POINTWISE ESTIMATES AND STABILITY FOR DISPERSIVE–DIFFUSIVE SHOCK WAVES

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Abstract. We study the stability and pointwise behavior of perturbed viscous shock waves for a general scalar conservation law with constant diffusion and dispersion. Along with the usual Lax shocks, such equations are known to admit undercompressive shocks. We unify the treatment of these two cases by introducing a new wave-tracking method based on "instantaneous projection," giving improved estimates even in the Lax case. Another important feature connected with the introduction of dispersion is the treatment of a non-sectorial operator. An immediate consequence of our pointwise estimates is a simple spectral criterion for stability in all L^p norms, $p \geq 1$ for the Lax case and $p > 1$ for the undercompressive case.

Our approach extends immediately to the case of certain scalar equations of higher order, and would also appear suitable for extension to systems.

1. Introduction

We consider the scalar viscous conservation law

$$\begin{aligned} u_t + f(u)_x &= (b(u)u_x)_x + u_{xxx}, \quad f, u, x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where the constant dispersion has been scaled to unity, $b \in C^N(\mathbb{R}), N \geq 2$, such that $b(\bar{u}(x)) \geq b_0 > 0$, $f \in C^N(\mathbb{R})$ and $u_0(x) \to u_\pm$ as $x \to \pm\infty$. We will be concerned with the stability of traveling wave solutions to (1.1), that is, solutions of the form $\bar{u}(x - st)$, which satisfy $\bar{u}(\pm\infty) = u_\pm$ and the Rankine–Hugoniot condition

$$s(u_+ - u_-) = f(u_+) - f(u_-).$$

By a shift of coordinates we may take without loss of generality $s = 0$. We will assume non-sonic shocks, that is $f'(u_-)$, $f'(u_+) \neq s$, but note that *undercompressive shocks*—viscous profiles for $n \times n$ systems of conservation laws having fewer than the $n+1$ entering characteristics of the Lax case—are allowed, and as they contain both an incoming and an outgoing characteristic, will be the focus of the analysis. For convenience, we will refer to the above hypotheses together as $(\mathcal{H})$.

A number of the preliminary results presented here are valid for the case $b(\cdot)$ nonconstant, but our nonlinear stability result is only valid in the case $b(\cdot)$ constant. The difficulty we encounter in extending this result to the case of nonconstant diffusion lies in the small time behavior of the Green's function of the linearized

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operator. For equations with odd order, the Green's function tends to oscillate with increasing rapidity as time goes to zero (see, e.g., [KF]). Such oscillatory behavior is difficult to take advantage of within the framework or our analysis. In the case of even order equations, however, our methods will extend to nonconstant diffusion (see Section 7).

While a number of works regarding the stability of Lax shocks in the presence of diffusion only have appeared [G, H.2, L.1, MN, SX, ZH], to date relatively few results have been obtained either on the stability of undercompressive waves or for shock waves in the presence of dispersion. For a history and discussion of undercompressive waves, we refer the reader to [LZ.1–2], and for shock waves in the presence of dispersion to [D]. We mention here only a few results that seem most relevant to this paper. In 1987, Shearer et al [SSMP] solved the Riemann problem for a non-strictly hyperbolic 2×2 system of conservation laws with an undercompressive shock, indicating that in certain cases such non-Lax waves may be necessary to consider. Zumbrun et al [ZPM] gave numerical evidence in 1992 that undercompressive shocks for certain 2×2 systems are stable, and in 1991 Wu [Wu] showed that undercompressive shocks for the single conservation law

$$u_t + (u^3)_x = 0$$

can be approximated by smooth traveling wave solutions to the modified Korteweg–de Vries equation,

$$u_t + (u^3)_x = \nu u_{xxx} + \mu u_{xx}, \tag{1.2}$$

a model equation for Magnetohydrodynamics (MHD). Wu further observed numerically that the corresponding undercompressive shocks appeared stable. Working with 2×2 systems, Liu and Zumbrun [LZ.1-2] provided the first analytic stability results for undercompressive shocks in 1995. Dodd then showed in 1996 that for μ sufficiently small in (1.2), certain undercompressive viscous shock solutions to (1.2) are stable [D]. An important aspect of Dodd's analysis is the incorporation of additional effects of dispersion not considered in [LZ.1-2], a feature that has received relatively little study; to our knowledge, the only other analytic shock stability result for combined dispersion–diffusion is a much earlier analysis of scalar Lax shocks for the KdV–Burgers equation carried out by Khodja [Ko], via energy methods .

In this paper we study the general dispersive–diffusive conservation law (1.1), and put forth a method which appears suitable for considerable extension. In particular, we employ the pointwise approach developed in [L.2–3, LZ.1–2, SX, SZ, ZH] to establish a spectral criterion for evaluating the stability of equations of form (1.1), extending the analysis of [H, ZH] to the case with dispersion. The most striking new development is the small time analysis in which dispersion plays a key role. In this case, the operator is no longer sectorial, but rather generates a C^0 semigroup. We must then take a higher order expansion in the rescaling argument of [AGJ, GZ] and take advantage of oscillations (dispersion) in our estimates.

A difficulty in the analysis of both the Lax and undercompressive cases is that the perturbation will not generally approach the traveling wave itself, but rather will approach a translate. In the Lax case it is well known that this translate can be determined through conservation of mass [LZ.2]. In the undercompressive case, however, this is not possible, and we employ the *instantaneous projection* of [ZH] in order to track the perturbation's location as it evolves in time. Applying this approach to the Lax case also, we unify the treatment of Lax and undercompressive waves, in addition showing that convergence along the instantaneous translate is faster than convergence to the time-asymptotic translate. (This can be seen by comparing the results here with those of [H.2] and Nishihara's exact analysis of Burgers equation [N]). Indeed, we obtain in the Lax case a convergence result even for data not in L^1 (specifically, for data decaying as $\zeta(1+|x|)^{-r}, r > 1/2, \zeta \ll 1$), in which case the time-asymptotic translate is not well-defined.

As opposed to the weighted-norm approach of Dodd, our method of analysis gives detailed information in the far field, and develops a qualitative behavioral picture of the wave interaction. For instance, we observe directly that oscillatory diffusion waves are swept through the shock—shifting it—and out to the far field at $+\infty$, where they decay like heat kernels. Further, while approaches involving energy methods appear generally unsatisfactory for dealing with the outgoing diffusion waves that arise in the case of undercompressive shocks arising in systems (unless combined with the pointwise approach as in [SX]), methods such as those developed here have previously been shown suitable for the analysis of systems [ZH].

That extension of our approach to systems is of interest is clear, for example, from [SSMP, ZPM, LZ.1–2], while the importance of extension to higher order scalar equations is amply demonstrated by recent developments in this context. For example, it has been shown that for the fourth order scalar equation

$$h_t + (h^2 - h^3)_x = -\epsilon^3 (h^3 h_{xxx})_x, \tag{1.3}$$

viscous shocks arise and display a variety of intriguing behavior [BMS]. Numerical evidence indicates that (1.3)—governing thin liquid films—admits a countable family of viscous shock solutions with alternating stability, accumulating at an undercompressive viscous shock. This robust stability behavior is indicative of the need for a readily verifiable stability criterion, such as that of [ZH].

In general, we see from equations such as (1.2) and (1.3) that high-order terms affect solutions in a fundamental manner, due both to the delicate nature of singular perturbation problems, and the large high-order coefficients that arise in nature, and so cannot be ignored [W, SMP, LH, BMS]. By reducing the issue of stability to the Evans function criterion of [ZH], we develop a context in which these issues may be studied.

Before turning to the statements of our main two results, we take a moment to set the stage. When (1.1) is linearized about a viscous profile we arrive at the convection–diffusion–dispersion equation with constant dispersion

$$v_t + (a(x)v)_x = (b(x)v_x)_x + v_{xxx}, \tag{1.4}$$

where $a(x) = f'(\bar{u}(x)) - b'(\bar{u}(x))\bar{u}_x$, $b(x) = b(\bar{u}(x))$, and higher order error terms have for the moment been omitted. The eigenvalue equation associated with (1.4) is

$$v_{xxx} + (b(x)v_x)_x - (a(x)v)_x = \lambda v. \tag{1.5}$$

From $(\mathcal{H})$, and the above definitions, we obtain the following consequences, $(\mathcal{C})$:

$$
\begin{aligned}
&(i) \quad a(x) \in C^{N-1}(\mathbb{R}), b(x) \in C^N(\mathbb{R}),\\
&(ii) \quad |\frac{\partial^{N-1}}{\partial x^{N-1}}(a(x) - a_\pm)|, |\frac{\partial^N}{\partial x^N}(b(x) - b_\pm)| = \mathbf{O}(e^{-\alpha|x|}) \quad \alpha > 0,\\
&(iii) \quad a_\pm \neq 0,\\
&(iv) \quad b(x) \geq b_0 > 0,
\end{aligned}
$$

where $\lim_{x\to\pm\infty} a(x) = a_\pm$ *and* $\lim_{x\to\pm\infty} b(x) = b_\pm$.

As equations of form (1.4) are of interest in their own right, it is useful to think of these consequences as assumptions, in which case we obtain a result independent of (1.1). We remark that (ii) and (iii) are results of our assumption that $\bar{u}(\cdot)$ is not a sonic shock. Following the notation of [ZH], we denote our stability criterion by

$(\mathcal{D})$: *For the operator* $Lv := v_{xxx} + (b(x)v_x)_x - (a(x)v)_x$, *a simple eigenvalue at* $\lambda = 0$ *is the only effective eigenvalue with nonnegative real part.*

Here, effective eigenvalues are defined as zeroes of the *Evans function* $W_0(\lambda)$, defined below (see (2.4) and (2.5)), and we may take as our space of eigenfunctions any L^p space, $p < \infty$, so long as the eigenfunctions decay at $\pm\infty$. These coincide with standard eigenvalue–eigenfunction pairs away from essential spectrum. It will often be useful to divide $(\mathcal{D})$ into the following two conditions, which taken together are equivalent to $(\mathcal{D})$.

(I) Excepting the origin, all eigenvalues of the operator L lie in the strict negative-real half-plane, $Re(\lambda) < 0$.

(II) The eigenvalue at the origin arising from translation invariance gives rise to a simple zero of the Evans function.

Typically, one employs energy estimates to show that there are no standard eigenvalues $\lambda \neq 0$ with non-negative real part, then determines the order of the zero at $\lambda = 0$ by an Evans function calculation [AGJ, D, GZ, ZH]. Alternatively, Brin has developed a technique for numerically evaluating $(\mathcal{D})$ [B]. In the present paper $(\mathcal{D})$ will be assumed to hold true.

A consequence of $(\mathcal{D})$ is that, excepting the origin, the entire point spectrum of L, and indeed all zeroes of the Evans function $W_0(\lambda)$, must lie strictly to the left of a contour in the complex plane described through

$$\lambda_d(k) := -id_3k^3 - d_2k^2 - id_1k - d, \tag{1.6}$$

where $d, d_i \in \mathbb{R}_+$ and will be chosen sufficiently small in the forthcoming analysis. We will refer to this contour as Γ_d (see Figure 5.1). This assertion is proved in Lemma 3.3.

We now state the first of two theorems.

Theorem 1.1. *Under Conditions ($\mathcal{C}$) and ($\mathcal{D}$), and for some constants $C, \tilde{C}$, M, $n+m \leq N+1$, and $\eta > 0$ depending on the asymptotic behavior of $a(x)$ and $b(x)$ and also on the eigenvalues of L, that is the values of d, d_1, d_2 and d_3 in (1.6), the Green's function, $G(t,x;y)$, for (1.4) satisfies the following estimates:*

(I) For $|x-y| \geq Kt$, K sufficiently large,

(i) $x - y \leq 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-\frac{n+m+1}{3}}) e^{-\frac{|x-y|^{3/2}}{M\sqrt{t}}}$$

(ii) $x - y \geq 0$

$$G(t,x;y) = \mathbf{O}(t^{-1/3}) e^{-\eta|x-y|}$$

and for $n+m \geq 1$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \begin{cases} \mathbf{O}(t^{-\frac{n+m+1}{3}}) e^{-\eta|x-y|} & \frac{x-y}{\sqrt[3]{t}} \leq \tilde{C} \\ \mathbf{O}(t^{-\frac{n+m+1}{3}}) \Big(\frac{x-y}{\sqrt[3]{t}}\Big)^{\frac{n+m}{2}-\frac{1}{4}} e^{-\eta|x-y|} & \frac{x-y}{\sqrt[3]{t}} \geq \tilde{C} \end{cases}$$

(II) For $|x-y| \leq Kt$, K as above,

(L+) Lax Case ($a_+ < 0 < a_-, x \geq 0$)

(i) $y \leq 0 \leq x$,

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-m/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x)\Big) I_{\{|x-y| \leq |a_+|t\}}$$

(ii) $0 \leq y \leq x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}}$$
$$+ \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x)\Big) I_{\{|x-y| \leq |a_+|t\}}$$

(iii) $0 \leq x \leq y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-m/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}} + \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x)\Big) I_{\{|x-y| \leq |a_+|t\}}$$

(L-) Lax Case ($a_+ < 0 < a_-, x \leq 0$) (Estimates Symmetric)

(U+) Undercompressive Case ($a_-, a_+ > 0, x \geq 0$)

(i) $y \le 0 \le x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x - \frac{a_+}{a_-} y - a_+ t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-m/2}) e^{-\frac{(x - \frac{a_+}{a_-} y - a_+ t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|y|}) \mathbf{O}(t^{-(n+1)/2}) e^{-\frac{(x - \frac{a_+}{a_-} y - a_+ t)^2}{Mt}}$$
$$+ \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x - \frac{a_+}{a_-} y| \le |a_+|t\}}$$

(ii) $0 \le y \le x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|y|}) \mathbf{O}(t^{-(n+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}} + \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x-y| \le |a_+|t\}}$$

(iii) $0 \le x \le y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}}$$
$$+ \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x-y| \le |a_+|t\}}$$

(U-) Undercompressive Case ($a_-, a_+ > 0, x \le 0$)

(i) $y \le x \le 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \mathbf{O}(e^{-\eta|x|}) \mathbf{O}(t^{-m/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}} + \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x-y| \le |a_-|t\}}$$

(ii) $x \le y \le 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_- t)^2}{Mt}}$$
$$+ \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x-y| \le |a_-|t\}}$$

(iii) $x \le 0 \le y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \mathbf{O}(t^{-(n+m+1)/2}) e^{-\frac{(x-y-a_+ t)^2}{Mt}}$$
$$+ \Big(\frac{\partial^{n+m}}{\partial x^n \partial y^m} P(x,y) \Big) I_{\{|x-y| \le |a_+|t\}}.$$

Here, the projection kernel P is defined by

$$P(x,y)=\bar{u}_x(x)\pi(y),$$

where $\pi(y)$ is the eigenfunction for $\lambda=0$ of the linear operator L^* associated with the problem adjoint to (1.4), and can be written in terms of the asymptotically decaying solutions of (1.5). For the undercompressive case, we have $(\partial^n/\partial y^n)\pi(y)=\mathbf{O}(e^{-\alpha|y|})$, for $n\geq 1$, and for the Lax case $\pi(y)$ is constant.

We observe that the estimates of Theorem 1.1 match those of Theorem 8.3 of [ZH], except for the dispersive effect of $t^{-1/3}$ blow-up as $t\to 0$ (for $|x-y|\geq Kt$), rather than $t^{-1/2}$ blow-up as occurs in the purely diffusive case. For $x\leq y$, this small-time behavior matches what we would expect for a sectorial operator, but for $x\geq y$, the lack of time dependence in the exponential decay is a result of our non-sectorial operator. We remark that these estimates match the exact Green's function of the Airy equation $u_t=u_{xxx}$ (see [KF]), except that our positive diffusion gives, additionally, exponential $|x-y|$-decay. Further comments regarding the type of Green's function estimates of Theorem 1.1 can be found in [H.1] and [ZH].

We turn now to developing the framework in which our stability result will lie, outlining the class of initial data we will be concerned with (decaying algebraically and slower) and the type of stability we will be concerned with (L^p orbital stability for all $1\leq p\leq\infty$ norms in the case of Lax shocks and $1<p\leq\infty$ norms in the case of undercompressive shocks). We shall establish this result through detailed pointwise bounds on the solution $v(t,x)$ of (1.4), which are of considerable interest in their own right.

