THE EVANS FUNCTION AND STABILITY CRITERIA FOR DEGENERATE VISCOUS SHOCK WAVES

Peter Howard and Kevin Zumbrun

July 7, 2003

Abstract. It is well known that the stability of certain distinguished waves arising in evolutionary PDE can be determined by the spectrum of the linear operator found by linearizing the PDE about the wave. Indeed, work over the last fifteen years has shown that *spectral stability* implies nonlinear stability in a broad range of cases, including asymptotically constant traveling waves in both reaction–diffusion equations and viscous conservation laws. A critical step toward analyzing the spectrum of such operators was taken in the late eighties by Alexander, Gardner, and Jones, whose *Evans function* (generalizing earlier work of John W. Evans) serves as a characteristic function for the above-mentioned operators. Thus far, results obtained through working with the Evans function have made critical use of the function's analyticity at the origin (or its analyticity over an appropriate Riemann surface). In the case of degenerate (or sonic) viscous shock waves, however, the Evans function is certainly not analytic in a neighborhood of the origin, and does not appear to admit analytic extension to a Riemann manifold. We surmount this obstacle by dividing the Evans function (plus related objects) into two pieces: one analytic in a neighborhood of the origin, and one sufficiently small.

1. Introduction

We consider degenerate viscous shock waves arising in the system,

$$\begin{aligned} u_t + f(u)_x &= u_{xx}, \quad u, f \in \mathbb{R}^2, \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where $u_0(\pm\infty) = u_\pm$ and $f \in C^2(\mathbb{R})$; that is, solutions of the form $(\bar{u}_1(x-st), \bar{u}_2(x-st))^{\text{tr}}$ that satisfy the Rankine–Hugoniot condition,

$$s = \frac{f_k(u_1^+, u_2^+) - f_k(u_1^-, u_2^-)}{u_k^+ - u_k^-}, \quad k = 1, 2,$$

and for which $s \in \text{Spectrum}(df(u_\pm))$. (Typically, s is only in $\text{Spectrum}(df(u_\pm))$ on one side, which we refer to as the degenerate side.) Letting $a_k^\pm$ $(k = 1, 2)$ represent the eigenvalues of $df(u_\pm)$, we will restrict our attention to the case

$$a_1^- < s < a_2^- \quad \text{and} \quad a_1^+ < s = a_2^+$$

or, symmetrically, $a_1^- = s < a_2^-$ and $a_1^+ < s < a_2^+$. In either case, we have strict hyperbolicity at the endpoints, and make no requirements regarding hyperbolicity along the wave. Under the additional, generic, assumption of first order degeneracy, which corresponds with $\bar{u}_1(\xi)$ and $\bar{u}_2(\xi)$

This research was partially supported by the NSF under grant No. DMS-0230003 (Howard) and grant nos. DMS–0070765 and DMS–0300487 (Zumbrun).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

both decaying to endstates at rate $|\xi|^{-1}$ (see Section 2) we show that the Evans function can be constructed in this case (though not analytically) and that it can be analyzed in a manner similar to that of the non-degenerate case (see [GZ,ZH]).

For a general discussion of degenerate viscous shock waves and the contexts in which they arise, the reader is referred to [H.1] and the references therein. Here, we mention only that the analyses of Howard were both limited to single equations [H.1–2], that degenerate waves are not considered in the general systems analyses of Gardner and Zumbrun [GZ], or Zumbrun and Howard [ZH], and that Nishihara's analysis of degenerate waves in systems, which was carried out in the case of the p-system (see [N])

$$\begin{aligned} u_{1t} - u_{2x} &= 0 \\ u_{2t} + p(u_1)_x &= u_{2xx}, \end{aligned}$$

is based on energy estimates and limited to the case of arbitrarily weak shock strength and zero-mass initial perturbations. Our result, then, applicable to shocks of arbitrary strength and general (sufficiently small) initial perturbations is the first general stability analysis for degenerate viscous shock waves arising in systems of conservation laws.

In a recent analysis of the single conservation law

$$u_t + f(u)_x = (b(u)u_x)_x,$$

Howard has observed that the pointwise Green's function approach of [ZH] can be extended to the case of degenerate viscous shock waves so long as sufficiently sharp estimates can be obtained on solutions to the associated eigenvalue ODE

$$(b(x)v_x)_x - (a(x)v)_x = \lambda v, \tag{1.2}$$

where $a(x) := f'(\bar{u}(x)) - b'(\bar{u}(x))\bar{u}_x(x)$, and $b(x) := b(\bar{u}(x))$ [H.1–2]. Such analyses are complicated by two critical features of (1.2). First, whereas in the case of non-degenerate waves the coefficients $a(x)$ and $b(x)$ decay to end-states at exponential rate, in the case of degenerate waves they decay at rate $|x|^{-1}$. Exacerbating this situation is the further fact that when (1.2) is written as a first order system, the *ODE* eigenvalues coalesce as $\lambda \to 0$, $x \to \pm\infty$, (whichever is the degenerate side). It follows from these properties that asymptotically decaying solutions $v^+(x,\lambda)$ of (1.2) take the form (see [H.2])

$$v^+(x,\lambda) = e^{-\int_0^x \sqrt{\lambda/b(s)}ds}(\bar{u}(x) - u_+)\Big(- \sqrt{\lambda/b(x)} + \frac{\bar{u}_x(x)}{\bar{u}(x) - u_+} + e(x,\lambda)\Big),$$

where $e(x,\lambda) = \mathbf{O}(|x|^{-1})\mathbf{O}(\sqrt{\lambda}\log\lambda)$. Since the Evans function is typically built from these asymptotically decaying solutions, it cannot be constructed analytically (or, due to the $\log\lambda$ behavior, readily extended analytically on a Riemann manifold, though see [SS]). Hence, the Taylor approximation techniques of e.g. [GZ, KR] near the critical point $\lambda = 0$ cannot be used. And while solutions of (1.2) are straightforward to analyze either for the case $\lambda = 0$ or the case $|\lambda| \geq \delta_0 > 0$, the transition from one of these cases to the other as λ goes to zero (which is our main concern) is singular and can generally be quite subtle. In particular, solutions $v^+(x,\lambda)$ heuristically behave as functions of the combined variable $\sqrt{\lambda}x$ (if $a(x), b(x) \sim x^{-1}$, then (1.2) can be approximated by a Bessel equation), so tracing asymptotically decaying solutions back to $x = 0$, where the Evans function is typically defined, is rather delicate. In fact, we find that estimates depend not only on the general properties of (1.2), as is the case with non-degenerate waves, but indeed on the structure of the underlying degenerate wave. Consequently, standard ODE estimates cannot possibly suffice (the situation is simply too specialized). In the yet more complicated setting here of systems, a number of further difficulties arise, including the intermingling of slow and degenerate decay modes as well as a considerably more cumbersome Evans function.

Plan of the paper. In Section 2 we discuss the general behavior of degenerate viscous shock waves, while in Section 3 we set a context for the Evans function analysis to follow by reviewing the basic ideas behind the pointwise Green's function approach to stability. Following [H.1–2], we then develop the required system ODE estimates (Section 4), and following [GZ] we extend the Evans function framework to this complicated setting, establishing general criteria for instability (Sections 5 and 6). A stability criterion is also given, though its full proof will be developed in a companion paper [H.3]. We remark at the outset that Sections 2 and 4 are somewhat technical, Section 3 is mostly expository, and the results are all collected in Sections 5 and 6.

2. Structure of Degenerate Viscous Shock Waves

Since $f(u)$ is a general nonlinearity in (1.1), we may shift without loss of generality to a moving coordinate frame in which $s = 0$. Suppose, then, that $(\bar{u}_1(x), \bar{u}_2(x))^{\mathrm{tr}}$ represents a standing wave solution to (1.1), hence satisfying

$$\begin{aligned} \bar{u}_{1xx} &= f_1(\bar{u}_1, \bar{u}_2)_x \\ \bar{u}_{2xx} &= f_2(\bar{u}_1, \bar{u}_2)_x. \end{aligned}$$

Integrating once and expanding about the asymptotic state $(u_1^+, u_2^+)^{\mathrm{tr}}$, we have

$$\begin{aligned} (\bar{u}_1 - u_1^+)_x &= \partial_{u_1} f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+) + \partial_{u_2} f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) \\ &\quad + \frac{1}{2}\partial_{u_1 u_1} f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)^2 + \partial_{u_1 u_2} f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)(\bar{u}_2 - u_2^+) \\ &\quad + \frac{1}{2}\partial_{u_2 u_2} f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+)^2 + \dots \\ (\bar{u}_2 - u_2^+)_x &= \partial_{u_1} f_2(u_1^+, u_2^+)(\bar{u}_1 - u_1^+) + \partial_{u_2} f_2(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) \\ &\quad + \frac{1}{2}\partial_{u_1 u_1} f_2(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)^2 + \partial_{u_1 u_2} f_2(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)(\bar{u}_2 - u_2^+) \\ &\quad + \frac{1}{2}\partial_{u_2 u_2} f_2(u_1^+, u_2^+)(\bar{u}_2 - u_2^+)^2 + \dots. \end{aligned} \tag{2.1}$$

