our boundary conditions for (5.6) as $\alpha y(0) = 0$ and $\beta y(1) = 0$.

As a specific case, we will take

$$v_0(x) = -2 + 10\sin(12x)$$
$$v_2(x) = 10\cos(10x),$$

along with

$$\alpha, \beta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.7}$$

These boundary conditions correspond with $\phi'(0) = 0$, $\phi'''(0) = 0$, $\phi'(1) = 0$, and $\phi'''(1) = 0$. The associated target Langrangian subspace ℓ_1 has frame $\mathbf{X}_1 = J\beta^*$, and an equivalent frame for this space is the Neumann frame $\mathbf{X}_N = \binom{I_2}{0_2}$ (i.e., we will take $\mathbf{X}_1 = \mathbf{X}_N$). (Similarly as with the matrix $V(x)$ in Section 5.1, the functions $v_0(x)$ and $v_2(x)$ have no particular significance, and rather have been chosen to provide a specific example that clearly illustrates the nature and implementation of the results.)

For (5.6), we can check that the flow will be monotonic if the target is taken as the Lagrangian subspace ℓ_2 associated with the boundary matrix

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

To see this, we note that the target frame associated with γ is

$$\mathbf{X}_2 = J\gamma^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.8}$$

Any element of ℓ_2 will have the form $u = (0, u_2, u_3, 0)^T$, so that

$$(\mathbb{B}(x;\lambda)u, u) = |u_2|^2 \geq 0.$$

In this way, we see that $\mathbb{B}(x;\lambda)|_{\ell_2} \geq 0$ for all $x \in [0,1]$. This establishes the first part of **(B2)**. For the second, suppose $y(x;\lambda)$ solves $Jy' = \mathbb{B}(x;\lambda)y$ in some interval $[a,b] \subset [0,1]$, $a < b$, and $y(x;\lambda) \in \ell_2$ for all $x \in [a,b]$. Then, in particular, we must have $y_1(x;\lambda) = 0$ for all $x \in [a,b]$. But since $y_1(x;\lambda) = \phi(x;\lambda)$, this implies $y(x;\lambda) = 0$ for all $x \in [a,b]$, so that y must be trivial. This establishes the second condition in **(B2)**, allowing us to use Theorem 1.2 with the target space ℓ_2.

As in the previous example, we can use Theorems 1.1 and 1.2 to compute $\mathcal{N}([-3,0))$. For each index on the right-hand side of (5.2), we can write

$$\operatorname{Mas}(\ell_0(\cdot;\lambda_i), \ell_1; [0,1]) - \operatorname{Mas}(\ell_0(\cdot;\lambda_i), \ell_2; [0,1]) = s(\ell_2, \ell_1; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i)),$$

and we can compute $s(\ell_2, \ell_1; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i))$ with

$$s(\ell_2, \ell_1; \ell_0(0;\lambda_i), \ell_0(1;\lambda_i)) = \mathcal{I}(\ell_0(1;\lambda_i); \mathbf{X}_2, \mathbf{X}_1) - \mathcal{I}(\ell_0(0;\lambda_i); \mathbf{X}_2, \mathbf{X}_1).$$

Since ℓ_1 is the Neumann Lagrangian subspace, we can apply the development from Section 3.4. The relevant frames are $\mathbf{X}_0(0;\lambda) = \binom{I_2}{0_2}$, $\mathbf{X}_1 = \binom{I_2}{0_2}$, and $\mathbf{X}_2$ from (5.8), along with the following two frames, obtained by numerical computation,

$$\mathbf{X}_0(1;\lambda_1) = \begin{pmatrix} M(1;\lambda_1) \\ I_2 \end{pmatrix}; \quad M(1;\lambda_1) = \begin{pmatrix} 1.0263 & -.7332 \\ -.7332 & .0301 \end{pmatrix}$$
$$\mathbf{X}_0(1;\lambda_2) = \begin{pmatrix} M(1;\lambda_2) \\ I_2 \end{pmatrix}; \quad M(1;\lambda_2) = \begin{pmatrix} -.5326 & -.1073 \\ -.1073 & -.1613 \end{pmatrix}.$$

Referring now to Section 3.4, we see that the Lagrangian property is clear since Y_2 is self-adjoint, and dimensionality is clear since $E_{(-\infty,0)}(X_1) = \{0\}$. Starting with

$$\mathcal{I}(\ell_0(0;\lambda_i);\mathbf{X}_2,\mathbf{X}_1),$$

since $Y_0(0;\lambda_i) = 0$, a value $t_* \in [0,1)$ is a crossing in this case if and only if $\ker \mathbf{X}_0(0;\lambda_i)^* J\mathbf{X}_2 \neq \{0\}$. We have

$$\mathbf{X}_0(0;\lambda_i)^* J\mathbf{X}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

so in fact every $t_* \in [0,1)$ is a crossing with order 1. Moreover, since $\mathbf{X}_0(0;\lambda_i)^* J\mathbf{X}_1 = 0$, $t_* = 1$ is a crossing with order 2. We can conclude that the only possible contribution to the Maslov index will occur at $t_* = 1$ with this change in order. The direction associated with this arrival is determined in the usual way by the restriction of Y_2^* to

$$\ker \mathbf{X}_0(0;\lambda_i)^* J\mathbf{X}_2 = \mathbb{C}^2.$$

The eigenvalue residing at -1 for all $t_* \in [0,1]$ corresponds with the neutral direction, and the eigenvalue arriving at -1 at $t_* = 1$ corresponds with the positive direction. We conclude that

$$\mathcal{I}(\ell_0(0;\lambda_i);\mathbf{X}_2,\mathbf{X}_1) = +1,$$

for both $\lambda_1 = -3$ and $\lambda_2 = 0$.