Let $\bar{u}(x-st)$ be a traveling wave solution to (1.1), and without loss of generality take $s=0$ to get the standing wave $\bar{u}(x)$. Since solutions with initial data near $\bar{u}(x)$ will typically approach a translate of $\bar{u}(x)$ rather than $\bar{u}(x)$ itself, we will introduce a tracking mechanism $\delta(t)$ that will be determined in the course of the analysis. Let $u(t,x)$ be another solution to (1.1) and define

$$v(t,x):=u(t,x+\delta(t))-\bar{u}(x), \tag{1.7}$$

to be the *perturbation* of u from the viscous shock. We will choose $\delta(t)$ in such a way that $u(t,x+\delta(t))$ will remain near $\bar{u}(x)$ (in an appropriate sense to be discussed) at each time t. In this manner we will always compare $u(t,x)$ against the shape of $\bar{u}(x)$ rather than its position. We assume $\delta(0)=0$, that is that we indeed begin with a perturbation to the viscous shock.

Substituting $u(t,x+\delta(t))$ into (1.1) yields the perturbation equation

$$v_t-v_{xxx}-(b(x)v_x)_x+(a(x)v)_x=Q(v,v_x)_x-\dot{\delta}(t)(\bar{u}_x+v_x), \tag{1.8}$$

where

$$a(x):=f'(\bar{u}(x))-b'(\bar{u}(x))\bar{u}_x(x),\quad b(x)=b(\bar{u}(x))$$

and

$$Q(v,v_x):=\mathbf{O}(v^2)+\mathbf{O}(vv_x)$$

is a smooth function of its arguments. In the case that $b(\cdot)$ is constant, $Q(v)=\mathbf{O}(v^2)$, and we do not need an estimate on v_x.

We note that the Green's function estimates of Theorem 1.1 correspond to those for the equation found by setting the right-hand side of (1.8) to zero. Letting $G(t,x;y)$ continue to represent that Green's function, Duhamel's Principle gives the integral representation of v

(1.9)
$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} G(t-s,x;y)\Big[Q(v(s,y))_y - \dot{\delta}(s)(\bar{u}_y(y)+v_y(s,y))\Big]dyds. \end{aligned}$$

Using the fact that $\bar{u}_x$ is an eigenfunction of the linearized eigenvalue equation (for $\lambda = 0$) we see that $e^{Lt}\bar{u}_x = \bar{u}_x$, so that (after integration by parts of the nonlinear term)

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy - \delta(t)\bar{u}_x \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) - \dot{\delta}(s)v(s,y)\Big]dyds. \end{aligned} \tag{1.10}$$

We now turn to the critical task of choosing $\delta(t)$. Ignoring the higher order contribution of the term $\dot{\delta}v$ in (1.10), we see that the principal effect of δ (in the more general framework of systems) is to shift v along the direction $\bar{u}_x$, that is, in the direction tangent to the manifold of translates of $\bar{u}$. Note that $Span\{\bar{u}_x\}$ comprises the nondecaying modes of the linearized solution operator e^{Lt}, i.e., the effective eigenspace of L at $\lambda = 0$ (see [ZH] and the remarks following the statement of Theorem 1.1 here). Following the usual strategy, therefore, we choose δ so as to annihilate an appropriately chosen projection onto $Span\{\bar{u}_x\}$. The standard choice of projection, motivated by finite-dimensional ODE analysis, would be the zero eigenprojection of L, for which the analog in our case is the effective eigenprojection $\mathcal{P}f := \int P(\cdot,y)f(y)dy$; this choice, for example, was the basis of the stability analysis in [LZ.2]. This choice, however, is not optimal in the present setting, due to the appearance of the indicator function multiplying $P(x,y)$ in the Green's function bounds of Theorem 1.1, a purely PDE phenomenon related to the accumulation of essential spectrum at $\lambda = 0$. Instead, we employ a nonlinear version of the *instantaneous projection*, defined in [ZH].

Definition 1.1. *The instantaneous projection of v is given as*

$$\begin{aligned} \varphi(t,x,v,\delta) &:= \varphi_L(t,x,v_0) - \varphi_N(t,x,v,\dot{\delta}) \\ &= \Big(\int_0^{|a_+|t} P(x,y)v_0(y)dy + \int_{-|a_-|t}^0 P(x,y)v_0(y)dy - \delta(t)\bar{u}_x\Big) \\ &\quad - \Big(\int_0^t \int_0^{|a_+|(t-s)} P_y(x,y)\Big[Q(v(s,y)) - \dot{\delta}(s)v(s,y)\Big]dyds \\ &\quad - \int_0^t \int_{-|a_-|(t-s)}^0 P_y(x,y)\Big[Q(v(s,y)) - \dot{\delta}(s)v(s,y)\Big]dyds\Big), \end{aligned}$$

where φ_L represents the linear part of φ and φ_N represents the nonlinear part.

Note that φ, up to exponentially small tail errors (resulting from approximation by $I_{\{|y|\le|a_\pm t|\}}$ of the exact quantity $I_{\{|x-y|\le|a_\pm t|\}}$), is exactly the contribution to v in (1.10) of terms in the Green's function involving the projection kernel P. It has a physical interpretation as the superposition of all (stationary) time-asymptotic states that have been excited up to time t by the arrival at $x=0$ of a signal from the far field at y.

In the scalar Lax setting, the instantaneous projection lends itself to a particuarly direct interpretation. The excited terms $E(t,x;y)=P(x,y)I(t)$ of Theorem 1.1 are indicative of the mass of $G(t,x;y)$ that does not decay in time. Comparing this observation with our definition $v(t,x)=u(t,x+\delta(t))-\bar{u}(x)$, we see that this non-decaying mass is directly connected to our shift from the stationary shock: mass that fails to decay in time forces $u(t,x)$ toward a translate of $\bar{u}(x)$ rather than $\bar{u}(x)$ itself. More precisely, $\int u(0,x)-\bar{u}(x)dx$ will (in general) not be zero. The difference between $u(0,x)$ and $\bar{u}(x)$ is principally a difference of shape. As $t\to\infty$, however, the shape of $u(t,x)$ (in the case of stability) converges to that of $\bar{u}(x)$. The mass between $u(t,x)$ and $\bar{u}(x)$ must be conserved, forcing a shift. What the instantaneous projection measures is the v_0-weighted contribution of these excited terms at time t. We may heuristically think of $\varphi(t,x,v,\delta)$, then, as the mass that has accumulated at the origin at time t. With this observation in mind, we choose $\delta(t)$ so that $\varphi(t,x,v,\delta)\equiv 0$. Persuasive as this motivating argument may or may not be, the wisdom of this choice of $\delta(t)$ will ultimately be determined by the size of $v(t,x)$ (see Theorem 1.2).

Since the estimate we require in order to obtain a bound on $v(t,x)$ is on $\dot{\delta}(t)$ we consider the relation $(\partial/\partial t)\varphi(t,x)\equiv 0$. This yields (recalling that $P(x,y)=\bar{u}_x(x)\pi(y)$)

(1.11)
$$\begin{aligned}\dot{\delta}(t)\bar{u}_x &= \bar{u}_x(x)\pi(|a_+|t)v_0(|a_+|t)+\bar{u}_x(x)\pi(-a_-t)v_0(-a_-t)\\ &\quad+\int_0^t \bar{u}_x(x)\pi_y(|a_+|(t-s))\Big[Q(v(s,|a_+|(t-s)))-\dot{\delta}(s)v(s,|a_+|(t-s))\Big]ds\\ &\quad+\int_0^t \bar{u}_x(x)\pi_y(-a_-(t-s))\Big[Q(v(s,-a_-(t-s)))-\dot{\delta}(s)v(s,-a_-(t-s))\Big]ds,\end{aligned}$$

where we need not put $|\cdot|$ on a_- because it will be positive in both the Lax and undercompressive case. We remark here that in the Lax case $P_y\equiv 0$ so that the integrals in (1.11) are both zero and the properties of $\dot{\delta}(t)$ are determined without further work (see [HZ]).

In general, we have the estimate

$$\begin{aligned}|\dot{\delta}(t)| &\le |\pi(|a_+|t)v_0(|a_+|t)|+|\pi(-a_-t)v_0(-a_-t)|\\ &\quad+\Big|\int_0^t \pi_y(|a_+|(t-s))\Big[Q(v(s,|a_+|(t-s)))-\dot{\delta}(s)v(s,|a_+|(t-s))\Big]ds\Big|\\ &\quad+\Big|\int_0^t \pi_y(-a_-(t-s))\Big[Q(v(s,-a_-(t-s)))-\dot{\delta}(s)v(s,-a_-(t-s))\Big]ds\Big|.\end{aligned}\tag{1.12}$$

We proceed now with some definitions and our results on stability.

Definition 1.2. *(Class of initial data.) Denote by* Δ_r, $r>0$, *the space of functions* $d \geq 0$ *such that* $d(x) \leq C(1+|x|)^{-r}$.

We remark that our analysis for the undercompressive case can accomodate data in Δ_r for any $r>1$ (thus integrable), and indeed for the slightly larger set Δ, defined as in [H.2] (Definition 1.3). In particular, we can accomodate integrable data that decays at the non-integrable rate $(1+|x|)^{-1}$.

Definition 1.3. *Denote by* Δ *the space of function* $d \geq 0$ *such that* $d \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $d(\cdot)$ *nonincreasing on* $x \geq 0$, *nondecreasing on* $x \leq 0$, $d(\cdot)$ *even, and* $d(\gamma t) \leq C(\gamma) d(t)$ $\forall \gamma > 0$, $C(\gamma)$ *constant in* t.

For the Lax case, our analysis can accomodate data in Δ_r for any $r > 1/2$ (thus not necessarily integrable), with an extended class similar to that of Definition 1.3.

Definition 1.4. *(Asymptotic stability.) We say that a traveling wave solution* $\bar{u}$ *to (1.1) is asymptotically stable in norm* $||\cdot||$ *if there exists an* $\epsilon > 0$ *such that if another solution,* u, *to (1.1) satisfies* $||u(0,x)-\bar{u}(x)|| < \epsilon$, *then* $||u(t,x)-\bar{u}(x-st)||$ *decays to zero in time.*

Definition 1.5. *(Orbital stability.) We say that a traveling wave solution* $\bar{u}$ *to (1.1) is orbitally stable in norm* $||\cdot||$ *if there exists an* $\epsilon > 0$ *and a translate of* $\bar{u}$, *say* $\bar{u}_l := \bar{u}(x-l)$, *such that if another solution,* u, *to (1.1) satisfies* $||u(0,x)-\bar{u}(x)|| < \epsilon$, *then* $||u(t,x) - \bar{u}_l(x-st)||$ *decays to zero in time.*

Theorem 1.2. *Suppose* $\bar{u}(x-st)$ *is a traveling wave solution to (1.1), with* $b(u) = b_0 > 0$ *constant. If Assumptions* $(\mathcal{H})$ *and Condition* $(\mathcal{D})$ *hold, then we obtain the following results. For data*

$$u_0(x) - \bar{u}(x) \in \mathcal{A}_\zeta := \{v_0(x) : |v_0(x)| \leq \zeta d(x),\ d \in \Delta_r\},$$

ζ *sufficiently small,* $r > 1/2$ *in the Lax case,* $r > 1$ *in the undercompressive case, we have:*

(I) (Incoming waves) If $f'(u_-) > s, x \leq 0$, *or* $f'(u_+) < s, x \geq 0$,

$$|u(t,x+\delta(t)) - \bar{u}(x-st)| \leq C\zeta\Big[t^{-1/3}e^{-\eta t}d(x) + e^{-\frac{\eta}{2}|x-st|}t^{1/2}d(t) + d(|x-st|+t)\Big].$$

(O) (Outgoing waves) If $f'(u_+) > s, x \geq 0$,

$$|u(t,x+\delta(t)) - \bar{u}(x-st)| \leq C\zeta\Big[t^{-1/3}e^{-\eta t}d(x) + e^{-\frac{\eta}{2}|x-st|}t^{1/2}d(t) \\ + \Big[t^{-1/2} \wedge d((x-st)-a_+t) + K(t,(x-st)-a_+t)\Big],$$

where K *is the heat kernel*

$$K(t,x) := (1+t)^{-1/2}e^{-\frac{x^2}{2Mt}},$$

and $|\dot{\delta}(t)| \leq C\zeta d(t)$.

Remark. We note that our proof can be altered to accomodate data that is integrable, but decays only as fast as $(1+|x|)^{-1}$. In this case, we find that the shift

$\delta(t)$ is bounded by $C \ln t$ and thus may be unbounded as $t \to \infty$. Such an effect occurs in general for systems; a similar phenomenon can be seen in the quarter-plane analysis of Liu and Yu [LY]. We remark that such sublinear drift is not inconsistent with hyperbolic rescaling, since the $t \to \infty$ limit still leads to the Riemann solution.

The term $t^{-1/3}e^{-\eta t}d(x)$, which clearly blows up as $t \to 0$, arises from the small time estimates on $G(t, x; y)$ in Theorem 1.1. This term is indicative of the behavior of the Green's function for the Airy equation ($u_t = u_{xxx}$) as $t \to 0$. For non-smooth data (e.g., a delta function), we do not expect good small-time behavior. Our focus here, however, is on large time behavior, and so we allow this blow-up. In Section 7 we outline a method by which it can be dealt with through additional regularity assumptions on $v_0(x)$.

Theorem 1.2 provides two immediate corollaries on stability.

Corollary 1.3. *(Linear Stability.) Under Assumptions ($\mathcal{H}$), linearized L^p orbital stability of viscous shock solutions to (1.1) with respect to perturbations in $\mathcal{A} = L^1$ is equivalent to ($\mathcal{D}$). In the case of stability*

$$v(t, \cdot) - \varphi_L(t, \cdot) \to 0, \quad as \quad t \to \infty$$

with no rate given. For

$$\mathcal{A} := \{v_0(x) : |v_0(x)| \le Cd(x), d \in \Delta_r\},$$

$r > 1/2$ in the Lax case, $r > 1$ in the undercompressive case, we have the following rates ($t \ge T$, some $T > 0$):

(I) (Lax case, $a_- > s > a_+$)

$$\|v(t, \cdot) - \varphi_L(t, \cdot)\|_{L^p} \le Ct^{1/2-r},$$

(II) (Undercompressive case, , a_-, $a_+ > s$)

$$\|v(t, \cdot) - \varphi_L(t, \cdot)\|_{L^p} \le Ct^{-1/2(1-1/p)}.$$

Corollary 1.4. *(Nonlinear stability.) Suppose $\bar{u}(x-st)$ is a traveling wave solution to (1.1), with $b(u) = b_0 > 0$ constant. Then under Assumptions ($\mathcal{H}$) and Condition ($\mathcal{D}$), $\bar{u}(x - st)$ is nonlinearly stable in L^p with respect to data in $u(0, x) - \bar{u}(x) \in \mathcal{A}$,*

$$\mathcal{A} := \{v_0(x) : |v_0(x)| \le \zeta d(x), d \in \Delta_r\},$$

$r > 1/2$ in the Lax case, $r > 1$ in the undercompressive case, with rates of decay in L^p given in Corollary 1.3

Proof of Corollaries 1.3 and 1.4: Corollary 1.3 is proved during the nonlinear analysis of Theorem 1.2. For the Lax case the proof of Corollary 1.4 is immediate from the similar analysis in [H.2]. For the undercompressive case, the computation is routine, and we provide only one indicative example. For the term $t^{-1/2} \wedge d(x -$

$st - a_+t$), we note that clearly in L^∞ norm we have an estimate by $C(1+t^{1/2})^{-1}$. For $1 \le p < \infty$, we compute

$$\int_{-\infty}^{+\infty} \Big(t^{-1/2} \wedge d(y - st - a_+t)\Big)^p dy \le C \int_{-\infty}^{+\infty} (1+t)^{-(1-p)} d(y - st - a_+t) dy$$
$$\le C(1+t^{1/2})^{1-p}.$$

Hence

$$\|t^{-1/2} \wedge d(y - st - a_+t)\|_{L_p} \le C(1+t^{1/2})^{\frac{1-p}{p}}.$$

□

The implications of Theorem 1.2(I) are discussed at length in [H.2] following the statement of Theorem 1.1. We note here that extension to nonconstant diffusion and/or dispersion would *at least* require additional regularity on initial data (Hölder continuity, for example). Moreover, in the case of nonconstant dispersion, there appears a need for an additional restriction, such as $c'(x)+b(x) > \tilde{b}_0 > 0$. Extension to higher order scalar equations and systems are discussed in Section 7.