We assume that the linear matrices

$$A_\pm := \begin{pmatrix} \partial_{u_1} f_1(u_1^\pm, u_2^\pm) & \partial_{u_2} f_1(u_1^\pm, u_2^\pm) \\ \partial_{u_1} f_2(u_1^\pm, u_2^\pm) & \partial_{u_2} f_2(u_1^\pm, u_2^\pm) \end{pmatrix}$$

have eigenvalues $a_1^- < 0 < a_2^-$; $a_1^+ < 0 = a_2^+$. While, as in the case of single equations, degeneracy allows for the possibility of $(\bar{u}_1(x), \bar{u}_2(x))$ decaying to end-states at an algebraic rate, it does not necessitate it. For example, the wave $(-\tanh x/2 + 1, 0)$ serves as an exponentially decaying degenerate wave for the (admittedly contrived) system

$$\begin{aligned} u_{1t} + \frac{1}{2}(u_1^2 - u_2^2 - 2u_1)_x &= u_{1xx} \\ u_{2t} + (u_1 u_2)_x &= u_{2xx} \end{aligned} \tag{2.2}$$

(in this case, $a_1^-, a_2^- > 0$; $a_1^+ < 0 = a_2^+$). On the other extreme, the wave $(\bar{u}_1(x), \bar{u}_2(x))$ can decay to end-states as slowly as we like; i.e., at rate $|x|^{-1/k}$, for k (an integer) arbitrarily large. For example, the standing wave solution $(\bar{u}(x), 0)$ to

$$\begin{aligned} u_{1t} + (u_1^4 - u_1^3 - u_2^2)_x &= u_{1xx} \\ u_{2t} + (u_1 u_2)_x &= u_{2xx} \end{aligned} \tag{2.3}$$

(in this case $a_1^- = a_2^- = 1; a_1^+ = a_2^+ = 0$) given through the implicit formula

$$\frac{2\bar{u}(x)+1}{2\bar{u}(x)^2} + \log\Big|\frac{\bar{u}(x)-1}{\bar{u}(x)}\Big| = x,$$

clearly decays at rate $|x|^{-1/2}$ to $(u_1^+, u_2^+) = (0,0)$.

Definition. We will describe degenerate viscous shock waves that decay to endstate at rate $|x|^{-1/k}$ in both coordinates as k^{th}-order degenerate.

Our focus in this paper will be on the most generic type of degeneracy, first order, or $k = 1$. In the remainder of this section, we develop a criteria for distinguishing first order degeneracy.

In general, equations of form (2.1) can be analyzed by center manifold techniques [Carr, K]. Under the substitutions $w := \bar{u}_1(x) - u_1^+$ and $z = \bar{u}_2(x) - u_2^+$, and with α_{jk} and β_{jk} representing the Taylor coefficients of (2.1),we have

$$\begin{aligned}\dot{w} &= a_{11}^+ w + a_{12}^+ z + \sum_{j+k\geq 2} \alpha_{jk} w^j z^k \\ \dot{z} &= a_{21}^+ w + a_{22}^+ z \sum_{j+k\geq 2} \beta_{jk} w^j z^k.\end{aligned}$$

In the event that $a_{12}^+ = a_{21}^+ = 0$, we cannot have first-order degeneracy. In order to see this, we observe that our degeneracy condition $a_{11}^+ a_{22}^+ = a_{12}^+ a_{21}^+$ requires that in this case exactly one of a_{11}^+ and a_{22}^+ must also be 0. (They cannot both be zero by our assumption $a_1^+ < 0$.) Without loss of generality, take $a_{11}^+ = 0$, so that we have

$$\begin{aligned}\dot{w} &= \sum_{j+k\geq 2} \alpha_{jk} w^j z^k \\ \dot{z} &= a_{22}^+ z + \sum_{j+k\geq 2} \beta_{jk} w^j z^k.\end{aligned}$$

The Center Manifold Theorem ([Carr, K]) asserts the existence of a center manifold $z = h(w) = \mathbf{O}(w^2)$; that is, an invariant manifold locally tangent to the center eigenspace. Such a manifold contradicts our assumption of first order degeneracy ($\bar{u}_1(x) - u_1^+$ and $\bar{u}_2(x) - u_2^+$ cannot both decay at the same algebraic rate).

Assuming either a_{12}^+ or a_{21}^+ is not zero, we take without loss of generality $a_{12}^+ \neq 0$, and make the change of variables $W = PR$, where

$$W = \begin{pmatrix} w \\ z \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -\frac{a_{11}^+}{a_{12}^+} & \frac{a_{22}^+}{a_{12}^+} \end{pmatrix}, \quad R = \begin{pmatrix} r \\ s \end{pmatrix}. \tag{2.4}$$

We have

$$PR' = APR + \begin{pmatrix} \sum_{j+k\geq 2} \alpha_{jk} (r+s)^j (-\frac{a_{11}^+}{a_{12}^+} r + \frac{a_{22}^+}{a_{12}^+} s)^k \\ \sum_{j+k\geq 2} \beta_{jk} (r+s)^j (-\frac{a_{11}^+}{a_{12}^+} r + \frac{a_{22}^+}{a_{12}^+} s)^k \end{pmatrix},$$

so that

$$R' = \begin{pmatrix} 0 & 0 \\ 0 & a_{11}^+ + a_{22}^+ \end{pmatrix} R + \frac{a_{12}^+}{a_{11}^+ + a_{22}^+} \begin{pmatrix} \frac{a_{22}^+}{a_{12}^+} & -1 \\ \frac{a_{11}^+}{a_{12}^+} & 1 \end{pmatrix} \begin{pmatrix} \sum_{j+k\geq 2} \alpha_{jk} (r+s)^j (-\frac{a_{11}^+}{a_{12}^+} r + \frac{a_{22}^+}{a_{12}^+} s)^k \\ \sum_{j+k\geq 2} \beta_{jk} (r+s)^j (-\frac{a_{11}^+}{a_{12}^+} r + \frac{a_{22}^+}{a_{12}^+} s)^k \end{pmatrix}.$$

The Center Manifold Theorem now assures us of a center manifold $s = h(r) = \mathbf{O}(r^2)$, so that

$$\dot{r} = \gamma_2 r^2 + \mathbf{O}(r^3),$$

where

$$\gamma_2 := \frac{a_{22}^+}{a_{11}^+ + a_{22}^+}\Big(\alpha_{20} + \alpha_{02}\frac{(a_{11}^+)^2}{(a_{12}^+)^2} - \alpha_{11}\frac{a_{11}^+}{a_{12}^+}\Big) - \frac{a_{12}^+}{a_{11}^+ + a_{22}^+}\Big(\beta_{20} + \beta_{02}\frac{(a_{11}^+)^2}{(a_{12}^+)^2} - \beta_{11}\frac{a_{11}^+}{a_{12}^+}\Big).$$

For $\gamma_2 \neq 0$, $r(x) = \mathbf{O}(|x|^{-1})$, and according to (2.4) the standing-wave solution is first order degenerate.

For example, consider the viscous p-system

$$u_{1t} - u_{2x} = u_{1xx},$$
$$u_{2t} + p(u_1)_x = u_{2xx}.$$

In this case, $a_{11}^+ = s$, $a_{12}^+ = -1$, $a_{21}^+ = -s^2$, $a_{22}^+ = -s$, $\alpha_{20} = \alpha_{11} = \alpha_{02} = 0$, $\beta_{20} = p''(u_1^+)/2$, $\beta_{11} = \beta_{02} = 0$. Hence, our first-order degeneracy condition becomes simply $p''(u_1^+) \neq 0$.

A first order degenerate wave from the p-system with $p(u_1) = -u_1 - u_1^3$ is given in Figure 2.1, in which the exponential and algebraic decay rates can be observed clearly.

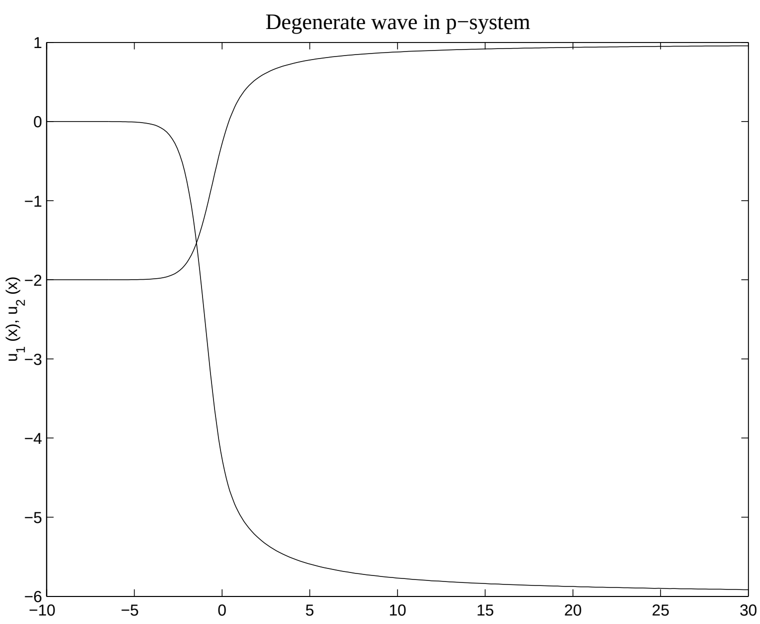

Figure 2.1. Degenerate wave from the viscous p-system

Finally, we remark that for both example equations (2.2) and (2.3) both a_{12}^+ and a_{21}^+ are 0, and so the wave cannot be first order degenerate.

