# The Maslov and Morse Indices for Sturm-Liouville Systems on the Half-Line

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September 10, 2019

**Abstract**

We show that for Sturm-Liouville Systems on the half-line $[0, \infty)$, the Morse index can be expressed in terms of the Maslov index and an additional term associated with the boundary conditions at $x = 0$. Relations are given both for the case in which the target Lagrangian subspace is associated with the space of $L^2((0, \infty), \mathbb{C}^n)$ solutions to the Sturm-Liouville System, and the case in which the target Lagrangian subspace is associated with the space of solutions satisfying the boundary conditions at $x = 0$. In the former case, a formula of Hörmander's is used to show that the target space can be replaced with the Dirichlet space, along with additional explicit terms. We illustrate our theory by applying it to an eigenvalue problem that arises when the nonlinear Schrödinger equation on a star graph is linearized about a half-soliton solution.

## 1 Introduction

We consider Sturm-Liouville systems

$$-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi, \quad x \in (0, \infty), \tag{1.1}$$

with the one-sided self-adjoint boundary conditions

$$\alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) = 0. \tag{1.2}$$

Here, $\phi = \phi(x; \lambda) \in \mathbb{C}^n$, and we assume:

**(A1)** The matrices $P(x)$, $V(x)$, and $Q(x)$ are defined and self-adjoint for a.e. $x \in (0, \infty)$, with also $P \in AC_{\text{loc}}([0, \infty), \mathbb{C}^{n\times n})$ and $V(\cdot), Q(\cdot) \in L^1_{\text{loc}}([0, \infty), \mathbb{C}^{n\times n})$. Moreover, there exist constants $\theta_P, \theta_Q > 0$ and $C_V \geq 0$ so that for any $v \in \mathbb{C}^n$

$$(P(x)v, v) \geq \theta_P|v|^2; \quad (Q(x)v, v) \geq \theta_Q|v|^2; \quad |(V(x)v, v)| \leq C_V|v|^2$$

for a.e. $x \in (0, \infty)$. Here, $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{C}^n$, and $|\cdot|$ denotes the standard norm on the same space. We emphasize that $x = 0$ is included in our local designations, so the boundary condition at $x = 0$ is regular.

**(A2)** We assume that $P$, $V$, and $Q$ all approach well-defined asymptotic endstates at exponential rate. That is, we assume there exist self-adjoint matrices $P_+, V_+, Q_+ \in \mathbb{C}^{n\times n}$, with $P_+, Q_+$ positive definite, and constants $C$ and $\eta > 0$ so that

$$|P(x) - P_+| \le Ce^{-\eta|x|}, \quad \text{a.e. } x \in (0,\infty),$$

and similarly for $V(x)$ and $Q(x)$. In addition, we assume $|P'(x)| \le Ce^{-\eta|x|}$ for a.e. $x \in (0,\infty)$.

**(A3)** For the boundary conditions, we take $\alpha_1, \alpha_2 \in \mathbb{C}^{n\times n}$, and for notational convenience, set $\alpha = (\alpha_1\ \alpha_2)$. We assume $\operatorname{rank}\alpha = n$, $\alpha J \alpha^* = 0$, which is equivalent to self-adjointness in this case. Here, $J$ denotes the standard symplectic matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with $I_n$ denoting the usual $n \times n$ identity matrix.

We can think of (1.1) in terms of the operator

$$\mathcal{L}\phi = Q(x)^{-1}\{-(P(x)\phi')' + V(x)\phi\},$$

with which we associate the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}) = \{\phi \in L^2((0,\infty),\mathbb{C}^n) : \phi, \phi' \in AC_{\text{loc}}([0,\infty),\mathbb{C}^n), \\ \mathcal{L}\phi \in L^2((0,\infty),\mathbb{C}^n),\ \alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) = 0\}, \end{aligned}$$

and the inner product

$$\langle \phi, \psi \rangle_Q := \int_0^\infty (Q(x)\phi(x), \psi(x))_{\mathbb{C}^n} dx.$$

With this choice of domain and inner product, it's verified in [26] that $\mathcal{L}$ is densely defined, closed, and self-adjoint, so $\sigma(\mathcal{L}) \subset \mathbb{R}$.

Our particular interest lies in counting the number of negative eigenvalues of $\mathcal{L}$ (i.e., the Morse index). We proceed by relating the Morse index to the Maslov index, which is described in Section 3. We find that the Morse index can be computed in terms of the Maslov index and an additional term associated with the boundary condition at $x = 0$.

Our relatively strong assumptions on the coefficient matrices $P(x)$, $V(x)$, and $Q(x)$ are motivated by two primary concerns. First, the immediate applications we have in mind arise when a nonlinear evolutionary PDE is linearized about a stationary solution that approaches a fixed endstate at exponential rate as $x$ increases to $\infty$. In this setting, our assumptions are often naturally met, as is the case for the application we discuss in Section 6. Second, in order to develop the straightforward *Maslov = Morse* results that we state in Theorems 1.1 and 1.2, we require well-defined frames (as described below) for our Lagrangian subspaces at the endpoints (i.e., at $x = 0$ and in the limit $x \to \infty$). Such well-defined frames are guaranteed respectively by the assumption that our problem is regular at $x = 0$, and by the asymptotic conditions stated in **(A2)**. These assumptions place our analysis directly in the framework of our motivating reference [16] and broadly in the framework of analyses based on the notion of exponential dichotomy (e.g., [1, 3, 22]).

More generally, it's well known that the spectrum of Sturm-Liouville systems on the half-line can be studied under substantially milder assumptions than those taken here (see, e.g., [9, 17, 26]). Of particular relevance to the current analysis, we mention the *renormalized oscillation* approach introduced in [23], which is well-suited for accommodating mild assumptions on the Sturm-Liouville coefficient matrices (see also [7, 24, 25]). As shown in [14], renormalized oscillation theory can be formulated in a natural way via the Maslov index. We won't pursue such considerations further in the current work.

As a starting point for our analysis, we define what we will mean by a *Lagrangian subspace* of $\mathbb{C}^{2n}$. For comments about working in $\mathbb{C}^{2n}$ rather than $\mathbb{R}^{2n}$, the reader is referred to Remark 1.1 of [14], and the references mentioned in that remark.

**Definition 1.1.** *We say $\ell \subset \mathbb{C}^{2n}$ is a Lagrangian subspace of $\mathbb{C}^{2n}$ if $\ell$ has dimension $n$ and*

$$(Ju, v)_{\mathbb{C}^{2n}} = 0, \tag{1.3}$$

*for all $u, v \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{C}^{2n}}$ denotes the standard inner product on $\mathbb{C}^{2n}$. In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{C}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.*

Any Lagrangian subspace of $\mathbb{C}^{2n}$ can be spanned by a choice of $n$ linearly independent vectors in $\mathbb{C}^{2n}$. We will generally find it convenient to collect these $n$ vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for $\ell$. Moreover, we will often coordinatize our frames as $\mathbf{X} = \binom{X}{Y}$, where $X$ and $Y$ are $n \times n$ matrices. Following [6] (p. 274), we specify a metric on $\Lambda(n)$ in terms of appropriate orthogonal projections. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for $\ell_i$, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^* \mathbf{X}_i)^{-1} \mathbf{X}_i^*$. We take our metric $d$ on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that a path of Lagrangian subspaces $\ell : \mathcal{I} \to \Lambda(n)$ is continuous provided it is continuous under the metric $d$.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote continuous paths of Lagrangian subspaces $\ell_i : \mathcal{I} \to \Lambda(n)$, for some parameter interval $\mathcal{I}$. The Maslov index associated with these paths, which we will denote $\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I})$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (In this setting, if we let $t_*$ denote the point of intersection (often referred to as a *conjugate point*), then multiplicity corresponds with the dimension of the intersection $\ell_1(t_*) \cap \ell_2(t_*)$; a precise definition of what we mean in this context by *direction* will be given in Section 3.)

In order to place our analysis in the usual Hamiltonian framework, we express (1.1) as a first order system for $y = \binom{y_1}{y_2}$, with $y_1 = \phi$ and $y_2 = P(x)\phi'$. We find

$$y' = \mathbb{A}(x; \lambda) y, \tag{1.4}$$

where

$$\mathbb{A}(x; \lambda) = \begin{pmatrix} 0 & P(x)^{-1} \\ V(x) - \lambda Q(x) & 0 \end{pmatrix},$$

which can be expressed in the standard linear Hamiltonian form

$$Jy' = \mathbb{B}(x; \lambda) y,$$

with

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V(x) & 0 \\ 0 & P(x)^{-1} \end{pmatrix}.$$

Let $\mathbf{X}_1(x;\lambda) \in \mathbb{C}^{2n\times n}$ denote the matrix solution to

$$\begin{aligned} J\mathbf{X}_1' &= \mathbb{B}(x;\lambda)\mathbf{X}_1 \\ \mathbf{X}_1(0;\lambda) &= J\alpha^*. \end{aligned} \tag{1.5}$$

We will verify in Section 4 that for each $(x,\lambda) \in [0,\infty) \times \mathbb{R}$, $\mathbf{X}_1(x;\lambda)$ is the frame for a Lagrangian subspace of $\mathbb{C}^{2n}$, $\ell_1(x;\lambda)$. Likewise, let $\mathbf{X}_2(x;\lambda) \in \mathbb{C}^{2n\times n}$ denote a matrix solution to

$$\begin{aligned} J\mathbf{X}_2' &= \mathbb{B}(x;\lambda)\mathbf{X}_2 \\ \mathbf{X}_2(\cdot;\lambda) &\in L^2((0,\infty),\mathbb{C}^{2n}), \end{aligned} \tag{1.6}$$

such that $\ker \mathbf{X}_2(x;\lambda) = \{0\}$ for all $x \in [0,\infty)$. We will verify in Section 2 that for

$$\kappa := \inf_{r\in\mathbb{C}^n\setminus\{0\}} \frac{(V_+r,r)}{(Q_+r,r)}, \tag{1.7}$$

we have $\sigma_{\text{ess}}(\mathcal{L}) = [\kappa,\infty)$, where $\sigma_{\text{ess}}(\cdot)$ denotes essential spectrum, as defined in Section 2. Subsequently, we verify in Section 4 that for each $(x,\lambda) \in [0,\infty) \times (-\infty,\kappa)$, $\mathbf{X}_2(x;\lambda)$ is the frame for a Lagrangian subspace of $\mathbb{C}^{2n}$, $\ell_2(x;\lambda)$, and moreover that for any $\lambda \in (-\infty,\kappa)$, the asymptotic space

$$\ell_2^+(\lambda) := \lim_{x\to\infty} \ell_2(x;\lambda)$$

is well-defined and Lagrangian (with convergence in the metric $d$ described above). Finally, we will establish that the map $\ell_2^+ : (-\infty,\kappa) \to \Lambda(n)$ is continuous.

There are two different ways in which we can formulate a relation between the Maslov index and the Morse index, depending upon whether we view $x = 0$ as our target or $x = +\infty$ as our target. We state these results respectively as Theorems 1.1 and 1.2. Prior to these statements, we set some terminology with the following lemma.

**Lemma 1.1.** *Let Assumptions* **(A1)***,* **(A2)***, and* **(A3)** *hold, and let $\Lambda_\infty \in \mathbb{R}$. Then there exists $\lambda_\infty > \Lambda_\infty$ so that*

$$\ell_1(0;-\lambda_\infty) \cap \ell_2^+(-\lambda_\infty) = \{0\}.$$

*In this case, we refer to $\lambda_\infty$ as boundary inconjugate.*

We emphasize that for any $\lambda_\infty \in \mathbb{R}$, $\ell_1(0;-\lambda_\infty)$ is the Lagrangian subspace with frame $\mathbf{X}_1(0;-\lambda_\infty) = J\alpha^*$, independent of $\lambda_\infty$. Likewise, in Theorems 1.1 and 1.2 below, the Lagrangian subspace $\ell_1(0;\lambda_0)$ is independent of $\lambda_0$. In all such cases, the appearance of a spectral coordinate is only for notational consistency, since $\ell_1(x;\lambda)$ does in general depend on $\lambda$ for all $x > 0$.

In the following statements, we use the notation $\text{Mor}(\mathcal{L};\lambda_0)$ to indicate the number of eigenvalues that $\mathcal{L}$ has, including multiplicities, on the interval $(-\infty,\lambda_0)$.

**Theorem 1.1.** *Let Assumptions* **(A1)**, **(A2)**, *and* **(A3)** *hold, and fix any $\lambda_0 < \kappa$ (with $\kappa$ defined in (1.7)). Then there exists a value $\Lambda_\infty$ sufficiently large so that for any boundary inconjugate $\lambda_\infty > \Lambda_\infty$, we have*

$$\operatorname{Mor}(\mathcal{L}; \lambda_0) = \operatorname{Mas}(\ell_1(0; \lambda_0), \ell_2(\cdot; \lambda_0); [0, \infty]) - \operatorname{Mas}(\ell_1(0; \cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0]).$$

**Theorem 1.2.** *Let Assumptions* **(A1)**, **(A2)**, *and* **(A3)** *hold, and fix any $\lambda_0 < \kappa$ (with $\kappa$ defined in (1.7)). Then there exists a value $\Lambda_\infty$ sufficiently large so that for any boundary inconjugate $\lambda_\infty > \Lambda_\infty$, we have*

$$\operatorname{Mor}(\mathcal{L}; \lambda_0) = -\operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2^+(\lambda_0); [0, \infty]) - \operatorname{Mas}(\ell_1(0; \cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0]).$$

**Remark 1.1.** *Our notation $[0, \infty]$ in the specification of the first Maslov index on the right-hand side in each theorem indicates that we allow the possibility for a conjugate point to be obtained in the limit as $x \to \infty$. We note that this limiting point will be conjugate if and only if $\lambda_0$ is an eigenvalue for the equation.*

*For Theorem 1.2, the target space $\ell_2^+(\lambda_0)$ can be replaced by the Dirichlet space $\ell_D$ (with frame $\mathbf{X}_D = \binom{0}{I_n}$), at the cost of additional terms that can be expressed explicitly. See Corollary 5.1 in Section 5.1. We also note that by combining Theorems 1.1 and 1.2 we see that*

$$\operatorname{Mas}(\ell_1(0; \lambda_0), \ell_2(\cdot; \lambda_0); [0, \infty]) = -\operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2^+(\lambda_0); [0, \infty]).$$

# 2 ODE Preliminaries

In this section, we develop preliminary ODE results that will serve as the foundation of our analysis. This development is standard, and follows [27], pp. 779-781 (see, e.g., [4] for similar analyses). We begin by clarifying our terminology.