We remark finally that for the case of systems, there will generally be both impinging characteristics and characteristics passing through the origin so that all terms above arise. To complicate matters more, diffusion waves can signal back to the shock giving rise to *resonant waves* [SX,GSZ]. Though there are still other factors (see, e.g., [L.3]) it would appear that a similar nonlinear analysis could be extended to this crucial case.

Plan of the paper. In Section 2 we provide a basic framework for the analysis, which essentially consists of four tiers of estimates, on: (1) the growth and decay modes for the eigenvalue ODE $Lv = \lambda v$, (2) the Green's function $G_\lambda(x,y)$ for the operator $L - \lambda I$, (3) the time-propagating Green's function $G(t,x;y)$ and (4) the perturbation $v(t,x)$. In Section 3 we carry out estimates (1) and (2), while in Section 4 we carry out estimate (3) for small time. In Section 5 we make estimate (3) for large time, and in Section 6 we estimate the perturbation $v(t,x)$. In the final section, Section 7, we discuss related work, applications and open problems.

2. Preliminary Observations

Our approach to the Green's function estimates will follow [H.1, ZH]. We consider the eigenvalue equation

$$Lv = \lambda v, \tag{2.1}$$

or (1.5) written in terms of L, defined in Condition ($\mathcal{D}$). In particular, we solve the associated Green's function equation

$$(L - \lambda)v = -\delta_y(x).$$

If we let $R(\lambda) := (\lambda - L)^{-1}$ denote the *resolvent operator*, then (2.1) is solved by the Green's function

$$G_\lambda(x,y) = R(\lambda)\delta_y(x)$$

wherever $R(\lambda)$ is defined (whenever $\lambda \notin \sigma(L) :=$ spectrum of L).

We will carry out the computation of $G_\lambda(x,y)$ in terms of the solutions of (2.1). Our notation will be to let ϕ denote the (unique) decay modes associated with (2.1), so that ϕ^+ decays at $+\infty$ and ϕ^- decays at $-\infty$. On the other hand, ψ will denote the growth modes associated with (2.1), so that ψ^+ becomes unbounded at $+\infty$ and ψ^- becomes unbounded at $-\infty$. (Note that away from essential spectrum solutions always either grow or decay exponentially at $\pm\infty$ and that, for example, it may be the case that $\phi^+ = \psi^-$. The essential spectrum boundary is precisely that contour along which these solutions change from decaying to growing and vice versa.)

The first difficulty we face in our analysis is that the asymptotic growth and decay rates of ϕ and ψ are not easily computed in closed form. We note that at $\pm\infty$ (1.5) becomes

$$v_{xxx} + b_\pm v_{xx} - a_\pm v_x - \lambda v = 0,$$

so that solutions of the form $v \sim e^{\mu x}$ give rise to the cubic equation

$$\mu^3 + b_\pm \mu^2 - a_\pm \mu - \lambda = 0, \tag{2.2}$$

which we solve for λ near 0 by Taylor expansion. (We note before doing so that from (2.2) it is straightforward to obtain the essential spectrum boundary contour ($\mu = ik$), denoted Γ_0 (see Figure 5.1), which is the right-most of the contours

$$\lambda_0^\pm(k) = -ik^3 - b_\pm k^2 - ia_\pm k.) \tag{2.3}$$

We find a Taylor expansion for $\mu(\lambda)$ to have the following three expressions:

$$\begin{aligned}
\mu_0(\lambda) &:= -\frac{1}{a_\pm}\lambda + \frac{b_\pm}{a_\pm^3}\lambda^2 + \mathbf{O}(\lambda^3)\\
\mu_-(\lambda) &:= \frac{-b_\pm - \sqrt{b_\pm^2 + 4a_\pm}}{2} + \frac{\lambda}{\frac{1}{4}(b_\pm - \sqrt{b_\pm^2 + 4a_\pm})^2 + a_\pm} + \mathbf{O}(\lambda^2)\\
\mu_+(\lambda) &:= \frac{-b_\pm + \sqrt{b_\pm^2 + 4a_\pm}}{2} + \frac{\lambda}{\frac{1}{4}(b_\pm + \sqrt{b_\pm^2 + 4a_\pm})^2 + a_\pm} + \mathbf{O}(\lambda^2).
\end{aligned}$$

Our notation will be $\mu_j^\pm$, where $\pm$ indicates which asymptotic value of $a(x)$ and $b(x)$ to use, and $Re(\mu_j^\pm) \le Re(\mu_{j+1}^\pm)$ (away from essential spectrum and in a sufficiently small ball around the origin).

For example, in the Lax case ($a_+ < 0 < a_-$) we have $\mu_1^+(\lambda) = \mu_-(\lambda)$, $\mu_2^+(\lambda) = \mu_+(\lambda)$ and $\mu_3^+(\lambda) = \mu_0(\lambda)$, with $Re(\mu_1^+) \le Re(\mu_2^+) \le 0 \le Re(\mu_3^+)$. On the other hand, we have $\mu_1^-(\lambda) = \mu_-(\lambda)$, $\mu_2^-(\lambda) = \mu_0(\lambda)$ and $\mu_3^-(\lambda) = \mu_+(\lambda)$, with $Re(\mu_1^-) \le Re(\mu_2^-) \le 0 \le Re(\mu_3^-)$.

In particular, we see that we have two decay modes and one growth mode at $+\infty$, and two growth modes and one decay mode at $-\infty$. We will denote these by $\phi_1^+, \phi_2^+, \psi_3^+, \phi_3^-, \psi_1^-$, and ψ_2^-, each associated with μ of the same label. Further, we can observe directly from $\mu_0(\lambda), \mu_-(\lambda)$ and $\mu_+(\lambda)$ that in all cases we similarly have this same arrangement of growth and decay modes (the same number and labeling at each end), though the definitions of the $\mu_j^\pm$ change. We note that while $Re(\mu_1^\pm) < 0$ and $Re(\mu_3^\pm) > 0$ in all cases, $Re(\mu_2^\pm)$ will depend upon the case under consideration. We remark that our *spectral gap*, as defined in [GZ], will always be positive so that the Evans function (Wronskian) can be extended (constructed) analytically and uniquely into the essential spectrum [AGJ, GZ, J, KS]. Hence, our analysis will not employ the Gap Lemma of [GZ, KS], but will follow in the spirit of analyses that do.

Under this notation, we can derive a general expression for the Green's function of (2.1) similar to that of [CH].

We write

$$G_\lambda(x,y) = \begin{cases} \phi_1^+(x)A(y) + \phi_2^+(x)B(y) & x \geq y, \\ \phi_3^-(x)C(y) & x \leq y. \end{cases}$$

and employ the continuity of $G_\lambda(x,y)$ and $(\partial/\partial x)G_\lambda(x,y)$, and the jump at $x = y$ in $\partial^2/\partial x^2 G_\lambda(x,y)$, to compute the coefficients $A(y), B(y)$ and $C(y)$ to be

$$\begin{aligned} A(y) &= \frac{\phi_3^-(y)\phi_2^+{}'(y) - \phi_2^+(y)\phi_3^-{}'(y)}{W_y(\lambda)} \\ B(y) &= \frac{\phi_1^+(y)\phi_3^-{}'(y) - \phi_3^-(y)\phi_1^+{}'(y)}{W_y(\lambda)} \qquad {}' := \frac{d}{dy} \\ C(y) &= \frac{\phi_1^+(y)\phi_2^+{}'(y) - \phi_2^+(y)\phi_1^+{}'(y)}{W_y(\lambda)}. \end{aligned}$$

We arrive, then, at the representation

$$G_\lambda(x,y) = \begin{cases} \frac{\phi_1^+(x)[\phi_3^-(y)\phi_2^+{}'(y) - \phi_2^+(y)\phi_3^-{}'(y)]}{W_y(\lambda)} + \frac{\phi_2^+(x)[\phi_1^+(y)\phi_3^-{}'(y) - \phi_3^-(y)\phi_1^+{}'(y)]}{W_y(\lambda)} & x \geq y, \\ \frac{\phi_3^-(x)[\phi_1^+(y)\phi_2^+{}'(y) - \phi_2^+(y)\phi_1^+{}'(y)]}{W_y(\lambda)} & x \leq y, \end{cases}$$

where

$$\begin{aligned} W_\lambda(y) = {} & \phi_1^+\phi_3^-{}'\phi_2^+{}'' + \phi_2^+\phi_1^+{}'\phi_3^-{}'' + \phi_3^-\phi_2^+{}'\phi_1^+{}'' \\ & - \phi_1^+\phi_2^+{}'\phi_3^-{}'' - \phi_2^+\phi_3^-{}'\phi_1^+{}'' - \phi_3^-\phi_1^+{}'\phi_2^+{}'', \end{aligned} \tag{2.4}$$

from which we can conclude $W_y(\lambda) = \mathbf{O}(|\lambda|^{-1})$ for y fixed. Also useful will be Abel's representation of the Wronskian as a solution to the ODE

$$W_\lambda'(y) = -b(y)W_\lambda(y). \tag{2.5}$$

In Sections 4 and 5 we will achieve the estimates of Theorem 1.1 on $G(t,x;y)$ from Dunford's Integral (the resolvent formula for the semigroup, or in many cases simply the inverse Laplace transform of $G_\lambda(x,y)$) [Y], which gives

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)\,d\lambda,$$

where Γ is a contour enclosing the entire spectrum of L (possibly passing through the point at infinity). Dunford's Integral is justified in our case in Lemma 3.7, which relies on estimates made on $G_\lambda(x, y)$ establishing the integrability of $e^{\lambda t} G_\lambda(x, y)$ into the essential spectrum. This computation is of particular importance in the presence of dispersion, where L is non-sectorial. We find that $G_\lambda(x, y)$ remains analytic to the right of a sector. It is this analysis, which we do here directly, that generally involves the Gap Lemma.

Before beginning the analysis we make a brief remark about notation. In all that follows, the terms $\mathbf{O}(\cdot)$ will be uniform in all variables other than the argument. Constants C will be independent of x, y, t and λ, but will often change without mention from one expression to the next. We also note that the values of $d, d_j, j = 1, 2, 3$ for the contour Γ_d will be chosen during the course of the proof of Theorem 1.1. Finally, our notation for the Wronskian will vary between $W_\lambda(y)$ and $W_y(\lambda)$, depending upon which variable is under discussion.

3. Estimates on $\mathbf{G}_\lambda(\mathbf{x}, \mathbf{y})$

In this section we prove a number of lemmas regarding the behavior of solutions of (2.1). These results are all in the context of the consequences $(\mathcal{C})$ taken as assumptions, and L assumed to satisfy Condition $(\mathcal{D})$.

Lemma 3.1. *(Small $|\lambda|$ ODE estimates on solutions of (2.1).) Let $|\lambda| \le M_s$ for some constant M_s, and also let λ lie on or to the right of Γ_d. Under Assumptions $(\mathcal{C})$ and Condition $(\mathcal{D})$, there exist solutions of (1.5), ϕ and ψ, satisfying the following asymptotic estimates ($n, m \le N + 1$, N as in $(\mathcal{C})$; ϕ^+, ψ^+ for $x > 0, \phi^-, \psi^-$ for $x < 0$; and $\mu_j^\pm$ defined as in Section 2 for $j = 1, 2, 3$):*
(i) If $Re(\mu_j^\pm) \lessgtr 0$, then

$$\frac{\partial^n}{\partial x^n} \phi_j^\pm(x) = e^{\mu_j^\pm x}((\mu_j^\pm)^n + \mathbf{O}(e^{-\alpha|x|})),$$

(ii) If $Re(\mu_j^\pm) \gtrless 0$, then

$$\frac{\partial^n}{\partial x^n} \psi_j^\pm(x) = e^{\mu_j^\pm x}((\mu_j^\pm)^n + \mathbf{O}(e^{-\alpha|x|})),$$

(iii) For $i \ne j \ne k, i, j, k = 1, 2, 3$,

$$\frac{\partial^m}{\partial y^m} \frac{W(\theta_i^\pm, \theta_k^\pm)}{W_\lambda(y)} = \frac{\mathbf{O}(1)}{W_\lambda(0)} e^{-\mu_j^\pm y}((-\mu_j^\pm)^m + \mathbf{O}(e^{-\alpha|y|})),$$

where $\theta_i^\pm$ represent either $\phi_j^\pm$ or $\psi_j^\pm$. Moreover, $\phi_j^\pm$ and $\psi_j^\pm$ are analytic in λ for all λ on or to the right of Γ_d.

Proof. The method of proof of (i) and (ii) consists of writing (2.1) as a first order system and setting up a Duhamel's Principle iteration for its solution. As the analysis is similar to that of [C, H] we omit it here. For (iii) we note that according to (i) and (ii) we can write

$$\frac{\partial^m}{\partial y^m} \theta_i^\pm(y) = e^{\mu_i^\pm y}((\mu_i^\pm)^m + \mathbf{O}(e^{-\alpha|y|}))$$
$$\frac{\partial^m}{\partial y^m} \theta_k^\pm(y) = e^{\mu_k^\pm y}((\mu_k^\pm)^m + \mathbf{O}(e^{-\alpha|y|})),$$

for any $\theta_i^{\pm}, \theta_k^{\pm}$.

For $m = 1$ we compute

$$\begin{aligned}
&\frac{\partial}{\partial y}\frac{W(\theta_i^{\pm},\theta_k^{\pm})}{W_\lambda(y)} = \frac{1}{W_\lambda(y)}\Big[\theta_i^{\pm}(y)\theta_k^{\pm}{}''(y) - \theta_i^{\pm}(y){}''\theta_k^{\pm}(y) + b(y)W(\theta_i,\theta_k)\Big] \\
&= \frac{1}{W_\lambda(y)}\Big[e^{\mu_i^{\pm}y}e^{\mu_k^{\pm}y}\Big(((\mu_k^{\pm})^2 + \mathbf{O}(e^{-\alpha|y|})) - ((\mu_i^{\pm})^2 + \mathbf{O}(e^{-\alpha|y|})) \\
&\qquad + b(y)(\mu_k^{\pm} + \mathbf{O}(e^{-\alpha|y|})) - b(y)(\mu_i^{\pm} + \mathbf{O}(e^{-\alpha|y|}))\Big)\Big] \\
&= \frac{e^{(\mu_i^{\pm}+\mu_k^{\pm})y}}{W_\lambda(y)}\Big[(\mu_k^{\pm})^2 - (\mu_i^{\pm})^2 + b(y)(\mu_k^{\pm} - \mu_i^{\pm}) + \mathbf{O}(e^{-\alpha|y|})\Big].
\end{aligned}$$

We recall here that Abel's representation of $W_\lambda(y)$ (2.5) yields

$$W_\lambda(y) = W_\lambda(0)e^{-\int_0^y b(s)ds}.$$

Further, since $\mu_1^{\pm}, \mu_2^{\pm}$ and $\mu_3^{\pm}$ are roots of the asymptotic eigenvalue equation

$$\mu^3 + b_{\pm}\mu^2 - a_{\pm}\mu - \lambda = 0,$$

we have

$$\mu_1^{\pm} + \mu_2^{\pm} + \mu_3^{\pm} = -b_{\pm}.$$

We find that

$$\begin{aligned}
\frac{e^{(\mu_i^{\pm}+\mu_k^{\pm})y}}{W_\lambda(y)} &= \frac{1}{W_\lambda(0)}e^{(\mu_i^{\pm}+\mu_k^{\pm})y}e^{\int_0^y b(s)ds} \\
&= \frac{1}{W_\lambda(0)}e^{-\mu_j^{\pm}y}e^{\int_0^y (b(s)-b_{\pm})ds} = \frac{\mathbf{O}(1)}{W_\lambda(0)}e^{-\mu_j^{\pm}y}.
\end{aligned}$$

Also,

$$\begin{aligned}
(\mu_k^{\pm})^2 - (\mu_i^{\pm})^2 + b(y)(\mu_k^{\pm} - \mu_i^{\pm}) &= (\mu_k^{\pm} - \mu_i^{\pm})(\mu_i^{\pm} + \mu_k^{\pm} + b(y)) \\
&= (\mu_k^{\pm} - \mu_i^{\pm})(-\mu_j^{\pm} + \mathbf{O}(e^{-\alpha|y|})).
\end{aligned}$$

Combining these last two observations, we have the claim for $m = 1$. A counting argument that keeps track of derivatives reveals that more generally

$$\frac{\partial^m}{\partial y^m}\frac{W(\theta_i^{\pm},\theta_k^{\pm})}{W_\lambda(y)} = \frac{1}{W_\lambda(y)}\Big[(\mu_k^{\pm} - \mu_i^{\pm})(\mu_i^{\pm} + \mu_k^{\pm} + b(y))^m + \mathbf{O}(e^{-\alpha|y|})\Big],$$

which yields the claimed result directly for arbitrary m. □

In the next lemma we employ a scaling argument similar to that of [GZ, JGK, ZH], though here we must expand to higher order. As this extension is crucial to the analysis, we provide some detail in the proof. An important element of the small time (large $|\lambda|$) analysis is that even though L is not a sectorial operator, large-$|\lambda|$ solutions to (2.1) and consequently $G_\lambda(x, y)$ are analytic to the right of a sector. We will later refer to this critical property as *quasi-sectorality* of L.