3. The pointwise Green's function approach to stability

It will be useful to set a context for the following analysis by providing a brief overview of the pointwise Green's function approach to the study of stability. Suppose $\bar{u}(x)$ is a standing-wave solution of (1.1). It is well known that solutions $u(t,x)$ of (1.1), initialized by $u(0,x)$ near $\bar{u}(x)$ will not generally approach $\bar{u}(x)$, but rather will approach a translate of $\bar{u}(x)$ determined by the amount of mass (measured by $\int_{\mathbb{R}} u(0,x) - \bar{u}(x)dx$) carried into the shock as well as the amount carried out to the far field. In our framework, a local tracking function $\delta(t)$ will serve to approximate this shift at each time t. Following [HZ], we build this shift into our model by defining our perturbation $v(t,x)$ as $v(t,x) = u(t, x+\delta(t)) - \bar{u}(x)$. We will say that our wave is stable with respect to some measure if $v(0,x)$ sufficiently small implies that $v(t,x) \to 0$ as $t \to \infty$. A full discussion of local tracking is beyond the scope of this paper, and the reader is referred to [H.2] and the references therein.

Substituting $v(t,x) = u(t, x+\delta(t)) - \bar{u}(x)$ into (1.1), we obtain the perturbation equation

$$v_t = Lv + Q(v)_x + \dot{\delta}(t)(\bar{u}_x + v_x), \tag{3.1}$$

where $Lv := v_{xx} - (A(x)v)_x$, $A(x) = df(\bar{u}(x))$, and $Q(v) = \mathbf{O}(v^2)$ is a smooth function of v. Integrating (3.1), we have (after integration by parts on the second integral and observing that $e^{Lt}\bar{u}_x(x) = \bar{u}_x(x)$)

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} G(t,x;y)v_0(y)dy + \delta(t)\bar{u}_x(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t-s,x;y)\Big[Q(v(s,y)) + \dot{\delta}(s)v(s,y)\Big]dyds, \end{aligned} \tag{3.2}$$

where $G(t,x;y)$ represents a (matrix) Green's function for the linear part of (3.1):

$$G_t + (A(x)G)_x = G_{xx}; \quad G(0,x;y) = \delta_y(x)I.$$

The idea behind the pointwise Green's function approach to stability is to obtain estimates on $G(t,x;y)$ sharp enough so that an iteration on (3.2) can be closed (see especially [H.1, HZ, MZ, Z] for a full nonlinear analysis). Typically, we analyze $G(t,x;y)$ through its Laplace transform, $G_\lambda(x,y)$, which satisfies the ODE ($t \to \lambda$)

$$G_{\lambda xx} - (A(x)G_\lambda)_x - \lambda G_\lambda = -\delta_y(x)I,$$

and can be estimated by standard methods. Letting φ_1^+, φ_2^+ represent the (necessarily) two linearly independent asymptotically decaying solutions at $+\infty$ of the eigenvalue ODE

$$L\varphi = \lambda\varphi \tag{3.3}$$

and φ_1^-, φ_2^- similarly the two linearly independent asymptotically decaying solutions at $-\infty$, we follow (for example) [CH] and write $G_\lambda(x,y)$ as a linear combination of decaying solutions:

$$G_\lambda(x,y) = \begin{cases} \varphi_1^+(x)N_1^-(y) + \varphi_2^+(x)N_2^-(y) & x > y, \\ \varphi_1^-(x)N_1^+(y) + \varphi_2^-(x)N_2^+(y) & x < y, \end{cases}$$

where we observe the notation

$$\varphi_1^+ N_1^- = \begin{pmatrix} \varphi_{11}^+ \\ \varphi_{12}^+ \end{pmatrix} \begin{pmatrix} N_{11}^- & N_{12}^- \end{pmatrix} = \begin{pmatrix} \varphi_{11}^+ N_{11}^- & \varphi_{11}^+ N_{12}^- \\ \varphi_{12}^+ N_{11}^- & \varphi_{12}^+ N_{12}^- \end{pmatrix}.$$

Insisting on the continuity and jump of $G_\lambda(x,y)$ and $\partial_x G_\lambda(x,y)$ (respectively) across $x=y$, we have

$$(\varphi_1^+ N_1^- + \varphi_2^+ N_2^- - \varphi_1^- N_1^+ - \varphi_2^- N_2^+)(y) = 0 \\ (\varphi_1^{+\prime} N_1^- + \varphi_2^{+\prime} N_2^- - \varphi_1^{-\prime} N_1^+ - \varphi_2^{-\prime} N_2^+)(y) = -I. \tag{3.4}$$

Equations (3.4) represent eight equations and eight unknowns, which decouple into two sets of four equations and four unknowns. Solving by Cramer's rule, we have, for example,

$$N_{11}^-(y;\lambda) = -\frac{\det\begin{pmatrix} \varphi_{21}^+ & \varphi_{11}^- & \varphi_{21}^- \\ \varphi_{22}^+ & \varphi_{12}^- & \varphi_{22}^- \\ \varphi_{22}^{+\prime} & \varphi_{12}^{-\prime} & \varphi_{22}^{-\prime} \end{pmatrix}}{\det\begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \varphi_1^{+\prime} & \varphi_2^{+\prime} & \varphi_1^{-\prime} & \varphi_2^{-\prime} \end{pmatrix}}.$$

Clearly, then, $G_\lambda(x,y)$ will be well behaved so long as

$$D(\lambda,x) := \det\begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \varphi_1^{+\prime} & \varphi_2^{+\prime} & \varphi_1^{-\prime} & \varphi_2^{-\prime} \end{pmatrix} \neq 0.$$

Following Jones et al [AGJ, E.1–4, GZ, J, KS], we define the *Evans function* as $D(\lambda) := D(\lambda,0)$.

In order to understand the behavior of the Evans function, consider an eigenvector, $V(x,\lambda)$, of the operator

$$Lv := v_{xx} - (A(x)v)_x.$$

Since $V(x,\lambda)$ must decay at both $\pm\infty$, it must be a linear combination of φ_1^+, φ_2^+ at $+\infty$ and φ_1^-, φ_2^- at $-\infty$; thus these four solutions must be linearly dependent, leading to a zero of the Evans function. *In general, zeros of the Evans function correspond with eigenvalues of the operator L,* an observation that has been made precise in [AGJ] in the case pertaining to reaction–diffusion equations of standard isolated eigenvalues and in [ZH] in the case pertaining to conservation laws of nonstandard "effective" eigenvalues embedded in essential spectrum of L. (The latter correspond with resonant poles of L, as examined in the scalar context in [PW].)

4. ODE Estimates

The primary difficulty in analyzing the stability of degenerate viscous shock waves with the pointwise Green's function method lies in obtaining sufficiently sharp estimates on the growth and decay solutions of (3.3). Loss of analyticity at the critical point $\lambda = 0$ requires that in lieu of the Taylor expansion of previous analyses, we develop higher order (in λ) ODE estimates. As discussed in Section 1, however, these estimates arise in the critical case that ODE eigenvalues coalesce as λ goes to 0: a situation not covered by standard analyses such as [C]. We find, in fact, that our estimates depend not only on the general properties of the eigenvalue ODE (as is the case in the context of non-degenerate waves), but also on the particular structure of the underlying degenerate wave. Indeed, the specialized tricks that will be effective here fail for higher order degeneracies, leaving those cases as interesting open problems.