Turning now to the evaluation of $\mathcal{I}(\ell_0(1;\lambda_i);\mathbf{X}_2,\mathbf{X}_1)$, we note from (3.24) that crossing points correspond with non-positive eigenvalues of the matrix

$$\mathbf{X}_0(1;\lambda_i)^* J\mathbf{X}_2 = M(1;\lambda_i)\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -M_{11}(1;\lambda_i) & 0 \\ -M_{21}(1;\lambda_i) & 1. \end{pmatrix} \tag{5.9}$$

For $\lambda_1 = -3$, $-M_{11}(1;-3) = -1.0263$, and so this matrix has exactly one non-positive eigenvalue. For $\lambda_2 = 0$, $-M_{11}(1;0) = .5326$, and this matrix has no non-positive eigenvalues. We can conclude immediately that for $\lambda_2 = 0$,

$$\mathcal{I}(\ell_0(1;0);\mathbf{X}_2,\mathbf{X}_1) = 0,$$

while for $\lambda_1 = -3$, we need to consider the direction of the crossing.

Although we don't strictly require this much detail, we observe that the precise crossing point $t_* \in (0,1)$ can be obtained by solving

$$\frac{t_*}{1-t_*} = 1.0263 \implies t_* = .5065.$$

The direction associated with this crossing point is determined by restricting the matrix Y_2^* to the subspace $\ker \mathbf{X}_0(1;-3)^* J\tilde{\mathbf{X}}(t_*)$. We can compute

$$\mathbf{X}_0(1;-3)^* J\tilde{\mathbf{X}}(t_*) = \begin{pmatrix} 0 & 0 \\ -(1-t_*)M_{21}(1;-3) & 1 \end{pmatrix}.$$

We see that due to the form of Y_2^*, its restriction to $\ker \mathbf{X}_0(1;-3)J\tilde{\mathbf{X}}(t_*)$ will only be non-positive if $(1-t_*)M_{21}(1;-3) = 0$. This is clearly not the case for any $t_* \in (0,1)$, which is why we didn't need the specific crossing point in this case. Since the restricted matrix is positive, the crossing point is counted positively, and we find

$$\mathcal{I}(\ell_0(1;-3);\mathbf{X}_2,\mathbf{X}_1) = 1.$$

With these values in place, we see that

$$\begin{aligned} s(\ell_2,\ell_1;\ell_0(0;-3),\ell_0(1;-3)) &= 0 \\ s(\ell_2,\ell_1;\ell_0(0;0),\ell_0(1;0)) &= -1, \end{aligned}$$

and consequently

$$\begin{aligned} \operatorname{Mas}(\ell_0(\cdot;-3),\ell_1;[0,1]) &= \operatorname{Mas}(\ell_0(\cdot;-3),\ell_2;[0,1]) \\ \operatorname{Mas}(\ell_0(\cdot;0),\ell_1;[0,1]) &= \operatorname{Mas}(\ell_0(\cdot;0),\ell_2;[0,1]) - 1. \end{aligned}$$

By numerical computation, we find

$$\begin{aligned} \operatorname{Mas}(\ell_0(\cdot;-3),\ell_2;[0,1]) &= -1 \\ \operatorname{Mas}(\ell_0(\cdot;0),\ell_2;[0,1]) &= -1, \end{aligned}$$

and from these we compute

$$\begin{aligned} \operatorname{Mas}(\ell_0(\cdot;-3),\ell_1;[0,1]) &= -1 \\ \operatorname{Mas}(\ell_0(\cdot;0),\ell_1;[0,1]) &= -2. \end{aligned}$$

Using (5.2) from [21], we conclude that $\mathcal{N}([-3,0)) = -(-2) - 1 = 1$.

As with the previous example, we illustrate the process by depicting the full Maslox boxes associated with these calculations. In Figure 4 the Maslov box for the target space ℓ_1 is on the left, while the Maslov box for the target space ℓ_2 is on the right. In the latter case, no spectral curves enter the Maslov box, though the entire bottom shelf is comprised of crossings.