Lemma 3.2. *(Large $|\lambda|$ ODE estimates on solutions of (2.1).) Under Assumptions $(\mathcal{C})$ and Condition $(\mathcal{D})$, decaying solutions of (2.1) satisfy the following estimates: For λ on or to the right of a sector $\mathcal{S} = \{\lambda : Re\lambda = 2M_l - |Im\lambda|\}$, we have, for some $k_i^{\pm}(x)$ bounded in λ*

$$(i)\frac{\partial^n}{\partial x^n}\phi_1^+(x) = k_1^+(x)(\sqrt[3]{\lambda}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) - \frac{b(x)}{3})^n(1 + \mathbf{O}(|\lambda|^{-2/3}))$$

$$(ii)\frac{\partial^n}{\partial x^n}\phi_2^+(x) = k_2^+(x)(\sqrt[3]{\lambda}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) - \frac{b(x)}{3})^n(1 + \mathbf{O}(|\lambda|^{-2/3}))$$

$$(iii)\frac{\partial^n}{\partial x^n}\phi_3^-(x) = k_3^-(x)(\sqrt[3]{\lambda} - \frac{b(x)}{3})^n(1 + \mathbf{O}(|\lambda|^{-2/3}))$$

$$(iv)\quad \frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} = \frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}\Big((-\sqrt[3]{\lambda} + b(y)/3)^m + \mathbf{O}(|\lambda|^{\frac{m-2}{3}})\Big)$$

$$(v)\quad \frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_3^-)}{W_\lambda(y)} = \frac{W(\phi_1^+,\phi_3^-)}{W_\lambda(y)}\Big((\sqrt[3]{\lambda}(+\frac{1}{2} - i\frac{\sqrt{3}}{2}) + b(y)/3)^m + \mathbf{O}(|\lambda|^{\frac{m-2}{3}})\Big)$$

$$(vi)\quad \frac{\partial^m}{\partial y^m}\frac{W(\phi_2^+,\phi_3^-)}{W_\lambda(y)} = \frac{W(\phi_2^+,\phi_3^-)}{W_\lambda(y)}\Big((\sqrt[3]{\lambda}(+\frac{1}{2} + i\frac{\sqrt{3}}{2}) + b(y)/3)^m + \mathbf{O}(|\lambda|^{\frac{m-2}{3}})\Big).$$

Before proceeding with the proof of Lemma 3.2 we note that a useful representation of the first three estimates is:

$$(i)'\quad \frac{\partial^n}{\partial x^n}\phi_1^+(x) = \phi_1^+(x)\Big((\sqrt[3]{\lambda}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) - b(x)/3)^n + \mathbf{O}(|\lambda|^{\frac{n-2}{3}})\Big)$$

$$(ii)'\quad \frac{\partial^n}{\partial x^n}\phi_2^+(x) = \phi_2^+(x)\Big((\sqrt[3]{\lambda}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) - b(x)/3)^n + \mathbf{O}(|\lambda|^{\frac{n-2}{3}})\Big)$$

$$(iii)'\quad \frac{\partial^n}{\partial x^n}\phi_3^-(x) = \phi_3^-(x)\Big((\sqrt[3]{\lambda} - b(x)/3)^n + \mathbf{O}(|\lambda|^{\frac{n-2}{3}})\Big).$$

Proof. We give the analysis in detail for Cases (iii) and (iv) only, beginning with Case (iii). Writing (2.1) as a first order system of ODE's with $v_1 = v$, $v_2 = v_x$, $v_3 = v_{xx}$, we obtain the matrix equation

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda + a'(x) & a(x) - b'(x) & -b(x) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

We let $A(x,\lambda)$ represent this matrix. For the large $|\lambda|$ case we make the scale change $x \mapsto x/\sqrt[3]{|\lambda|}$, which yields the matrix equation

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{\lambda} + \frac{a'(x/\sqrt[3]{|\lambda|})}{|\lambda|} & \frac{a(x/\sqrt[3]{|\lambda|}) - b'(x/\sqrt[3]{|\lambda|})}{|\lambda|^{2/3}} & -\frac{b(x/\sqrt[3]{|\lambda|})}{|\lambda|^{1/3}} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

where $\tilde{\lambda} := \lambda/|\lambda|$ and we let $\tilde{A}(x,\lambda)$ represent this matrix. We write

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{\lambda} & 0 & -\frac{b(x/\sqrt[3]{|\lambda|})}{|\lambda|^{1/3}} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a'(x/\sqrt[3]{|\lambda|})}{|\lambda|} & \frac{a(x/\sqrt[3]{|\lambda|}) - b'(x/\sqrt[3]{|\lambda|})}{|\lambda|^{2/3}} & 0 \end{pmatrix},$$

where we let $M(x,\lambda)$ denote the first matrix and note that the second is $\mathbf{O}(|\lambda|^{-2/3})$. The eigenvalues of $M(x,\lambda)$ can readily be found to be (given so that on our λ domain $Re(\tilde{\mu}_1), Re(\tilde{\mu}_2) \leq 0 \leq Re(\tilde{\mu}_3)$)

$$\begin{aligned}
\tilde{\mu}_1 &= \sqrt[3]{\tilde{\lambda}}(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) - \frac{b(x/\sqrt[3]{|\lambda|})}{3|\lambda|^{1/3}} + \mathbf{O}(|\lambda|^{-2/3})) \\
\tilde{\mu}_2 &= \sqrt[3]{\tilde{\lambda}}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) - \frac{b(x/\sqrt[3]{|\lambda|})}{3|\lambda|^{1/3}} + \mathbf{O}(|\lambda|^{-2/3}) \\
\tilde{\mu}_3 &= \sqrt[3]{\tilde{\lambda}} - \frac{b(x/\sqrt[3]{|\lambda|})}{3|\lambda|^{1/3}} + \mathbf{O}(|\lambda|^{-2/3})).
\end{aligned}$$

Such a computation is easily made by viewing $b(x/\sqrt[3]{|\lambda|})/|\lambda|^{1/3}$ as an independent variable of μ (which is then small for large $|\lambda|$) and considering its Tayor expansion around $b(x/\sqrt[3]{|\lambda|})/|\lambda|^{1/3} = 0$.

The eigenvectors associated with $\tilde{\mu}_1, \tilde{\mu}_2$ and $\tilde{\mu}_3$ are respectively

$$\begin{pmatrix} 1 \\ \tilde{\mu}_1 \\ \tilde{\mu}_1^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_2^2 \end{pmatrix} \; and \; \begin{pmatrix} 1 \\ \tilde{\mu}_3 \\ \tilde{\mu}_3^2 \end{pmatrix}.$$

We let

$$P := \begin{pmatrix} 1 & 1 & 1 \\ \tilde{\mu}_3 & \tilde{\mu}_2 & \tilde{\mu}_1 \\ \tilde{\mu}_3^2 & \tilde{\mu}_2^2 & \tilde{\mu}_1^2 \end{pmatrix},$$

so that

$$P^{-1}MP = D = \begin{pmatrix} \tilde{\mu}_3 & 0 & 0 \\ 0 & \tilde{\mu}_2 & 0 \\ 0 & 0 & \tilde{\mu}_1 \end{pmatrix}.$$

Now, we make the tranformation $W := P^{-1}(x)V(x)$ and compute

$$\begin{aligned}
W' &= (P^{-1}V)' = P^{-1}V' + (P^{-1})'V \\
&= P^{-1}(MV + \mathbf{O}(|\lambda|^{-2/3})V) + (P^{-1})'V \\
&= P^{-1}MPW + P^{-1}\mathbf{O}(|\lambda|^{-2/3})PW + (P^{-1})'PW \\
&= DW + \mathbf{O}(|\lambda|^{-2/3})W.
\end{aligned}$$

We let $z_2 := W_2/W_1$ and $z_3 := W_3/W_1$ and compute

$$\begin{aligned}
z_2' &= \frac{W_1W_2' - W_2W_1'}{W_1^2} = \frac{W_2'}{W_1} - \frac{W_2}{W_1}\frac{W_1'}{W_1} \\
&= \frac{\mu_2 W_2 + \mathbf{O}(|\lambda|^{-2/3})W_1 + \mathbf{O}(|\lambda|^{-2/3})W_2 + \mathbf{O}(|\lambda|^{-2/3})W_3}{W_1} \\
&\quad - z_2\frac{\tilde{\mu}_3 W_1 + \mathbf{O}(|\lambda|^{-2/3})W_1 + \mathbf{O}(|\lambda|^{-2/3})W_2 + \mathbf{O}(|\lambda|^{-2/3})W_3}{W_1} \\
&= (\tilde{\mu}_2 - \tilde{\mu}_3)z_2 + \mathbf{O}(|\lambda|^{-2/3}) + z_2\mathbf{O}(|\lambda|^{-2/3}) \\
&\quad + z_2^2\mathbf{O}(|\lambda|^{-2/3}) + z_3\mathbf{O}(|\lambda|^{-2/3}) + z_2z_3\mathbf{O}(|\lambda|^{-2/3}).
\end{aligned}$$

Similarly,

$$\begin{aligned} z_3' &= (\tilde\mu_1 - \tilde\mu_3) z_3 + \mathbf{O}(|\lambda|^{-2/3}) + z_2 \mathbf{O}(|\lambda|^{-2/3}) \\ &\quad + z_3 \mathbf{O}(|\lambda|^{-2/3}) + z_3^2 \mathbf{O}(|\lambda|^{-2/3}) + z_2 z_3 \mathbf{O}(|\lambda|^{-2/3}). \end{aligned}$$

Thus the key observation is the gap between modes

$$\tilde\mu_2 - \tilde\mu_3 = \sqrt[3]{\tilde\lambda}\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right) + \mathbf{O}(|\lambda|^{-2/3}).$$

Similarly,

$$\tilde\mu_1 - \tilde\mu_3 = \sqrt[3]{\tilde\lambda}\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) + \mathbf{O}(|\lambda|^{-2/3}).$$

Now, let $Z := \begin{pmatrix} z_2(x) \\ z_3(x) \end{pmatrix}$ and let $F(z_2, z_3)$ be the nonlinear term:

$$\begin{aligned} F(z_2,z_3) &= \begin{pmatrix} f_1(z_2,z_3) \\ f_2(z_2,z_3) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_2\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_3\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_2^2\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_2 z_3\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \\ \mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_2\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_3\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_3^2\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + z_2 z_3\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \end{pmatrix}. \end{aligned}$$

We have the matrix equation

$$Z'(x) = \begin{pmatrix} \tilde\mu_2 - \tilde\mu_3 & 0 \\ 0 & \tilde\mu_1 - \tilde\mu_3 \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} + F(z_2, z_3).$$

We denote this diagonal matrix L_d and employ Duhamel's Principle to arrive at the integral equation

$$Z(x) = \int_{-\infty}^{x} e^{\int_\xi^x L_d(s)ds} F(z_2(\xi), z_3(\xi)) d\xi,$$

where we have used that $Z(-\infty) = 0$. We define the operator T by

$$TZ := \int_{-\infty}^{x} e^{\int_\xi^x L_d(s)ds} F(z_2(\xi), z_3(\xi)) d\xi.$$

Taking $Z, \tilde Z \in L^\infty(-\infty, +\infty)$, we show that T is a contraction mapping on the space $L^\infty(-\infty, +\infty)$ by computing

$$\begin{aligned} TZ - T\tilde Z &= \int_{-\infty}^{x} e^{\int_\xi^x L_d(s)ds}\Big[F(z_2(\xi), z_3(\xi)) - F(\tilde z_2(\xi), \tilde z_3(\xi))\Big] d\xi \\ &= \int_{-\infty}^{x} e^{\int_\xi^x L(s)ds} \begin{pmatrix} \mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_2 - \tilde z_2)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_3 - \tilde z_3)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \\ \mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_2 - \tilde z_2)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \end{pmatrix} d\xi \\ &+ \int_{-\infty}^{x} e^{\int_\xi^x L(s)ds} \\ &\times \begin{pmatrix} (z_2^2 - \tilde z_2^2)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_2 z_3 - \tilde z_2 \tilde z_3)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \\ (z_3 - \tilde z_3)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_3^2 - \tilde z_3^2)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) + (z_2 z_3 - \tilde z_2 \tilde z_3)\mathbf{O}(|\lambda|^{-\frac{2}{3}}) \end{pmatrix} d\xi. \end{aligned}$$

For the terms $z_2 z_3 - \tilde{z}_2 \tilde{z}_3$, we use the representation

$$z_2 z_3 - \tilde{z}_2 \tilde{z}_3 = \frac{1}{2}(z_2 - \tilde{z}_2)(z_3 + \tilde{z}_3) + \frac{1}{2}(z_3 - \tilde{z}_3)(z_2 + \tilde{z}_2).$$

Since $Z, \tilde{Z} \in L^\infty(-\infty, +\infty)$ we have

$$\begin{aligned} ||TZ - T\tilde{Z}||_{L^\infty} &\le C||Z - \tilde{Z}||_{L^\infty} \Big\| \int_{-\infty}^{x} e^{\int_\xi^x L(s)ds} \mathbf{O}(|\lambda|^{-2/3}) d\xi \Big\| \\ &\le \mathbf{O}(|\lambda|^{-2/3}) \|Z - \tilde{Z}\| \Big\| \int_{-\infty}^{x} e^{\int_\xi^x L(s)ds} d\xi \Big\|. \end{aligned}$$

Thus the idea is to show that this normed integral is bounded so that for $|\lambda|$ large enough a contraction mapping is obtained. From our representations of $\mu_2 - \mu_3$ and $\mu_1 - \mu_3$ we have

$$\begin{aligned} &\int_\xi^x \begin{pmatrix} \mu_2 - \mu_3 & 0 \\ 0 & \mu_1 - \mu_3 \end{pmatrix} \\ &= \int_\xi^x \begin{pmatrix} \sqrt[3]{\tilde{\lambda}}(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + \mathbf{O}(|\lambda|^{-2/3}) & 0 \\ 0 & \sqrt[3]{\tilde{\lambda}}(-\frac{3}{2} - i\frac{\sqrt{3}}{2}) + \mathbf{O}(|\lambda|^{-2/3}) \end{pmatrix} ds \\ &= \begin{pmatrix} \sqrt[3]{\tilde{\lambda}}(-\frac{3}{2} + i\frac{\sqrt{3}}{2})(x - \xi) & 0 \\ 0 & \sqrt[3]{\tilde{\lambda}}(-\frac{3}{2} - i\frac{\sqrt{3}}{2})(x - \xi) \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{O}(|\lambda|^{-2/3})(x - \xi) & 0 \\ 0 & \mathbf{O}(|\lambda|^{-2/3})(x - \xi) \end{pmatrix}, \end{aligned}$$

where $\mathbf{O}(|\lambda|^{-2/3})$ depends on s, but in a bounded manner. Hence, integrating $\|e^{\int_\xi^x L_d(s)ds}\|$ yields a bound of the form

$$\Big| Re(\sqrt[3]{\tilde{\lambda}(-\frac{3}{2} + i\frac{\sqrt{3}}{2})}) + \mathbf{O}(|\lambda|^{-2/3}) \Big|^{-1} + \Big| Re(\sqrt[3]{\tilde{\lambda}(-\frac{3}{2} - i\frac{\sqrt{3}}{2})}) + \mathbf{O}(|\lambda|^{-2/3}) \Big|^{-1},$$

so that

$$\begin{aligned} &\|TZ - T\tilde{Z}\|_{L^\infty} \\ &\quad \le \|Z - \tilde{Z}\| \Big[\mathbf{O}(|\lambda|^{-2/3}) \Big| Re(\sqrt[3]{\tilde{\lambda}(-\frac{3}{2} + i\frac{\sqrt{3}}{2})}) + \mathbf{O}(|\lambda|^{-2/3}) \Big|^{-1} \\ &\quad + \mathbf{O}(|\lambda|^{-2/3}) \Big| Re(\sqrt[3]{\tilde{\lambda}(-\frac{3}{2} - i\frac{\sqrt{3}}{2})}) + \mathbf{O}(|\lambda|^{-2/3}) \Big|^{-1} \Big]. \end{aligned}$$

We can conclude the sought estimate if we can show that both

$$\frac{|\lambda|^{-1/3}}{|Re(\sqrt[3]{\lambda}(-\frac{3}{2} + i\frac{\sqrt{3}}{2}))|}$$

and the complex conjugate term can be made arbitrarily small by taking $|\lambda|$ arbitrarily large in $\mathcal{S}$. That is, we must remain to the right of a contour such that

$$\Big| Re(\sqrt[3]{\lambda}(\frac{3}{2}+i\frac{\sqrt{3}}{2}))\Big| \geq \delta > 0.$$

It is clear from the definitions of $\mathcal{S}$ and $\sqrt[3]{\lambda}$ (the value of $\lambda^{1/3}$ with largest positive real part) that inside $\mathcal{S}$ we have $Re\sqrt[3]{\lambda} \geq |Im\sqrt[3]{\lambda}|$. For $\sqrt[3]{\lambda} = a+ib$, we have, for example,

$$Re\sqrt[3]{\lambda}(-\frac{3}{2}+i\frac{\sqrt{3}}{2}) = -\frac{3}{2}a - \frac{\sqrt{3}}{2}b,$$

which is bounded away from zero for $a \geq |b|$.