**Definition 2.1.** *We define the point spectrum of $\mathcal{L}$, denoted $\sigma_{\rm pt}(\mathcal{L})$, as the set*

$$\sigma_{\rm pt}(\mathcal{L}) = \{\lambda \in \mathbb{R} : \mathcal{L}\phi = \lambda\phi \text{ for some } \phi \in \mathcal{D}(\mathcal{L})\backslash\{0\}\}.$$

*Elements of the point spectrum will be referred to as eigenvalues. We define the essential spectrum of $\mathcal{L}$, denoted $\sigma_{\rm ess}(\mathcal{L})$, as the values in $\mathbb{C}$ (and so $\mathbb{R}$, by self-adjointness) that are not in the resolvent set of $\mathcal{L}$ and are not isolated eigenvalues of finite multiplicity.*

We note that the total spectrum is $\sigma(\mathcal{L}) = \sigma_{\rm pt}(\mathcal{L}) \cup \sigma_{\rm ess}(\mathcal{L})$, and the *discrete* spectrum is defined as $\sigma_{\rm discrete}(\mathcal{L}) = \sigma(\mathcal{L})\backslash\sigma_{\rm ess}(\mathcal{L})$. Since our analysis takes place entirely away from essential spectrum, the eigenvalues we are counting are elements of the discrete spectrum.

If we consider (1.1) as $x \to \infty$, we obtain the asymptotic system

$$-P_+\phi'' + V_+\phi = \lambda Q_+\phi. \tag{2.1}$$

For operators such as $\mathcal{L}$ posed on $\mathbb{R}$, it's well-known that the essential spectrum is entirely determined by the associated asymptotic problems at $\pm\infty$ (see, for example, in [8, 15]). As we will verify at the end of this section, it's straightforward to show that a similar result holds true in the current setting. In particular, if we look for solutions of (2.1) of the form

$\phi(x) = e^{ikx}r$, for some scalar constant $k \in \mathbb{R}$ and (non-zero) constant vector $r \in \mathbb{C}^n$ then the essential spectrum will be confined to the allowable values of $\lambda$. For (2.1), we find

$$(k^2 P_+ + V_+)r = \lambda Q_+ r,$$

and upon taking an inner product with $r$ we see that

$$k^2(P_+ r, r) + (V_+ r, r) = \lambda(Q_+ r, r).$$

Since $P_+$ and $Q_+$ are positive definite, we see that

$$\lambda(k) \geq \frac{(V_+ r, r)}{(Q_+ r, r)},$$

for all $k \in \mathbb{R}$, and consequently $\sigma_{\text{ess}}(\mathcal{L}) \subset [\kappa, \infty)$, where

$$\kappa = \inf_{r \in \mathbb{C}^n \setminus \{0\}} \frac{(V_+ r, r)}{(Q_+ r, r)} > 0.$$

In order to describe the Lagrangian subspaces $\ell_2(x; \lambda)$, we need to characterize the solutions of (1.6) in $L^2((0, \infty), \mathbb{C}^{2n})$. As a starting point for this characterization, we fix any $\lambda < \kappa$ and look for solutions of (2.1) of the form $\phi(x; \lambda) = e^{\mu x} r$, where in this case $\mu$ is a scalar function of $\lambda$, and $r$ is a vector function of $\lambda$ (in $\mathbb{C}^n$). Computing directly, we obtain the relation

$$(-\mu^2 P_+ + V_+ - \lambda Q_+)r = 0,$$

which we can rearrange as

$$P_+^{-1}(V_+ - \lambda Q_+)r = \mu^2 r.$$

Since $P_+$ is positive definite, we can work with the inner product

$$(r, s)_+ := (P_+ r, s)_{\mathbb{C}^n}, \tag{2.2}$$

and it's clear that for $\lambda \in \mathbb{R}$, the operator $P_+^{-1}(V_+ - \lambda Q_+)$ is self-adjoint with respect to this inner product, and moreover positive definite for $\lambda < \kappa$. We conclude that for $\lambda < \kappa$, the eigenvalues $\mu^2$ will be positive real values, and that the associated eigenvectors can be chosen to be orthonormal with respect to the inner product (2.2). For each of the $n$ values of $\mu^2$ (counted with multiplicity), we can associate two values $\pm\sqrt{\mu^2}$. By a choice of labeling, we can split these values into $n$ negative values $\{\mu_k(\lambda)\}_{k=1}^n$ and $n$ positive values $\{\mu_k(\lambda)\}_{k=n+1}^{2n}$ with the correspondence (again, by labeling convention)

$$\mu_k(\lambda) = -\mu_{2n+1-k}(\lambda); \quad k = 1, 2, \dots, n.$$

For $k = 1, 2, \dots, n$, we denote by $r_k(\lambda)$ the eigenvector of $P_+^{-1}(V_+ - \lambda Q_+)$ with associated eigenvalue $\mu_k^2 = \mu_{2n+1-k}^2$. I.e.,

$$P_+^{-1}(V_+ - \lambda Q_+)r_k = \mu_k^2 r_k; \quad k = 1, 2, \dots, n.$$

Recalling (1.4), we note that under our asymptotic assumptions on $P(x)$, $Q(x)$, and $V(x)$, the limit

$$\mathbb{A}_+(\lambda) := \lim_{x \to +\infty} \mathbb{A}(x; \lambda)$$

is well-defined. The values $\{\mu_k\}_{k=1}^{2n}$ described above comprise a labeling of the eigenvalues of $\mathbb{A}_+(\lambda)$. By self-adjointness, each of these eigenvalues is semi-simple, and so we can associate them with a choice of eigenvectors $\{\mathbf{r}_k\}_{k=1}^{2n}$ so that

$$\mathbb{A}_+(\lambda)\mathbf{r}_k(\lambda) = \mu_k(\lambda)\mathbf{r}_k(\lambda), \quad k \in \{1, 2, \dots, 2n\}.$$

We see that for $k = 1, 2, \dots, n$, we have relations

$$\mathbf{r}_k = \begin{pmatrix} r_k \\ \mu_k P_+ r_k \end{pmatrix}; \quad \mathbf{r}_{n+k} = \begin{pmatrix} r_k \\ -\mu_k P_+ r_k \end{pmatrix}.$$

If we set

$$R(\lambda) := \begin{pmatrix} r_1(\lambda) & r_2(\lambda) & \dots & r_n(\lambda) \end{pmatrix}, \tag{2.3}$$

and

$$D(\lambda) = \operatorname{diag}\begin{pmatrix} \mu_1(\lambda) & \mu_2(\lambda) & \dots & \mu_n(\lambda) \end{pmatrix}, \tag{2.4}$$

then we can express a frame for the eigenspace of $\mathbb{A}_+(\lambda)$ associated with negative eigenvalues as $\mathbf{X}_2^+ = \binom{R}{P_+ RD}$, and likewise we can express a frame for the eigenspace of $\mathbb{A}_+(\lambda)$ associated with positive eigenvalues as $\tilde{\mathbf{X}}_2^+ = \binom{R}{-P_+ RD}$.

**Lemma 2.1.** *Assume* **(A1)** *and* **(A2)**, *and let* $\{\mu_k(\lambda)\}_{k=1}^{2n}$ *and* $\{\mathbf{r}_k(\lambda)\}_{k=1}^{2n}$ *be as described above. Then there exists a* $\lambda$*-dependent family of bases* $\{\mathbf{y}_k(\cdot;\lambda)\}_{k=1}^{n}$, $\lambda \in (-\infty, \kappa)$, *for the spaces of* $L^2((0,\infty), \mathbb{C}^{2n})$ *solutions of (1.4), chosen so that*

$$\mathbf{y}_k(x;\lambda) = e^{\mu_k(\lambda)x}(\mathbf{r}_k(\lambda) + \mathbf{E}_k(x;\lambda)); \quad k = 1, 2, \dots, n,$$

*where*

$$\mathbf{E}_k(x;\lambda) = \mathbf{O}(e^{-\tilde{\eta}x})$$

*for some* $\tilde{\eta} > 0$, *and where the* $\mathbf{O}(\cdot)$ *term is uniform for* $\lambda \in (-\infty, \tilde{\kappa}]$ *for any* $\tilde{\kappa} < \kappa$.

*Moreover, a basis* $\{\mathbf{y}_k(\cdot;\lambda)\}_{k=n+1}^{2n}$ *for the space of non-*$L^2((0,\infty), \mathbb{C}^n)$ *solutions of (1.4) can be chosen so that*

$$\mathbf{y}_k(x;\lambda) = e^{\mu_k(\lambda)x}(\mathbf{r}_{3n+1-k}(\lambda) + \mathbf{E}_k(x;\lambda)); \quad k = n+1, n+2, \dots, 2n$$

*with* $\{\mathbf{E}_k(x;\lambda))\}_{k=n+1}^{2n}$ *satisfying the same properties as* $\{\mathbf{E}_k(x;\lambda))\}_{k=1}^{n}$.

*Finally, for each* $k \in \{1, 2, \dots, 2n\}$ *and each* $x > 0$, $\mathbf{y}_k(x;\cdot) \in C^1((-\infty,\kappa), \mathbb{C}^{2n})$.

*Proof.* For any $\lambda < \kappa$, we follow [27] and write (1.4) as

$$y' = \mathbb{A}_+ y + \mathbb{E}(x;\lambda)y, \tag{2.5}$$

where

$$\mathbb{E}(x;\lambda) = \mathbb{A}(x;\lambda) - \mathbb{A}_+(\lambda) = \mathbf{O}(e^{-\eta x}).$$

We can now fix a particular index $k \in \{1, 2, \dots, n\}$, and look for solutions to (2.5) of the form

$$y(x;\lambda) = e^{\mu_k(\lambda)x} z(x;\lambda),$$

for which

$$z' = (\mathbb{A}_+(\lambda) - \mu_k(\lambda))z + \mathbb{E}(x;\lambda)z. \tag{2.6}$$

Based on $\eta$, let $\eta_1, \eta_2 \in \mathbb{R}_+$ satisfy $0 < \eta_1 < \eta_2 < \eta$. Then there exists a neighborhood of $\lambda$ on which we can define a continuous projector $\mathcal{P}_k(\lambda)$ onto the direct sum of all eigenspaces of $\mathbb{A}_+(\lambda)$ with eigenvalues $\tilde{\mu}$ satisfying $\tilde{\mu} < \mu_k - \eta_1$, and likewise a projector $\mathcal{Q}_k(\lambda) = I - \mathcal{P}_k(\lambda)$ projecting onto the direct sum of all eigenspaces of $\mathbb{A}_+(\lambda)$ with eigenvalues $\tilde{\mu}$ satisfying

$$\tilde{\mu} \geq \mu_k - \eta_1 > \mu_k - \eta_2.$$

For some fixed value $M > 0$ taken sufficiently large, we will look for solutions to (2.6) of the form

$$\begin{aligned} z(x;\lambda) = \mathbf{r}_k(\lambda) &+ \int_M^x e^{(\mathbb{A}_+(\lambda) - \mu_k(\lambda)I)(x-y)} \mathcal{P}_k(\lambda)\mathbb{E}(y;\lambda)z(y;\lambda)dy \\ &- \int_x^{+\infty} e^{(\mathbb{A}_+(\lambda) - \mu_k(\lambda)I)(x-y)} \mathcal{Q}_k(\lambda)\mathbb{E}(y;\lambda)z(y;\lambda)dy. \end{aligned} \tag{2.7}$$

We proceed by contraction mapping, defining an operator action $\mathcal{T}z$ as the right-hand side of (2.7). For this, we use the following fact, which is immediate from the definitions of $\mathcal{P}_k$ and $\mathcal{Q}_k$ : there exist constants $C_1$ and $C_2$ so that

$$\begin{aligned} &|e^{(\mathbb{A}_+(\lambda) - \mu_k(\lambda)I)(x-y)} \mathcal{P}_k(\lambda)| \leq C_1 e^{-\eta_1(x-y)}, \quad x > y \\ &|e^{(\mathbb{A}_+(\lambda) - \mu_k(\lambda)I)(x-y)} \mathcal{Q}_k(\lambda)| \leq C_2 e^{+\eta_2(y-x)}, \quad x < y. \end{aligned} \tag{2.8}$$

We check that for $M$ sufficiently large $\mathcal{T}$ is a contraction on the space $L^\infty((M,\infty),\mathbb{C}^{2n})$. To see this, we note that given any $z, w \in L^\infty((M,\infty),\mathbb{C}^{2n})$, there exist constants $C_3$ and $C_4$ so that

$$\begin{aligned} |\mathcal{T}(z-w)| &\leq C_3 \int_M^x e^{-\eta_1(x-y)} e^{-\eta y} |z(y) - w(y)| dy + C_4 \int_x^\infty e^{-\eta_2(x-y)} e^{-\eta y} |z(y) - w(y)| dy \\ &\leq \|z-w\|_{L^\infty((M,\infty),\mathbb{C}^{2n})} \Big\{ C_3 \int_M^x e^{-\eta_1(x-y)} e^{-\eta y} dy + C_4 \int_x^\infty e^{-\eta_2(x-y)} e^{-\eta y} dy \Big\} \\ &\leq \|z-w\|_{L^\infty((M,\infty),\mathbb{C}^{2n})} \Big\{ C_3 \frac{e^{-\eta_1(x-M) - \eta M} - e^{-\eta x}}{\eta - \eta_1} + C_4 \frac{e^{-\eta x}}{\eta - \eta_2} \Big\}. \end{aligned}$$

Combining terms, we see that for some constant $C_5$,

$$\|\mathcal{T}(z-w)\|_{L^\infty((M,\infty),\mathbb{C}^{2n})} \leq \|z-w\|_{L^\infty((M,\infty),\mathbb{C}^{2n})} C_5 e^{-\eta M},$$

from which it's clear that by taking $M$ sufficiently large, we can ensure that we have a contraction. Invariance of $\mathcal{T}$ on $L^\infty((M,\infty),\mathbb{C}^{2n})$ follows similarly, and we conclude that there exists a unique $z \in L^\infty((M,\infty),\mathbb{C}^{2n})$ satifying (2.7). Upon direct differentiation of (2.7), we see that $z$ solves (2.6). Solutions to (2.6) are absolutely continuous, so in fact $z \in AC_{\text{loc}}([M,\infty),\mathbb{C}^n)$. But then we can continue $z$ from $M$ back to 0 by standard ODE continuation, so that we have $z \in AC_{\text{loc}}([0,\infty),\mathbb{C}^n)$.

We can now substitute $z$ back into (2.7) to obtain the asymptotic estimates we're after. Proceeding similarly as in our verification that $\mathcal{T}$ is a contraction, we find that

$$z(x) = \mathbf{r}_k(\lambda) + \mathbf{O}(e^{-\eta_1 x}).$$

Finally, differentiability in $\lambda$ is obtained by differentiating (2.7) with respect to $\lambda$ and proceeding with a similar argument for the resulting integral equation. $\square$

We see from Lemma 2.1 that for each fixed $\lambda \in (-\infty, \kappa)$, we can create a frame for the Lagrangian subspace of $L^2((0,\infty), \mathbb{C}^{2n})$ solutions of (1.4), namely

$$\mathbf{X}_2(x;\lambda) = \begin{pmatrix} \mathbf{y}_1(x;\lambda) & \mathbf{y}_2(x;\lambda) & \cdots & \mathbf{y}_n(x;\lambda) \end{pmatrix}.$$

If we set

$$\mathcal{D}(x;\lambda) = \operatorname{diag}\begin{pmatrix} e^{\mu_1(\lambda)x} & e^{\mu_2(\lambda)x} & \dots & e^{\mu_n(\lambda)x} \end{pmatrix},$$

then $\mathbf{X}_2(x;\lambda)$ can be replaced by the frame $\mathbf{X}_2(x;\lambda)\mathcal{D}(x;\lambda)^{-1}$. From this latter frame, it's clear that we can take $x \to \infty$ to obtain an asymptotic frame $\mathbf{X}_2^+$ comprising the eigenvectors $\{\mathbf{r}_k\}_{k=1}^n$ as its columns.