5.3 Schrödinger Systems on $\mathbb{R}$

In this section, we apply our theory in the setting of [23]. In that reference, the authors consider eigenvalue problems

$$H\phi := -\phi'' + V(x)\phi = \lambda\phi; \quad \operatorname{dom}(H) = H^2(\mathbb{R}), \tag{5.10}$$

and also (for any $s \in \mathbb{R}$)

$$H_s\phi := -\phi'' + s\phi' + V(x)\phi = \lambda\phi; \quad \operatorname{dom}(H_s) = H^2(\mathbb{R}), \tag{5.11}$$

where $\lambda \in \mathbb{R}$, $\phi(x) \in \mathbb{R}^n$ and $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ is a real-valued symmetric matrix potential satisfying Condition **(S)** from the introduction.

In this setting, the essential spectrum plays a role, so we briefly recall how it can be defined and identified.

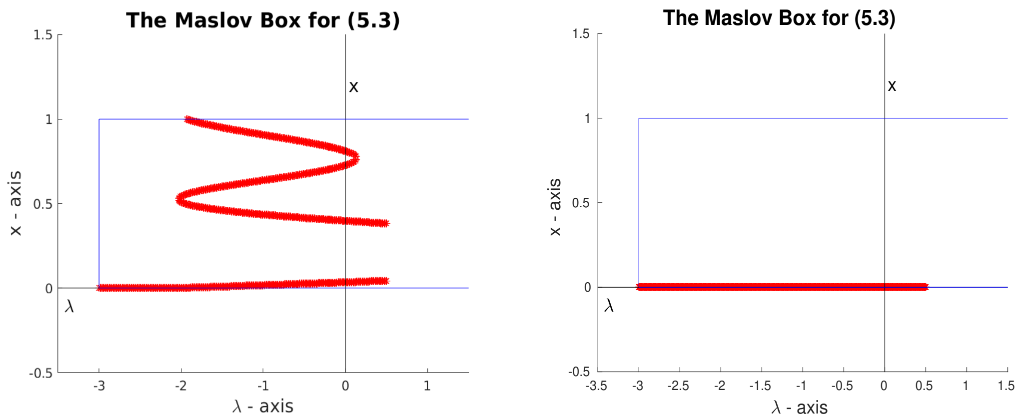

Figure 4: Spectral curves for (5.1): the Neumann target on the left; the Dirichlet target on the right.

Definition 5.1. *We define the point spectrum of H, denoted $\sigma_{\rm pt}(H)$, as the set*

$$\sigma_{\rm pt}(H) = \{\lambda \in \mathbb{R} : H\phi = \lambda\phi \text{ for some } \phi \in H^2(\mathbb{R})\backslash\{0\}\}.$$

We define the essential spectrum of H, denoted $\sigma_{\rm ess}(H)$, as the values in $\mathbb{R}$ that are not in the resolvent set of H and are not isolated eigenvalues of finite multiplicity.

We note that the total spectrum is $\sigma = \sigma_{\rm pt}(H) \cup \sigma_{\rm ess}(H)$, and the *discrete* spectrum is defined as $\sigma_{\rm discrete}(H) = \sigma\backslash\sigma_{\rm ess}(H)$. Since our analysis takes place entirely away from essential spectrum, the eigenvalues we are counting are elements of the discrete spectrum.

As discussed, for example, in [20, 33], the essential spectrum of H is determined by the asymptotic equations

$$-\phi'' + V_\pm\phi = \lambda\phi. \tag{5.12}$$

In particular, if we look for solutions of the form $\phi(x) = e^{ikx}r$, for some scalar constant $k \in \mathbb{R}$ and (non-zero) constant vector $r \in \mathbb{R}^n$ then the essential spectrum will be confined to the allowable values of λ. For (5.12), we find

$$(k^2 I + V_\pm)r = \lambda r,$$

so that

$$\lambda(k) \geq \frac{(V_\pm r, r)_{\mathbb{R}^n}}{\|r\|^2}.$$

Applying the min-max principle, we see that $\sigma_{\rm ess}(H) \subset [\kappa_{\min}, \infty)$, where as specified in the introduction $\kappa_{\min}$ denotes the minimum of all eigenvalues of the matrices $V_\pm$.

Away from essential spectrum, construction of the asymptotically growing and decaying solutions to (5.10) is standard and carried out in detail in [23]. Briefly reviewing that development, we proceed by looking for solutions of (5.12) of the form $\phi(x;\lambda) = e^{\mu x}r$, where in this case μ is a scalar function of λ, and r is again a constant vector in $\mathbb{R}^n$. We obtain the relation

$$(-\mu^2 I + V_\pm - \lambda I)r = 0,$$

from which we see that the values of $\mu^2+\lambda$ will correspond with eigenvalues of $V_\pm$, and the vectors r will be eigenvectors of $V_\pm$. We denote the spectrum of $V_\pm$ by $\sigma(V_\pm)=\{\kappa_j^\pm\}_{j=1}^n$, ordered so that $j<k$ implies $\kappa_j^\pm \le \kappa_k^\pm$, and we order the eigenvectors correspondingly so that $V_\pm r_j^\pm = \kappa_j^\pm r_j^\pm$ for all $j\in\{1,2,\dots,n\}$. Moreover, since $V_\pm$ are symmetric matrices, we can choose the set $\{r_j^-\}_{j=1}^n$ to be orthonormal, and similarly for $\{r_j^+\}_{j=1}^n$.