Finally, we note the computation

$$V = PW = W_1(x)\begin{pmatrix}1\\ \tilde{\mu}_3\\ \tilde{\mu}_3^2\end{pmatrix} + W_2(x)\begin{pmatrix}1\\ \tilde{\mu}_2\\ \tilde{\mu}_2^2\end{pmatrix} + W_3(x)\begin{pmatrix}1\\ \tilde{\mu}_1\\ \tilde{\mu}_1^2\end{pmatrix},$$

so that

$$\begin{pmatrix}V_1\\ V_2\\ V_3\end{pmatrix} = W_1(x)\Big[\begin{pmatrix}1\\ \tilde{\mu}_3\\ \tilde{\mu}_3^2\end{pmatrix} + \mathbf{O}(|\lambda|^{-2/3})\Big],$$

which yields the result after reverting to the original coordinates. Cases (i) and (ii) follow similarly.

As the proof of each of (iv)–(vi) is similar, we carry out the details only for (iv). The proof of (iv) is similar in nature to the dual eigenfunction estimates of Lemma 3.1. We begin by making the definitions

$$\begin{aligned}\mu_1 &:= \sqrt[3]{\lambda}(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) - b(y)/3\\ \mu_2 &:= \sqrt[3]{\lambda}(-\frac{1}{2}+i\frac{\sqrt{3}}{2}) - b(y)/3\\ \mu_3 &:= \sqrt[3]{\lambda} - b(y)/3.\end{aligned}$$

From the proof of Lemma 3.1 (iii) we have

$$\begin{aligned}&\frac{\partial}{\partial y}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} = \frac{1}{W_\lambda(y)}\Big[\phi_1^+\phi_2^{+\,\prime\prime} - \phi_1^{+\,\prime\prime}\phi_2^+ + b(y)(\phi_1^+\phi_2^{+\,\prime} - \phi_1^{+\,\prime}\phi_2^+)\Big]\\ &= \frac{\phi_1^+(y)\phi_2^+(y)}{W_\lambda(y)}\Big[(\mu_2^2+\mathbf{O}(1))\\ &\quad - (\mu_1^2+\mathbf{O}(1)) + b(y)(\mu_2+\mathbf{O}(|\lambda|^{-1/3})) - b(y)(\mu_1+\mathbf{O}(|\lambda|^{-1/3}))\Big]\\ &= \frac{\phi_1^+\phi_2^+}{W_\lambda(y)}\Big[(\mu_2-\mu_1)(\mu_1+\mu_2+b(y)+\mathbf{O}(|\lambda|^{-1/3}))\Big].\end{aligned}$$

We now make two observations. First

$$W(\phi_1^+,\phi_2^+) = \phi_1^+\phi_2^{+\,\prime} - \phi_1^{+\,\prime}\phi_2^+ = \phi_1^+\phi_2^+(\mu_2-\mu_1+\mathbf{O}(|\lambda|^{-1/3})).$$

Also,

$$\mu_1 + \mu_2 + \mu_3 = -b(y) \quad \Rightarrow \mu_1 + \mu_2 + b(y) = -\mu_3,$$

so that

$$\begin{aligned}\frac{\partial}{\partial y}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} &= \frac{W(\phi_1^+,\phi_2^+) + \mathbf{O}(|\lambda|^{-1/3})}{W_\lambda(y)}\Big(-\mu_3^+ + \mathbf{O}(|\lambda|^{-1/3})\Big)\\ &= \frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}\Big(-\mu_3 + \mathbf{O}(|\lambda|^{-1/3})\Big),\end{aligned}$$

where we have observed that

$$\mathbf{O}(|\lambda|^{-1/3})\mu_3 = W(\phi_1^+,\phi_2^+)\mathbf{O}(|\lambda|^{-1/3}).$$

As in the proof of Lemma 2.1, we use a counting argument for higher order derivatives to see that

$$\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} = \frac{\phi_1^+\phi_2^+}{W_\lambda(y)}\Big[(\mu_2-\mu_1)(\mu_1+\mu_2+b(y)+\mathbf{O}(|\lambda|^{-1/3}))^m\Big],$$

which yields the claimed estimate. □

Lemma 3.3. *Under Assumptions* $(\mathcal{C})$ *and Condition* $(\mathcal{D})$*, we have: (1) for* λ *on or to the right of* Γ_d*, and moreover for* λ *on or to the right of the sector* $\mathcal{S} = \{\lambda : Re\lambda = 2M_l - |Im\lambda|\}$*,* $W_y(\lambda)$ *is analytic in* λ*; and (2) for* d, d_i *appropriately chosen in* Γ_d*, the set* $\{\lambda : \lambda \neq 0, W_y(\lambda) = 0\}$ *lies strictly to the left of* Γ_d*.*

Proof. We note that (1) is immediate, as analyticity of $W_y(\lambda)$ follows directly from (2.4) and the analyticity in λ of $\phi_i^\pm$, $\phi_i^{\pm\,\prime}$, and $\phi_i^{\pm\,\prime\prime}$. As the proof of Lemma 3.1 was not provided in its entirety, we mention that this is a result of the observation that $\mu_j - \mu_i\Big|_{\lambda=0} \neq 0$ for $i \neq j$, since only one can vanish at $\lambda = 0$.

As for (2) the essential spectrum is bounded on or to the left of the contour Γ_0, so that any zeros of the Wronskian lying to the right of this contour must be point spectrum, limiting them to the negative real half-plane, by Assumption (II). Further, there can be only finitely many of these zeros in a ball around the origin, because in such a ball the Wronskian is a non-trivial analytic function of λ and hence can have only isolated zeros in any bounded neighborhood. An energy estimate, or the large $|\lambda|$ estimates of Lemma 3.6 below, suffices to show that all such zeros are confined to a bounded domain. Consequently, we can enclose all zeros of $W_y(\lambda)$ within a contour of the form of Γ_d, for d, d_i appropriately chosen. □

The following proof follows closely the proof of Lemma 3.4 of [H.1].

Lemma 3.4. *(Small* $|\lambda|$ *Green's function estimates.) Let* $|\lambda| \leq r$ *for* r *sufficiently small. Under Assumptions* $(\mathcal{C})$ *and Condition* $(\mathcal{D})$*, we have the following estimates on the Green's function* $G_\lambda(x,y)$ *for (2.1):*

(L+) Lax Case ($a_+ < 0 < a_-, x \geq 0$)

(i) $y \leq 0 \leq x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^n e^{\mu_2^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}$$

(ii) $0 \leq y \leq x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1) e^{\mu_2^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}(\mu_3^+)^m e^{\mu_2^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}$$

(iii) $0 \leq x \leq y$

$$\begin{aligned}\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) &= \mathbf{O}(1)(\mu_3^+)^{n+m} e^{\mu_3^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_3^+)^m e^{\mu_3^+(x-y)} \\ &\quad + \mathbf{O}(e^{-\eta|y|})(\mu_3^+)^n e^{\mu_3^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}\end{aligned}$$

(L-) Lax Case ($a_+ < 0 < a_-, x < 0$)

(i) $y \leq x \leq 0$

$$\begin{aligned}\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) &= \mathbf{O}(1)(\mu_2^-)^{n+m} e^{\mu_2^-(x-y)} + \mathbf{O}(e^{-\eta|x-y|}) e^{\mu_2^-(x-y)} \\ &\quad + \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^m e^{\mu_2^-(x-y)} + \mathbf{O}(e^{-\eta|y|})(\mu_2^-)^n e^{\mu_2^-(x-y)} \\ &\quad + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}\end{aligned}$$

(ii) $x \leq y \leq 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1) e^{\mu_3^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}(\mu_2^-)^n e^{\mu_3^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}$$

(iii) $x \leq 0 \leq y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_3^+)^n e^{\mu_3^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}$$

(U+) Undercompressive Case ($a_-, a_+ > 0, x > 0$)

(i) $y \leq 0 \leq x$

$$\begin{aligned}\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) &= \mathbf{O}(1)(\mu_2^-)^m (\mu_2^+)^n e^{\mu_2^+ x} e^{-\mu_2^- y} + \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^m e^{\mu_2^+ x} e^{-\mu_2^- y} \\ &\quad + \mathbf{O}(e^{-\eta|y|})(\mu_2^+)^n e^{\mu_2^+ x} e^{-\mu_2^- y} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}\end{aligned}$$

(ii) $0 \leq y \leq x$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1)(\mu_2^+)^{n+m} e^{\mu_2^+(x-y)} + \mathbf{O}(e^{-\eta|x|})(\mu_2^+)^m e^{\mu_2^+(x-y)} + \mathbf{O}(e^{-\eta|y|})(\mu_2^+)^n e^{\mu_2^+(x-y)} + \mathbf{O}(e^{-\eta|x-y|}) e^{\mu_2^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}$$

(iii) $0 \leq x \leq y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1) e^{\mu_3^+(x-y)} + \mathbf{O}(1)(\mu_2^+)^n e^{\mu_2^+ x} e^{-\mu_3^+ y} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}$$

(U-) Undercompressive Case ($a_-, a_+ > 0, x \leq 0$)

(i) $y \leq x \leq 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1)(\mu_2^-)^{n+m} e^{\mu_2^-(x-y)} + \mathbf{O}(e^{-\eta|x-y|}) e^{\mu_2^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^m e^{\mu_2^-(x-y)} + \mathbf{O}(e^{-\eta|y|})(\mu_2^-)^n e^{\mu_2^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|y|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}$$

(ii) $x \leq y \leq 0$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(1) e^{\mu_3^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|y|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}$$

(iii) $x \leq 0 \leq y$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_3^- x} e^{-\mu_3^+ y}.$$

Proof. Aside from some technical details the analysis separates into two cases: incoming and outgoing waves. Hence, it will suffice to study the Lax case with $x \geq 0$ and the undercompressive case with $x \geq 0$.

Incoming waves. We begin with the Lax case, and observe that here a key observation is that $\mu_1^+, \mu_2^+, \mu_1^-, \mu_3^- = \mathbf{O}(1)$ and $\mu_2^-, \mu_3^+ = \mathbf{O}(\lambda)$. For the first Lax subcase, *L+(i)*, $y \leq 0 \leq x$, we have from Section 2

$$G_\lambda(x,y) = \frac{\phi_1^+(x) W(\phi_3^-, \phi_2^+)}{W_\lambda(y)} + \frac{\phi_2^+(x) W(\phi_1^+, \phi_3^-)}{W_\lambda(y)}, \tag{3.1}$$

where for $y \leq 0$ we must write

$$\begin{aligned} \phi_1^+(y) &= A(\lambda)\psi_1^-(y) + B(\lambda)\psi_2^-(y) + C(\lambda)\phi_3^-(y) \\ \phi_2^+(y) &= D(\lambda)\psi_1^-(y) + E(\lambda)\psi_2^-(y) + F(\lambda)\phi_3^-(y), \end{aligned} \tag{3.2}$$

and analyze the expansion coefficients. Augmenting our expansion of $\phi_1^+(y)$ with first and second derivatives, we arrive at the matrix equation

$$\begin{pmatrix} \psi_1^-(y) & \psi_2^-(y) & \phi_3^-(y) \\ \psi_1^{-\prime}(x) & \psi_2^{-\prime}(x) & \phi_3^{-\prime}(x) \\ \psi_1^{-\prime\prime}(x) & \psi_2^{-\prime\prime}(x) & \phi_3^{-\prime\prime}(x) \end{pmatrix} \begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} = \begin{pmatrix} \phi_1^+(x) \\ \phi_1^{+\prime}(x) \\ \phi_1^{+\prime\prime}(x) \end{pmatrix}.$$

which can be evalutated for $A(\lambda), B(\lambda)$ and $C(\lambda)$ through Cramer's Rule of determinants. Since we are in the regime of bounded λ where these coefficients are clearly bounded, the essential issue becomes whether or not they vanish at $\lambda = 0$, and hence eliminate the pole in $G_\lambda(x, y)$ caused by the Wronskian becoming zero at $\lambda = 0$. In this case, we find that none do. Our expression for $G_\lambda(x, y)$ becomes

$$\begin{aligned} G_\lambda(x, y) = {} & \frac{\phi_1^+(x) W(\phi_3^-, D(\lambda)\psi_1^- + E(\lambda)\psi_2^-)}{W_\lambda(y)} \\ & + \frac{\phi_2^+(x) W(A(\lambda)\psi_1^- + B(\lambda)\psi_2^-, \phi_3^-)}{W_\lambda(y)}, \end{aligned}$$

so that (as a consequence of Lemma 3.1)

$$\begin{aligned} & \frac{\partial^{n+m}}{\partial x^n y^m} G_\lambda(x, y) = \\ & \quad D(\lambda)\phi_1^{+\,(n)}(x) \frac{\partial^m}{\partial y^m} \frac{W(\phi_3^-, \psi_1^-)}{W_\lambda(y)} + E(\lambda)\phi_1^{+\,(n)}(x) \frac{\partial^m}{\partial y^m} \frac{W(\phi_3^-, \psi_2^-)}{W_\lambda(y)} \\ & \quad + A(\lambda)\phi_2^{+\,(n)}(x) \frac{\partial^m}{\partial y^m} \frac{W(\psi_1^-, \phi_3^-)}{W_\lambda(y)} + B(\lambda)\phi_2^{+\,(n)}(x) \frac{\partial^m}{\partial y^m} \frac{W(\psi_2^-, \phi_3^-)}{W_\lambda(y)} \\ & \quad = \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^- y}((\mu_2^-)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^- y}((\mu_1^-)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^- y}((\mu_2^-)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^- y}((\mu_1^-)^m + \mathbf{O}(e^{-\alpha|y|})). \end{aligned}$$

Noting that $e^{\mu_1^+ x}, e^{\mu_2^+ x} = \mathbf{O}(e^{-\eta|x|})$ and $e^{-\mu_1^- y} = \mathbf{O}(e^{-\eta|y|})$, some η, we obtain

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x, y) = \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)} (\mu_2^-)^m e^{\mu_2^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.$$

We next consider $L+$ *(ii)*, $0 \le y \le x$. Again, we begin with (3.1), but for $y \ge 0$ we must write

$$\phi_3^-(y) = A(\lambda)\phi_1^+(y) + B(\lambda)\phi_2^+(y) + C(\lambda)\psi_3^+(y), \tag{3.3}$$

where $A(\lambda), B(\lambda)$ and $C(\lambda)$ have been recycled. We recall here the important fact about conservation laws that due to translation invariance, $\bar{u}_x$ is an eigenfunction for the eigenvalue $\lambda = 0$. Therefore at $\lambda = 0$ we must be able to write $\bar{u}_x$ as a linear combination of decaying modes at both $\pm\infty$. That is,

$$D(\lambda)\phi_3^-(y) = \bar{u}_x = E(\lambda)\phi_1^+(y) + F(\lambda)\phi_2^+(y).$$

Analyzing $A(\lambda), B(\lambda)$ and $C(\lambda)$ as before, we find that $C(\lambda)$ in this case is proportional to the Wronskian, or Evans function, and so vanishes at $\lambda = 0$.