Our eigenvalue ODE (3.3) takes the form

$$v_{1xx} - (a_{11}(x)v_1)_x - (a_{12}(x)v_2)_x = \lambda v_1 \\ v_{2xx} - (a_{21}(x)v_1)_x - (a_{22}(x)v_2)_x = \lambda v_2, \tag{4.1}$$

where $a_{ij}(x) = \partial_{u_j} f_i(\bar{u}_1(x), \bar{u}_2(x))$. Equations of form (4.1) admit four linearly dependent solutions: two degenerate and two non-degenerate (details on this terminology below). These two sets must be treated in significantly different manners. We consider the integrated equation

$$\begin{aligned} w_{1xx} - a_{11}(x)w_{1x} - a_{12}(x)w_{2x} &= \lambda w_1 \\ w_{2xx} - a_{21}(x)w_{1x} - a_{22}(x)w_{2x} &= \lambda w_2. \end{aligned} \tag{4.2}$$

We will take advantage of the observation that the derivative of a decaying solution of (4.2) is certainly a decaying solution of (4.1). Writing (4.2) as a first order system, we have

$$W' = \mathbb{A}(x, \lambda)W, \tag{4.3}$$

where

$$\mathbb{A}(x,\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & a_{11}(x) & a_{12}(x) \\ 0 & \lambda & a_{21}(x) & a_{22}(x) \end{pmatrix},$$

which has four eigenvalues $\mu_k(x;\lambda)$, satisfying

$$\begin{aligned} \mu_1(x;\lambda) &= \frac{a_1(x) - \sqrt{a_1(x)^2 + 4\lambda}}{2}; \quad \mu_2(x;\lambda) = \frac{a_2(x) - \sqrt{a_2(x)^2 + 4\lambda}}{2} \\ \mu_3(x;\lambda) &= \frac{a_1(x) + \sqrt{a_1(x)^2 + 4\lambda}}{2}; \quad \mu_4(x;\lambda) = \frac{a_2(x) + \sqrt{a_2(x)^2 + 4\lambda}}{2}, \end{aligned}$$

where $a_1(x)$ and $a_2(x)$ are the eigenvalues of

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix},$$

namely,

$$a_1(x) = \frac{\mathrm{tr}A - \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}; \quad a_2(x) = \frac{\mathrm{tr}A + \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2},$$

with associated eigenvectors

$$r_k(x) = \begin{pmatrix} 1 \\ \frac{a_k(x) - a_{11}(x)}{a_{12}(x)} \end{pmatrix}, \quad k = 1, 2$$

At $x = +\infty$, $\mu_1^+(\lambda) = \mathbf{O}(1)$, $\mu_2^+(\lambda) = -\sqrt{\lambda}$, $\mu_3^+(\lambda) = \mathbf{O}(\lambda)$, $\mu_4^+(\lambda) = \mathbf{O} + \sqrt{\lambda}$, prompting our designation of μ_1, μ_3 as non-degenerate modes and μ_2, μ_4 as degenerate modes.

We are now in a position to state the main result of this section.

Lemma 4.1. *Suppose $\bar{u}(x) = (\bar{u}_1(x), \bar{u}_2(x))^{\mathrm{tr}}$ represents a first-order degenerate standing wave solution to (1.1) (that is, $\bar{u}_1(x) - u_1^+$ and $\bar{u}_2(x) - u_2^+$ are both $\mathbf{O}(|x|^{-1})$) with $f \in C^2(\mathbb{R})$ and $a_1^- < 0 < a_2^-$, $a_1^+ < 0 = a_2^+$. Then for some constant M_s sufficiently small, and $|\lambda| \leq M_s$, there exist constants L sufficiently large and $\alpha > 0$ so that the following estimates hold for solutions of (4.3):*

(i) (Non-degenerate solutions) For $x \geq L$

$$\begin{aligned} W_1^+(x,\lambda) &= e^{\int_L^x \mu_1(s,\lambda)ds}(V_1^+(\lambda) + \mathbf{O}(|x|^{-1})) \\ W_1^+(0,\lambda) &= W_1^+(0,0) + \mathbf{O}(|\lambda|) \\ W_3^+(x,\lambda) &= e^{\int_L^x \mu_3(s,\lambda)ds}(V_3^+(\lambda) + \mathbf{O}(|x|^{-1})), \end{aligned}$$

where $\mu_1^+(x;\lambda)$ and $\mu_3^+(x;\lambda)$ represent the eigenvalues of $\mathbb{A}(x,\lambda)$ and $V_1^+(\lambda), V_3^+(\lambda)$ represent the associated asymptotic eigenvectors. Further, for $0 \leq x \leq L$, $W_1^+(x,\lambda)$ and $W_2^+(x,\lambda)$ are both $\mathbf{O}(1)$ by continuous dependence.

(ii) (Degenerate solutions) For $\lambda \notin \mathbb{R}_-$, $x \geq 0$, and $k = 1,2$

$$\begin{aligned} W_{2k}^+(x,\lambda) &= e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(1 + E_{2k}(x,\lambda)\Big), \\ W_{2(k+2)}^+(x,\lambda) &= e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(-\sqrt{\lambda} + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+} + E_{2(k+2)}(x,\lambda) + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+}E_{2k}(x,\lambda)\Big), \\ W_{4k}^+(x,\lambda) &= e^{\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(1 + E_{4k}(x,\lambda)\Big), \\ W_{4(k+2)}^+(x,\lambda) &= e^{\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(\sqrt{\lambda} + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+} + E_{4(k+2)}(x,\lambda) + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+}E_{4k}(x,\lambda)\Big). \end{aligned}$$

where ($\wedge$ represents min)

$$\begin{aligned} E_{2k}(x,\lambda), E_{4k}(x,\lambda) &= \mathbf{O}(\sqrt{\lambda}\log\lambda) \wedge \mathbf{O}(|x|^{-1}), \quad k = 1,2 \\ E_{2k}(x,\lambda), E_{4k}(x,\lambda) &= \mathbf{O}(\sqrt{\lambda})\mathbf{O}(|x|^{-1}), \quad k = 3,4. \end{aligned}$$

Proof. We specify at the outset that the ODE analysis is carried out for $x > L$, L sufficiently large, with estimates obtained down to $x = 0$ by standard continuous dependence. In particular, we take L large enough so that $(\mathrm{tr}A(x))^2 - 4\det A(x) > 0$ for all $x \geq L$.

Non-degenerate solutions. For the non-degenerate solutions, we proceed via a standard calculation, a few details of which will suffice to indicate the argument and to reveal why such a calculation does not extend to the degenerate solutions. We begin by looking for solutions of the form $W(x) = e^{\int_L^x \mu_1(s,\lambda)ds}Z(x)$, for which $Z(x)$ satisfies $Z'(x) = (\mathbb{A}(x,\lambda) - \mu_1 I)Z(x)$. Let the 4×4 matrix $P(x,\lambda)$ represent the matrix of eigenvectors associated with the eigenvalues μ_k,

$$P(x,\lambda) = \begin{pmatrix} r_1(x) & r_2(x) & r_1(x) & r_2(x) \\ \mu_1(x,\lambda)r_1(x) & \mu_2(x,\lambda)r_2(x) & \mu_3(x,\lambda)r_1(x) & \mu_4(x,\lambda)r_2(x) \end{pmatrix},$$

with inverse

$$P(x,\lambda)^{-1} = \frac{1}{r_{22} - r_{12}} \begin{pmatrix} -\frac{r_{22}\mu_3}{\mu_1 - \mu_3} & \frac{\mu_3}{\mu_1-\mu_3} & \frac{r_{22}}{\mu_1-\mu_3} & -\frac{1}{\mu_1-\mu_3} \\ \frac{r_{12}\mu_4}{\mu_2-\mu_4} & -\frac{\mu_4}{\mu_2-\mu_4} & -\frac{r_{12}}{\mu_2-\mu_4} & \frac{1}{\mu_2-\mu_4} \\ \frac{r_{22}\mu_1}{\mu_1-\mu_3} & -\frac{\mu_1}{\mu_1-\mu_3} & -\frac{r_{22}}{\mu_1-\mu_3} & \frac{1}{\mu_1-\mu_3} \\ -\frac{r_{12}\mu_2}{\mu_2-\mu_4} & \frac{\mu_2}{\mu_2-\mu_4} & \frac{r_{12}}{\mu_2-\mu_4} & -\frac{1}{\mu_2-\mu_4} \end{pmatrix}.$$

We note that $\lim_{x\to\infty,\lambda\to 0} P(x,\lambda)^{-1}$ has infinite entries (due to the coalescence of ODE eigenvalues). Making the standard substitution $Z = PY$, we have

$$Y'(x) = D(x,\lambda)Y(x) - P(x,\lambda)^{-1}P'(x,\lambda)Y(x) \tag{4.4}$$

where

$$D(x,\lambda)=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & \mu_2-\mu_1 & 0 & 0\\ 0 & 0 & \mu_3-\mu_1 & 0\\ 0 & 0 & 0 & \mu_4-\mu_1\end{pmatrix}.$$

As an integral equation, (4.4) becomes

$$Y(x)=Y(+\infty)+\int_x^{+\infty} e^{\int_\xi^x D(s,\lambda)ds}P(\xi)^{-1}P'(\xi)Y(\xi)d\xi.$$

We proceed via standard iteration, beginning with the eigenvector $Y(+\infty)=(1,0,0,0)^{\mathrm{tr}}$. Since $e^{\int_\xi^x D(s,\lambda)ds}$ decays at exponential rate in each diagonal entry except the first, and since $Y(+\infty)$ has only one non-zero entry, the only critical entry of the matrix $P(\xi)^{-1}P'(\xi)$ is the first row, first column, $\{P(\xi)^{-1}P'(\xi)\}_{11}$. Observing that $r'_{12}(x)$ and $r'_{22}(x)$ are both $\mathbf{O}(|x|^{-2})$ we find by direct calculation,

$$\{P(\xi)^{-1}P'(\xi)\}_{11}=\frac{\mu_3 r'_{12}+\mu'_1 r_{22}-\mu_1 r'_{12}-\mu'_1 r_{12}}{(r_{22}-r_{12})(\mu_1-\mu_3)}=\mathbf{O}(|x|^{-2}).$$

Integration on $[x,+\infty)$, for $x>L$, of $\mathbf{O}(|x|^{-2})$ yields decay at rate $\mathbf{O}(|x|^{-1})$, which suffices to close a standard iteration, for example by contraction mapping. (The strict separation between μ_1^+ and the other modes guarantees this result by standard arguments; see, for example, Coppel [C].) The second estimate, $W_1^+(0,\lambda)=W_1^+(0,0)+\mathbf{O}(|\lambda|)$ follows from the exponential decay of $e^{\int_L^x \mu_1(s,\lambda)ds}$, $x>L$, and the analyticity of $\mu_1(x,\lambda)$ in λ.