We have

$$\mu^2+\lambda=\kappa_j^\pm \implies \mu=\pm\sqrt{\kappa_j^\pm-\lambda}.$$

We will denote the admissible values of μ by $\{\mu_j^\pm\}_{j=1}^{2n}$, and for consistency we choose our labeling scheme so that $j<k$ implies $\mu_j^\pm \le \mu_k^\pm$ (for $\lambda\le\kappa_{\min}$). This leads us to the specifications

$$\begin{aligned}\mu_j^\pm(\lambda) &= -\sqrt{\kappa_{n+1-j}^\pm-\lambda}\\ \mu_{n+j}^\pm(\lambda) &= \sqrt{\kappa_j^\pm-\lambda},\end{aligned}$$

for $j=1,2,\dots,n$.

We now express (5.10) as a first order system, with $y=\binom{y_1}{y_2}$, $y_1=\phi$, $y_2=\phi'$. We find

$$\frac{dy}{dx}=\mathbb{A}(x;\lambda)y;\quad \mathbb{A}(x;\lambda)=\begin{pmatrix}0 & I\\ V(x)-\lambda I & 0\end{pmatrix}, \tag{5.13}$$

and we additionally set

$$\mathbb{A}_\pm(\lambda):=\lim_{x\to\pm\infty}\mathbb{A}(x;\lambda)=\begin{pmatrix}0 & I\\ V_\pm-\lambda I & 0\end{pmatrix}.$$

We note that the eigenvalues of $\mathbb{A}_\pm$ are precisely the values $\{\mu_j^\pm\}_{j=1}^{2n}$, and the associated eigenvectors are $\{\mathscr{r}_j^\pm\}_{j=1}^n=\{\binom{r_{n+1-j}^\pm}{\mu_j^\pm r_{n+1-j}^\pm}\}_{j=1}^n$ and $\{\mathscr{r}_{n+j}^\pm\}_{j=1}^n=\{\binom{r_j^\pm}{\mu_{n+j}^\pm r_j^\pm}\}_{j=1}^n$.

We have now established the notation we need to state Lemma 2.2 from [23].

Lemma 5.1. *Let $V\in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose* **(S)** *holds. Then for any $\lambda<\kappa_{\min}$ there exist n linearly independent solutions of (5.13) that decay to zero as $x\to-\infty$ and n linearly independent solutions of (5.13) that decay to zero as $x\to+\infty$. Respectively, we can choose these so that they can be expressed as*

$$\begin{aligned}y_{n+j}^-(x;\lambda) &= e^{\mu_{n+j}^-(\lambda)x}(\mathscr{r}_{n+j}^- + \mathbf{E}_{n+j}^-(x;\lambda));\quad j=1,2,\dots,n,\\ y_j^+(x;\lambda) &= e^{\mu_j^+(\lambda)x}(\mathscr{r}_j^+ + \mathbf{E}_j^+(x;\lambda));\quad j=1,2,\dots,n,\end{aligned}$$

where for any fixed $\lambda_0<\kappa_{\min}$ and $\lambda_\infty>0$ (with $-\lambda_\infty<\lambda_0$), $\mathbf{E}_{n+j}^-(x;\lambda)=\mathbf{O}((1+|x|)^{-1})$, uniformly for $\lambda\in[-\lambda_\infty,\lambda_0]$, and similarly for $\mathbf{E}_j^+(x;\lambda)$.

Moreover, there exist n linearly independent solutions of (5.13) that grow to infinity as $x\to-\infty$ and n linearly independent solutions of (5.13) that grow to infinity as $x\to+\infty$. Respectively, we can choose these so that they can be expressed as

$$\begin{aligned}y_j^-(x;\lambda) &= e^{\mu_j^-(\lambda)x}(\mathscr{r}_j^- + \mathbf{E}_j^-(x;\lambda));\quad j=1,2,\dots,n,\\ y_{n+j}^+(x;\lambda) &= e^{\mu_{n+j}^+(\lambda)x}(\mathscr{r}_{n+j}^+ + \mathbf{E}_{n+j}^+(x;\lambda));\quad j=1,2,\dots,n,\end{aligned}$$

where for any fixed $\lambda_0<\kappa_{\min}$ and $\lambda_\infty>0$ (with $-\lambda_\infty<\lambda_0$), $\mathbf{E}_j^-(x;\lambda)=\mathbf{O}((1+|x|)^{-1})$, uniformly for $\lambda\in[-\lambda_\infty,\lambda_0]$, and similarly for $\mathbf{E}_{n+j}^+(x;\lambda)$.