Our expression for $G_\lambda(x, y)$ becomes

$$\begin{aligned} G_\lambda(x,y) = & \frac{\phi_1^+(x)W(A(\lambda)\phi_1^+ + C(\lambda)\psi_3^+, \phi_2^+)}{W_\lambda(y)} \\ & + \frac{\phi_2^+(x)W(\phi_1^+(x), B(\lambda)\phi_2^+ + C(\lambda)\psi_3^+)}{W_\lambda(y)}, \end{aligned}$$

which yields (as a consequence of Lemma 3.1 and $C(\lambda) \sim W_0(\lambda)$)

$$\begin{aligned} & \frac{\partial^{n+m}}{\partial x^n y^m} G_\lambda(x,y) = \\ & \quad A(\lambda)\phi_1^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} + C(\lambda)\phi_1^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\psi_3^+,\phi_2^+)}{W_\lambda(y)} \\ & \quad + B(\lambda)\phi_2^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} + C(\lambda)\phi_2^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\psi_3^+)}{W_\lambda(y)} \\ & \quad = \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \mathbf{O}(1)e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^+ y}((\mu_1^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\ & \quad + \mathbf{O}(1)e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^+ y}((\mu_2^+)^m + \mathbf{O}(e^{-\alpha|y|})). \end{aligned}$$

We observe that $\mu_1^+, \mu_2^+ = \mathbf{O}(1)$, while $\mu_3^+ = \mathbf{O}(\lambda)$, so that we obtain

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \frac{\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}(\mu_2^+)^m e^{\mu_3^+(x-y)} + \mathbf{O}(1)e^{\mu_2^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.$$

The final Lax case we consider is *L+(iii)*, $0 \le x \le y$. Here, we use the representation for $G_\lambda(x,y)$

$$G_\lambda(x,y) = \frac{\phi_3^-(x)W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}, \tag{3.4}$$

where for $x \ge 0$ we must write

$$\phi_3^-(x) = A(\lambda)\phi_1^+(x) + B(\lambda)\phi_2^+(x) + C(\lambda)\psi_3^+(x). \tag{3.5}$$

As in the case *L+ (ii)* $C(\lambda)=0$ for $\lambda=0$. We wrote $G_\lambda(x,y)$ as

$$G_\lambda(x,y)=\frac{(A(\lambda)\phi_1^+(x)+B(\lambda)\phi_2^+(x)+C(\lambda)\psi_3^+(x))W(\phi_1^+,\phi_2^+)}{W_\lambda(y)},$$

so that

$$\begin{aligned}
&\frac{\partial^{n+m}}{\partial x^n y^m}G_\lambda(x,y)=\\
&\quad A(\lambda)\phi_1^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}+B(\lambda)\phi_2^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}\\
&\quad+C(\lambda)\psi_3^{+\,(n)}(x)\frac{\partial^m}{\partial y^m}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)}\\
&\quad=\frac{\mathbf{O}(1)}{W_0(\lambda)}e^{\mu_1^+x}((\mu_1^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+y}((\mu_3^+)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad+\frac{\mathbf{O}(1)}{W_0(\lambda)}e^{\mu_2^+x}((\mu_2^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+y}((\mu_3^+)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad+\mathbf{O}(1)e^{\mu_3^+x}((\mu_3^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+y}((\mu_3^+)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad=\frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_3^+)^me^{\mu_3^+(x-y)}+\mathbf{O}(1)(\mu_3^+)^{n+m}e^{\mu_3^+(x-y)}\\
&\quad+\mathbf{O}(e^{-\eta|y|})(\mu_3^+)^ne^{\mu_3^+(x-y)}+\frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.
\end{aligned}$$

Outgoing waves. We now consider outgoing waves, in our case the undercompressive case with $x\geq 0$. Beginning with *U+ (i)*, $y\leq 0\leq x$, we note that (3.1) and (3.2) hold. An important difference, however, is that in the undercompressive case, $\phi_2^+\Big|_{\lambda=0}$ does not decay as $x\to+\infty$. Therefore we must be able to write $\bar{u}_x$ as $\sim\phi_1^+(x)$ at $+\infty$ and $\sim\phi_3^-(x)$ at $-\infty$, so that ϕ_1^+ and ϕ_3^- are linearly independent. From (3.2) we see that the crucial consequence of this is that $A(\lambda)$ and $B(\lambda)$ both vanish at $\lambda=0$.

Computing from there we realize

$$\begin{aligned}
&\frac{\partial^{n+m}}{\partial x^n y^m}G_\lambda(x,y)=\\
&\quad\frac{\mathbf{O}(1)}{W_0(\lambda)}e^{\mu_1^+x}((\mu_1^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^-y}((\mu_2^-)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad+\frac{\mathbf{O}(1)}{W_0(\lambda)}e^{\mu_1^+x}((\mu_1^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^-y}((\mu_1^-)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad+\mathbf{O}(1)e^{\mu_2^+x}((\mu_2^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^-y}((\mu_2^-)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad+\mathbf{O}(1)e^{\mu_2^+x}((\mu_2^+)^n+\mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^-y}((\mu_1^-)^m+\mathbf{O}(e^{-\alpha|y|}))\\
&\quad=\mathbf{O}(1)(\mu_2^+)^n(\mu_2^-)^me^{\mu_2^+x}e^{-\mu_2^-y}+\mathbf{O}(e^{-\eta|y|})(\mu_2^+)^ne^{\mu_2^+x}e^{\mu_2^-y}\\
&\quad+\frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^me^{\mu_2^+x}e^{-\mu_2^-y}+\frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}.
\end{aligned}$$

In the case *U+ (ii)*, $0 \leq y \leq x$, we again begin with (3.1) but with (3.3), for which in the undercompressive case $B(\lambda)$ and $C(\lambda)$ vanish at $\lambda = 0$. Proceeding as there we have

$$\begin{aligned}
\frac{\partial^{n+m}}{\partial x^n y^m} G_\lambda(x,y) = \\
&\frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&+ \mathbf{O}(1) e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_1^+ y}((\mu_1^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&+ \mathbf{O}(1) e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&+ \mathbf{O}(1) e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_2^+ y}((\mu_2^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&= \mathbf{O}(1)(\mu_2^+)^{n+m} e^{\mu_2^+(x-y)} + \mathbf{O}(e^{-\eta|x|})(\mu_2^+)^m e^{\mu_2^+(x-y)} \\
&\quad + \mathbf{O}(e^{-\eta|y|})(\mu_2^+)^n e^{\mu_2^+(x-y)} + \mathbf{O}(e^{-\eta|x-y|}) e^{\mu_2^+(x-y)} \\
&\qquad + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.
\end{aligned}$$

It is worth mentioning that the difference between the term

$$\frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^m e^{\mu_2^+ x} e^{-\mu_2^- y}$$

for $y \leq 0$ and

$$\mathbf{O}(e^{-\eta|x|})(\mu_2^+)^m e^{\mu_2^+(x-y)}$$

for $y \geq 0$ is a result of the exponential y-decay of the dual eigenfunction in the undercompressive case for $y \geq 0$.

We conclude with the proof of Case *U+ (iii)*, $0 \leq x \leq y$, for which we have (3.4) along with (3.5)—$B(\lambda), C(\lambda)$ both vanishing at $\lambda = 0$. We find

$$\begin{aligned}
\frac{\partial^{n+m}}{\partial x^n y^m} G_\lambda(x,y) = \\
&\frac{\mathbf{O}(1)}{W_0(\lambda)} e^{\mu_1^+ x}((\mu_1^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&+ \mathbf{O}(1) e^{\mu_2^+ x}((\mu_2^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&+ \mathbf{O}(1) e^{\mu_3^+ x}((\mu_3^+)^n + \mathbf{O}(e^{-\alpha|x|}))e^{-\mu_3^+ y}((\mu_3^+)^m + \mathbf{O}(e^{-\alpha|y|})) \\
&= \mathbf{O}(1) e^{\mu_3^+(x-y)} + \mathbf{O}(e^{-\eta|x|})(\mu_3^+)^n e^{\mu_3^+(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.
\end{aligned}$$

□

Proposition 3.5. *For $\lambda = 0$ and $W_b(\cdot,\cdot)$ defined as*

$$W_b(\phi, \psi) := \phi\psi'' - \phi''\psi + W(\phi, \psi),$$

the following relations hold:

(L) (Lax case)

$$W_b(\phi_1^+,\phi_2^+)\Big|_{\lambda=0} = W_b(\phi_3^-,\phi_2^+)\Big|_{\lambda=0} = W_b(\phi_1^+,\phi_3^-)\Big|_{\lambda=0} \equiv 0$$

(U) (Undercompressive case)

$$W_b(\phi_1^+,\phi_3^-)\Big|_{\lambda=0} \equiv 0.$$

Before proving Proposition 3.5 we remark that it is simply a convenient manner of expressing part of Proposition 10.3 from [ZH] in the notation of this paper. Its direct implication is that in the Lax case the effective eigenspace of the equation adjoint to (2.1) at $\lambda = 0$ (which looks like $Res(W(\phi_i,\phi_j))/W_y(\lambda), \lambda = 0)$) are constants. For instance, from the proof of Lemma 3.1 we see that

$$\frac{\partial}{\partial y}\frac{W(\phi_1^+,\phi_2^+)}{W_\lambda(y)} = 0,$$

so that from Proposition 3.5 no pole occurs here at $\lambda = 0$.

We note that $W_0(W(\cdot,\cdot)/W_y)$ are the dual eigenfunctions for L^*, the adjoint operator for L. As discussed in [ZH] $\langle W_0(W(\cdot,\cdot)/W_y), *\rangle$ gives the projection describing the time-asymptotic state in the near-field, that is the shock shift. The observation that $W(\cdot,\cdot)/W_y$ is constant in the Lax case is tantamount to the well-known fact that the shift in Lax shocks can be determined by inner products against constant functions, thus by mass of the perturbation alone. The observation for undercompressive shocks that $W(\cdot,\cdot)/W_y$ decays exponentially as $y \to +\infty$ indicates that signals in the positive far field do not affect shock shift. These observations are critical in the nonlinear analysis of Section 6.

The immediate consequence of Proposition 3.5 is that y-derivatives of $G(t,x;y)$ in Theorem 1.1 for the Lax case contain no contribution from the eigenvalue at the origin.

Proof. Let π_1, π_2 be two solutions of

$$v_{xxx} + (b(x)v_x)_x - (a(x)v)_x = 0, \tag{3.6}$$

i.e. (2.1) with $\lambda = 0$, which both decay at the same infinity, for definiteness, say $-\infty$. Then we may integrate (3.6) from $-\infty$ up to x to obtain the relations

$$\pi_i''(x) + b(x)\pi_i'(x) = a(x)\pi_i(x), \quad i = 1,2.$$

Then we have

$$W_b(\pi_1,\pi_2) = \pi_1\pi_2'' - \pi_1''\pi_2 + b(y)\pi_1\pi_2' - b(y)\pi_1'\pi_2 = \pi_1(a(y)\pi_2) - \pi_2(a(y)\pi_1) = 0.$$

Since for $\lambda = 0$, ϕ_1^+ and ϕ_2^+ both decay at $+\infty$, $W_b(\phi_1^+,\phi_2^+)\Big|_{\lambda=0} = 0$ follows immediately. Further, in the Lax case with $\lambda = 0$ ϕ_3^- is a linear combination of ϕ_1^+ and ϕ_2^+ (because $\bar{u}_x$ is an exponentially decaying eigenfunction at $\lambda = 0$), yielding the second and third Lax-case assertions.

The undercompressive case assertion is a result of the linear dependence in that case (at $\lambda = 0$) of ϕ_1^+ and ϕ_3^-. □

Lemma 3.6. *(Large $|\lambda|$ estimates for the Green's function.) Under Assumptions $(\mathcal{C})$ and Condition $(\mathcal{D})$, and for λ on or to the right of a sector $\mathcal{S} = \{\lambda : Re\lambda = 2M_l - |Im\lambda|\}$, we have the following estimates on the solution $G_\lambda(x,y)$ of (2.1):*

(i) $(x \le y)$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}})(y) e^{\int_y^x \sqrt[3]{\lambda} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})(s) ds}$$

(ii) $(x \ge y)$

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}})(y) e^{\int_y^x \sqrt[3]{\lambda}(-\frac{1}{2}+i\frac{\sqrt{3}}{2}) - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})(s) ds}$$
$$+ \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}})(y) e^{\int_y^x \sqrt[3]{\lambda}(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})(s) ds}.$$

Proof. As the proof of each case is similar we will carry out the analysis only for $x \le y$. In this case we can rewrite $G_\lambda(x,y)$ as

$$G_\lambda(x,y) = \frac{\phi_3^-(x)}{\phi_3^-(y)} \cdot \frac{\phi_3^-(y) W(\phi_1^+, \phi_2^+)}{W_\lambda(y)}.$$

Thus we need separately to bound each of these terms. A straightforward computation gives

$$\frac{\phi_3^-(y) W(\phi_1^+, \phi_2^+)}{W_\lambda(y)} = \mathbf{O}(|\lambda|^{-2/3})(y),$$

where dependence on y is $\mathbf{O}(1)$ and only explicity noted because y-derivatives of this term will be taken later in the analysis. This relation is easy to observe on a formal level, keeping in mind that derivatives on the decay modes introduce factors of order $|\lambda|^{1/3}$. A rigorous proof can be made of this claim through the estimates of Lemma 3.2 and (2.4) as long as $|\lambda|$ is as specified.

We can estimate $\phi_3^-(x)/\phi_3^-(y)$ by noticing that Lemma 3.2 yields the relation

$$\phi_3^{-\,\prime}(x) = (\sqrt[3]{\lambda} - b(x)/3)\phi_3^-(x)(1 + \mathbf{O}(|\lambda|^{-2/3})). \tag{3.7}$$

This is a simple ODE for $\phi_3^-(x)$ and we can solve it with initial data $\phi_3^-(y)$ to get

$$\phi_3^-(x) = \phi_3^-(y) e^{\int_y^x (\sqrt[3]{\lambda} - b(s)/3)(1 + \mathbf{O}(|\lambda|^{-2/3})) ds}, \tag{3.8}$$

where $\mathbf{O}(|\lambda|^{-2/3})$ may depend on s but will do so in a bounded manner. We get

$$\frac{|\phi_3^-(x)|}{|\phi_3^-(y)|} \le e^{Re \int_y^x (\sqrt[3]{\lambda} - b(s)/3)(1 + \mathbf{O}(|\lambda|^{-2/3})) ds}$$
$$\le e^{-Re(\sqrt[3]{\lambda} - b_s/3)(1 + \mathbf{O}(|\lambda|^{-2/3}))|x-y|}.$$

By taking $|\lambda|$ suffiently large, the term $\mathbf{O}(|\lambda|^{-2/3})$ goes to zero, leaving a bound by $e^{-\frac{1}{2} Re(\sqrt[3]{\lambda} - b_s/3)|x-y|}$, where the 2 could be any constant larger than 1.

Combining the last two observations, we have

$$|G_\lambda(x,y)| \leq \mathbf{O}(|\lambda|^{-2/3}) e^{-\frac{1}{2}Re(\sqrt[3]{\lambda} - b_s/3)|x-y|},$$

where on our contours, we will have $Re(\sqrt[3]{\lambda} - b_s/3) > 0$ for $|\lambda|$ sufficiently large.