For the second non-degenerate mode, μ_3, which is $\mathbf{O}(|\lambda|)$, the calculation is more subtle. Searching for solutions of the form $W(x)=e^{\int_L^x \mu_3(s,\lambda)ds}$ Z(x), $Z(x)=PY$, we obtain (4.4) with diagonal matrix

$$D(x,\lambda)=\begin{pmatrix}\mu_1-\mu_3 & 0 & 0 & 0\\ 0 & \mu_2-\mu_3 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu_4-\mu_3\end{pmatrix}.$$

The critical new issue is that only the first diagonal entry of $e^{\int_\xi^x D(s,\lambda)ds}$ decays at exponential rate. In fact, since μ_2,μ_4 are both (asymptotically) $\mathbf{O}(\sqrt{\lambda})$ and $\mu_3=\mathbf{O}(\lambda)$, the dominating mode in $\mu_2-\mu_3$ and $\mu_4-\mu_3$ changes: μ_3 dominates for λ large and μ_2 or μ_4 dominates when λ is sufficiently small. Hence, in our first iteration, with $Y(+\infty)=(0,0,1,0)^{\mathrm{tr}}$, we are concerned with the behavior of three critical terms in the matrix $P(\xi)^{-1}P'(\xi)$, $\{P(\xi)^{-1}P'(\xi)\}_{23,33,43}$. We have, for example,

$$\begin{aligned}\{P(\xi)^{-1}P'(\xi)\}_{23}&=\frac{-\mu_4 r'_{12}-\mu'_3 r_{12}+\mu_3 r'_{12}+\mu'_3 r_{12}}{(r_{22}-r_{12})(\mu_2-\mu_4)}\\ &=\frac{-\mu_4 r'_{12}+\mu_3 r'_{12}}{(r_{22}-r_{12})(\mu_2-\mu_4)}=\mathbf{O}(|x|^{-2})\end{aligned}$$

The analyses of $\{P(\xi)^{-1}P'(\xi)\}_{33}$ and $\{P(\xi)^{-1}P'(\xi)\}_{43}$ are similar.

Finally, it is enlightening to consider why this argument cannot be carried through for the degenerate modes. In the case of μ_2, for example, the critical entries in $P^{-1}(\xi)P'(\xi)$ become $\{P(\xi)^{-1}P'(\xi)\}_{22,32,42}$. We have, for the first

$$\begin{aligned}\{P(\xi)^{-1}P'(\xi)\}_{22}&=\frac{-\mu_4 r'_{22}-\mu'_2 r_{12}+\mu_2 r'_{22}+\mu'_2 r_{12}}{(r_{22}-r_{12})(\mu_2-\mu_4)}\\ &=\frac{r'_{22}(\mu_2-\mu_4)+\mu'_2(r_{22}-r_{12})}{(r_{22}-r_{12})(\mu_2-\mu_4)}=\frac{r'_{22}}{r_{22}-r_{12}}+\frac{\mu'_2}{\mu_2-\mu_4}.\end{aligned}$$

The critical term is

$$\frac{\mu_2'}{\mu_2-\mu_4} = -\frac{\frac{a_2'(x)}{2}\big(1-\frac{a_2(x)}{\sqrt{a_2(x)^2+4\lambda}}\big)}{\sqrt{a_2(x)^2+4\lambda}}. \tag{4.5}$$

In order to close our iteration, we require asymptotic decay in x of order $\mathbf{O}(|x|^{-r}), r>1$, uniform in λ. We observe immediately, however, that for any x, if $\lambda = a_2(x)$, the right-hand side of (4.5) decays as $\mathbf{O}(|x|^{-1})$, eliminating the possibility of uniform integrable decay.

Degenerate modes. We turn now to the degenerate modes. Here, we follow [H.1–2], and beginning again with the integrated ODE (4.2), make the critical substitution

$$\begin{aligned} w_1(x) &= (\bar u_1(x) - u_1^+)u_1(x)\\ w_2(x) &= (\bar u_2(x) - u_2^+)u_2(x). \end{aligned}$$

We find that our equation becomes

$$\begin{aligned} u_{1xx} &+ \Big(\frac{2\bar u_{1x}}{\bar u_1 - u_1^+} - a_{11}(x)\Big)u_{1x}(x) + \frac{a_{12}(x)\bar u_{2x}}{\bar u_1-u_1^+}u_1\\ &- a_{12}(x)\frac{\bar u_2 - u_2^+}{\bar u_1 - u_1^+}u_{2x} - a_{12}(x)\frac{\bar u_{2x}}{\bar u_1 - u_1^+}u_2 = \lambda u_1\\ u_{2xx} &+ \Big(\frac{2\bar u_{2x}}{\bar u_2 - u_2^+} - a_{22}(x)\Big)u_{2x}(x) + \frac{a_{21}(x)\bar u_{1x}}{\bar u_2-u_2^+}u_2\\ &- a_{21}(x)\frac{\bar u_1 - u_1^+}{\bar u_2 - u_2^+}u_{1x} - a_{21}(x)\frac{\bar u_{1x}}{\bar u_2 - u_2^+}u_1 = \lambda u_2. \end{aligned}$$

We write this ODE as the system

$$U'(x) = \mathcal{A}_+(\lambda)U + E(x)U(x),$$

where

$$\mathcal{A}_+(\lambda) := \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \lambda & 0 & a_{11}^+ & -a_{11}^+\\ 0 & \lambda & -a_{22}^+ & a_{22}^+ \end{pmatrix}$$

and the non-zero entries of E are:

$$\begin{aligned} &E_{31}(x) = -\frac{a_{12}(x)\bar u_{2x}}{\bar u_1 - u_1^+};\quad E_{32}(x) = \frac{a_{12}(x)\bar u_{2x}}{\bar u_1 - u_1^+};\\ &E_{33}(x) = -\Big(\frac{2\bar u_{1x}}{\bar u_1 - u_1^+} - a_{11}(x) + a_{11}^+\Big);\quad E_{34}(x) = a_{12}(x)\frac{\bar u_2 - u_2^+}{\bar u_1 - u_1^+} + a_{11}^+;\\ &E_{41}(x) = \frac{a_{21}(x)\bar u_{1x}}{\bar u_2 - u_2^+};\quad E_{42}(x) = -\frac{a_{21}(x)\bar u_{1x}}{\bar u_2 - u_2^+};\\ &E_{43}(x) = a_{21}(x)\frac{\bar u_1 - u_1^+}{\bar u_2 - u_2^+} + a_{22}^+;\quad E_{44}(x) = -\Big(\frac{2\bar u_{2x}}{\bar u_2 - u_2^+} - a_{22}(x) + a_{22}^+\Big). \end{aligned}$$

Critically, $\mathcal{A}_+(\lambda)$ has the same eigenvalues as $\mathbb{A}_+(\lambda)$ $(:= \lim_{x\to\infty}\mathbb{A}(x;\lambda))$, while the degenerate mode eigenvectors select decay modes in the direction $(1,1)^{\mathrm{tr}}$, so that in the original (unintegrated) coordinates the solutions approach $(\bar{u}_{1x}, \bar{u}_{2x})$ as $\lambda \to 0$. In this way, we choose $\varphi_2^+(x;0) = \bar{u}_x$ here rather than (as in [GZ], for example) by a change of basis during the Evans function calculation.