As noted in our introduction, the authors of [23] verify that if we create a frame $\mathbf{X}^-(x;\lambda) = \binom{X^-(x;\lambda)}{Y^-(x;\lambda)}$ by taking $\{y^-_{n+j}(x;\lambda)\}_{j=1}^n$ as the columns of $\mathbf{X}^-(x;\lambda)$, then $\mathbf{X}^-(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^-(x;\lambda)$. Likewise, we can create a frame $\mathbf{X}^+(x;\lambda) = \binom{X^+(x;\lambda)}{Y^+(x;\lambda)}$ by taking $\{y^+_j(x;\lambda)\}_{j=1}^n$ as the columns of $\mathbf{X}^+(x;\lambda)$. Then $\mathbf{X}^+(x;\lambda)$ is a frame for a Lagrangian subspace, which we will denote $\ell^+(x;\lambda)$.

In constucting our Lagrangian frames, we can view the exponential multipliers $e^{\mu^\pm_j x}$ as expansion coefficients, and if we drop these off we retain frames for the same spaces. That is, we can create an alternative frame for $\ell^-(x;\lambda)$ by taking the expressions $\mathbf{r}^-_{n+j} + \mathbf{E}^-_{n+j}(x;\lambda)$ as the columns for a new frame, which we will again denote $\mathbf{X}^-(x;\lambda)$. We see that in the limit as x tends to $-\infty$ (of the resulting modified frames) we obtain the frame $\mathbf{R}^-(\lambda) = \binom{R^-}{R^-D^-(\lambda)}$, where

$$\begin{aligned} R^- &= \begin{pmatrix} r^-_1 & r^-_2 & \dots & r^-_n \end{pmatrix}, \\ D^-(\lambda) &= \operatorname{diag}\begin{pmatrix} \mu^-_{n+1}(\lambda) & \mu^-_{n+2}(\lambda) & \dots & \mu^-_{2n}(\lambda) \end{pmatrix}. \end{aligned}$$

In [23], the authors verify that $\mathbf{R}^-(\lambda)$ is the frame for a Lagrangian subspace, and we denote this space $\ell^-_{\mathbf{R}}(\lambda)$. We notice that since V_- is self-adjoint, we can choose R^- so that $(R^-)^{-1} = (R^-)^*$.

Proceeding similarly with $\ell^+(x;\lambda)$, we obtain the asymptotic Lagrangian subspace $\ell^+_{\mathbf{R}}(\lambda)$ with frame $\mathbf{R}^+(\lambda) = \binom{R^+}{R^+D^+(\lambda)}$, where

$$\begin{aligned} R^+ &= \begin{pmatrix} r^+_n & r^+_{n-1} & \dots & r^+_1 \end{pmatrix}, \\ D^+(\lambda) &= \operatorname{diag}\begin{pmatrix} \mu^+_1(\lambda) & \mu^+_2(\lambda) & \dots & \mu^+_n(\lambda) \end{pmatrix}. \end{aligned} \tag{5.14}$$

(The ordering of the columns of $\mathbf{R}^+$ is simply a convention, which follows naturally from our convention for indexing $\{\phi^+_j\}_{j=1}^n$.) Since V_+ is self-adjoint, we can normalize R^+ so that $(R^+)^{-1} = (R^+)^*$.

The main result of [23] can now be stated as follows (Theorem 1.2 in [23], slightly adapted for consistency with the current discussion):

Theorem 5.1. *Let $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose* **(S)** *holds. Then for any $\lambda < \kappa_{\min}$,*

$$\operatorname{Mor}(H;\lambda) = -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell^+_{\mathbf{R}}(\lambda); \bar{\mathbb{R}}).$$

In addition, the authors obtain the following theorem (Theorem 1.6 in [23], slightly adapted for consistency with the current discussion):

Theorem 5.2. *Let $V \in C(\mathbb{R};\mathbb{R}^{n\times n})$ be a real-valued symmetric matrix potential, and suppose* **(S)** *holds. Let $s \in \mathbb{R}$, and let $\ell^-(x;\lambda)$ and $\ell^+_{\mathbf{R}}(\lambda)$ denote Lagrangian subspaces developed for (5.11). Then for any $\lambda < \kappa_{\min}$*

$$\operatorname{Mor}(H_s;\lambda) = -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell^+_{\mathbf{R}}(\lambda); \bar{\mathbb{R}}).$$

Remark 5.1. *In both theorems, our notation $\bar{\mathbb{R}}$ indicates that we allow the possibility for a crossing point to be obtained in either limit, $x \to \pm\infty$. We note that for the case of $+\infty$, the limiting point will be a crossing if and only if λ is an eigenvalue for the equation, and it's straightforward to verify that for the case $-\infty$ the limiting point cannot be a crossing (see [23]).*

In order to prove Theorems 1.3 and 1.4, we need to show respectively that in Theorems 5.1 and 5.2, the target Lagrangian subspace $\ell^+_{\mathbf{R}}(\lambda)$ can be replaced with the Dirichlet Lagrangian subspace ℓ_D (with frame $\mathbf{X}_D = \binom{0}{I}$), for which the resulting spectral flow is monotonic. In order to accomplish this, we will use the relation

$$\begin{aligned} \operatorname{Mas}(\ell^-(\cdot;\lambda), \ell^+_{\mathbf{R}}(\lambda); \bar{\mathbb{R}}) &- \operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_D; \bar{\mathbb{R}}) \\ &= s(\ell_D, \ell^+_{\mathbf{R}}(\lambda); \ell^-(-\infty;\lambda), \ell^-(+\infty;\lambda)). \end{aligned} \tag{5.15}$$