We can now verify directly that

$$\sigma_{\text{ess}}(\mathcal{L}) = [\kappa, \infty).$$

First, for $\lambda < \kappa$, we can directly construct a Green's function $G_\lambda(x,\xi)$ satisfying $\mathcal{L}G_\lambda(x,\xi) = \delta_\xi(x)$. In particular, we obtain

$$G_\lambda(x,\xi) = \begin{cases} X_1(x;\lambda)\mathcal{M}(\lambda)X_2(\xi;\lambda)^*Q(\xi) & 0 < x < \xi \\ X_2(x;\lambda)\mathcal{M}(\lambda)^*X_1(\xi;\lambda)^*Q(\xi) & 0 < \xi < x, \end{cases}$$

where

$$\mathcal{M}(\lambda) = -(\mathbf{X}_2(\xi;\lambda)^* J \mathbf{X}_1(\xi;\lambda))^{-1}.$$

(The verification that $\mathcal{M}(\lambda)$ is independent of $\xi$ proceeds almost precisely as the verification below that $\mathbf{X}_1(\xi;\lambda)$ and $\mathbf{X}_2(\xi;\lambda)$ are Lagrangian subspaces.)

According to Lemma 2.2 in [14], for $\lambda < \kappa$, $\mathcal{M}(\lambda)$ exists if and only if the Lagrangian subspaces $\ell_1(\xi;\lambda)$ and $\ell_2(\xi;\lambda)$ do not intersect, and these Lagrangian subspaces intersect if and only if $\lambda$ is an eigenvalue of $\mathcal{L}$ (i.e., an element of the point spectrum). Moreover, for $\lambda < \kappa$, the frames $\mathbf{X}_1(\xi;\lambda)$ and $\mathbf{X}_2(\xi;\lambda)$ are analytic in $\lambda$ (see, e.g., Theorem 2.1 in [26], and this can also be seen with an approach essentially identical to our proof of Lemma 2.1). It follows that $\mathcal{M}(\lambda)$ is meromorphic in $\lambda < \kappa$, and so there can be no accumulation of eigenvalues on this interval. This allows us to conclude in fact that for $\lambda < \kappa$, $\mathcal{M}(\lambda)$ can only fail to exist if $\lambda \in \sigma_{\text{discrete}}(\mathcal{L})$.

In the case that $\mathcal{M}(\lambda)$ exists, it can be shown (e.g., as in the proof of Proposition 7.1 in [27]) that there exist constants $C(\lambda) > 0$, $c(\lambda) > 0$ so that

$$|G_\lambda(x,\xi)| \le C(\lambda)e^{-c(\lambda)|x-\xi|}$$

for all $0 \le x, \xi < \infty$. We can conclude that for any $\lambda < \kappa$ that is not an eigenvalue of $\mathcal{L}$, the resolvent map

$$(\mathcal{L} - \lambda I)^{-1} f = \int_0^\infty G_\lambda(x,\xi) f(\xi) d\xi$$

defines a bounded, linear operator on $L^2((0,\infty), \mathbb{C}^n)$. In particular, $(-\infty, \kappa) \cap \sigma_{\text{ess}}(\mathcal{L}) = \emptyset$. We conclude that $\sigma_{\text{ess}}(\mathcal{L}) \subset [\kappa, \infty)$.

To see that $[\kappa, \infty) \subset \sigma_{\text{ess}}(\mathcal{L})$, we first note that for any $\lambda \geq \kappa$, the matrix $P_+^{-1}(V_+ - \lambda Q_+)$ will have one or more non-positive eigenvalues. It follows that $\mathbb{A}_+(\lambda)$ will have two or more eigenvalues with zero real part. The proof of Lemma 2.1 proceeds essentially unchanged in this case, and we see that for $\lambda \geq \kappa$ the space of $L^2((0,\infty), \mathbb{C}^n)$ solutions of $\mathcal{L}\phi = \lambda\phi$ has dimension less than $n$. It follows immediately from Theorem 11.4.c of [26] that $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$ in these cases.

# 3 The Maslov Index

Our framework for computing the Maslov index is adapted from Section 2 of [14], and we briefly sketch the main ideas here. Given any pair of Lagrangian subspaces $\ell_1$ and $\ell_2$ with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we consider the matrix

$$\tilde{W} := -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{3.1}$$

In [14], the authors establish: (1) the inverses appearing in (3.1) exist; (2) $\tilde{W}$ is independent of the specific frames $\mathbf{X}_1$ and $\mathbf{X}_2$ (as long as these are indeed frames for $\ell_1$ and $\ell_2$); (3) $\tilde{W}$ is unitary; and (4) the identity

$$\dim(\ell_1 \cap \ell_2) = \dim(\ker(\tilde{W} + I)). \tag{3.2}$$

Given two continuous paths of Lagrangian subspaces $\ell_i : [0,1] \to \Lambda(n)$, $i = 1, 2$, with respective frames $\mathbf{X}_i : [0,1] \to \mathbb{C}^{2n \times n}$, relation (3.2) allows us to compute the Maslov index $\text{Mas}(\ell_1, \ell_2; [0,1])$ as a spectral flow through $-1$ for the path of matrices

$$\tilde{W}(t) := -(X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_2(t) - iY_2(t))(X_2(t) + iY_2(t))^{-1}. \tag{3.3}$$

In [14], the authors provide a rigorous definition of the Maslov index based on the spectral flow developed in [21]. Here, rather, we give only an intuitive discussion. As a starting point, if $-1 \in \sigma(\tilde{W}(t_*))$ for some $t_* \in [0,1]$, then we refer to $t_*$ as a conjugate point, and its multiplicity is taken to be $\dim(\ell_1(t_*) \cap \ell_2(t_*))$, which by virtue of (3.2) is equivalent to its multiplicity as an eigenvalue of $\tilde{W}(t_*)$. We compute the Maslov index $\text{Mas}(\ell_1, \ell_2; [0,1])$ by allowing $t$ to increase from 0 to 1 and incrementing the index whenever an eigenvalue crosses $-1$ in the counterclockwise direction, while decrementing the index whenever an eigenvalue crosses $-1$ in the clockwise direction. These increments/decrements are counted with multiplicity, so for example, if a pair of eigenvalues crosses $-1$ together in the counterclockwise direction, then a net amount of $+2$ is added to the index. Regarding behavior at the endpoints, if an eigenvalue of $\tilde{W}$ rotates away from $-1$ in the clockwise direction as $t$ increases from 0, then the Maslov index decrements (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates away from $-1$ in the counterclockwise direction as $t$ increases from 0, then the Maslov index does not change. Likewise, if an eigenvalue of $\tilde{W}$ rotates into $-1$ in the counterclockwise direction as $t$ increases to 1, then the Maslov index increments (according to multiplicity), while if an eigenvalue of $\tilde{W}$ rotates into $-1$ in the clockwise direction as $t$ increases to 1, then the Maslov index does not change. Finally, it's possible that an eigenvalue of $\tilde{W}$ will arrive at $-1$ for $t = t_*$ and stay. In these cases, the Maslov index only increments/decrements upon arrival or departure, and the increments/decrements are determined as for the endpoints (departures determined as with $t = 0$, arrivals determined as with $t = 1$).

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(\mathcal{I})$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t, s) \in \mathcal{I} \times [0, 1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties.

**(P1)** (Path Additivity) If $\mathcal{L} \in \mathcal{P}(\mathcal{I})$ and $a, b, c \in \mathcal{I}$, with $a < b < c$, then

$$\operatorname{Mas}(\mathcal{L}; [a, c]) = \operatorname{Mas}(\mathcal{L}; [a, b]) + \operatorname{Mas}(\mathcal{L}; [b, c]).$$

**(P2)** (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\operatorname{Mas}(\mathcal{L}; [a, b]) = \operatorname{Mas}(\mathcal{M}; [a, b]).$$

Straightforward proofs of these properties appear in [10] for Lagrangian subspaces of $\mathbb{R}^{2n}$, and proofs in the current setting of Lagrangian subspaces of $\mathbb{C}^{2n}$ are essentially identical.

### 3.1 Exchanging Target Spaces

Suppose we have a continuous path of Lagrangian subspaces $\ell : [0, 1] \to \Lambda(n)$, along with two fixed target Lagrangian subspaces $\ell_1$ and $\ell_2$. Our goal in this section is to relate the two Maslov indices $\operatorname{Mas}(\ell, \ell_1; [0, 1])$ and $\operatorname{Mas}(\ell, \ell_2; [0, 1])$. This question goes back at least to Hörmander [12], and has also been discussed in our primary references [5] and [28].

As a starting point, it's straightforward to check that the difference

$$\operatorname{Mas}(\ell, \ell_2; [0, 1]) - \operatorname{Mas}(\ell, \ell_1; [0, 1]),$$

does not depend on the specific path $\ell : [0, 1] \to \Lambda(n)$ (see, e.g., [5, 12, 28]), and we define the *Hörmander index* $s(\ell_1, \ell_2; \ell(0), \ell(1))$ by the relation

$$s(\ell_1, \ell_2; \ell(0), \ell(1)) := \operatorname{Mas}(\ell, \ell_2; [0, 1]) - \operatorname{Mas}(\ell, \ell_1; [0, 1]). \tag{3.4}$$

With slight adjustments for notation, this is equation (2.9) in [5] and Definition 3.9 in [28]. We will evaluate the Hörmander index with an expression from [12], and for this we need to define an associated bilinear form.

**Definition 3.1.** *Fix any $\ell_1, \ell_2 \in \Lambda(n)$ with $\ell_1 \cap \ell_2 = \{0\}$. Then any $n$-dimensional linear subspace $\ell_0 \subset \mathbb{C}^{2n}$ (i.e., $\ell_0$ not necessarily Lagrangian) with $\ell_0 \cap \ell_2 = \{0\}$ can be expressed as*

$$\ell_0 = \{u + Cu : u \in \ell_1\}$$

*for some $2n \times 2n$ matrix $C$ that maps $\ell_1$ to $\ell_2$. We define a bilinear form*

$$Q = Q(\ell_1, \ell_2; \ell_0) : \ell_1 \times \ell_1 \to \mathbb{C}$$

*by the relation*

$$Q(u, v) := -(JCu, v),$$

*for all $u, v \in \ell_1$.*

**Remark 3.1.** *Although we will only utilize the bilinear forms $Q$ in combination, it's worth noting how we should interpret the meaning of an individual form. Given three Lagrangian subspaces $\ell_0$, $\ell_1$, and $\ell_2$, $Q(\ell_1, \ell_2; \ell_0)$ provides information about the relative orientation of these three spaces. For the case $n = 1$, the nature of this information is particularly clear. In that setting, we can associate to any Lagrangian subspace $\ell_j$ with frame $\mathbf{X}_j = \binom{X_j}{Y_j}$ a unique point on $S^1$,*

$$\tilde{W}_j^D = (X_j + iY_j)(X_j - iY_j)^{-1}.$$

*If $\ell_1 \cap \ell_2 = \{0\}$, then $\tilde{W}_1^D$ and $\tilde{W}_2^D$ correspond with distinct points on $S^1$. Given any third Lagrangian plane $\ell_0$ distinct from both $\ell_1$ and $\ell_2$, $\tilde{W}_0^D$ will lie either on the arc going from $\tilde{W}_1^D$ to $\tilde{W}_2^D$ in the clockwise direction or on the arc going from $\tilde{W}_1^D$ to $\tilde{W}_2^D$ in the counterclockwise direction. In the former case, we will have $\operatorname{sgn} Q(\ell_1, \ell_2; \ell_0) = -1$, while in the latter case we will have $\operatorname{sgn} Q(\ell_1, \ell_2; \ell_0) = +1$. Using this observation, we can readily derive Hörmander's formula ((3.5), just below) for the case $n = 1$, and it can subsequently be established that (3.5) is valid for $n > 1$ as well.*

*The negative sign in our specification of $Q$ is an artifact of convention: while we're taking our symplectic form to be $(Ju, v)$, our reference [5] uses $(u, Jv)$, which simply has the opposite sign.*

Hörmander's Q-form is precisely the form defined in [5], and aside from a sign convention is also the same form specified in Section 3.1 of [28]. Suppose $\ell(0)$ intersects neither $\ell_1$ nor $\ell_2$ and likewise $\ell(1)$ intersects neither $\ell_1$ nor $\ell_2$. Then if $\ell_1 \cap \ell_2 = \{0\}$, Hörmander's formula for $s(\ell_1, \ell_2; \ell(0), \ell(1))$ can be expressed as (equation (2.10) of [5])

$$s(\ell_1, \ell_2; \ell(0), \ell(1)) = \frac{1}{2}\Big( \operatorname{sgn} Q(\ell_1, \ell_2; \ell(0)) - \operatorname{sgn} Q(\ell_1, \ell_2; \ell(1))\Big), \tag{3.5}$$

where $\operatorname{sgn}(\cdot)$ denotes the usual signature of a bilinear form (number of positive eigenvalues minus the number of negative eigenvalues).

One immediate consequence of this formula is that if $\ell : [0, 1] \to \Lambda(n)$ is a closed path (i.e., $\ell(0) = \ell(1)$), then $s(\ell_1, \ell_2; \ell(0), \ell(1)) = 0$. We see that if $\ell_1 \cap \ell_2 = \{0\}$, then for any closed path so that $\ell(0)$ intersects neither $\ell_1$ nor $\ell_2$ (and so automatically $\ell(1)$ intersects neither $\ell_1$ nor $\ell_2$), the target space can be changed from $\ell_1$ to $\ell_2$ without affecting the Maslov index.

In practice, we would often prefer the Dirichlet plane $\ell_D$ as our target (e.g., for Sturm-Liouville systems, when the target is Dirichlet, all crossings will necessarily be in the same direction), and so let's check the calculation associated with exchanging a general Lagrangian target space $\ell_G$ with the Dirichlet plane. For notational convenience, we will think of this the other way around, taking $\ell_1 = \ell_D$ and $\ell_2 = \ell_G$ in our general formulation. Since the analysis of $\ell(0)$ and $\ell(1)$ are the same, we will proceed with each replaced by the general notation $\ell_0$. I.e., we will compute $Q(\ell_D, \ell_G; \ell_0)$, for which we must have $\ell_D \cap \ell_G = \{0\}$ and $\ell_0 \cap \ell_G = \{0\}$.

