Oscillation Theory and Instability of Nonlinear Waves

Peter Howard

Texas A&M University

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Abstract

In recent work, Baird et al. have introduced a generalized Maslov index which allows oscillation techniques that have previously been restricted to eigenvalue problems with underlying Hamiltonian structure to be extended to the non-Hamiltonian setting [T. J. Baird, P. Cornwell, G. Cox, C. Jones, and R. Marangell, *Generalized Maslov indices for non-Hamiltonian systems*, SIAM J. Math. Anal. **54** (2022) 1623-1668]. We show that this approach can be implemented in the analysis of spectral instability for nonlinear waves, taking as our setting a class of equations previously investigated by Pego and Weinstein via the Evans function [R. L. Pego and M. I. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Phil. Trans. R. Soc. Lond. A **340** (1992) 47-94].

1 Introduction

For values of λ in a real open interval $I \subset \mathbb{R}$, we consider first-order ODE systems

$$\frac{dy}{dx} = A(x;\lambda)y, \quad x \in \mathbb{R}, \quad y(x;\lambda) \in \mathbb{R}^n, \quad n \in \{2,3,\dots\}, \tag{1.1}$$

for which we make the following assumptions, adapted from [33]:

(A) We assume that for some open set $\Omega \subset \mathbb{C}$ containing I, we have $A \in C(\mathbb{R} \times \Omega, \mathbb{C}^{n \times n})$, with $A(x;\lambda) \in \mathbb{R}^{n\times n}$ for all $(x,\lambda) \in \mathbb{R} \times I$, and that the map $\lambda \mapsto A(x;\lambda)$ is analytic on Ω for every $x \in \mathbb{R}$.

(B) For each $\lambda \in \Omega$, the limits

$$A_\pm(\lambda) := \lim_{x \to \pm\infty} A(x;\lambda)$$

exist and are obtained uniformly on compact subsets of Ω.

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(C) For each $\lambda \in \Omega$, each of the matrices $A_\pm(\lambda)$ has a unique (and so necessarily real for $\lambda \in I$) eigenvalue of largest real part, which is simple. We denote this eigenvalue $\mu_\pm(\lambda)$ and denote by $\mu^*_\pm(\lambda)$ the largest real part of any other eigenvalue of $A_\pm(\lambda)$, so that

$$\operatorname{Re}\mu_\pm(\lambda) > \mu^*_\pm(\lambda) := \max\{\operatorname{Re}\nu : \nu \neq \mu_\pm(\lambda),\ \nu \in \sigma(A_\pm(\lambda))\}.$$

Moreover, $\mu_\pm(\lambda)$ is analytic on Ω, and there exists an analytic (in Ω) choice of left and right eigenvectors of $A_\pm(\lambda)$, satisfying

$$(A_\pm(\lambda) - \mu_\pm(\lambda)I)v^\pm(\lambda) = 0, \quad w^\pm(\lambda)(A_\pm(\lambda) - \mu_\pm(\lambda)I) = 0, \quad w^\pm(\lambda)v^\pm(\lambda) = 1.$$

(D) The function

$$E(x;\lambda) := \begin{cases} A(x;\lambda) - A_-(\lambda) & x < 0 \\ A(x;\lambda) - A_+(\lambda) & x > 0 \end{cases}$$

satisfies the following two conditions: (i) $\int_{\mathbb{R}} |E(x;\lambda)|dx$ is finite for all $\lambda \in \Omega$; and (ii) this integral converges uniformly on compact subsets of Ω.

As with [33], our analysis is primarily motivated by consideration of stability for traveling-wave solutions $\bar{u}(x - st)$ arising in the context of single higher order nonlinear evolutionary PDE,

$$u_t + g(u) + f(u)_x = \sum_{k=2}^{n} (b_k(u)\partial_x^{k-1}u)_x, \quad n \in \{2, 3, \dots\}. \tag{1.2}$$

By replacing f with $f - su$, we can regard such solutions as stationary solutions $\bar{u}(x)$ to the same equation, and upon linearization (with $u = \bar{u} + v$), we obtain

$$v_t + a_0(x)v + (a_1(x)v)_x = \sum_{k=2}^{n} (a_k(x)\partial_x^{k-1}v)_x, \tag{1.3}$$

with

$$a_0(x) = g'(\bar{u}(x)), \quad a_1(x) = f'(\bar{u}(x)) - \sum_{k=2}^{n} b_k{}'(\bar{u}(x))\partial_x^{k-1}\bar{u}(x),$$

and

$$a_k(x) = b_k(\bar{u}(x)), \quad k = 2, 3, \dots, n.$$

The associated eigenvalue problem can be expressed as

$$L\phi := -\sum_{k=2}^{n} (a_k(x)\phi^{(k-1)})' + (a_1(x)\phi)' + a_0(x)\phi = \lambda\phi,$$

where our sign convention is taken to be consistent with cases in which (1.2) is second order and the negative sign corresponds with a positive operator. We now obtain (1.1) by expressing (1) as a first-order system with $y = (y_1\ y_2\ \dots\ y_n)^T$, $y_k = \phi^{(k-1)}$, $k = 1, 2, \dots, n-1$, $y_n = a_n(x)\phi^{(n-1)}$. Specific matrices arising in this way will be considered in the applications discussed in Section 6.