Using 3.9, we can compute

$$\begin{aligned} s(\ell_D, &\ell^+_{\mathbf{R}}(\lambda); \ell^-(-\infty;\lambda), \ell^-(+\infty;\lambda)) \\ &= \mathcal{I}(\ell^-(+\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) - \mathcal{I}(\ell^-(-\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)). \end{aligned}$$

Since R_+ is invertible, we can take our frame for $\ell^+_{\mathbf{R}}(\lambda)$ to be $\mathbf{R}^+(\lambda) = \binom{I}{R^+D^+(\lambda)(R^+)^*}$

As emphasized in [23], the exponential approach to endstates of the solutions comprising the frames $\mathbf{X}^\pm(x;\lambda)$ allows us to compactify $\mathbb{R}$ via (e.g.) the standard map

$$x = \ln(\frac{1+\tau}{1-\tau}), \quad \tau \in [-1, 1]. \tag{5.16}$$

In this way, our development of the Maslov index on finite intervals can be applied on $\bar{\mathbb{R}}$, as described in Remark 5.1.

To start, we will suppose that some $\lambda < \kappa_{\min}$ is not an eigenvalue of (5.10). The advantage to this case is that if it holds, then we know both $\ell^-(-\infty;\lambda)$ and $\ell^-(+\infty;\lambda)$ explicitly, and we can evaluate Hörmander's index directly. First, for any $\lambda < \kappa_{\min}$, we can take $\mathbf{R}^-(\lambda) = \binom{I}{R^-D^-(\lambda)(R^-)^*}$ as our frame for $\ell^-(-\infty;\lambda)$. In addition, if λ is not an eigenvalue of (5.10), then the Lagrangian subspace $\ell^-(+\infty;\lambda)$ will be the (unique) Lagrangian subspace associated with solutions of (5.10) that grow as $x \to +\infty$. We can take the frame for this space to be $\tilde{\mathbf{R}}^+(\lambda) = \binom{I}{-R^+D^+(\lambda)(R^+)^*}$.

This places us precisely in the setting of (3.22), and using additionally (3.7), we can write

$$\begin{aligned} \mathcal{I}(\ell^-(+\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) &= n_-(R^+D^+(\lambda)(R^+)^* - (-R^+D^+(\lambda)(R^+)^*)) \\ &\quad + n_0(R^+D^+(\lambda)(R^+)^* - (-R^+D^+(\lambda)(R^+)^*)) \\ \mathcal{I}(\ell^-(-\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) &= n_-(R^+D^+(\lambda)(R^+)^* - (R^-D^-(\lambda)(R^-)^*)) \\ &\quad + n_0(R^+D^+(\lambda)(R^+)^* - (R^-D^-(\lambda)(R^-)^*)). \end{aligned}$$

The matrix $R^+D^+(\lambda)(R^+)^*$ is negative definite, since the diagonals of $D^+(\lambda)$ are all negative, and likewise the matrix $R^-D^-(\lambda)(R^-)^*$ is positive definite. We can conclude

$$\begin{aligned} \mathcal{I}(\ell^-(+\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) &= n \\ \mathcal{I}(\ell^-(-\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) &= n, \end{aligned}$$

so that

$$s(\ell_D, \ell^+_{\mathbf{R}}(\lambda); \ell^-(-\infty;\lambda), \ell^-(+\infty;\lambda)) = 0.$$

Using (5.15), we see that

$$\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell^+_{\mathbf{R}}(\lambda); \bar{\mathbb{R}}) = \operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_D; \bar{\mathbb{R}}).$$

Recalling that we have monotonicity in this case when the target space is ℓ_D, and that $\ell^-(x;\lambda) \cap \ell_D = \{0\}$ for $|x|$ sufficiently large, we can conclude Theorem 1.3 in the case that λ is not an eigenvalue of (5.10).

Remark 5.2. *We have observed here that $\ell_{\mathbf{R}} \cap \ell_D = \{0\}$, and noted in addition that if λ_i is not an eigenvalue of H, then as $x \to +\infty$, $\ell^-(x;\lambda_i)$ will approach the Lagrangian subspace with frame $\tilde{\mathbf{R}}^+(\lambda_i) = \binom{I}{-R^+ D^+(\lambda_i)(R^+)^*}$, and this Lagrangian subspace does not intersect ℓ_D.*

In the event that λ is an eigenvalue of (5.10), the only part of our calculation that changes is that we no longer have an explicit expression for a frame for $\ell^-(+\infty;\lambda)$, which now comprises a combination of growing and decaying solutions. Nonetheless, we always have

$$-n \le \mathcal{I}(\ell^-(+\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) \le n,$$

so that

$$s(\ell_D, \ell_{\mathbf{R}}^+(\lambda); \ell^-(-\infty;\lambda), \ell^-(+\infty;\lambda)) = \mathcal{I}(\ell^-(+\infty;\lambda); \mathbf{X}_D, \mathbf{R}_+(\lambda)) - n \le 0.$$