Our starting point is to characterize the maps $C : \ell_D \to \ell_G$. If $u \in \ell_D$, then $u = \binom{0}{u_2}$ for some $u_2 \in \mathbb{C}^n$, and consequently $Cu = \binom{C_{12}}{C_{22}} u_2$. In particular, if $C$ maps onto $\ell_G$, then $\binom{C_{12}}{C_{22}}$ will be a frame for $\ell_G$. We denote the set of all such maps $C$ by $\mathcal{C}$. Next, we must be able to find some $C^{(0)} \in \mathcal{C}$ so that given any $w \in \ell_0$ there will exist $u \in \ell_D$ so that $w = u + C^{(0)}u$. I.e., we must have $w = \binom{0}{u_2} + \binom{C_{12}^{(0)}}{C_{22}^{(0)}} u_2$.

For $u \in \ell_D$, we can now compute

$$Q(\ell_D, \ell_G; \ell_0)(u, u) = -(JC^{(0)}u, u)_{\mathbb{C}^{2n}} = -(C_{12}^{(0)} u_2, u_2)_{\mathbb{C}^n},$$

from which it's clear that

$$\operatorname{sgn} Q(\ell_D, \ell_G; \ell_0) = - \operatorname{sgn} C_{12}^{(0)},$$

and moreover if $C_{12}^{(0)}$ is invertible

$$\operatorname{sgn} Q(\ell_D, \ell_G; \ell_0) = - \operatorname{sgn}(C_{12}^{(0)})^{-1}.$$

Since $\binom{C_{12}^{(0)}}{C_{22}^{(0)}}$ is a frame for $\ell_G$, we must have that for any other frame $\mathbf{X}_G = \binom{X_G}{Y_G}$, there exists an invertible matrix $M \in \mathbb{C}^{n \times n}$ so that

$$\begin{pmatrix} C_{12}^{(0)} \\ C_{22}^{(0)} \end{pmatrix} = \begin{pmatrix} X_G \\ Y_G \end{pmatrix} M.$$

Likewise, if $\mathbf{X}_0 = \binom{X_0}{Y_0}$ is any frame for $\ell_0$, then any other frame for $\ell_0$ can be expressed as $\binom{X_0}{Y_0}\mathcal{M}$ for some invertible matrix $\mathcal{M} \in \mathbb{C}^{n \times n}$. In this way, we can express the relation

$$\ell_0 = \{u + Cu : u \in \ell_D\}$$

in terms of frames

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \mathcal{M} = \begin{pmatrix} 0 \\ I \end{pmatrix} + \begin{pmatrix} C_{12}^{(0)} \\ C_{22}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} + \begin{pmatrix} X_G \\ Y_G \end{pmatrix} M. \tag{3.6}$$

Since $\ell_G \cap \ell_D = \{0\}$, $X_G$ must be invertible, allowing us to write $M = X_G^{-1} X_0 \mathcal{M}$, and subsequently

$$Y_0 \mathcal{M} = I + Y_G X_G^{-1} X_0 \mathcal{M}.$$

I.e., we have $(Y_0 - Y_G X_G^{-1} X_0)\mathcal{M} = I$, from which we see that $Y_0 - Y_G X_G^{-1} X_0$ is the inverse of $\mathcal{M}$, and so $\mathcal{M} = (Y_0 - Y_G X_G^{-1} X_0)^{-1}$. We conclude,

$$C_{12}^{(0)} = X_0 \mathcal{M} = X_0 (Y_0 - Y_G X_G^{-1} X_0)^{-1}. \tag{3.7}$$

**Remark 3.2.** *We note that in the event that $X_0$ is also invertible, we obtain the expression*

$$C_{12}^{(0)} = (Y_0 X_0^{-1} - Y_G X_G^{-1})^{-1}, \tag{3.8}$$

*so that*

$$\operatorname{sgn} C_{12}^{(0)} = \operatorname{sgn}(C_{12}^{(0)})^{-1} = \operatorname{sgn}(Y_0 X_0^{-1} - Y_G X_G^{-1}).$$

We summarize these observations in the following lemma.

**Lemma 3.1.** *Suppose $\ell_D$ is the Dirichlet Lagrangian subspace, and $\ell_G, \ell_0 \in \Lambda(n)$ are such that $\ell_G \cap \ell_D = \{0\}$ and $\ell_0 \cap \ell_G = \{0\}$. Then*

$$\operatorname{sgn} Q(\ell_D, \ell_G; \ell_0) = -\operatorname{sgn} C_{12}^{(0)},$$

*where $C_{12}^{(0)}$ is specified in (3.7). Moreover, if $X_0$ is invertible then $C_{12}^{(0)}$ is given in (3.8).*

Returning to Hörmander's formula with $\ell_1 = \ell_D$ and $\ell_2 = \ell_G$, we have

$$\begin{aligned}\operatorname{Mas}(\ell, \ell_G; [0,1]) - \operatorname{Mas}(\ell, \ell_D; [0,1]) &= s(\ell_D, \ell_G; \ell(0), \ell(1)) \\ &= \frac{1}{2}\Big( \operatorname{sgn} Q(\ell_D, \ell_G; \ell(0)) - \operatorname{sgn} Q(\ell_D, \ell_G; \ell(1))\Big),\end{aligned}$$

provided the following conditions hold: $\ell_D \cap \ell_G = \{0\}$, $\ell(0) \cap \ell_D = \{0\}$, $\ell(0) \cap \ell_G = \{0\}$, $\ell(1) \cap \ell_D = \{0\}$, and $\ell(1) \cap \ell_G = \{0\}$. Since $\ell_D \cap \ell_G = \{0\}$, $\ell(0) \cap \ell_D = \{0\}$, and $\ell(1) \cap \ell_D = \{0\}$, we can conclude that $X_G$, $X(0)$, and $X(1)$ are all invertible. In this way, we can express the relation above as

$$\begin{aligned}&\operatorname{Mas}(\ell, \ell_G; [0,1]) - \operatorname{Mas}(\ell, \ell_D; [0,1]) \\ &\quad = \frac{1}{2}\Big( \operatorname{sgn}(Y_G X_G^{-1} - Y(0)X(0)^{-1}) - \operatorname{sgn}(Y_G X_G^{-1} - Y(1)X(1)^{-1})\Big).\end{aligned}$$

We will use these considerations in Section 5 to establish Corollary 5.1.

# 4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Our starting point is to verify that $\mathbf{X}_1(0;\lambda)$ and $\mathbf{X}_2(x;\lambda)$ are indeed frames for Lagrangian subspaces. According to Proposition 2.1 of [14], a matrix $\mathbf{X} \in \mathbb{C}^{2n \times n}$ is the frame for a Lagrangian subspace if and only if the following two conditions both hold: (1) $\operatorname{rank}(\mathbf{X}) = n$; and (2) $\mathbf{X}^* J \mathbf{X} = 0$.

For $\mathbf{X}_1(0;\lambda)$, we have $\mathbf{X}_1(0;\lambda) = J\alpha^*$. According to **(A3)**, $\operatorname{rank} \alpha = n$, and it follows immediately that $\operatorname{rank} J\alpha^* = n$. Moreover,

$$\mathbf{X}_1(0;\lambda)^* J \mathbf{X}_1(0;\lambda) = (\alpha J^*) J (J\alpha^*) = \alpha J \alpha^* = 0.$$

For $\mathbf{X}_2(x;\lambda)$, we fix $\lambda \in (-\infty, \kappa)$ and temporarily set $\mathcal{A}(x) := \mathbf{X}_2(x;\lambda)^* J \mathbf{X}_2(x;\lambda)$. (Our notation here doesn't assert that $\mathcal{A}$ is independent of $\lambda$, but rather that $\lambda$ is fixed in the ensuing calculations.) Since $\mathbf{X}_2(\cdot;\lambda) \in L^2((0,\infty), \mathbb{C}^{2n \times n})$, we see that

$$\lim_{x \to +\infty} \mathcal{A}(x) = 0.$$

In addition, we can compute directly to find,

$$\begin{aligned}\mathcal{A}'(x) &= \mathbf{X}_2'(x;\lambda)^* J \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* J \mathbf{X}_2'(x;\lambda) \\ &= -(J\mathbf{X}_2'(x;\lambda))^* \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* J \mathbf{X}_2'(x;\lambda) \\ &= -(\mathbb{B}(x;\lambda)\mathbf{X}_2(x;\lambda))^* \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* \mathbb{B}(x;\lambda) \mathbf{X}_2(x;\lambda) \\ &= 0,\end{aligned}$$

where in obtaining the final equality to 0 we have used the fact that $\mathbb{B}(x;\lambda)$ is self-adjoint. Combining these observations, we can conclude that $\mathcal{A}(x) \equiv 0$ on $[0,\infty)$. Since this argument holds for any $\lambda \in (-\infty,\kappa)$, we conclude that $\mathbf{X}_2(x;\lambda)$ is the frame for a Lagrangian subspace for any $(x,\lambda) \in [0,\infty) \times (-\infty,\kappa)$.

Finally, we recall from Section 2 that the Lagrangian subspaces $\ell_2(x;\lambda)$ with frames $\mathbf{X}_2(x;\lambda)$ can be extended as $x$ tends to infinity to the Lagrangian subspaces $\ell_2^+(\lambda)$ with frames $\mathbf{X}_2^+(\lambda) = \binom{R(\lambda)}{P_+R(\lambda)D(\lambda)}$. Here, $R(\lambda)$ and $D(\lambda)$ are specified respectively in (2.3) and (2.4). In order to verify that $\ell_2^+(\lambda)$ is indeed Lagrangian, we compute

$$\begin{aligned}\mathbf{X}_2^+(\lambda)^* J \mathbf{X}_2^+(\lambda) &= \begin{pmatrix} R(\lambda)^* & (P_+R(\lambda)D(\lambda))^* \end{pmatrix} \begin{pmatrix} -P_+R(\lambda)D(\lambda) \\ R(\lambda) \end{pmatrix} \\ &= -R(\lambda)^*P_+R(\lambda)D(\lambda) + D(\lambda)R(\lambda)^*P_+R(\lambda),\end{aligned}$$

where we have observed that $P_+$ and $D(\lambda)$ are self-adjoint. Recalling the normalization identity $R(\lambda)^*P_+R(\lambda) = I$, we see that $\mathbf{X}_2^+(\lambda)^*J\mathbf{X}_2^+(\lambda) = 0$ for all $\lambda < \kappa$, from which we can conclude that $\ell_2^+(\lambda)$ is Lagrangian.

We proceed now by considering the *Maslov box*, for which we fix $\lambda_0 < \kappa$, and work with a value $\lambda_\infty$ that will be chosen sufficiently large during the proof, and certainly large enough so that $-\lambda_\infty < \lambda_0$. The Maslov box in this case will refer to the following sequence of four lines, creating a rectangle in the $(\lambda, x)$-plane: we fix $x = 0$ and let $\lambda$ increase from $-\lambda_\infty$ to $\lambda_0$ (the bottom shelf); we fix $\lambda = \lambda_0$ and let $x$ increase from 0 to $+\infty$ (the right shelf); we fix $x = +\infty$ and let $\lambda$ decrease from $\lambda_0$ to $-\lambda_\infty$ (the top shelf); and we fix $\lambda = -\lambda_\infty$ and let $x$ decrease from $+\infty$ to 0 (the left shelf).

For Theorem 1.1, we view the bottom shelf at $x = 0$ as our target, and the Lagrangian subspace we associate with the target is $\ell_1(0;\lambda)$, with frame $\mathbf{X}_1(0;\lambda) = J\alpha^*$. Clearly, $\ell_1(0;\lambda)$ does not depend on $\lambda$, and $\lambda$ only appears as an argument for notational consistency. In this case, the evolving Lagrangian subspace is $\ell_2(x;\lambda)$, which we recall corresponds with the space of solutions that decay as $x \to +\infty$. As our frame for $\ell_2(x;\lambda)$, we use the matrix $\mathbf{X}_2(x;\lambda)$ constructed in (1.6). We set

$$\begin{aligned}\tilde{W}(x;\lambda) = &-(X_1(0;\lambda) + iY_1(0;\lambda))(X_1(0;\lambda) - iY_1(0;\lambda))^{-1} \\ &\times (X_2(x;\lambda) - iY_2(x;\lambda))(X_2(x;\lambda) + iY_2(x;\lambda))^{-1}.\end{aligned} \tag{4.1}$$

The Maslov index computed with $\tilde{W}(x;\lambda)$ will detect intersections between $\ell_1(0;\lambda)$ and $\ell_2(x;\lambda)$. For expositional convenience, we consider the sides of the Maslov box in the following order: bottom, top, left, right.

*Bottom shelf.* Beginning with the bottom shelf, we observe that our Lagrangian subspaces have been constructed in such a way that conjugate points correspond with eigenvalues of $\mathcal{L}$, with the multiplicity of $\lambda$ as an eigenvalue of $\mathcal{L}$ matching the multiplicity of the intersection. This means that if each conjugate point along the bottom shelf has the same direction then the Maslov index along the bottom shelf will be (up to a sign) a count of the total number of eigenvalues that $\mathcal{L}$ has between $-\lambda_\infty$ and $\lambda_0$. We will show below that as $\lambda$ increases from $-\lambda_\infty$ toward $\lambda_0$ on the bottom shelf, the conjugate points are all negatively directed, and so the corresponding Maslov index is a negative of this count. In addition, we will show during our discussion of the left shelf that we can choose $\lambda_\infty$ sufficiently large so that $\mathcal{L}$

has no eigenvalues on the interval $(-\infty, -\lambda_\infty]$. We will be able to conclude, then, that the Maslov index along the bottom shelf is negative a count of the total number of eigenvalues, including multiplicity, that $\mathcal{L}$ has below $\lambda_0$; i.e.,

$$\operatorname{Mas}(\ell_1(0;\cdot), \ell_2(0;\cdot); [-\lambda_\infty, \lambda_0]) = -\operatorname{Mor}(\mathcal{L}; \lambda_0). \tag{4.2}$$

According to Lemma 3.1 of [14] (also Lemma 4.2 of [10]), rotation of the eigenvalues of $\tilde{W}(x;\lambda)$ as $\lambda$ varies—for any fixed $x \in [0,\infty)$—can be determined from the matrix $\mathbf{X}_2(x;\lambda)^* J \partial_\lambda \mathbf{X}_2(x;\lambda)$ in the following sense: If this matrix is positive definite at some point $(x_0, \lambda_0)$, then as $\lambda$ increases through $\lambda_0$ (with $x = x_0$ fixed), all $n$ eigenvalues of $\tilde{W}(x;\lambda)$ will monotonically rotate in the counterclockwise direction.