Looking for solutions of the form $U(x) = e^{-\sqrt{\lambda}x} Z$, we have

$$Z'(x) = (\mathcal{A}_+ + \sqrt{\lambda})Z(x) + E(x)Z(x),$$

or as an integral equation

$$Z(x) = Z(x_0) + \int_{x_0}^{x} e^{(\mathcal{A}_+ + \sqrt{\lambda})(x-\xi)} E(\xi)Z(\xi)d\xi. \tag{4.6}$$

If we let $P(\lambda)$ represent the matrix of eigenvectors of $\mathcal{A}_+$, we have

$$P(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{a_{22}^+}{a_{11}^+} & 1 & -\frac{a_{22}^+}{a_{11}^+} & 1 \\ \mu_1^+(\lambda) & -\sqrt{\lambda} & \mu_3^+(\lambda) & \sqrt{\lambda} \\ -\frac{a_{22}^+}{a_{11}^+}\mu_1^+(\lambda) & -\sqrt{\lambda} & -\frac{a_{22}^+}{a_{11}^+}\mu_3^+(\lambda) & \sqrt{\lambda} \end{pmatrix},$$

$$\det P = 2\sqrt{\lambda}(1 + \frac{a_{22}^+}{a_{11}^+})^2(\mu_3^+ - \mu_1^+),$$

and

$$P(\lambda)^{-1} = (1 + \frac{a_{22}^+}{a_{11}^+})^{-1} \begin{pmatrix} \frac{\mu_3^+}{\mu_3^+ - \mu_1^+} & -\frac{\mu_3^+}{\mu_3^+ - \mu_1^+} & -\frac{1}{\mu_3^+ - \mu_1^+} & \frac{1}{\mu_3^+ - \mu_1^+} \\ \frac{a_{22}^+/a_{11}^+}{2} & \frac{1}{2} & -\frac{a_{22}^+/a_{11}^+}{2\sqrt{\lambda}} & -\frac{1}{2\sqrt{\lambda}} \\ -\frac{\mu_1^+}{\mu_3^+ - \mu_1^+} & \frac{\mu_1^+}{\mu_3^+ - \mu_1^+} & \frac{1}{\mu_3^+ - \mu_1^+} & -\frac{1}{\mu_3^+ - \mu_1^+} \\ \frac{a_{22}^+/a_{11}^+}{2} & \frac{1}{2} & \frac{a_{22}^+/a_{11}^+}{2\sqrt{\lambda}} & \frac{1}{2\sqrt{\lambda}} \end{pmatrix},$$

where $\mu_k^+(\lambda) := \lim_{x\to\infty} \mu_k(x,\lambda)$ and $e^{(\mathcal{A}_+ + \sqrt{\lambda})(x-\xi)} = P(\lambda)e^{D(\lambda)(x-\xi)}P(\lambda)^{-1}$, with

$$D(\lambda) = \begin{pmatrix} \mu_1^+ + \sqrt{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3^+ + \sqrt{\lambda} & 0 \\ 0 & 0 & 0 & 2\sqrt{\lambda} \end{pmatrix}.$$

In the analysis that follows, we take advantage of the symmetry of $P(\lambda)^{-1}$; namely,

$$\begin{aligned} \{P(\lambda)^{-1}\}_{14} &= \{P(\lambda)^{-1}\}_{33} = -\{P(\lambda)^{-1}\}_{13} = -\{P(\lambda)^{-1}\}_{34}, \\ \{P(\lambda)^{-1}\}_{44} &= -\{P(\lambda)^{-1}\}_{24} = -\frac{a_{11}^+}{a_{22}^+}\{P(\lambda)^{-1}\}_{23} = \frac{a_{11}^+}{a_{22}^+}\{P(\lambda)^{-1}\}_{43}. \end{aligned}$$

Computing directly, we have

$$\begin{aligned} &e^{(\mathcal{A}_+ + \sqrt{\lambda})(x-\xi)}E(\xi)Z(\xi) \\ &= \begin{pmatrix} R_1 e^{\sqrt{\lambda}(x-\xi)}(e^{\mu_1^+(x-\xi)} - e^{\mu_3^+(x-\xi)}) + R_2(1 - e^{2\sqrt{\lambda}(x-\xi)}) \\ -\frac{a_{22}^+}{a_{11}^+}R_1 e^{\sqrt{\lambda}(x-\xi)}(e^{\mu_1^+(x-\xi)} - e^{\mu_3^+(x-\xi)}) + R_2(1 - e^{2\sqrt{\lambda}(x-\xi)}) \\ R_1 e^{\sqrt{\lambda}(x-\xi)}(\mu_1^+ e^{\mu_1^+(x-\xi)} - \mu_3^+ e^{\mu_3^+(x-\xi)}) - \sqrt{\lambda}R_2(1 + e^{2\sqrt{\lambda}(x-\xi)}) \\ -\frac{a_{22}^+}{a_{11}^+}R_1 e^{\sqrt{\lambda}(x-\xi)}(\mu_1^+ e^{\mu_1^+(x-\xi)} - \mu_3^+ e^{\mu_3(x-\xi)}) - \sqrt{\lambda}R_2(1 + e^{2\sqrt{\lambda}(x-\xi)}) \end{pmatrix}, \end{aligned}$$

where (summation assumed)

$$R_1 = (P^{-1})_{13}(E_{3k}Z_k - E_{4k}Z_k)$$
$$R_2 = (P^{-1})_{23}(E_{3k}Z_k + \frac{a_{11}^+}{a_{22}^+}E_{4k}Z_k).$$

We would like to iterate (4.6) now, beginning with $(Z_1, Z_2, Z_3, Z_4)^{\mathrm{tr}} = (1, 1, -\sqrt{\lambda}, -\sqrt{\lambda})^{\mathrm{tr}}$, corresponding to the asymptotically decaying degenerate mode. We must observe certain cancellations, gathered in the following claim (cf. Proposition 2.2 of [H.1]).

Claim 4.1.

$$(i) E_{31}(x) + E_{32}(x) = 0,$$
$$(ii) E_{41}(x) + E_{42}(x) = 0,$$
$$(iii) E_{33}(x) + E_{34}(x) + \frac{a_{11}^+}{a_{22}^+}E_{43}(x) + \frac{a_{11}^+}{a_{22}^+}E_{44}(x) = \mathbf{O}(|x|^{-2}).$$

Remark. It is in the cancellation of Claim 4.1 that we explicitly make use of the structure of our degenerate wave. We would point out that for degeneracies of order 2 or higher the result of Claim 4.1 does not hold, and a different analysis is required. In particular, in the case of higher order degeneracies, Taylor expansions at the endpoints must be taken to higher order, and the critical cancellation that leads to Claim 4.1 (see below) no longer holds.

Proof. We first observe that (i) and (ii) are trivial. For (iii), we begin by showing that

$$E_{33}(x) + E_{34}(x) = -a_{11}^+ - a_{12}^+\frac{\bar{u}_2 - u_2^+}{\bar{u}_1 - u_1^+} + \mathbf{O}(|x|^{-2}).$$

In order to establish this, we begin with the Taylor expansion:

$$\begin{aligned}\bar{u}_{1x} &= f_1(\bar{u}_1, \bar{u}_2) - f_1(u_1^+, u_2^+)\\ &= a_{11}^+(\bar{u}_1 - u_1^+) + a_{12}^+(\bar{u}_2 - u_2^+) + \frac{1}{2}\partial_{u_1u_1}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)^2\\ &+ \partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)(\bar{u}_2 - u_2^+) + \frac{1}{2}\partial_{u_2u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+)^2 + \mathbf{O}(|x|^{-3}),\end{aligned}$$

with also

$$a_{11}(x) = a_{11}^+ + \partial_{u_1u_1}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+) + \partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) + \mathbf{O}(|x|^{-2}),$$

and

$$a_{12}(x) = a_{12}^+ + \partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+) + \partial_{u_2u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) + \mathbf{O}(|x|^{-2}).$$

Combining, we have

$$\begin{aligned}E_{33}(x) + E_{34}(x) &= -\Big(\frac{2\bar{u}_{1x}}{\bar{u}_1 - u_1^+} - a_{11}(x) + a_{11}^+\Big) + a_{12}(x)\frac{\bar{u}_2 - u_2^+}{\bar{u}_1 - u_1^+} + a_{11}^+\\ &= -2a_{11}^+ - 2a_{12}^+\frac{\bar{u}_2 - u_2^+}{\bar{u}_1 - u_1^+} - \partial_{u_1u_1}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+)\\ &- 2\partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) - \partial_{u_2u_2}f_1(u_1^+, u_2^+)\frac{(\bar{u}_2 - u_2^+)^2}{\bar{u}_1 - u_1^+}\\ &+ a_{11}^+ - a_{11}^+ + \partial_{u_1u_1}f_1(u_1^+, u_2^+)(\bar{u}_1 - u_1^+) + \partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+)\\ &+ a_{12}^+\frac{\bar{u}_2^+ - u_2^+}{\bar{u}_1^+ - u_1^+} + \partial_{u_1u_2}f_1(u_1^+, u_2^+)(\bar{u}_2 - u_2^+) + \partial_{u_2u_2}f_1(u_1^+, u_2^+)\frac{(\bar{u}_2 - u_2^+)^2}{\bar{u}_1^+ - u_1^+}\\ &+ a_{11}^+ + \mathbf{O}(|x|^{-2}) = -a_{12}^+\frac{\bar{u}_2 - u_2^+}{\bar{u}_1 - u_1^+} - a_{11}^+ + \mathbf{O}(|x|^{-2}).\end{aligned}$$

Similarly, we have

$$E_{43}(x) + E_{44}(x) = -a_{22}^{+} - a_{21}^{+}\frac{\bar{u}_1 - u_1^{+}}{\bar{u}_2 - u_2^{+}} + \mathbf{O}(|x|^{-2}).$$