We see that in this case,

$$\operatorname{Mor}(H;\lambda) = -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_{\mathbf{R}}^+(\lambda); \bar{\mathbb{R}}) \ge -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_D; \bar{\mathbb{R}}). \tag{5.17}$$

For the reverse inequality, we recall from Section 2.3 that when the Dirichlet Lagrangian plane ℓ_D is taken as the target for flow under the Schrödinger equation, the spectral curves will be monotonic, and in particular strictly decreasing when viewed in the (λ, x)-plane. It follows that for any $\epsilon > 0$ we have the inequality

$$-\operatorname{Mas}(\ell^-(\cdot;\lambda-\epsilon), \ell_D; \bar{\mathbb{R}}) \le -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_D; \bar{\mathbb{R}});$$

i.e., the number of crossings along the vertical line at $\lambda - \epsilon$ is less than or equal to the number of crossings along the vertical line at λ (see Figure 1). Since we're away from essential spectrum, the eigenvalues are discrete, and we can choose $\epsilon > 0$ small enough so that (5.10) has no eigenvalues on the interval $[\lambda - \epsilon, \lambda)$. Since $\lambda - \epsilon$ is not an eigenvalue, we see that

$$\begin{aligned} \operatorname{Mor}(H;\lambda) = \operatorname{Mor}(H;\lambda-\epsilon) &= -\operatorname{Mas}(\ell^-(\cdot;\lambda-\epsilon), \ell_D; \bar{\mathbb{R}}) \\ &\le -\operatorname{Mas}(\ell^-(\cdot;\lambda), \ell_D; \bar{\mathbb{R}}). \end{aligned} \tag{5.18}$$

Combining (5.17) and (5.18), we obtain the claimed equality, and this completes the proof of Theorem 1.3. □

The proof of Theorem 1.4 is essentially identical, based on Theorem 1.6 from [23].

5.3.1 Reduction to the Case of Large Bounded Intervals

We conclude this section on the Schrödinger equation on $\mathbb{R}$ by observing a connection between the current analysis and results such as [5, 41], in which it is shown that in certain cases the eigenvalues for a linear Hamiltonian system on $\mathbb{R}$ can be approximated by the eigenvalues of the same linear Hamiltonian system posed on a large bounded interval with appropriate boundary conditions. Since this is effectively just a long remark, we only sketch the argument.

First, using Theorem 1.3, we see that if $\lambda_1 < \lambda_2 < \kappa_{\min}$, then the number of eigenvalues that H has on the interval $[\lambda_1, \lambda_2)$ can be computed as

$$\mathcal{N}([\lambda_1, \lambda_2)) = - \operatorname{Mas}(\ell^-(\cdot; \lambda_2), \ell_D; \bar{\mathbb{R}}) + \operatorname{Mas}(\ell^-(\cdot; \lambda_1), \ell_D; \bar{\mathbb{R}}).$$

Suppose neither λ_1 nor λ_2 is an eigenvalue of H. Then, recalling Remark 5.2, we can find $L > 0$ sufficiently large so that for each $i \in \{1, 2\}$, $\ell^-(x; \lambda_i) \cap \ell_D = \{0\}$ for all $x \geq L$. It follows that

$$\mathcal{N}((\lambda_1, \lambda_2)) = - \operatorname{Mas}(\ell^-(\cdot; \lambda_2), \ell_D; (-\infty, +L]) + \operatorname{Mas}(\ell^-(\cdot; \lambda_1), \ell_D; (-\infty, +L]), \qquad (5.19)$$

for all L sufficiently large. By an argument similar to the one used to establish Theorem 5.1, we can show that the right-hand side of (5.19) is precisely a count of the number of eigenvalues on the interval (λ_1, λ_2) for the half-line problem

$$-\phi'' + V(x)\phi = \lambda\phi; \quad x \in (-\infty, L); \quad \phi(L) = 0. \qquad (5.20)$$

Next, for each of $i \in \{1, 2\}$, let $\mathbf{X}_L(x; \lambda_i)$ solve the matrix equation

$$J\mathbf{X}_L'(x; \lambda_i) = \mathbb{B}(x; \lambda_i)\mathbf{X}_L(x; \lambda_i); \quad x \in (-\infty, L); \quad \mathbf{X}_L(L; \lambda_i) = \begin{pmatrix} 0_n \\ I_n \end{pmatrix}.$$

By homotopy invariance, we have the relation

$$\begin{aligned} \operatorname{Mas}(\ell^-(\cdot; \lambda_i), \ell_D; (-\infty, +L]) &+ \operatorname{Mas}(\ell^-(-\infty; \lambda_i), \ell_L(\cdot; \lambda_i); (-\infty, +L]) \\ &= \operatorname{Mas}(\ell^-(\cdot; \lambda_i), \ell_L(\cdot; \lambda_i); (-\infty, +L]). \end{aligned}$$