For this calculation, we temporarily set

$$\mathcal{B}(x;\lambda) = \mathbf{X}_2(x;\lambda)^* J \partial_\lambda \mathbf{X}_2(x;\lambda),$$

for which we can compute (with prime denoting differentiation with respect to $x$)

$$\begin{aligned}
\mathcal{B}'(x;\lambda) &= \mathbf{X}_2'(x;\lambda)^* J \partial_\lambda \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* J \partial_\lambda \mathbf{X}_2'(x;\lambda) \\
&= -(J\mathbf{X}_2'(x;\lambda))^* \partial_\lambda \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* \partial_\lambda (J\mathbf{X}_2'(x;\lambda)) \\
&= -(\mathbb{B}(x;\lambda)\mathbf{X}_2(x;\lambda))^* \partial_\lambda \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* \partial_\lambda (\mathbb{B}(x;\lambda)\mathbf{X}_2(x;\lambda)) \\
&= -\mathbf{X}_2(x;\lambda)^* \mathbb{B}(x;\lambda) \partial_\lambda \mathbf{X}_2(x;\lambda) + \mathbf{X}_2(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}_2(x;\lambda) \\
&\quad + \mathbf{X}_2(x;\lambda)^* \mathbb{B}(x;\lambda) \partial_\lambda \mathbf{X}_2(x;\lambda) \\
&= \mathbf{X}_2(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}_2(x;\lambda).
\end{aligned}$$

Integrating, we see that

$$\mathcal{B}(x;\lambda) = -\int_x^{+\infty} \mathbf{X}_2(y;\lambda)^* \mathbb{B}_\lambda(y;\lambda) \mathbf{X}_2(y;\lambda) dy,$$

where convergence of the integral is assured by the exponential decay of the elements in our frame $\mathbf{X}_2$. In this case,

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} Q(x) & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\mathcal{B}(x;\lambda) = -\int_x^{+\infty} X_2(y;\lambda)^* Q(y) X_2(y;\lambda) dy.$$

This matrix is clearly non-positive (since $Q$ is positive definite), and moreover it cannot have 0 as an eigenvalue, because the associated eigenvector $v \in \mathbb{C}^n$ would necessarily satisfy $X_2(y;\lambda)v = 0$ for all $y \in [x,\infty)$, and this would contradict linear independence of the columns of $X_2(y;\lambda)$ (as solutions of (1.1)).

Since $\mathcal{B}(x;\lambda)$ is negative definite, we can conclude that as $\lambda$ increases, the eigenvalues of $\tilde{W}(x;\cdot)$ rotate monotonically clockwise. It follows immediately that for the bottom shelf, (4.2) holds.

*Top shelf.* For the top shelf (obtained in the limit as $x \to +\infty$), we set

$$\tilde{W}^+(\lambda) := \lim_{x \to +\infty} \tilde{W}(x;\lambda),$$

and note that $\tilde{W}^+(\lambda)$ detects intersections between $\ell_1(0;\lambda)$ and $\ell_2^+(\lambda)$. Our frames for these Lagrangian subspaces are explicit, $\mathbf{X}_1(0;\lambda) = J\alpha^*$ and $\mathbf{X}_2^+(\lambda) = \binom{R(\lambda)}{P_+ R(\lambda) D(\lambda)}$, and we can use these frames to explicitly compute $\text{Mas}(\ell_1(0;\cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0])$.

We observe that the monotonicity that we found along horizontal shelves does not immediately carry over to the top shelf (since that calculation is only valid for $x \in [0,\infty)$). Nonetheless, we can conclude monotonicity along the top shelf in the following way: by continuity of our frames, we know that as $\lambda$ increases along the top shelf the eigenvalues of $\tilde{W}^+(\lambda)$ cannot rotate in the counterclockwise direction. Moreover, eigenvalues of $\tilde{W}^+(\lambda)$ cannot remain at $-1$ for any interval of $\lambda$ values. In order to clarify this last statement, we observe that the Lagrangian subspaces $\ell_1(0;\lambda)$ and $\ell_2^+(\lambda)$ intersect if and only if $\lambda$ is an eigenvalue for the constant coefficient equation

$$\begin{aligned} -P_+\phi'' + V_+\phi &= \lambda Q_+\phi \\ \alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) &= 0. \end{aligned} \tag{4.3}$$

(Due to the appearance of $P(0)$ in the boundary condition rather than $P_+$, this equation may not be self-adjoint, but that doesn't affect this argument.) If $\lambda$ is an eigenvalue of (4.3) that is not isolated from the rest of the spectrum, then it must be in the essential spectrum of (4.3), but by an argument essentially identical to the one given at the end of Section 2, we see that the essential spectrum for (4.3) is confined to the interval $[\kappa, \infty)$, so there can be no interval of eigenvalues below $\kappa$.

*Left shelf.* For the left shelf, intersections between $\ell_1(0;\lambda)$ and $\ell_2(x;\lambda)$ at some value $x = s$ will correspond with one or more non-trivial solutions to the truncated boundary value problem

$$\begin{aligned} \mathcal{L}_s\phi := Q(x)^{-1}\{-(P(x)\phi')' + V(x)\phi\} = \lambda\phi, \quad x \in (s,\infty) \\ \alpha_1\phi(s) + \alpha_2 P(s)\phi'(s) = 0, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \phi(\cdot;\lambda) \in \mathcal{D}(\mathcal{L}_s) := \{\phi \in L^2((s,\infty),\mathbb{C}^n) : \phi, \phi' \in AC_{\text{loc}}([s,\infty),\mathbb{C}^n), \\ \mathcal{L}\phi \in L^2((s,\infty),\mathbb{C}^n),\ \alpha_1\phi(s) + \alpha_2 P(s)\phi'(s) = 0\}. \end{aligned}$$

For this calculation, it's useful to use the projector formulation of our boundary conditions, developed in [2, 19] (see also [13] for an implentation of this formulation in circumstances quite similar to those of the current analysis). Briefly, there exist three orthogonal (and mutually orthogonal) projection matrices $P_D$ (the Dirichlet projection), $P_N$ (the Neumann projection), and $P_R = I - P_D - P_N$ (the Robin projection), and an invertible self-adjoint operator $\Lambda$ acting on the space $P_R\mathbb{C}^n$ such that the boundary condition

$$\alpha_1\phi(s) + \alpha_2 P(s)\phi'(s) = 0$$

can be expressed as

$$\begin{aligned} P_D\phi(s) &= 0 \\ P_N P(s)\phi'(s) &= 0 \\ P_R P(s)\phi'(s) &= P_R\Lambda P_R\phi(s). \end{aligned} \tag{4.5}$$

Moreover, $P_D$ can be constructed as the projection onto the kernel of $\alpha_2$ and $P_N$ can be constructed as the projection onto the kernel of $\alpha_1$.

Suppose $\lambda$ is an eigenvalue for (4.4), with corresponding eigenvector $\phi(\cdot;\lambda) \in \mathcal{D}(\mathcal{L}_s)$, and consider an $L^2((s,\infty),\mathbb{C}^n)$ inner product of $\phi(\cdot;\lambda)$ with (4.4). Integrating once by parts, we obtain (suppressing dependence on $\lambda$ for notational brevity)

$$\int_s^{+\infty} (P\phi', \phi')dx - (P(s)\phi'(s), \phi(s)) + \int_s^{+\infty} (V\phi, \phi)dx = \lambda \int_s^{+\infty} (Q\phi, \phi)dx. \tag{4.6}$$

Using uniform positivity of the matrices $P$ and $Q$, we can assert that for the positive constants $\theta_P$ and $\theta_Q$ described in **(A1)**, we have

$$\int_s^{+\infty} (P\phi', \phi')dx \geq \theta_P \|\phi'\|^2_{L^2((s,\infty),\mathbb{C}^n)}$$
$$\int_s^{+\infty} (Q\phi, \phi)dx \geq \theta_Q \|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)}.$$

In addition, with $C_V$ as described in **(A1)**, we have

$$\Big| \int_s^{+\infty} (V\phi, \phi)dx \Big| \leq C_V \|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)}.$$

For the boundary term, we can use our projection formulation to write

$$\begin{aligned} (P(s)\phi'(s), \phi(s)) &= (P(s)\phi'(s), P_D\phi(s) + P_N\phi(s) + P_R\phi(s)) \\ &= (P(s)\phi'(s), P_N\phi(s) + P_R\phi(s)) \\ &= (P_N P(s)\phi'(s), \phi(s)) + (P_R P(s)\phi'(s), \phi(s)) \\ &= (P_R \Lambda P_R \phi(s), \phi(s)). \end{aligned}$$

We have, then,

$$|(P(s)\phi'(s), \phi(s))| = |(P_R \Lambda P_R \phi(s), \phi(s))| \leq C_b |\phi(s)|^2,$$

where $C_b$ depends only on the boundary matrices $\alpha_1$ and $\alpha_2$. For $\phi(\cdot;\lambda) \in \mathcal{D}(\mathcal{L}_s)$, we can write

$$|\phi(s)|^2 = -\int_s^\infty \frac{d}{dx} |\phi(x)|^2 dx = -\int_s^\infty (\phi'(x), \phi(x)) + (\phi(x), \phi'(x)) dx,$$

so that the Cauchy-Schwarz inequality leads to

$$\begin{aligned} |\phi(s)|^2 &\leq \int_s^\infty 2|\phi'(x)||\phi(x)|dx \leq \int_s^\infty \epsilon |\phi'(x)|^2 + \frac{1}{\epsilon}|\phi(x)|^2 dx \\ &= \epsilon \|\phi'\|_{L^2((s,\infty),\mathbb{C}^n)} + \frac{1}{\epsilon} \|\phi\|_{L^2((s,\infty),\mathbb{C}^n)}, \end{aligned}$$

for any $\epsilon > 0$.

Combining these observations, we see that (4.6) leads, for any $\lambda < 0$, to the inequality

$$\begin{aligned} \lambda \theta_Q \|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)} \geq \lambda \int_s^{+\infty} (Q\phi, \phi)dx &\geq \theta_P \|\phi'\|^2_{L^2((s,\infty),\mathbb{C}^n)} - C_V \|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)} \\ &- C_b \Big( \epsilon \|\phi'\|_{L^2((s,\infty),\mathbb{C}^n)} + \frac{1}{\epsilon} \|\phi\|_{L^2((s,\infty),\mathbb{C}^n)} \Big). \end{aligned}$$

We choose $\epsilon$ so that $\theta_P - C_b\epsilon \geq 0$ to obtain the inequality

$$\lambda\theta_Q\|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)} \geq -\Big(C_V + \frac{C_b}{\epsilon}\Big)\|\phi\|^2_{L^2((s,\infty),\mathbb{C}^n)},$$

from which we conclude the lower bound

$$\lambda \geq -\Big(\frac{C_V}{\theta_Q} + \frac{C_b}{\epsilon\theta_Q}\Big). \tag{4.7}$$

We see that we can choose $\lambda_\infty$ sufficiently large so that $\mathcal{L}_s$ has no eigenvalues $\lambda$ on the interval $(-\infty, -\lambda_\infty]$ for any $s \in [0,\infty)$. Consequently, there can be no conjugate points $s \in [0,\infty)$ along a left shelf at $\lambda = -\lambda_\infty$.

**Remark 4.1.** *We contrast this observation with the case of Sturm-Liouville systems on* $[0,1]$*, for which conjugate points are possible on the left shelf. In the* $[0,1]$*-setting, if the boundary conditions at either* $0$ *or* $1$ *are Dirichlet, then there are no crossings along the left shelf (for* $\lambda_\infty$ *sufficiently large). The boundary condition* $\phi \in L^2((0,\infty),\mathbb{C}^n)$ *often has the same effect on unbounded domains as Dirichlet conditions have on bounded domains, and this is an example of that observation.*

We note that this analysis leaves open the possibility that the asymptotic point at $+\infty$ is conjugate. In the event that it is conjugate, $\lambda_\infty$ can be increased slightly to break the conjugacy. This is an immediate consequence of monotonicity along the top shelf, and serves to establish Lemma 1.1.

*Right shelf.* For the right shelf, we leave the Maslov index as a computation,

$$\text{Mas}(\ell_1(0;\lambda_0), \ell_2(\cdot;\lambda_0); [0,\infty]).$$

Combining these observations, and using catenation of paths along with homotopy invariance, we find that the sum

$$\text{bottom shelf} + \text{right shelf} + \text{top shelf} + \text{left shelf} = 0,$$

respectively becomes

$$-\text{Mor}(\mathcal{L};\lambda_0) + \text{Mas}(\ell_1(0;\lambda_0), \ell_2(\cdot;\lambda_0); [0,\infty]) - \text{Mas}(\ell_1(0;\cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0]) - 0 = 0,$$

and Theorem 1.1 is a rearrangement of this equality. □

# 5 Proof of Theorem 1.2

We established in our proof of Theorem 1.1 that $\ell_2(x;\lambda)$ is Lagrangian for all $(x,\lambda) \in [0,\infty) \times (-\infty,\kappa)$, and we can proceed similarly to verify that the same is true for $\ell_1(x;\lambda)$. We omit the details.

As with our proof of Theorem 1.1, we work with the Maslov box, but in this case, we place the top shelf at $x = x_\infty$, for $x_\infty$ chosen sufficiently large during the analysis. We proceed in this way, because the Lagrangian subspace

$$\ell_1^+(\lambda) := \lim_{x\to+\infty} \ell_1(x;\lambda)$$

(which is well-defined for each $\lambda < \kappa$) is not generally continuous as a function of $\lambda$. In particular, it is discontinuous at each eigenvalue of $\mathcal{L}$ (see [11] for a discussion in the context of Schrödinger operators on $\mathbb{R}$).

We will use the Maslov index to detect intersections between our evolving Lagrangian subspace $\ell_1(x;\lambda)$ and our target Lagrangian subspace $\ell_2(x_\infty;\lambda)$. Re-defining $\tilde{W}$ for this section, we now set

$$\begin{aligned}\tilde{W}(x;\lambda) = -(X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1} \\ \times (X_2(x_\infty;\lambda) - iY_2(x_\infty;\lambda))(X_2(x_\infty;\lambda) + iY_2(x_\infty;\lambda))^{-1}.\end{aligned} \tag{5.1}$$

For expositional convenience, we consider the sides of the Maslov box in the following order: left, top, bottom/right (together).

*Left shelf.* In this case, conjugate points $x = s$ along the left shelf correspond with values $s$ for which $\lambda = -\lambda_\infty$ is an eigenvalue for the ODE

$$\begin{aligned}-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi; \quad \text{in } (0,s) \\ \alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) = 0 \\ Y_2(x_\infty;\lambda)^*\phi(s) - X_2(x_\infty;\lambda)^* P(s)\phi'(s) = 0,\end{aligned} \tag{5.2}$$

where for notational brevity we are suppressing dependence of $\phi$ on $\lambda$. By taking $x_\infty$ sufficiently large, we can make $X_2(x_\infty;\lambda)$ as close as we like to the invertible matrix $R(\lambda)$, so that in this case $X_2(x_\infty;\lambda)$ is also invertible, and we can write,

$$P(s)\phi'(s) = (X_2(x_\infty;\lambda)^*)^{-1} Y_2(x_\infty;\lambda)^*\phi(s). \tag{5.3}$$

Moreover, we have

$$(X_2(x_\infty;\lambda)^*)^{-1} Y_2(x_\infty;\lambda)^* \approx (R(\lambda)^*)^{-1} D(\lambda) R(\lambda)^* P_+ = P_+ R(\lambda) D(\lambda) R(\lambda)^* P_+, \tag{5.4}$$

where the error on this approximation is $\mathbf{O}(e^{-\eta x_\infty})$ for some $\eta > 0$. The matrix

$$P_+ R(\lambda) D(\lambda) R(\lambda)^* P_+$$

is self-adjoint, and since the entries of $D(\lambda)$ are the negative eigenvalues of $\mathbb{A}_+(\lambda)$, it is negative definite. Also, the entries of $D(\lambda)$ approach $-\infty$ as $\lambda$ approaches $-\infty$, so the eigenvalues of $P_+ R(\lambda) D(\lambda) R(\lambda)^* P_+$ approach $-\infty$ as $\lambda$ approaches $-\infty$.