Finally, we compute

$$\begin{aligned}
&E_{33}(x) + E_{34}(x) + \frac{a_{11}^{+}}{a_{22}^{+}}(E_{43}(x) + E_{44}(x)) \\
&= -a_{11}^{+} - a_{12}^{+}\frac{\bar{u}_2 - u_2^{+}}{\bar{u}_1 - u_1^{+}} - a_{11}^{+} - \frac{a_{11}^{+}a_{21}^{+}}{a_{22}^{+}}\frac{\bar{u}_1 - u_1^{+}}{\bar{u}_2 - u_2^{+}} + \mathbf{O}(|x|^{-2}) \\
&= -\frac{1}{a_{12}^{+}}\frac{{a_{11}^{+}}^2(\bar{u}_1 - u_1^{+})^2 + 2a_{11}^{+}a_{12}^{+}(\bar{u}_1 - u_1^{+})(\bar{u}_2 - u_2^{+}) + {a_{12}^{+}}^2(\bar{u}_2 - u_2^{+})^2}{(\bar{u}_1 - u_1^{+})(\bar{u}_2 - u_2^{+})} + \mathbf{O}(|x|^{-2}) \\
&= -\frac{1}{a_{12+}}\frac{(a_{11}^{+}(\bar{u}_1 - u_1^{+}) + a_{12}^{+}(\bar{u}_2 - u_2^{+}))^2}{(\bar{u}_1 - u_1^{+})(\bar{u}_2 - u_2^{+})} + \mathbf{O}(|x|^{-2}).
\end{aligned}$$

But according to our center manifold development of Section 2, $\bar{u}_{1x} = \mathbf{O}(|x|^{-2})$ and hence $a_{11}^{+}(\bar{u}_1 - u_1^{+}) + a_{12}^{+}(\bar{u}_2 - u_2^{+}) = \mathbf{O}(|x|^{-2})$, establishing Claim 1. □

With algebraic decay of rate $|x|^{-2}$ established, we may proceed as in [H.1–2] and [ZH, pp. 779–780] to obtain

$$\begin{aligned}
Z_{21}(x,\lambda) &= 1 + E_{21}(x,\lambda) \\
Z_{22}(x,\lambda) &= 1 + E_{22}(x,\lambda) \\
Z_{23}(x,\lambda) &= -\sqrt{\lambda} + E_{23}(x,\lambda) \\
Z_{24}(x,\lambda) &= -\sqrt{\lambda} + E_{24}(x,\lambda),
\end{aligned}$$

where

$$E_{21}(x,\lambda), E_{22}(x,\lambda) = \mathbf{O}(\sqrt{\lambda}\log\lambda) \wedge \mathbf{O}(|x|^{-1}),$$

and

$$E_{23}(x,\lambda), E_{24}(x,\lambda) = \mathbf{O}(\sqrt{\lambda})\mathbf{O}(|x|^{-1}).$$

Following our substitutions back now, we obtain Lemma 4.1. □

Solutions of the unintegrated eigenvalue equation (4.1),

$$(\varphi_k, \varphi_k')^{\mathrm{tr}} = (\phi_k) = (\phi_{k1}, \phi_{k2}, \phi_{k3}, \phi_{k4})^{\mathrm{tr}}$$

can now be obtained through appropriate differentiation. For convenience, we collect this result as Lemma 4.2. Following [ZH], we denote growth solutions here by ψ.

Lemma 4.2. *Under the hypotheses of Lemma 4.1, we have the following estimates on solutions to the unintegrated eigenvalue equation (4.1)*

(i) $(x \leq 0)$

$$\begin{aligned}
\phi_j^{-}(x,\lambda) &= e^{\mu_j^{-}(\lambda)x}(V_j^{-}(\lambda) + \mathbf{O}(e^{-\alpha|x|})) \\
\phi_j^{-}(0,\lambda) &= \phi_j^{-}(0,0) + \mathbf{O}(|\lambda|),
\end{aligned}$$

where $\mu_j^{-}(\lambda)$ and $V_j^{-}(\lambda)$ are eigenvalue–eigenvector pairs of the matrix $\mathbb{A}_{-}(\lambda)(:= \lim_{x\to}\mathbb{A}(x,\lambda))$.

(ii) (Non-degenerate solutions) For $x \geq L$

$$\begin{aligned}
\varphi_1^+(x,\lambda) &= e^{\int_L^x \mu_1(s,\lambda)ds}(\mu_1^+(\lambda)r_1^+ + \mathbf{O}(|x|^{-1})) \\
\varphi_1^+(0,\lambda) &= \varphi_1^+(0,0) + \mathbf{O}(|\lambda|) \\
\psi_1^+(x,\lambda) &= e^{\int_L^x \mu_3(s,\lambda)ds}(\mu_3^+(\lambda)r_1^+ + \mathbf{O}(|x|^{-1})).
\end{aligned}$$

Further, for $0 \leq x \leq L$, $\varphi_1^+(x,\lambda)$ and $\psi_1^+(x,\lambda)$ are both $\mathbf{O}(1)$.
(iii) $(x \geq 0)$ (Degenerate solutions) For $\lambda \notin \mathbb{R}_-$ and $k = 1, 2$

$$\begin{aligned}
\varphi_{2k}^+(x,\lambda) &= e^{-\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(-\sqrt{\lambda} + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+} + E_{2(k+2)}(x,\lambda) + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+}E_{2k}(x,\lambda)\Big) \\
\varphi_2^{+\prime}(x,\lambda) &= A(x)\varphi_2^+(x,\lambda) + \lambda\mathcal{W}_2^+(x,\lambda) \\
\psi_{2k}^+(x,\lambda) &= e^{\sqrt{\lambda}x}(\bar{u}_k(x) - u_k^+)\Big(\sqrt{\lambda} + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+} + E_{4(k+2)}(x,\lambda) + \frac{\bar{u}_{kx}}{\bar{u}_k - u_k^+}E_{4k}(x,\lambda)\Big)
\end{aligned}$$

where $\mathcal{W}_2(x,\lambda) = (W_{21}, W_{22})^{\mathrm{tr}}$ and E_{jk} are exactly the same terms as in Lemma 4.1.

Proof. For $x \leq 0$ the analysis is identical to the proof of Proposition 2.1 of [ZH]. For $x \geq 0$ the estimates follow from Lemma 4.1 by direct calculation. $\square$

5. Analyzing the Evans function

For values of λ bounded away from 0, the Evans function can be analyzed as in [GZ] (see especially pp. 826–827 on Lax shocks). Here, we shall focus on its behavior near $\lambda = 0$. We begin by developing useful expressions for the $\varphi_k^{\pm\prime}$.

Fast decay ODE solutions ($\varphi_k^{\pm}(x;0) = \mathbf{O}(e^{-\alpha|x|})$, $\varphi_2^-(x;0) = c\bar{u}_x$). The fast decay solutions in this analysis are φ_2^- and φ_1^+. For the first, proceeding as in [GZ], we integrate

$$\varphi_2^{-\prime\prime}(x) - (A(x)\varphi_2^-)_x = \lambda\varphi_2^-$$

on $(-\infty, x]$ to obtain

$$\varphi_2^{-\prime}(x) - A(x)\varphi_2^-(x) = \lambda\int_{-\infty}^x \varphi_2^-(y)dy =: \lambda\mathcal{W}_2^-(x;\lambda),$$

where $\mathcal{W}_2^-(x;\lambda)$ is analytic in λ. Similarly for φ_1^+ we have

$$\varphi_1^{+\prime}(x) - A(x)\varphi_1^+(x) = \lambda\mathcal{W}_1^+(x;\lambda),$$

where $\mathcal{W}_1^+(x;\lambda) = (W_{11}^+, W_{12}^+)^{\mathrm{tr}}$ and by Lemma 4.1 satisfies $\mathcal{W}_1^+(0;\lambda) = \mathcal{W}_1^+(0;0) + \mathbf{O}(\lambda)$.

Slow decay ODE solutions ($\varphi_1^-(x;0) = \mathbf{O}(1)$). The only slow decay solution in this analysis is $\varphi_1^-(x;0)$. Integrating again on $(-\infty, x]$ and taking into account the loss of decay at $\lambda = 0$, we have

$$\varphi_1^{-\prime}(x) - A(x)\varphi_1^-(x) + A_-r_1^- = \lambda\mathcal{W}_1^-(x;\lambda),$$

where $\mathcal{W}_1^-(x;\lambda)$ is analytic in λ, and r_1^- is the eigenvector of A_- associated with the eigenvalue a_1^-.

Degenerate ODE solutions ($\varphi_2^+(x;0) = \bar{u}_x(x)$). Integrating our degenerate ODE solution on $[x, +\infty)$, we have

$$\varphi_2^{+\prime}(x) - A(x)\varphi_2^+(x) = \lambda \mathcal{W}_2^+(x;\lambda),$$

where by Lemma 4.1, $\mathcal{W}_2^+(0;\lambda) = \mathcal{W}_2^+(0;0) + \mathbf{O}(\sqrt{\lambda}\log\lambda)$.