For the Maslov index on the right-hand side of this relation, crossing points indicate that λ_i is an eigenvalue for the Schrödinger system on the half-line with Dirichlet boundary condition at $x = L$. However, by monotonicity for the Dirichlet boundary condition, we've seen that the spectral curves associated with this problem will have asymptotes at the eigenvalues of H and consequently we see that if λ_i is not an eigenvalue of H we can take L sufficiently large so that λ_i is not an eigenvalue of this half-line problem. We can conclude that

$$\operatorname{Mas}(\ell^-(\cdot; \lambda_i), \ell_L(\cdot; \lambda_i); (-\infty, +L]) = 0,$$

so that

$$\operatorname{Mas}(\ell^-(\cdot; \lambda_i), \ell_D; (-\infty, +L]) = - \operatorname{Mas}(\ell^-(-\infty; \lambda_i), \ell_L(\cdot; \lambda_i); (-\infty, +L]).$$

To complete the argument, we now use Hörmander's index again to replace the target $\ell^-(-\infty; \lambda_i)$ in this last Maslov index with ℓ_D. For this, we have, as usual (i.e., using (3.3)),

$$\begin{aligned} \operatorname{Mas}(\ell^-(-\infty; \lambda_i), \ell_L(\cdot; \lambda_i); (-\infty, +L]) &- \operatorname{Mas}(\ell_D, \ell_L(\cdot; \lambda_i); (-\infty, +L]) \\ &= s(\ell_L(-\infty; \lambda_i), \ell_D; \ell_D, \ell^-(-\infty; \lambda_i)). \end{aligned}$$

We can compute Hörmander's index with

$$\begin{aligned} s(\ell_L(-\infty; \lambda_i), &\ell_D; \ell_D, \ell^-(-\infty; \lambda_i)) \\ &= \mathcal{I}(\ell^-(-\infty; \lambda_i); \mathbf{X}_L(-\infty; \lambda_i), \mathbf{X}_D) - \mathcal{I}(\ell_D; \mathbf{X}_L(-\infty; \lambda_i), \mathbf{X}_D). \end{aligned}$$

Using the frames

$$\mathbf{X}^-(-\infty;\lambda_i) = \begin{pmatrix} I \\ R^- D^-(\lambda_i)(R^-)^* \end{pmatrix}; \quad \text{and} \quad \mathbf{X}_L(-\infty;\lambda_i) = \begin{pmatrix} I \\ -R^- D^-(\lambda_i)(R^-)^* \end{pmatrix},$$

we compute

$$\mathcal{I}(\ell^-(-\infty;\lambda_i); \mathbf{X}_L(-\infty;\lambda_i), \mathbf{X}_D) = -n_-(2R^- D^-(\lambda_i)(R^-)^*) - n_0(2R^- D^-(\lambda_i)(R^-)^*) = 0,$$

and likewise

$$\mathcal{I}(\ell_D; \mathbf{X}_L(-\infty;\lambda_i), \mathbf{X}_D) = -n_-(R^- D^-(\lambda_i)(R^-)^*) - n_0(R^- D^-(\lambda_i)(R^-)^*) = 0.$$

(In both cases, the result follows from the positivity of the matrices $R^- D^-(\lambda_i)(R^-)^*$.)

In this way, we see that for each of $i \in \{1,2\}$ we have the equality

$$\operatorname{Mas}(\ell^-(-\infty;\lambda_i), \ell_L(\cdot;\lambda_i); (-\infty,+L]) = \operatorname{Mas}(\ell_D, \ell_L(\cdot;\lambda_i); (-\infty,+L]),$$

so that (5.19) can be expressed as

$$\mathcal{N}((\lambda_1,\lambda_2)) = \operatorname{Mas}(\ell_D, \ell_L(\cdot;\lambda_2); (-\infty,+L]) - \operatorname{Mas}(\ell_D, \ell_L(\cdot;\lambda_1); (-\infty,+L]). \tag{5.21}$$

Last, we can choose $K > 0$ sufficiently large so that for each of $i \in \{1,2\}$, $\ell_D \cap \ell_L(x;\lambda_i) = \{0\}$ for all $x < -K$. This allows us to write

$$\mathcal{N}((\lambda_1,\lambda_2)) = \operatorname{Mas}(\ell_D, \ell_L(\cdot;\lambda_2); [-K,+L]) - \operatorname{Mas}(\ell_D, \ell_L(\cdot;\lambda_1); [-K,+L]). \tag{5.22}$$

The right-hand side of (5.22) is now a count of the number of eigenvalues on (λ_1,λ_2) for the bounded-interval problem

$$-\phi'' + V(x)\phi = \lambda\phi; \quad x \in (-K,L) \quad \phi(-K) = 0,\, \phi(L) = 0. \tag{5.23}$$

Much more general results along these lines have been obtained in [5, 41], and we only give the analysis here as an illustration of our general theory.

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