We now turn our attention to derivatives, continuing to work in the case $x \leq y$. From our representation for $G_\lambda(x,y)$ we can compute

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \phi_3^{-\,(n)}(x) \cdot \frac{\partial^m}{\partial y^m} \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)}.$$

Following the above analysis, we write

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \frac{\phi_3^{-\,(n)}(x)}{\phi_3^{-\,(n)}(y)} \cdot \phi_3^{-\,(n)}(y) \frac{\partial^m}{\partial y^m} \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)},$$

and estimate each term separately. First, we may observe from Lemma 3.2 that

$$\phi_3^{-\,(n+1)}(x) = \Big(\sqrt[3]{\lambda} - b(x)/3 + \mathbf{O}(|\lambda|^{-1/3})\Big) \phi_3^{-\,(n)}(x)$$

so that

$$\frac{\phi_3^{-\,(n)}(x)}{\phi_3^{-\,(n)}(y)} = e^{\int_y^x \sqrt[3]{\lambda} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds}. \tag{3.9}$$

Similarly, we have

$$\frac{\partial^m}{\partial y^m} \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)} = \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)} \Big((-\sqrt[3]{\lambda} + b(y)/3)^m + \mathbf{O}(|\lambda|^{\frac{m-2}{3}})\Big) \tag{3.10}$$

We then have

$$\begin{aligned} \phi_3^{-\,(n)}(y) \frac{\partial^m}{\partial y^m} \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)} &= \phi_3^-(y) \Big((\sqrt[3]{\lambda} - b(y)/3)^n + \mathbf{O}(|\lambda|^{\frac{n-2}{3}})\Big) \\ &\times \frac{W(\phi_1^+, \phi_2^+)}{W_\lambda(y)} \Big((-\sqrt[3]{\lambda} + b(y)/3)^m + \mathbf{O}(|\lambda|^{\frac{m-2}{3}})\Big) \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}}). \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.10) we have

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}}) e^{\int_y^x \sqrt[3]{\lambda} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds},$$

as claimed. The proof of Case (ii) follows similarly. □

Lemma 3.7. *Under Assumptions ($\mathcal{C}$) and condition ($\mathcal{D}$), and for any contour Γ to the right of Γ_d, enclosing the spectrum of L and parametrized by k, with large-k behavior $Re\Gamma \sim -k^2, Im\Gamma \sim k^3$, (1.4) has a Green's function $G(t,x;y)$ given in terms of $G_\lambda(x,y)$ by (Dunford's Integral)*

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y) d\lambda.$$

Proof. Though L is not sectorial in this case, it is clear from the estimates of Lemma 3.6 that over such a contour Γ $e^{\lambda t}G_\lambda(x,y)$ is integrable for all t, x. (In particular, this integrability is brought out in the forthcoming analysis.) An application of Lebesgue Dominated convergence then gives that distributional x- and t-derivatives commute with the λ integration to give

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - L\Big)G(t,x;y) &= \frac{1}{2\pi i}\int_\Gamma \Big[\lambda e^{\lambda t}G_\lambda(x,y) - (\lambda e^{\lambda t}G_\lambda(x,y) - e^{\lambda t}\delta_y(x))\Big]d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\delta_y(x) d\lambda = \delta_y(x)\delta_0(t),\end{aligned}$$

where the last equality is a standard result from the theory of Laplace transforms.
□

4. Small time Green's Function Estimates

We now convert the pointwise estimates on $G_\lambda(x,y)$ into pointwise estimates on $G(t,x;y)$, beginning with the case of small time, which corresponds with large $|\lambda|$. We proceed through the representation

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y) d\lambda.$$

In the case $|x-y| \geq Kt$, K to be chosen during the analysis, we expect the Green's function to principally be governed by the high-order effects of dispersion. Indeed, the estimates we find are similar in form to those for the exact Green's function of the Airy equation, $u_t = u_{xxx}$ [KF]. For $x - y \leq 0$ the Green's function for the Airy equation has the scaling of the Green's function for a sectorial operator, namely for $(x-y)/t^{1/3} \to -\infty$

$$\frac{\partial^n}{\partial x^n} G_A(t,x;y) \sim t^{-(n+1)/3}\Big(\frac{x-y}{\sqrt[3]{3t}}\Big)^{n/2-1/4} e^{-\frac{2(x-y)^{3/2}}{3\sqrt{3t}}} \leq Ct^{-\frac{n+1}{3}} e^{-\frac{(x-y)^{3/2}}{2\sqrt{t}}}. \tag{4.1}$$

For $x - y \geq 0$ the representation takes the oscillatory form (for $(x-y)/t^{1/3} \to +\infty$)

$$\frac{\partial^n}{\partial x^n} G_A(t,x;y) \sim t^{-(n+1)/3}\Big(\frac{x-y}{\sqrt[3]{3t}}\Big)^{n/2-1/4} \sin\Big(-\frac{2(x-y)^{3/2}}{3\sqrt{3t}} + c_n\Big). \tag{4.2}$$

In our case, this oscillation is difficult to quantify, so we will develop only normed estimates. In the case $x - y \geq 0$ significant information is lost, but the stability results of Section 1 can still be recovered as a consequence of the exponential $|x-y|$-decay arising from the diffusion (see the estimates of Theorem 1.1 and below). It would seem plausible that more detailed oscillatory estimates on the Green's function could lead to a stability result for non-constant diffusion. We note, however, that Kruzhkov and Faminskii's analysis of $u_t + (u^2)_x = u_{xxx}$ proceeds similarly to ours even though they had access to an exact representation of the linear Green's function (for $u_t = u_{xxx}$). The difficulty in extending the nonlinear portion of our analysis to non-contant $b(\cdot)$ is that second derivatives of G have the form $(t-s)^{-5/4}$, which we cannot integrate from 0 to t in the absence of either exponential scaling or oscillation.

We begin with the case $x - y \le 0$, for which we have from Lemma 3.6

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G_\lambda(x,y) = \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}}) e^{\int_y^x \sqrt[3]{\lambda} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds},$$

so that by dominated convergence and the proof of Lemma 3.7

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} G(t,x;y) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}}) e^{\int_y^x \sqrt[3]{\lambda} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds} d\lambda.$$

We take the contour Γ_0, defined through

$$\lambda(k) = \lambda_0 - \theta k^2 - ik^3,$$

where $\lambda_0 > 0$ and $\theta > 0$ are to be chosen, with λ_0 chosen to insure the proper scaling and so that $|\lambda(k)| \ge M_l$ for all k. (Recall that the large-$|\lambda|$ estimates of Lemma 3.2 and Lemma 3.6 hold for $|\lambda| \ge M_l$, as well as to the right of the sector $\mathcal{S}$.) We note that

$$|\lambda(k)|^2 = (\lambda_0 - \theta k^2)^2 + k^6 \Rightarrow |\lambda(k)| \ge \frac{1}{2}\lambda_0 + \frac{1}{2}k^3,$$

and

$$Re \sqrt[3]{\lambda_0 - \theta k^2 - ik^3} \ge \frac{1}{2} \sqrt[6]{(\lambda_0 - \theta k^2)^2 + k^6} \ge \frac{1}{4}\lambda_0^{1/3} + \frac{1}{4}k, \tag{4.3}$$

where our convention will be that while $\lambda(\xi)^{1/3}$ will represent three values, $\sqrt[3]{\lambda(\xi)}$ will represent only the third root of λ with largest positive real part. Along this contour Γ_0 we have

$$\int_{-\infty}^{+\infty} e^{\lambda_0 t - \theta k^2 t - ik^3 t} \mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) e^{\int_y^x \sqrt[3]{\lambda(k)} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds} (-2\theta k - 3ik^2) dk.$$

Evaluation of this integral on $(-\infty, 0]$ is similar to evaluation on $[0, +\infty)$, so we consider only $[0, +\infty)$. We divide this integral into two subintervals, $[0, t^{-1/3}]$ and $[t^{-1/3}, +\infty)$. On $[0, t^{-1/3}]$ we have (with $|\lambda(k)| \le C(|\lambda_0| + k)$ and $k \le t^{-1/3}$)

$$\begin{aligned} \Big| \int_0^{t^{-1/3}} & e^{\lambda_0 t - \theta k^2 t - ik^3 t} \mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) \\ & \times e^{\int_y^x \sqrt[3]{\lambda(k)} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3}) ds} (-2\theta k - 3ik^2) dk \Big| \\ & \le C t^{-1/3} (t^{-2/3} \lambda_0^{\frac{n+m-2}{3}} + t^{-\frac{n+m}{3}}) e^{\lambda_0 t - \frac{1}{4}\sqrt[3]{\lambda_0}|x-y| + \bar{b}|x-y|}, \end{aligned}$$

where $\bar{b} = \mathbf{O}(1)$ and so will later be absorbed by the exponential scaling. We now choose λ_0 so as to insure the correct scaling. That is, we take

$$\lambda_0 := \frac{(x-y)^{3/2}}{64t^{3/2}},$$

so that

$$\lambda_0 t - \frac{1}{4}\sqrt[3]{\lambda_0}|x-y| = \frac{(x-y)^{3/2}}{64t^{1/2}} - \frac{1}{4}\frac{(x-y)^{3/2}}{4t^{1/2}} = -\frac{3(x-y)^{3/2}}{64t^{1/2}}. \tag{4.4}$$

We note that since $|x-y|/t \geq K$, with K to be taken arbitrarily large, we can insure $|\lambda(k)| \geq \frac{1}{4}\lambda_0 \geq M_l$. For the algebraic decay, we observe that

$$\begin{aligned} t^{-1}\lambda_0^{\frac{n+m-2}{3}} e^{-3\frac{(x-y)^{3/2}}{64\sqrt{t}}} &= t^{-1}\frac{(x-y)^{\frac{n+m-2}{2}}}{t^{\frac{n+m-2}{2}}} e^{-3\frac{(x-y)^{3/2}}{64\sqrt{t}}} \\ &= t^{-\frac{n+m+1}{3}}\Big(\frac{(x-y)^{3/2}}{\sqrt{t}}\Big)^{\frac{n+m-2}{3}} e^{-3\frac{(x-y)^{3/2}}{64\sqrt{t}}} \\ &\leq Ct^{-\frac{n+m+1}{3}} e^{-\frac{(x-y)^{3/2}}{64\sqrt{t}}}, \end{aligned} \tag{4.5}$$

as claimed in Theorem 1.1.

On the infinite integral we have

$$\int_{t^{-1/3}}^{+\infty} e^{\lambda_0 t - \theta k^2 t - ik^3 t}\mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) e^{\int_y^x \sqrt[3]{\lambda(k)} - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})ds}(-2\theta k - 3ik^2)dk.$$

The exponential decay on this interval will be obtained as before (under norm), but as the algebraic decay is a dispersive phenomenon, it will be most readily noted through integration by parts. Since we would like to put integration on the term e^{-ik^3t} we make the change of variable $\xi = k^3 t$, so that $\lambda(\xi) = \lambda_0 - \theta\xi^{2/3}t^{-2/3} - i\xi t^{-1}$. We have, then

$$\begin{aligned} &\int_1^{+\infty} e^{\lambda_0 t - \theta\xi^{2/3}t^{1/3} - i\xi}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}}) \\ &\quad \times e^{\int_x^y \sqrt[3]{\lambda(\xi)} - b(s)/3 + \mathbf{O}(|\lambda(\xi)|^{-1/3})ds}(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3} - it^{-1})d\xi, \end{aligned} \tag{4.6}$$

which we integrate by parts, putting integrals on $e^{-i\xi}$. In order to take derivatives of the $\mathbf{O}(\cdot)$ terms, we will employ the quasi-sectorality of L (see remarks prior to the statement of Lemma 2.2) and the following lemma, stated from Olver [Ol, p. 9].

Lemma 4.1. *Let $f(z)$ be analytic in a region containing a closed annular sector $\mathbf{S}$, and*

$$f(z) = \mathbf{O}(z^p) \quad or \quad f(z) = \mathbf{o}(z^p)$$

as $z \to \infty$ in $\mathbf{S}$, where p is any fixed real number. Then

$$f^{(m)}(z) = \mathbf{O}(z^{p-m}) \quad or \quad f^{(m)}(z) = \mathbf{o}(z^{p-m})$$

as $z \to \infty$ in any closed annular sector $\mathbf{C}$ properly interior to $\mathbf{S}$ and having the same vertex.

Corollary 4.2. *The terms of form* $\mathbf{O}(|\lambda|^p)$ *arising in* $G_\lambda(x,y)$ *satisfy*

$$\frac{\partial}{\partial\lambda}\mathbf{O}(|\lambda|^p) = \mathbf{O}(|\lambda|^{p-1}).$$

Proof. Lemma 4.1 is proved in Olver through interior estimates for such a sector $\mathbf{S}$. In our case (Corollary 4.2), we have that the $\mathbf{O}(\cdot)$ terms appearing in (4.6) are analytic in the sector consisting of $\mathcal{S}$ from Lemma 3.2 and thus in any annular sector contained therein. In particular, Γ lies in such an annular sector. □

Integrating (4.6) by parts, we have

(4.7)

$$\begin{aligned}
&ie^{-i\xi}e^{\lambda_0 t-\theta\xi^{2/3}t^{1/3}}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}})\\
&\qquad\times e^{\int_y^x \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3}-it^{-1}\Big)\Big|_1^{+\infty}\\
&-i\int_1^{+\infty} e^{\lambda_0 t-i\xi}\frac{\partial}{\partial\xi}\Big[e^{-\theta\xi^{2/3}t^{1/3}}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}})\\
&\qquad\times e^{\int_y^x \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3}-it^{-1}\Big)\Big]d\xi\\
&=-ie^{-i}e^{\lambda_0 t-\theta t^{1/3}}\mathbf{O}(|\lambda(1)|^{\frac{n+m-2}{3}})\\
&\qquad\times e^{\int_x^y \sqrt[3]{\lambda(1)}-b(s)/3+\mathbf{O}(|\lambda(1)|^{-1/3})ds}\Big(-\frac{2}{3}\theta t^{-2/3}-it^{-1}\Big)\\
&\overset{1}{-}i\int_1^{+\infty} e^{\lambda_0 t-i\xi}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{1/3}\Big)\Big[e^{-\theta\xi^{2/3}t^{1/3}}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}})\\
&\qquad\times e^{\int_x^y \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3}-it^{-1}\Big)\Big]d\xi\\
&\overset{2}{-}i\int_1^{+\infty} e^{\lambda_0 t-i\xi}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-5}{3}})\lambda'(\xi)\Big[e^{-\theta\xi^{2/3}t^{1/3}}\\
&\qquad\times e^{\int_x^y \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3}-it^{-1}\Big)\Big]d\xi\\
&\overset{3}{-}i\int_1^{+\infty} e^{\lambda_0 t-i\xi}\Big(\int_y^x \frac{1}{3}\lambda(\xi)^{-2/3}\lambda'(\xi)+\mathbf{O}(|\lambda(\xi)|^{-4/3})\lambda'(\xi)ds\Big)\Big[e^{-\theta\xi^{2/3}t^{1/3}}\\
&\times\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}})e^{\int_x^y \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3}-it^{-1}\Big)\Big]d\xi\\
&\overset{4}{-}i\int_1^{+\infty} e^{\lambda_0 t-i\xi}\Big[e^{-\theta\xi^{2/3}t^{1/3}}\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-2}{3}})\\
&\qquad\times e^{\int_x^y \sqrt[3]{\lambda(\xi)}-b(s)/3+\mathbf{O}(|\lambda(\xi)|^{-1/3})ds}\Big(\frac{2}{9}\theta\xi^{-4/3}t^{-2/3}\Big)\Big]d\xi.
\end{aligned}$$

All we require is an understanding of the correct algebraic decay under norm, as the exponential decay follows from the previous choice of λ_0. For the boundary expression, we may observe that $|\lambda(1)|\le C(\lambda_0+t^{-1})$ so that a computation similar to (4.5) yields the claim.