Let $\phi(x;\lambda)$ denote a solution to (5.2). Upon taking an $L^2((0,s),\mathbb{C}^n)$ inner product of $\phi$ with (5.2), we obtain

$$-\int_0^s ((P(x)\phi')', \phi)dx + \int_0^s (V(x)\phi,\phi)dx = \lambda \int_0^s (Q(x)\phi,\phi)dx.$$

For the first integral in this last expression, we compute

$$-\int_0^s ((P(x)\phi')', \phi)dx = \int_0^s (P(x)\phi',\phi')dx - (P(s)\phi'(s),\phi(s)) + (P(0)\phi'(0),\phi(0)).$$

Using (5.3), we see that

$$-(P(s)\phi'(s),\phi(s)) = -((X_2(x_\infty;\lambda)^*)^{-1} Y_2(x_\infty;\lambda)^*\phi(s),\phi(s)).$$

For the boundary term at $x = 0$, we proceed using the projectors $P_D$, $P_N$, and $P_R$ determined by $\alpha_1$ and $\alpha_2$ (as specified in (4.5)). Proceeding as in the proof of Theorem 1.1, we find

$$(P(0)\phi'(0), \phi(0)) = (P_R\Lambda P_R\phi(0), \phi(0)).$$

Combining these observations, we see that the boundary terms can be expressed as

$$\begin{aligned}
&-(P(s)\phi'(s), \phi(s)) + (P(0)\phi'(0), \phi(0)) \\
&\quad = -((X_2(x_\infty; \lambda)^*)^{-1}Y_2(x_\infty; \lambda)^*\phi(s), \phi(s)) + (P_R\Lambda P_R\phi(0), \phi(0)).
\end{aligned}$$

For $s$ sufficiently small, $\phi(s) = \phi(0) + \mathbf{O}(s)$, so that we approximately have

$$\Big(\Big((X_2(x_\infty; \lambda)^*)^{-1}Y_2(x_\infty; \lambda)^* - P_R\Lambda P_R\Big)\phi(0), \phi(0)\Big), \tag{5.5}$$

which is positive for $x_\infty$ and $\lambda_\infty$ both chosen sufficiently large (by the discussion following (5.4)). We conclude that there exists $s_0 > 0$ sufficiently small so that

$$-((X_2(x_\infty; \lambda)^*)^{-1}Y_2(x_\infty; \lambda)^*\phi(s), \phi(s)) + (P_R\Lambda P_R\phi(0), \phi(0)) \geq 0,$$

for all $0 < s \leq s_0$.

Similarly as in the proof of Theorem 1.1, we have

$$\begin{aligned}
\int_0^s (P\phi', \phi')dx &\geq \theta_P\|\phi'\|^2_{L^2((0,s),\mathbb{C}^n)}; \\
\int_0^s (Q\phi, \phi)dx &\geq \theta_Q\|\phi\|^2_{L^2((0,s),\mathbb{C}^n)}; \\
\Big|\int_0^s (V\phi, \phi)dx\Big| &\leq C_V\|\phi\|^2_{L^2((0,s),\mathbb{C}^n)}.
\end{aligned}$$

For $\lambda < 0$, this allows us to write (still for $0 < s \leq s_0$)

$$\begin{aligned}
\lambda\theta_Q\|\phi\|^2_{L^2((0,s),\mathbb{C}^n)} &\geq \lambda\int_0^s (Q\phi, \phi)dx \\
&\geq \theta_P\|\phi'\|^2_{L^2((0,s),\mathbb{C}^n)} - C_V\|\phi\|^2_{L^2((0,s),\mathbb{C}^n)},
\end{aligned}$$

from which we can immediately conclude

$$\lambda \geq -\frac{C_V}{\theta_Q},$$

for all $0 < s \leq s_0$.

For $s > s_0$, we scale the independent variable by setting

$$\xi = \frac{x}{s}; \quad \varphi(\xi) = \phi(x).$$

Our system becomes

$$\begin{aligned}
-(P(\xi s)\varphi')' + s^2V(\xi s)\varphi &= s^2\lambda Q(\xi s)\varphi; \quad \text{in } (0,1) \\
\alpha_1\varphi(0) + \frac{1}{s}\alpha_2 P(0)\varphi'(0) &= 0 \\
Y_2(x_\infty; \lambda)^*\varphi(1) - \frac{1}{s}X_2(x_\infty; \lambda)^*P(s)\varphi'(1) &= 0.
\end{aligned} \tag{5.6}$$

Suppose $\varphi$ solves (5.6) for $\lambda = -\lambda_\infty$. Taking an inner product of $\varphi$ with (5.6), we get

$$-\int_0^1 ((P(\xi s)\varphi')', \varphi) d\xi + s^2 \int_0^1 (V(\xi s)\varphi, \varphi) d\xi = s^2 \lambda \int_0^1 (Q(\xi s)\varphi, \varphi) d\xi.$$

For the first integral, we have

$$-\int_0^1 ((P(\xi s)\varphi')', \varphi) d\xi = \int_0^1 (P(\xi s)\varphi', \varphi') d\xi - (P(s)\varphi'(1), \varphi(1)) + (P(0)\varphi'(0), \varphi(0)).$$

For the boundary term at $\xi = 1$, we have

$$-(P(s)\varphi'(1), \varphi(1)) = -s((X_2(x_\infty; \lambda)^*)^{-1} Y_2(x_\infty; \lambda)^* \varphi(1), \varphi(1)) \geq 0,$$

where the inequality follows for $x_\infty$ sufficiently large from our prior discussion of

$$(X_2(x_\infty; \lambda)^*)^{-1} Y_2(x_\infty; \lambda)^*.$$

For the boundary term at $\xi = 0$, we have

$$(P(0)\varphi'(0), \varphi(0)) = s(P_R \Lambda P_R \varphi(0), \varphi(0)).$$

According to Lemma 1.3.8 in [2], we can compute the upper bound

$$|(P_R \Lambda P_R \varphi(0), \varphi(0))| \leq C_b |\varphi(0)|^2 \leq C_b(\epsilon \|\varphi'\|^2_{L^2((0,1),\mathbb{C}^n)} + \frac{2}{\epsilon} \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)}).$$

For $\lambda < 0$, this allows us to compute

$$\begin{aligned} s^2 \lambda \theta_Q \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)} &\geq s^2 \lambda \int_0^1 (Q(\xi s)\varphi, \varphi) d\xi \\ &\geq \theta_P \|\varphi'\|^2_{L^2((0,1),\mathbb{C}^n)} - s^2 C_V \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)} \\ &\quad - s C_b(\epsilon \|\varphi'\|^2_{L^2((0,1),\mathbb{C}^n)} + \frac{2}{\epsilon} \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)}). \end{aligned}$$

For each $s \in [s_0, x_\infty]$, we choose $\epsilon = \epsilon_s = \theta_P/(sC_b)$. This ensures

$$\theta_P - s C_b \epsilon = 0,$$

which leads immediately to

$$s^2 \lambda \theta_Q \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)} \geq -s^2 C_V \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)} - s^2 \frac{2C_b^2}{\theta_P} \|\varphi\|^2_{L^2((0,1),\mathbb{C}^n)}.$$

We conclude a lower bound on $\lambda$,

$$\lambda \geq -\frac{C_V}{\theta_Q} - \frac{2C_b^2}{\theta_P \theta_Q}.$$

Combining these observations, we can conclude that for any value $\lambda_\infty$ chosen so that

$$-\lambda_\infty < -\frac{C_V}{\theta_Q} - \frac{2C_b^2}{\theta_P \theta_Q},$$

we will have no crossings along the left shelf. Similarly as in the proof of Theorem 1.1, this leaves open the possibility of a conjugate point at $(0, -\lambda_\infty)$, corresponding with an intersection between $\ell_1(0; -\lambda_\infty)$ and $\ell_2(x_\infty, -\lambda_\infty)$. Precisely as in the proof of Theorem 1.1, we can increase $\lambda_\infty$ (if necessary) to ensure that $\ell_1(0; -\lambda_\infty) \cap \ell_2^+(-\lambda_\infty) = \{0\}$, and then we can choose $x_\infty$ sufficiently large to ensure that this implies $\ell_1(0; -\lambda_\infty) \cap \ell_2(x_\infty; -\lambda_\infty) = \{0\}$. For these choices of $x_\infty$ and $\lambda_\infty$, we have

$$\operatorname{Mas}(\ell_1(\cdot; -\lambda_\infty), \ell_2(x_\infty; -\lambda_\infty); [0, x_\infty]) = 0.$$

*Top shelf.* In the case of Theorem 1.2, $\tilde{W}(x; \lambda)$ has been constructed so that conjugate points along the top shelf correspond precisely with eigenvalues of $\mathcal{L}$. In order to verify that the Maslov index along the top shelf corresponds with a count of eigenvalues, we need to check that the eigenvalues of $\tilde{W}(x; \lambda)$ rotate monotonically counterclockwise as $\lambda$ decreases. In this case, both $\mathbf{X}_1$ and $\mathbf{X}_2$ depend on $\lambda$, so according to Lemma 3.1 of [14] (also Lemma 4.2 of [10]), rotation of the eigenvalues of $\tilde{W}(x; \lambda)$—for any $x \in [0, \infty)$—can be determined from the matrices $-\mathbf{X}_1(x; \lambda)^* J \partial_\lambda \mathbf{X}_1(x; \lambda)$ and $\mathbf{X}_2(x_\infty; \lambda)^* J \partial_\lambda \mathbf{X}_2(x_\infty; \lambda)$ in the following sense: If *both* of these matrices are non-positive, and at least one is negative definite at some point $(x_0, \lambda_0)$, then as $\lambda$ increases through $\lambda_0$ (with $x = x_0$ fixed), all $n$ eigenvalues of $\tilde{W}(x; \lambda)$ will monotonically rotate in the clockwise direction.

We have already established during the proof of Theorem 1.1 that the matrix

$$\mathbf{X}_2(x_\infty; \lambda)^* J \partial_\lambda \mathbf{X}_2(x_\infty; \lambda)$$

is negative definite, so we only need to check that $-\mathbf{X}_1(x; \lambda)^* J \partial_\lambda \mathbf{X}_1(x; \lambda)$ is non-positive. In fact, this latter matrix is negative definite as well, and since the proof is essentially identical to the proof for $\mathbf{X}_2(x_\infty; \lambda)^* J \partial_\lambda \mathbf{X}_2(x_\infty; \lambda)$, we omit the details.

We can conclude, similarly as for the bottom shelf in the proof of Theorem 1.1, that

$$\operatorname{Mas}(\ell_1(x_\infty; \cdot), \ell_2(x_\infty; \cdot); [-\lambda_\infty, \lambda_0]) = -\operatorname{Mor}(\mathcal{L}; \lambda_0).$$

*Bottom and right shelves.* We will need to compute Maslov indices along the bottom and right shelves, so it's natural to address the two of them together. Our approach is based substantially on the proofs of Claims 4.11 and 4.12 in [11].

As a starting point, we introduce the new unitary matrix

$$\begin{aligned} \tilde{\mathcal{W}}(x; \lambda) := -(X_1(x; \lambda) + iY_1(x; \lambda))(X_1(x; \lambda) - iY_1(x; \lambda))^{-1} \\ \times (R(\lambda) - iS(\lambda))(R(\lambda) + iS(\lambda))^{-1}, \end{aligned}$$

which detects intersections between $\ell_1(x; \lambda)$ and the asymptotic Lagrangian subspace

$$\ell_2^+(\lambda) := \lim_{x \to +\infty} \ell_2(x; \lambda).$$

Likewise, we specify the asymptotic matrix

$$\tilde{\mathcal{W}}^+(\lambda) := \lim_{x \to \infty} \tilde{\mathcal{W}}(x; \lambda),$$

which is well-defined for each $\lambda < \kappa$, but not generally continuous as a function of $\lambda$. (See the appendix in [11] for a discussion of this discontinuity.) Since $R(\lambda)$ and $S(\lambda)$ can be written down explicitly, it is much more convenient to work with $\tilde{\mathcal{W}}(x; \lambda)$ than it is to work with $\tilde{W}(x; \lambda)$. In light of this, we will show that our calculations can be carried out entirely in terms of the former matrix. In particular, we have the following claim:

**Claim 5.1.** *Under the assumptions of Theorem 1.2, we have the relation*

$$\begin{aligned} &\operatorname{Mas}(\ell_1(0; \cdot), \ell_2(x_\infty, \cdot); [-\lambda_\infty, \lambda_0]) + \operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2(x_\infty; \lambda_0); [0, x_\infty]) \\ &\quad = \operatorname{Mas}(\ell_1(0; \cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0]) + \operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2^+(\lambda_0); [0, \infty]). \end{aligned}$$

*Proof.* First, it's clear that we have the relation

$$\begin{aligned} \tilde{\mathcal{W}}(x; \lambda) = \tilde{W}(x; \lambda)&(X_2(x_\infty; \lambda) + iY_2(x_\infty; \lambda))(X_2(x_\infty; \lambda) - iY_2(x_\infty; \lambda))^{-1} \\ &\times (R(\lambda) - iS(\lambda))(R(\lambda) + iS(\lambda))^{-1}. \end{aligned}$$

Recalling from Lemma 2.1 that

$$\mathbf{X}_2(x_\infty; \lambda) = \begin{pmatrix} R(\lambda) \\ S(\lambda) \end{pmatrix} + \mathbf{O}(e^{-\tilde{\eta} x_\infty}),$$

for some $\tilde{\eta} > 0$, we see that by choosing $x_\infty$ sufficiently large, we can ensure that the eigenvalues of $\tilde{\mathcal{W}}(x; \lambda)$ are as close as we like to the eigenvalues of $\tilde{W}(x; \lambda)$ for all $(x, \lambda) \in [0, \infty) \times [-\lambda_\infty, \lambda_0]$. (Here, exponential decay in $x$ allows us to compactify $[0, \infty)$ with the usual one-point compactification.) In particular, we can ensure that no eigenvalue of $\tilde{W}(x; \lambda_0)$ can complete a loop of $S^1$ unless a corresponding eigenvalue of $\tilde{\mathcal{W}}(x; \lambda_0)$ completes a loop of $S^1$, with the converse holding as well.