Now, we compute

$$\begin{aligned}
D(\lambda) &= \det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \varphi_1^{+\prime} & \varphi_2^{+\prime} & \varphi_1^{-\prime} & \varphi_2^{-\prime} \end{pmatrix} \Big|_{y=0} \\
&= \det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ A\varphi_1^+ + \lambda\mathcal{W}_1^+ & A\varphi_2^+ + \lambda\mathcal{W}_2^+ & A\varphi_1^- - a_1^- r_1^- + \lambda\mathcal{W}_1^- & A\varphi_2^- + \lambda\mathcal{W}_2^- \end{pmatrix} \Big|_{y=0} \\
&= \det \Big[\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \lambda\mathcal{W}_1^+ & \lambda\mathcal{W}_2^+ & -a_1^- r_1^- + \lambda\mathcal{W}_1^- & \lambda\mathcal{W}_2^- \end{pmatrix} \Big] \Big|_{y=0} \\
&= \det \begin{pmatrix} \varphi_1^+ & \varphi_2^+ & \varphi_1^- & \varphi_2^- \\ \lambda\mathcal{W}_1^+ & \lambda\mathcal{W}_2^+ & -a_1^- r_1^- + \lambda\mathcal{W}_1^- & \lambda\mathcal{W}_2^- \end{pmatrix} \Big|_{y=0}.
\end{aligned}$$

From this final expression, we see immediately that $D(0) = 0$. The standard approach toward gaining higher order information on the Evans function at $\lambda = 0$ involves differentiating this final expression with respect to λ (see, for example, [GZ]). Since in the case of a degenerate wave, this Evans function is not analytic at $\lambda = 0$ (and cannot readily be extended analytically on a Riemann surface), our approach here will be to use our detailed estimates on $\varphi_k^\pm$ and $\mathcal{W}_k^\pm$ to write the Evans function as an analytic function plus a small error:

$$\begin{aligned}
D(\lambda) = \det &\begin{pmatrix} \varphi_1^+(0;0) & \varphi_2^+(0;0) & \varphi_1^-(0;0) & \varphi_2^-(0;0) \\ \lambda\mathcal{W}_1^+(0;0) & \lambda\mathcal{W}_2^+(0;0) & -a_1^- r_1^- & \lambda\mathcal{W}_2^-(0;0) \end{pmatrix} \\
&+ \mathbf{O}(|\lambda|^{3/2}|\log\lambda|) =: D_a(\lambda) + \mathbf{O}(|\lambda|^{3/2}|\log\lambda|).
\end{aligned}$$

The analytic function $D_a(\lambda)$ can now be analyzed directly as in [GZ].

6. Stability and Instability Criteria

Following [ZH, p. 760] we introduce the following stability condition $(\mathcal{D})$:

$(\mathcal{D})$: $D(\lambda)$ has precisely 1 zero in $\{\Re\lambda \geq 0\}$, necessarily at $\lambda = 0$, and $D_a'(0) \neq 0$.

While condition $(\mathcal{D})$ is generally quite difficult to study analytically (see, for example, [D]), it can be checked numerically [B, OZ]. We stress that analyticity of the Evans function for $\mathrm{Re}\lambda \geq 0$ is only lost at $\lambda = 0$, and consequently standard methods apply away from an arbitrarily small ball around the origin.

A condition that lends itself more readily to exact study is the *stability index*, typically defined as

$$\Gamma := \mathrm{sgn} D'(0) \times \mathrm{sgn} \lim_{\mathbb{R} \ni \lambda \to \infty} D(\lambda).$$

For $\lambda \in \mathbb{R}_+$, we have $D(\lambda) \in \mathbb{R}$, so that in the event that $\Gamma = -1$, $D(\cdot)$ must have a positive real root, which guarantees instability. In the case that $\Gamma = +1$, the question of stability remains undecided. Proceeding exactly as in [GZ], we find that in the present degenerate-wave setting, Γ becomes

$$\Gamma = \mathrm{sgn}\Big[\det(r_1^-, u_+ - u_-) \det(r_1^-, \lim_{x \to -\infty} \frac{\bar{u}_x(x)}{|\bar{u}_x(x)|}) \Big].$$

Consider, for example, the degenerate viscous shock wave depicted in Figure 2.1, for the viscous p-system

$$u_{1t} - u_{2x} = u_{1xx}$$
$$u_{2t} + p(u_1)_x = u_{2xx},$$

with $p(u_1) = -u_1 - u_1^3$. We have in this case, $u_1^- = -2, u_2^- = 0$, with $u_1^+ = 1, u_2^+ = -6$, and consequently $s = 2$, $r_1^- = (1, s)^{\mathrm{tr}} = (1, 2)^{\mathrm{tr}}$. Thus

$$\begin{aligned}\Gamma &= \operatorname{sgn}\Big[\det\begin{pmatrix}1 & 3\\ 2 & -6\end{pmatrix}\det\begin{pmatrix}1 & (+)\\ 2 & (-)\end{pmatrix}\Big]\\ &= \operatorname{sgn}\Big[(-6-6)(1(-) - 2(+))\Big] = +1,\end{aligned}$$

where $(+)$ and $(-)$ represent quantities for which only signs are known. Proceeding similarly, we find that any monotonic degenerate viscous shock arising in the p-system has a positive stability index.

References

[AGJ] J. Alexander, R. Gardner, and C. K. R. T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math. **410** (1990) 167–212.

[B] L. Brin, *Numerical testing of the stability of viscous shock waves*, Math. Cont. **70** (2001), no. 235, 1071–1088.

[C] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D.C. Heath and Co., Boston, 1965.

[Carr] J. Carr, *Applications of Centre Manifold Theory*, Applied Mathematical Sciences **35** (1981), Springer–Verlag, New York–Berlin.

[CH] R. Courant and D. Hilbert, *Methods of mathematical physics, Volume I*, Interscience, New York, 1953.

[D] J. Dodd, *Convection stability of shock profile solutions of a modified KdV–Burgers equation*, Thesis under Bob Pego, University of Maryland, 1996.

[E.1] J. W. Evans, *Nerve axon equations I: Linear approximations*, Indiana Univ. Math. J. **21** (1972) 877–885.

[E.2] J. W. Evans, *Nerve axon equations II: Stability at rest*, Indiana Univ. Math. J. **22** (1972) 75–90.

[E.3] J. W. Evans, *Nerve axon equations III: Stability of the nerve impulse*, Indiana Univ. Math. J. **22** (1972) 577–594.

[E.4] J. W. Evans, *Nerve axon equations IV: The stable and unstable impulse*, Indiana Univ. Math. J. **24** (1975) 1169–1190.

[G] R. Gardner, *Spectral analysis of long wavelength periodic waves and applications*, J. Reine Angew. Math. **491** (1997) 149–181.

[Go] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rat. Mech. Anal. **95** (1986) 325–344.

[GZ] R. Gardner and K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*, CPAM **51** (1998) 789–847.

[H.1] P. Howard, *Pointwise estimates and stability for degenerate viscous shock waves*, J. Reine Angew. Math. **545** (2002) 19–65.

[H.2] P. Howard, *Local tracking and stability for degenerate viscous shock waves*, JDE **186** (2002) 440–469.

[H.3] P. Howard, *Nonlinear stability for degenerate viscous shock waves*, in preparation.

[HZ] P. Howard and K. Zumbrun, *Pointwise estimates and stability for dispersive–diffusive shock waves*, Arch. Rat. Mech. Anal. **155** (2000) 85–169.

[J] C. K. R. T. Jones, *Stability of the traveling wave solution of the FitzHugh–Nagumo system*, Trans. Amer. Math. Soc. **286** (1984), no. 2, 431–469.

[K] A. Kelley, *The stable, center–stable, center, center–unstable, unstable manifolds*, JDE **3** (1967) 546–570.

[KR] T. Kapitula and J. Rubin, *Existence and stability of standing hole solutions to complex Ginzburg–Landau equations*, Nonlinearity **13** (2000), no. 1, 77–112.

[KS] T. Kapitula and B. Sandstede, *Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations*, Physica D **124** (1998) 58–103.

[MZ] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks*, Indiana Univ. Math. J. **51** (2002) 773–904.

[N] K. Nishihara, *Stability of traveling waves with degenerate shock for system of one-dimensional visco-elastic model*, J. Diff. Eqns. **120** (1995) 304–318.

[OZ] M. Oh, and K. Zumbrun, *Stability of periodic solutions of viscous conservation laws: Analysis of the Evans function*, Arch. Rat. Mech. Anal. **166** (2003) 167–196.

[PW] R. L. Pego and M. I. Weinstein, *Eigenvalues and instabilities of solitary waves*, Philos. Trans. Roy. Soc. London Ser. A **340** (1992) 47–94.

[SS] B. Sandstede and A. Scheel, *Evans function and blow-up methods in critical eigenvalue problems*, Preprint 2002.

[Z] K. Zumbrun, *Refined wave-tracking and nonlinear stability of viscous Lax shocks*, Methods Appl. Anal. **7** (2000) 747–768.

[ZH] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana U. Math. J. **47** (1998), no. 3, 741–871.