What we will observe on each of the four integrals in (4.7) is that as a consequence of Corollary 4.2 the ξ-derivative has the expected effect of decreasing the expression by a factor of ξ^{-1} yielding algebraic integrability so that we need not integrate

$e^{-\theta\xi^{2/3}t^{1/3}}$, which would return t blow-up of the from $t^{-1/2}$. For example, on the first integral on the right-hand side of (4.7) (reference numbers appear at the far left over the negative signs) we observe that

$$\xi^{-1/3}t^{1/3}e^{-\theta\xi^{2/3}t^{1/3}} \leq C\xi^{-1}e^{-\frac{\theta}{2}\xi^{2/3}t^{1/3}},$$

so that under norm we have an expression of the form

$$\int_1^{+\infty} e^{\lambda_0 t - \frac{1}{4}\sqrt[3]{\lambda_0}|x-y| + \bar{b}|x-y|}\xi^{\frac{n+m-5}{3}}t^{-\frac{n+m+1}{3}}d\xi. \tag{4.8}$$

For $n+m \leq 1$ this yields the same estimate as above by the integrability of $\xi^{\frac{n+m-5}{3}}$. On the second integral we have

$$\begin{aligned}|\mathbf{O}(|\lambda(\xi)|^{\frac{n+m-5}{3}})\lambda'(\xi)| &\leq C(\lambda_0 + \xi t^{-1})^{\frac{n+m-5}{3}}t^{-1}\\ &\leq \xi^{\frac{n+m-5}{3}}t^{-\frac{n+m-2}{3}}\end{aligned}$$

so that under norm we have again an expression of form (4.8). For the third integral we have, employing now our relation $Re\sqrt[3]{\lambda(\xi)} \geq \frac{1}{4}\lambda_0 + \frac{1}{4}\xi^{1/3}t^{-1/3}$

$$\begin{aligned}&\Big(|\lambda(\xi)|^{-2/3}|\lambda'(\xi)| + \mathbf{O}(|\lambda(\xi)|^{-4/3})|\lambda'(\xi)|\Big)|x-y|e^{-\frac{1}{4}\xi^{1/3}t^{-1/3}|x-y|}\\ &\leq C\xi^{-2/3}t^{-1/3}|x-y|e^{-\frac{1}{4}\xi^{1/3}t^{-1/3}|x-y|} \leq C\xi^{-1}e^{-\frac{1}{8}\xi^{1/3}t^{-1/3}|x-y|},\end{aligned}$$

from which the result follows. The final integral is trivial. Estimates on higher order derivatives may be made through further applications of integration by parts. That is to say, we would take each of the integrals 1–4 and integrate again by parts, putting integrals on $e^{-i\xi}$.

The case $x \geq y$ is more delicate, even if we content ourselves with an estimate in norm only. The principal difficulty is that the expected scaling $(|x-y|^{3/2}/t^{1/2})$ arises in an oscillatory fashion (see (4.2)). In this case, we have

$$\begin{aligned}\frac{\partial^{n+m}}{\partial x^n \partial y^m}G_\lambda(x,y) &= \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}})e^{\int_y^x \sqrt[3]{\lambda}(-\frac{1}{2}+i\frac{\sqrt{3}}{2}) - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})ds}\\ &\quad + \mathbf{O}(|\lambda|^{\frac{n+m-2}{3}})e^{\int_y^x \sqrt[3]{\lambda}(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) - b(s)/3 + \mathbf{O}(|\lambda|^{-1/3})ds},\end{aligned} \tag{4.9}$$

where it will be the case that over portions of our contour both $Re\sqrt[3]{\lambda}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})$ and $Re\sqrt[3]{\lambda}(-\frac{1}{2}-i\frac{\sqrt{3}}{2})$ will approach zero, so that (4.3) will no longer hold. As we no longer expect exponential scaling, we now take the fixed contour $\lambda(k) = M - \theta k^2 - ik^3$, where M is chosen sufficiently large so that we will have again $|\lambda| \geq M_l$. We denote this contour Γ_M. A straightforward computation reveals that along this contour we have

$$\sqrt[3]{\lambda(k)} = \begin{cases} (\frac{\sqrt{3}}{2} - \frac{i}{2})k + \frac{\theta}{3}(\frac{1}{2} + i\frac{\sqrt{3}}{2}) + \mathbf{O}((M^{1/3}+k)^{-1}), & k \geq 0\\ (-\frac{\sqrt{3}}{2} - \frac{i}{2})k + \frac{\theta}{3}(\frac{1}{2} - i\frac{\sqrt{3}}{2}) + \mathbf{O}((M^{1/3}+|k|)^{-1}), & k \leq 0.\end{cases}$$

The analysis of each term in (4.9) will be the same, so we consider only the first, which will have the form

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} e^{Mt-\theta k^2 t - ik^3 t}\mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) \\
&\qquad \times e^{\int_y^x \sqrt[3]{\lambda(k)}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})-b(s)/3+\mathbf{O}(|\lambda|^{-1/3})ds}(-2\theta k - 3ik^2)dk \\
&= \int_{-\infty}^{0} e^{Mt-\theta k^2 t - ik^3 t}\mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) \\
&\qquad \times e^{\int_y^x \sqrt[3]{\lambda(k)}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})-b(s)/3+\mathbf{O}(|\lambda|^{-1/3})ds}(-2\theta k - 3ik^2)dk \\
&+ \int_{0}^{+\infty} e^{Mt-\theta k^2 t - ik^3 t}\mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}})e^{ik(x-y)} \\
&\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|\lambda|^{-1/3})ds}(-2\theta k - 3ik^2)dk,
\end{aligned}
\tag{4.10}
$$

where $\tilde{b}(s) := b(s)/3 + \theta/3 \geq \tilde{b}_0 > 0$. In particular, we will be able to take K ($|x-y| \geq Kt$) sufficiently large so that

$$
e^{Mt-\frac{\tilde{b}_0}{2}|x-y|} \leq e^{Mt-\frac{\tilde{b}_0}{4}Kt-\frac{\tilde{b}_0}{4}|x-y|} \leq e^{-\eta|x-y|}.
$$

Observing that for $k \leq 0$ $Re(\sqrt[3]{\lambda(k)}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})) \geq \frac{1}{4}\sqrt[3]{M} + \frac{1}{4}k$ on this contour, we see that the first integral on the right-hand side of (4.10) can be analyzed in a manner similar to the case $x-y \leq 0$, except that without the exponential scaling introduced by λ_0 we obtain exponential decay of the form $e^{-\eta|x-y|}$ for some $\eta > 0$. That is, we gain an estimate of the form

$$
t^{-\frac{n+m+1}{3}}e^{-\eta|x-y|}.
$$

We observe that were we to take the contour Γ_0 we would obtain on this integral the same estimate as in the case $x-y \leq 0$. The contour Γ_0, however, would fail to yield a useful estimate on the second integral of (4.10).

For the second integral on the right-hand side of (4.10), we see that on the integral over $[0, t^{-1/3}]$ we have a trivial estimate by

$$
Ct^{-\frac{n+m+1}{3}}e^{-\eta|x-y|}.
$$

For $k \in [t^{-1/3}, +\infty)$, we make the usual change of variable $\xi = k^3 t$ to arrive at the representation

$$
\begin{aligned}
&\int_{1}^{+\infty} e^{Mt-\theta\xi^{2/3}t^{1/3}-i\xi+i\xi^{1/3}t^{-1/3}|x-y|}\mathbf{O}(|\lambda(\xi^{1/3}t^{-1/3})|^{\frac{n+m-2}{3}}) \\
&\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|\lambda(\xi^{1/3}t^{-1/3})|^{-1/3})}(-\frac{2}{3}\theta\xi^{-1/3}t^{-2/3} - it^{-1})d\xi.
\end{aligned}
$$

Integrating by parts as in (4.7), we observe that the only term that cannot be dealt with precisely as before is that arising from the derivative on $e^{i\xi^{1/3}t^{-1/3}}|x-y|$. Where in the previous analysis we had

$$
\Big|\frac{\partial}{\partial\xi}e^{-\xi^{1/3}t^{-1/3}|x-y|}\Big| = \Big|\frac{1}{3}\xi^{-2/3}t^{-1/3}|x-y|e^{-\xi^{1/3}t^{-1/3}|x-y|}\Big| \leq C\xi^{-1}e^{-\frac{1}{2}\xi^{1/3}t^{-1/3}|x-y|},
$$

we now have

$$\left|\frac{\partial}{\partial \xi} e^{i\xi^{1/3}t^{-1/3}|x-y|}\right| = \left|\frac{1}{3}\xi^{-2/3}t^{-1/3}|x-y|e^{i\xi^{1/3}t^{-1/3}|x-y|}\right| \leq \xi^{-2/3}t^{-1/3}|x-y|.$$

This estimate yields the desired algebraic integrability in ξ for $n+m=0$ *only if* $t^{-1/3}|x-y| \leq \tilde{C}$, for some constant $\tilde{C}$. For $n+m>0$, we may integrate repeatedly by parts to gain the required integrability for this scaling.

We are left with the case $|x-y|/t^{1/3} \geq \tilde{C}$, for which we will apply an asymptotic analysis similar to that found in [Ol, pp. 98–103] to obtain results similar to those of [FK] (cf. (4.1) and (4.2)). As our scaling will now change, we begin again with

$$\int_0^{+\infty} e^{Mt-\theta k^2 t - ik^3 t + ik(x-y)} \mathbf{O}(|\lambda(k)|^{\frac{n+m-2}{3}}) \\ \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|\lambda(k)|^{-1/3})ds}(-2\theta k - 3ik^2)dk.$$

We would like to integrate by parts, but vis-à-vis the previous analysis we now put anti-derivatives on the entire oscillating portion, namely

$$e^{ik(x-y)-ik^3 t}.$$

The point is that for $|x-y|/t^{1/3} \leq \tilde{C}$, t is sufficiently dominant that the ik^3t term controls the integrand's oscillation. For the case $|x-y|/t^{1/3} \geq \tilde{C}$, the terms compete. In fact, it is fairly clear heuristically that the main contribution arises when $k(x-y) = k^3 t$, or when $k = \sqrt{(x-y)/t}$. This observation motivates the forthcoming analysis.

We make the change of variable $(x-y \geq 0)$

$$k := \sqrt{\frac{x-y}{t}}\xi$$

to get

$$\sqrt{\frac{x-y}{t}} \int_0^{+\infty} e^{Mt-\theta|x-y|\xi^2 + i\alpha(\xi-\xi^3)} \mathbf{O}((M^{1/3} + (\sqrt{(x-y)/t})\xi)^{n+m}) \\ \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|\lambda((\sqrt{(x-y)/t})\xi)|^{-1/3})ds} d\xi,$$

where

$$\alpha := \frac{(x-y)^{3/2}}{\sqrt{t}}.$$

We expect the largest contribution to this integral to occur for $\xi - \xi^3 = 0$, where there is no oscillation. Accordingly, we make the definitions

$$p(\xi) := \xi - \xi^3 \quad \text{and} \quad v(\xi) := p(\xi) - p(1/\sqrt{3}),$$

where $p'(1/\sqrt{3}) = 0$. Splitting the integrand up we have

$$\begin{aligned}
& e^{i\alpha p(1/\sqrt{3})}\Big(\frac{x-y}{t}\Big)^{\frac{n+m+1}{2}} \int_{-\infty}^{0} e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m}{3}}) \\
& \qquad\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds}\frac{-dv}{p'(\xi(v))} \\
& + e^{i\alpha p(1/\sqrt{3})}\Big(\frac{x-y}{t}\Big)^{\frac{n+m+1}{2}} \int_{-p(1/\sqrt{3})}^{0} e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m}{3}}) \\
& \qquad\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds}\frac{dv}{p'(\xi(v))}.
\end{aligned} \tag{4.11}$$

In the absence of our $\mathbf{O}(\cdot)$ terms (for example in an analysis of the Airy equation) oscillation of the form $e^{i\alpha p(1/\sqrt{3})}$ is clearly obtained (see [Ol]).

We now collect some useful observations regarding

$$\xi(v)^2 \quad \text{and} \quad h(v) := \frac{1}{p'(\xi(v))}.$$

We have

$$\begin{aligned}
\frac{\partial^n}{\partial v^n}\xi(v)^2 &= \mathbf{O}(v^{2/3-n}), \quad v \geq v_0 > 0 \\
\frac{\partial^n}{\partial v^n}h(v) &= \mathbf{O}(v^{-2/3-n}), \quad v \geq v_0 > 0.
\end{aligned}$$

Also, $v(1/\sqrt{3}) = v'(1/\sqrt{3}) = 0$ so that $v(1/\sqrt{3}) \sim (\xi - 1/\sqrt{3})^2$ and hence $\xi - 1/\sqrt{3} \sim v^{1/2}$. Thus

$$p'(\xi) = 1 - 3\xi^2 = (1-\sqrt{3}\xi)(1+\sqrt{3}\xi) \sim v^{1/2}$$

and

$$h(v) = \mathbf{o}(v^{-1/2}), \quad \frac{\partial^n}{\partial v^n}h(v) = \mathbf{o}(v^{-1/2-n}). \tag{4.12}$$

We are now in a position to evaluate the first integral in (4.11) by splitting its analysis into regions of large and small v. In light of (4.12) we may choose $\tilde{C}$ $((x-y)/t^{1/3} \geq \tilde{C})$ large enough so that on $v \in [-\alpha^{-1}, 0]$ $|h(v)| < \epsilon|v|^{-1/2}$. That is to say, (4.12) gives that there exists some $\kappa > 0$ such that for $|v| \leq \kappa$, $|h(v)| < \epsilon v^{-1/2}$, and we may choose $\tilde{C}$ large enough so that $|\alpha^{-1}| \leq \kappa$. This yields

$$\begin{aligned}
& \Big| e^{i\alpha p(1/\sqrt{3})}\Big(\frac{x-y}{t}\Big)^{\frac{n+m+1}{2}} \int_{-\alpha^{-1}}^{0} e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m}{3}}) \\
& \qquad\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds} h(v)dv \Big| \\
& \leq Ce^{-\eta|x-y|}\Big(\frac{x-y}{t}\Big)^{\frac{n+m+1}{2}} \int_{-\alpha^{-1}}^{0} v^{-1/2}dv = Ce^{-\eta|x-y|}\Big(\frac{x-y}{t}\Big)^{\frac{n+m+1}{2}}\alpha^{-1/2} \\
& \qquad\qquad = Ct^{-\frac{n+m+1}{3}}\Big(\frac{x-y}{\sqrt[3]{t}}\Big)^{\frac{n+m}{2}-\frac{1}{4}}.
\end{aligned}$$

For $v \le -\alpha^{-1}$ we have

$$\begin{aligned}
&\Big|\int_{-\kappa}^{-\alpha^{-1}} e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})\\
&\qquad\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds}dv\Big|\\
&\le \Big|\alpha^{-1}e^{i\alpha v}e^{Mt-\theta|x-y|\xi(v)^2}\mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds}\Big|_{-\kappa}^{-\alpha^{-1}}\\
&+\alpha^{-1}\Big|\int_{-k}^{-\alpha^{-1}} e^{i\alpha v}\frac{\partial}{\partial v}\Big[e^{Mt-\theta|x-y|\xi(v)^2}\mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})\\
&\qquad\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|^{-1/3})ds}\Big]dv\Big|\\
&\qquad\qquad \le C\alpha^{-1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{-\kappa} e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})\\
&\qquad \times e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|)^{-1/3}ds}dv\Big|\\
&\overset{\text{parts}}{=} (i\alpha)^{-1}e^{Mt-\theta|x-y|\xi(v)^2+i\alpha v}\mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|)^{-1/3}ds}\Big|_{-\infty}^{-\alpha^{-1}}\\
&-\int_{-\infty}^{-\alpha^{-1}}(i\alpha)^{-1}e^{i\alpha v}\frac{\partial}{\partial v}\Big[e^{Mt-\theta|x-y|\xi(v)^2}\\
&\qquad \times \mathbf{O}(\tilde{M}+v^{\frac{n+m-2}{3}})e^{\int_y^x -\tilde{b}(s)+\mathbf{O}(|M+v|)^{-1/3}ds}\Big]dv,
\end{aligned}$$

which is better than the small-v term by a factor of $\alpha^{-1/2}$ (for $n+m \le 1$). For $n+m>1$ we integrate repeatedly by parts. The second integral in (4.11) may be analyzed in a similar fashion, as indeed can similar oscillatory integrals that arise in the analysis of equations of higher order (see Section 7).

5. Large time Green's Function Estimates

In the case $|x-y| \le Kt$, we follow very closely the analysis of [ZH]. In fact, the main point is that for $|\lambda|$ small (t large) we only see the effect of the first two orders of spatial derivatives, making the analysis effectively that of the case with convection and diffusion only. Numerous subcases arise here for which the analyses are almost identical. Our strategy will be to present the analysis for a few of these cases in detail, then list for subsequent subcases the previous analysis that pertains. At the end of the subsection we make a general comment about converting the estimates of Lemma 3.4 into those of Theorem 1.1.

Lax case. We begin the small $|\lambda|$ analysis with the Lax case for which we have $a_+ < 0 < a_-$. Here, by symmetry, we need consider only the subcase $x \ge 0$.

Case L+ (i) In the Lax case with $y \le 0 \le x$, we have from Lemma 3.4

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m}G_\lambda(x,y) = \frac{\mathbf{O}(e^{-\eta|x|})}{W_0(\lambda)}(\mu_2^-)^n e^{\mu_2^-(x-y)} + \frac{\mathbf{O}(e^{-\eta|x|})\mathbf{O}(e^{-\eta|y|})}{W_0(\lambda)}.$$