Following our discussion of the left shelf, we have chosen $\lambda_\infty$ so that

$$\ell_1(0; -\lambda_\infty) \cap \ell_2^+(-\lambda_\infty) = \{0\},$$

and $x_\infty$ sufficiently large to ensure that this implies

$$\ell_1(0; -\lambda_\infty) \cap \ell_2(x_\infty; -\lambda_\infty) = \{0\}.$$

With these choices, we see that $\tilde{W}(0; -\lambda_\infty)$ does not have $-1$ as an eigenvalue, and also $\tilde{\mathcal{W}}(0; -\lambda_\infty)$ does not have $-1$ as an eigenvalue.

*Case 1.* First, suppose $\lambda_0$ is not an eigenvalue for $\mathcal{L}$. Then $\tilde{W}(x_\infty; \lambda_0)$ does not have $-1$ as an eigenvalue, and also $\tilde{\mathcal{W}}^+(\lambda_0)$ does not have $-1$ as an eigenvalue. By continuity, we can take $x_\infty$ large enough so that $\tilde{\mathcal{W}}(x_\infty; \lambda_0)$ does not have $-1$ as an eigenvalue, and additionally so that $\tilde{\mathcal{W}}(x; \lambda_0)$ does not have $-1$ as an eigenvalue for any $x \geq x_\infty$. Since the eigenvalues of $\tilde{W}$ and $\tilde{\mathcal{W}}$ remain uniformly close, the total spectral flow associated with the bottom and right shelves for $\tilde{W}(x; \lambda)$ must be the same as for $\tilde{\mathcal{W}}(x; \lambda)$. Specifically, we have

$$\begin{aligned} &\operatorname{Mas}(\ell_1(0; \cdot), \ell_2(x_\infty, \cdot); [-\lambda_\infty, \lambda_0]) + \operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2(x_\infty; \lambda_0); [0, x_\infty]) \\ &\quad = \operatorname{Mas}(\ell_1(0; \cdot), \ell_2^+(\cdot); [-\lambda_\infty, \lambda_0]) + \operatorname{Mas}(\ell_1(\cdot; \lambda_0), \ell_2^+(\lambda_0); [0, x_\infty]), \end{aligned}$$

and the claim for Case 1 follows immediately from the specification that $x_\infty$ is taken large enough so that $\ell_1(x;\lambda_0)$ and $\ell_2^+(\lambda_0)$ do not intersect for $x \geq x_\infty$.

*Case 2.* Next, suppose $\lambda_0$ is an eigenvalue for $\mathcal{L}$. Then certainly $\tilde{W}(x_\infty;\lambda_0)$ has $-1$ as an eigenvalue, and its multiplicity corresponds with the multiplicity of $\lambda_0$ as an eigenvalue of $\mathcal{L}$. Likewise, $\tilde{\mathcal{W}}^+(\lambda_0)$ will have $-1$ as an eigenvalue, and its multiplicity corresponds with the multiplicity of $\lambda_0$ as an eigenvalue of $\mathcal{L}$. As in the case when $\lambda_0$ is not an eigenvalue, we can choose $x_\infty$ large enough so that for $x \geq x_\infty$ the eigenvalues of $\tilde{\mathcal{W}}(x;\lambda)$ that do not approach $-1$ as $x \to +\infty$ remain bounded away from $-1$ as $x \to +\infty$.

We now proceed precisely as in Case 1 for the eigenvalues of $\tilde{W}(x_\infty;\lambda_0)$ other than $-1$, and we note that an eigenvalue of $\tilde{W}(x;\lambda_0)$ will approach $-1$ as $x \to x_\infty$ if and only if an eigenvalue of $\tilde{\mathcal{W}}(x;\lambda)$ approaches $-1$ as $x \to +\infty$. Moreover, despite possible transient crossings, the net number of crossings associated with these eigenvalues must coincide, because otherwise, an eigenvalue of $\tilde{W}(x;\lambda)$ would complete a full loop of $S^1$ without a corresponding eigenvalue of $\tilde{\mathcal{W}}(x;\lambda)$ also completing such a loop (or vice versa). □

Combining now our observations for the four shelves, we find that the sum

$$\text{bottom shelf} + \text{right shelf} + \text{top shelf} + \text{left shelf} = 0,$$

respectively becomes

$$\operatorname{Mas}(\ell_1(0;\cdot),\ell_2^+(\cdot);[-\lambda_\infty,\lambda_0]) + \operatorname{Mas}(\ell_1(\cdot;\lambda_0),\ell_2^+(\lambda_0);[0,\infty]) + \operatorname{Mor}(\mathcal{L};\lambda_0) + 0 = 0,$$

and Theorem 1.2 is just a rearrangement of this equality. □

## 5.1 Changing the Target

In this section, we verify that under certain conditions the target frame $\ell_2^+(\lambda_0)$ in the calculation $\operatorname{Mas}(\ell_1(\cdot;\lambda_0),\ell_2^+(\lambda_0);[0,\infty])$ can be replaced with the Dirichlet plane $\ell_D$. As noted earlier, one advantage of this replacement is that for a Dirichlet target the rotation of eigenvalues of $\tilde{W}(x;\lambda)$ as $x$ increases is monotonically clockwise. (This is straightforward to show, e.g., with the methods of [14].) The key observation we take advantage of here is that if $\lambda_0$ is not an eigenvalue of $\mathcal{L}$, then we explicitly know both $\ell_1(0;\lambda_0)$ and

$$\ell_1^+(\lambda_0) = \lim_{x\to+\infty} \ell_1(x;\lambda_0) = \tilde{\ell}_2^+(\lambda_0),$$

where $\tilde{\ell}_2^+(\lambda_0)$ denotes the Lagrangian subspace associated with solutions that grow as $x$ tends to positive infinity. This allows us to compute both

$$\operatorname{sgn} Q(\ell_D,\ell_2^+(\lambda_0);\ell_1(0;\lambda_0)) \quad \text{and} \quad \operatorname{sgn} Q(\ell_D,\ell_2^+(\lambda_0);\ell_1^+(\lambda_0)),$$

and consequently we can compute the Hörmander index $s(\ell_D,\ell_2^+(\lambda_0);\ell_1(0;\lambda_0),\ell_1^+(\lambda_0))$.

In order to apply our development from Section 3.1, we need the following five conditions to hold: (i) $\ell_D \cap \ell_1(0;\lambda_0) = \{0\}$; (ii) $\ell_2^+(\lambda_0) \cap \ell_1(0;\lambda_0) = \{0\}$; (iii) $\ell_D \cap \ell_1^+(\lambda_0) = \{0\}$; (iv) $\ell_2^+(\lambda_0) \cap \ell_1^+(\lambda_0) = \{0\}$; and (v) $\ell_D \cap \ell_2^+(\lambda_0) = \{0\}$. We will check below that Items (iii), (iv), and (v) hold under our general assumptions, and we will take Items (i) and (ii) to be additional assumptions for this section (which hold for our application in Section 6).

The first items to check are (iii) and (iv), which (since $\ell_1^+(\lambda_0) = \tilde{\ell}_2^+(\lambda_0)$) we can express as the intersections $\ell_D \cap \tilde{\ell}_2^+(\lambda_0) = \{0\}$ and $\ell_2^+(\lambda_0) \cap \tilde{\ell}_2^+(\lambda_0) = \{0\}$. For these, we recall that our frame for $\tilde{\ell}_2^+(\lambda_0)$ is

$$\tilde{\mathbf{X}}_2^+(\lambda_0) = \begin{pmatrix} R(\lambda_0) \\ -PR(\lambda_0)D(\lambda_0) \end{pmatrix},$$

where $R(\lambda_0)$ and $D(\lambda_0)$ are as in (2.3) and (2.4). For the Dirichlet plane,

$$\tilde{\mathbf{X}}_2^+(\lambda_0)^* J \mathbf{X}_D = (R(\lambda_0)^* \quad -D(\lambda_0)^* R(\lambda_0)^* P) \begin{pmatrix} -I \\ 0 \end{pmatrix} = R(\lambda_0)^*,$$

and since $R(\lambda_0)$ is invertible we have $\ker(\tilde{\mathbf{X}}_2^+(\lambda_0)^* J \mathbf{X}_D) = \{0\}$. Likewise, the frame for $\ell_2^+(\lambda_0)$ is $\mathbf{X}_2^+(\lambda_0) = \binom{R(\lambda_0)}{PR(\lambda_0)D(\lambda_0)}$, so that

$$\begin{aligned} \tilde{\mathbf{X}}_2^+(\lambda_0)^* J \mathbf{X}_2^+(\lambda_0) &= (R(\lambda_0)^* \quad -D(\lambda_0)^* R(\lambda_0)^* P) \begin{pmatrix} -PR(\lambda_0)D(\lambda_0) \\ R(\lambda_0) \end{pmatrix} \\ &= -R(\lambda_0)^* PR(\lambda_0)D(\lambda_0) - D(\lambda_0)^* R(\lambda_0)^* PR(\lambda_0) \\ &= -2D(\lambda_0), \end{aligned}$$

which is positive definite. The verification that $\ell_D \cap \ell_2^+(\lambda_0) = \{0\}$ (i.e., Item (v) above) is essentially identical to the verification that $\ell_D \cap \tilde{\ell}_2^+(\lambda_0) = \{0\}$, and we omit the details.

By definition, the Hörmander index for these Lagrangian subspaces is

$$s(\ell_D, \ell_2^+(\lambda_0); \ell_1(0;\lambda_0), \tilde{\ell}_2^+(\lambda_0)) = \operatorname{Mas}(\ell_1(\cdot;\lambda_0), \ell_2^+(\lambda_0); [0,\infty]) - \operatorname{Mas}(\ell_1(\cdot;\lambda_0), \ell_D; [0,\infty]).$$

According to Hörmander's formula (3.5),

$$s(\ell_D, \ell_2^+(\lambda_0); \ell_1(0;\lambda_0), \tilde{\ell}_2^+(\lambda_0)) = \frac{1}{2}\Big( \operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \ell_1(0;\lambda_0)) - \operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \tilde{\ell}_2^+(\lambda_0)) \Big). \tag{5.7}$$

We can now use Lemma 3.1 to compute the two quantities $\operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \ell_1(0;\lambda_0))$ and $\operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \tilde{\ell}_2^+(\lambda_0))$. First, recalling that the frame for $\ell_1(0;\lambda_0)$ is $\mathbf{X}_1(0;\lambda_0) = \binom{-\alpha_2^*}{\alpha_1^*}$, and noting that the condition $\ell_1(0;\lambda_0) \cap \ell_D = \{0\}$ implies that $\alpha_2$ is invertible, we have (from Lemma 3.1)

$$\begin{aligned} \operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \ell_1(0;\lambda_0)) &= \operatorname{sgn}(Y_2^+(X_2^+)^{-1} + \alpha_1^*(\alpha_2^*)^{-1}) \\ &= \operatorname{sgn}(P_+ R(\lambda_0)D(\lambda_0)R(\lambda_0)^{-1} + \alpha_1^*(\alpha_2^*)^{-1}) \\ &= \operatorname{sgn}(P_+ R(\lambda_0)D(\lambda_0)R(\lambda_0)^* P_+ + \alpha_1^*(\alpha_2^*)^{-1}). \end{aligned}$$

Likewise,

$$\begin{aligned} \operatorname{sgn} Q(\ell_D, \ell_2^+(\lambda_0); \tilde{\ell}_2^+(\lambda_0)) &= \operatorname{sgn}(Y_2^+(\lambda_0)(X_2^+(\lambda_0))^{-1} + Y_2^+(\lambda_0)(X_2^+(\lambda_0))^{-1}) \\ &= \operatorname{sgn}(2P_+ R(\lambda_0)D(\lambda_0)R(\lambda_0)^* P_+) \\ &= -n, \end{aligned}$$

because $P_+ R(\lambda_0)D(\lambda_0)R(\lambda_0)^* P_+$ is negative definite.

Combining these observations, we see that

$$\begin{aligned}\mathrm{Mas}(\ell_1(\cdot;\lambda_0),\ell_2^+(\lambda_0);[0,\infty]) &= \mathrm{Mas}(\ell_1(\cdot;\lambda_0),\ell_D;[0,\infty]) \\ &\quad + \frac{1}{2}\Big(n + \mathrm{sgn}(\alpha_1^*(\alpha_2^*)^{-1} + P_+ R(\lambda_0) D(\lambda_0) R(\lambda_0)^* P)\Big).\end{aligned}$$

In this way, we obtain the following corollary to Theorem 1.2.

**Corollary 5.1.** *Let the assumptions of Theorem 1.2 hold, and suppose additionally that* $\lambda_0 \notin \sigma(\mathcal{L})$*,* $\ell_1(0;\lambda_0) \cap \ell_D = \{0\}$*, and* $\ell_1(0;\lambda_0) \cap \ell_2^+(\lambda_0) = \{0\}$*. Then*

$$\begin{aligned}\mathrm{Mor}(\mathcal{L};\lambda_0) = &- \mathrm{Mas}(\ell_1(\cdot;\lambda_0),\ell_D;[0,\infty]) \\ &- \frac{1}{2}\Big(n + \mathrm{sgn}(\alpha_1^*(\alpha_2^*)^{-1} + P_+ R(\lambda_0) D(\lambda_0) R(\lambda_0)^* P_+)\Big) \\ &- \mathrm{Mas}(\ell_1(0;\cdot),\ell_2^+(\cdot);[-\lambda_\infty,\lambda_0]).\end{aligned}$$

# 6 Application to Quantum Graphs

In this section, we apply our framework to an operator on the half-line that arises through consideration of nonlinear Schrödinger equations on quantum graphs with $n$ infinite edges extending from a single vertex (i.e., on star graphs). Our direct motivation for considering this example is the recent analysis of Kairzhan and Pelinovsky (see [16]), and we also note that Kostrykin and Schrader have shown how the symplectic framework fits well with such problems (see [18]) and that Latushkin and Sukhtaiev have recently developed this framework in the case of quantum graphs with edges of finite length (see [20]). Finally, we mention that our general approach to quantum graphs is adapted from the reference [2].

## 6.1 The Schrödinger Operator on Star Graphs

We consider a star graph with $n$ edges, which can be visualized as a single point with $n$ distinct half-lines emerging from it. We will associate with each edge of our graph the interval $[0,\infty)$, and our basic Hilbert space associated with the full graph will be

$$\mathcal{H} = \bigoplus_{j=1}^{n} L^2((0,\infty),\mathbb{C}).$$

We will view elements $\phi \in \mathcal{H}$ as vector functions $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^t$, and we specify the linear operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ by

$$(\mathcal{L}\phi)_j = -\phi_j'' + v(x)\phi_j,$$

where $v \in C([0,\infty),\mathbb{C})$ is a scalar potential for which we will assume the limit

$$\lim_{x\to\infty} v(x) = v_+$$

exists and satisfies the asymptotic relation

$$\int_0^\infty x(v(x) - v_+)dx < \infty.$$

(This is slightly weaker than our Assumption **(A2)**, but sufficient in the current setting (see [11]).) We specify boundary conditions at the vertex as

$$\alpha_1 \phi(0) + \alpha_2 \phi'(0) = 0, \tag{6.1}$$

with $\alpha_1$ and $\alpha_2$ satisfying the assumptions described in **(A3)**. Under these assumptions, we take as our domain for $\mathcal{L}$,

$$\mathcal{D}(\mathcal{L}) = \{\phi \in \mathcal{H} : \phi, \phi' \in AC_{\rm loc}([0,\infty), \mathbb{C}^n),\ \mathcal{L}\phi \in \mathcal{H}\}.$$

With this notation in place, we can consider the eigenvalue problem $\mathcal{L}\phi = \lambda\phi$ with boundary conditions (6.1). In order to place this system in the framework of our analysis, we set $y(x;\lambda) = \binom{y_1(x;\lambda)}{y_2(x;\lambda)}$, with $y_1(x;\lambda) = \phi(x;\lambda)$ and $y_2(x;\lambda) = \phi'(x;\lambda)$. In this way, we arrive at our standard Hamiltonian system

$$\begin{aligned} Jy' &= \mathbb{B}(x;\lambda) y \\ \alpha y(0) &= 0, \end{aligned} \tag{6.2}$$

where $\mathbb{B}(x;\lambda)$ denotes the diagonal matrix

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} (\lambda - v(x))I & 0 \\ 0 & I \end{pmatrix}.$$

Under our assumptions on the scalar potential $v$, it's well known that for each $\lambda < v_+$ the scalar equation

$$-z'' + v(x) z = \lambda z \tag{6.3}$$

has one non-trivial solution that decays as $x \to +\infty$ and one non-trivial solution that grows as $x \to +\infty$. (See, e.g., [11].) If we denote by $\zeta(x;\lambda)$ the solution that decays as $x \to +\infty$, then we can express our frame $\mathbf{X}_2(x;\lambda)$ of solutions of (6.2) decaying as $x \to +\infty$ as

$$\mathbf{X}_2(x;\lambda) = \begin{pmatrix} \zeta(x;\lambda) I \\ \zeta'(x;\lambda) I \end{pmatrix}.$$

We see that in this case, and in the context of Theorem 1.1,

$$\tilde{W}(x;\lambda) = -(-\alpha_2^* + i\alpha_1^*)(-\alpha_2^* - i\alpha_1^*)^{-1} \frac{\zeta(x;\lambda) - i\zeta'(x;\lambda)}{\zeta(x;\lambda) + i\zeta'(x;\lambda)}.$$

(I.e., this is (4.1) for the current case.) In particular, if we denote the eigenvalues of $(-\alpha_2^* + i\alpha_1^*)(-\alpha_2^* - i\alpha_1^*)^{-1}$ by $\{a_j\}_{j=1}^n$, then the eigenvalues of $\tilde{W}(x;\lambda)$ will be

$$\Big\{ -\frac{\zeta(x;\lambda) - i\zeta'(x;\lambda)}{\zeta(x;\lambda) + i\zeta'(x;\lambda)} a_j \Big\}_{j=1}^n.$$

**Remark 6.1.** *We distinguish the Neumann or Neumann-Kirchhoff boundary conditions as those specified by the relations*

$$\begin{aligned} \phi_1(0) = \phi_2(0) = \cdots &= \phi_n(0) \\ \sum_{j=1}^n \phi_j'(0) &= 0. \end{aligned}$$

*(See p. 14 of [2] for a discussion of terminology.) These correspond with*

$$\alpha_1 = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{6.4}$$

*and*

$$\alpha_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{6.5}$$

*In this case, the eigenvalues of* $(-\alpha_2^* + i\alpha_1^*)(-\alpha_2^* - i\alpha_1^*)^{-1}$ *are* $-1$ *and* $+1$*, with* $+1$ *simple and* $-1$ *occurring with multiplicity* $n-1$*. This fact is straightforward to verify directly, and is also an immediate consequence of Corollary 2.3 from [18].*

## 6.2 NLS on Star Graphs

We now consider the nonlinear Schrödinger equation

$$iu_t = -\Delta u - (p+1)|u|^{2p}u, \tag{6.6}$$

where $p > 0$ and $u \in \mathcal{H}$ with $u_j$ taking the values of $u$ on edge $j$ of the graph. We interpret the notation $\Delta u$ and $|u|^{2p}u$ in this setting as

$$\Delta u = (u_1'', u_2'', \dots, u_n'')^t$$
$$|u|^{2p}u = (|u_1|^{2p}u_1, |u_2|^{2p}u_2, \dots, |u_n|^{2p}u_n)^t.$$

Such equations are known to admit standing wave solutions

$$u(x,t) = e^{i\omega t}\tilde{u}_\omega(x),$$

for any $\omega > 0$. Upon direct substitution into (6.6), we see that

$$-\Delta\tilde{u}_\omega - (p+1)|\tilde{u}_\omega|^{2p}\tilde{u}_\omega = -\omega\tilde{u}_\omega.$$

In [16], the authors observe that by setting

$$z = \omega^{1/2}x; \quad \tilde{u}_\omega(x) = \omega^{\frac{1}{2p}}\tilde{u}(z),$$

we arrive at

$$-\Delta\tilde{u} - (p+1)|\tilde{u}|^{2p}\tilde{u} = -\tilde{u}. \tag{6.7}$$

This scaling justifies restricting our attention to the case $\omega = 1$. It's straightforward to verify that for any $p > 0$ (6.7) admits the explicit solution

$$\tilde{u}(x) = s(x)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \quad s(x) = \operatorname{sech}^{1/p}(px),$$

which also satisfies the Neumann boundary conditions described in Remark 6.1.

We linearize (6.6) about $e^{it}\tilde{u}(x)$, writing

$$u(x,t) = e^{it}\tilde{u}(x) + e^{it}(v(x,t) + iw(x,t)),$$

where $v(x,t)$ and $w(x,t)$ are both real-valued functions. Dropping off higher order terms, we obtain the linear system

$$\begin{aligned} v_t &= L_- w \\ w_t &= -L_+ v, \end{aligned}$$

where

$$\begin{aligned} L_- &= -\Delta + 1 - (p+1)\tilde{u}(x)^{2p} \\ L_+ &= -\Delta + 1 - (p+1)(2p+1)\tilde{u}(x)^{2p}. \end{aligned}$$

Our framework can now be used in order to determine the Morse indices of $L_\pm$ with Neumann–Kirchhoff boundary conditions. We focus on the slightly more interesting case, $L_+$. (The Morse index of $L_-$ with Neumann–Kirchhoff boundary conditions is 0.) The eigenvalue problem for $L_+$ can be expressed as

$$\begin{aligned} -\phi'' + (1-(p+1)(2p+1)s(x)^{2p})\phi = \lambda\phi; \quad x \in (0,\infty) \\ \alpha_1\phi(0) + \alpha_2\phi'(0) = 0, \end{aligned} \tag{6.8}$$

with $\alpha_1$ and $\alpha_2$ as in (6.4) and (6.5).

For this calculation, we will use Theorem 1.1 with $\lambda_0 = 0$. We observe that by construction,

$$\phi(x) = s'(x)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}; \quad s(x) = \operatorname{sech}^{1/p}(px),$$

solves (6.8) for $\lambda = 0$ (just differentiate (6.7) to see this; here, $\phi$ is not expected to satisfy the boundary condition at $x = 0$). This allows us to express our frame for solutions of (6.8) that decay as $x \to +\infty$ as

$$\mathbf{X}_2(x;\lambda) = \begin{pmatrix} s'(x)I \\ s''(x)I \end{pmatrix}.$$

We set $\mathbf{X}_1(0;\lambda) = \binom{-\alpha_2^*}{\alpha_1^*}$, so that

$$\tilde{W}(x;0) = -(-\alpha_2^* + i\alpha_1^*)(-\alpha_2^* - i\alpha_1^*)^{-1}\frac{s'(x) - is''(x)}{s'(x) + is''(x)}.$$

According to Remark 6.1, the eigenvalues of $\tilde{W}(x;0)$ are

$$q(x) := (s'(x) - is''(x))(s'(x) + is''(x))^{-1},$$

with multiplicity $n-1$ and the negative of this with multiplicity 1. (Here, the notation $q(x)$ has been introduced simply for expositional convenience).

In [14], the authors have developed a straightforward approach toward determining the direction of rotation for the eigenvalues of $\tilde{W}(x;\lambda)$ as $x$ varies, but in the current setting this rotation can be determined directly from the form of $s(x)$. We observe that

$$\begin{aligned} s'(x) &= -s(x)\tanh(px) \\ s''(x) &= s(x)\tanh^2(px) - s(x)p\,\mathrm{sech}^2(px). \end{aligned}$$

We can write

$$\frac{s'(x) - is''(x)}{s'(x) + is''(x)} = \frac{s'(x)^2 - s''(x)^2 - 2is'(x)s''(x)}{s'(x)^2 + s''(x)^2},$$

for which we focus on the real and imaginary parts of the numerator

$$\begin{aligned} s'(x)^2 - s''(x)^2 &= s(x)^2\Big(\tanh^2(px) - (\tanh^2(px) - p\,\mathrm{sech}^2(px))^2\Big) \\ -2s'(x)s''(x) &= 2s(x)^2\tanh(px)\Big(\tanh^2(px) - p\,\mathrm{sech}^2(px)\Big). \end{aligned}$$

We note that for any $x > 0$,

$$\begin{aligned} \mathrm{sgn}\,\mathrm{Re}\; q(x) &= \mathrm{sgn}\,\Big(\tanh^2(px) - (\tanh^2(px) - p\,\mathrm{sech}^2(px))^2\Big) \\ \mathrm{sgn}\,\mathrm{Im}\; q(x) &= \mathrm{sgn}\,\Big(\tanh^2(px) - p\,\mathrm{sech}^2(px)\Big). \end{aligned} \tag{6.9}$$

We now consider the motion of $q(x)$ as $x$ increases from 0 to $+\infty$. First, $s'(0) = 0$ and $s''(0) = -p$, so

$$q(0) = -1.$$

This means that $-1$ is an eigenvalue of $\tilde{W}(0;0)$ with multiplicity $n-1$, and $+1$ is an eigenvalue of $\tilde{W}(0;0)$ with multiplicity 1. (The fact that $-1$ is an eigenvalue of $\tilde{W}(0;0)$ with multiplicity $n-1$ corresponds with the fact that $\lambda_0 = 0$ is an eigenvalue of $L_+$ with multiplicity $n-1$.) As $x$ increases from 0, we see from (6.9) that the imaginary part of $q(x)$ becomes negative, so rotation is in the counterclockwise direction. Moreover, since $\tanh^2(px)$ and $\mathrm{sech}^2(px)$ are both monotonic in $x$ (for $x \geq 0$), we see that the imaginary part of $q(x)$ remains negative until $x$ arrives at the unique value $\bar{x}$ for which

$$\tanh^2(p\bar{x}) - p\,\mathrm{sech}^2(p\bar{x}) = 0.$$

We see from (6.9) that $\mathrm{sgn}\,\mathrm{Re}\; q(\bar{x}) > 0$, so $q(\bar{x}) = +1$. For $x > \bar{x}$, the imaginary part of $q(x)$ is positive, and by noting the asymptotic relations $s'(x) \sim -2^{1/p}e^{-x}$, $s''(x) \sim 2^{1/p}e^{-x}$, we see that as $x \to +\infty$, $q(x)$ approaches $i$. In summary, we see that as $x$ increases from 0 to $+\infty$, $q(x)$ rotates from $-1$ to $i$, leaving $-1$ in the counterclockwise direction and never crossing $-1$. Indeed, with a bit more work, we can verify that the rotation is entirely counterclockwise, but we don't require that much information to draw our conclusions.

Returning to the matrix $\tilde{W}(x;0)$, we can conclude that $n-1$ eigenvalues trace out precisely the path described in the previous paragraph, and the final eigenvalue begins at $+1$ when $x = 0$ and rotates in the counterclockwise direction, approaching $-i$ as $x \to +\infty$. We conclude that

$$\mathrm{Mas}(\ell_1(0;0), \ell_2(\cdot;0); [0,\infty]) = +1.$$

Finally, in order to use Theorem 1.1, we need to compute $\mathrm{Mas}(\ell_1(0;\cdot), \ell_2^+(\cdot); [-\lambda_\infty, 0])$. For this, we observe that if we set $y = \binom{y_1}{y_2}$, with $y_1 = \phi$ and $y_2 = \phi'$, then (6.8) can be expressed as $y' = \mathbb{A}(x;\lambda)y$, with

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I_n \\ ((1-(p+1)(2p+1)s(x)^{2p}) - \lambda)I_n & 0 \end{pmatrix}.$$

Since $s(x) \to 0$ as $x \to \infty$, we see that

$$\mathbb{A}_+(\lambda) := \lim_{x\to\infty} \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I_n \\ (1-\lambda)I_n & 0 \end{pmatrix}.$$

We can readily check that as a choice for the corresponding asymptotic frame $\mathbf{X}_2^+(\lambda) = \binom{R(\lambda)}{S(\lambda)}$, we can take $\mathbf{X}_2^+(\lambda) = \binom{I_n}{-\sqrt{1-\lambda} I_n}$. Thus for the top shelf, we have

$$\tilde{W}^+(\lambda) = -(-\alpha_2^* + i\alpha_1^*)(-\alpha_2^* - i\alpha_1^*)^{-1} \frac{1 + i\sqrt{1-\lambda}}{1 - i\sqrt{1-\lambda}}.$$

We conclude from Remark 6.1 that the eigenvalues of $\tilde{W}^+(\lambda)$ are $(1+i\sqrt{1-\lambda})(1-i\sqrt{1-\lambda})^{-1}$ with multiplicity $(n-1)$ and the negative of this with multiplicity 1. For $\lambda < 1$ the value of $(1+i\sqrt{1-\lambda})(1-i\sqrt{1-\lambda})^{-1}$ cannot be $\pm 1$, so there are no conjugate points along the top shelf. We conclude that in this case

$$\mathrm{Mas}(\ell_1(0;\cdot), \ell_2^+(\cdot); [-\lambda_\infty, 0]) = 0.$$

Applying Theorem 1.1, we find that

$$\mathrm{Mor}(L_+) = \mathrm{Mas}(\ell_1(0;0), \ell_2(\cdot;0); [0,\infty)) = +1.$$

I.e., $L_+$ has precisely one negative eigenvalue.

**Remark 6.2.** *For a more complete discussion of the instability of the half-soliton* $e^{i\omega t}\tilde{u}_\omega(x)$ *as a solution to (6.6), including a calculation of* $\mathrm{Mor}(L_+)$ *by other means, we refer the reader to [16].*

*Acknowledgments.* This work was initiated while P.H. was visiting Miami University in March, 2018. The authors are grateful to the Department of Mathematics at Miami University for supporting this trip.

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