Under Assumption **(C)**, there exists a one-dimensional subspace of solutions of (1.1) that decay at the maximal exponential rate $e^{\mu_-(\lambda)x}$ as x tends to $-\infty$, and likewise an $(n-1)$-dimensional subspace of solutions to (1.1) that fail to grow at the maximal rate $e^{\mu_+(\lambda)x}$ as x tends to $+\infty$. Following [33], our goal will be to identify values $\lambda \in I$ for which there exists a solution $y(\cdot;\lambda) \in C^1(\mathbb{R}, \mathbb{R}^n)$ of (1.1) that lies in the intersection of these two spaces. Although we will refer to such values as eigenvalues throughout the analysis, we observe here that they are only "genuinely" eigenvalues in the event that $\mu_-(\lambda)$ is strictly positive and is the only positive eigenvalue of $A_-(\lambda)$, and additionally $\mu_+(\lambda)$ is non-negative and is the only eigenvalue of $A_+(\lambda)$ with non-negative real part. In the usual way, if λ is an eigenvalue of (1.1), we will refer to the dimension of the space of all associated solutions as the geometric multiplicity of λ. Our main goal is to show that oscillation theory can be used to obtain a lower bound on the number of eigenvalues $\mathcal{N}_\#([\lambda_1, \lambda_2])$, counted *without* multiplicity, that (1.1) has on a given interval $[\lambda_1, \lambda_2] \subset I$, $\lambda_1 < \lambda_2$. Under our relatively weak assumptions on the dependence of $A(x;\lambda)$ on λ, it's possible that the eigenvalues of (1.1), as we've defined them, won't comprise a discrete set on the interval $[\lambda_1, \lambda_2]$. In this case, our convention will be to take $\mathcal{N}_\#([\lambda_1, \lambda_2]) = +\infty$, and to view our lower bounds on $\mathcal{N}_\#([\lambda_1, \lambda_2])$ as holding by convention. We will see that in many cases the value λ_1 can be taken sufficiently negative so that (1.1) has no eigenvalues below λ_1, and in such cases, we will often write $\mathcal{N}_\#([\lambda_1, \lambda_2]) = \mathcal{N}_\#((-\infty, \lambda_2])$ to emphasize that it becomes a count of the number of eigenvalues that (1.1) has at or below λ_2. For stability analyses, we are most interested in taking $\lambda_2 = 0$ (since negative eigenvalues correspond with instability by our sign conventions), so the count we will be most interested in is $\mathcal{N}_\#((-\infty, 0])$.

In [33], the authors address a closely related question via the Evans function, namely, determining conditions under which evaluation of the Evans function along with appropriate derivatives at $\lambda_2 = 0$ can be combined with asymptotic information as λ tends toward $-\infty$ to indicate the existence of at least one eigenvalue on the open half-line $(-\infty, 0)$. Our analysis puts the results of [33] in a broader context and adds an additional geometric criterion for the existence of such eigenvalues.

Our primary tool for this analysis will be a generalization of the Maslov index introduced in [2], and for the purposes of this introduction we will start with a brief, intuitive discussion of this object (see Section 2 for additional details and reference [2] for a full development). Precisely, we focus on the hyperplane setting discussed in Section 3.2 of [2].

To begin, for any $n \in \mathbb{N}$ we denote by $Gr_n(\mathbb{R}^{2n})$ the Grassmannian comprising the collection of all n-dimensional subspaces of $\mathbb{R}^{2n}$, and we let $\mathfrak{g}$ denote an element of $Gr_n(\mathbb{R}^{2n})$. The space $\mathfrak{g}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$, and we will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{G}$, which we will refer to as a *frame* for $\mathfrak{g}$. We specify a metric on $Gr_n(\mathbb{R}^{2n})$ in terms of appropriate orthogonal projections. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\mathfrak{g}_i \in Gr_n(\mathbb{R}^{2n})$ for $i = 1, 2$. I.e., if $\mathbf{G}_i$ denotes a frame for $\mathfrak{g}_i$, then $\mathcal{P}_i = \mathbf{G}_i(\mathbf{G}_i^* \mathbf{G}_i)^{-1} \mathbf{G}_i^*$. We take our metric d on $Gr_n(\mathbb{R}^{2n})$ to be defined by

$$d(\mathfrak{g}_1, \mathfrak{g}_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that a path of Grassmannian subspaces $\mathfrak{g} : [a, b] \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Remark 1.1. *Here, and throughout, we will be as consistent as possible with the following notational conventions: we will express Grassmannian subspaces with script letters such as* $\mathfrak{g}$ *or* $\mathfrak{h}$*, and we will denote a choice of basis elements for* $\mathfrak{g}$ *or* $\mathfrak{h}$ *respectively by* $\{g_i\}_{i=1}^n$ *or* $\{h_i\}_{i=1}^n$*. We will also collect these basis elements into associated frames designated with bold capital letters,*

$$\mathbf{G} = (g_1,\ g_2,\ \dots,\ g_n) \quad \textit{or} \quad \mathbf{H} = (h_1,\ h_2,\ \dots,\ h_n)$$

Given a continuous path of Grassmannian subspaces $\mathfrak{g} : [a,b] \to Gr_n(\mathbb{R}^{2n})$ and a fixed *target* space $\mathfrak{q} \in Gr_n(\mathbb{R}^{2n})$, the generalized Maslov index of [2] (under some additional conditions discussed below) provides a means of counting intersections between the subspaces $\mathfrak{g}(t)$ and $\mathfrak{q}$ as t increases from a to b, counted with direction, but not with multiplicity. (By multiplicity, we mean the dimension of the intersection; direction will be discussed in detail in Section 2). In order to understand how this works, we first recall the notion of a kernel for a skew-symmetric n-linear map ω.

Definition 1.1. *For a skew-symmetric* n*-linear map* $\omega : \mathbb{R}^{2n} \times \cdots \times \mathbb{R}^{2n} \to \mathbb{R}$ *(*$\mathbb{R}^{2n}$ *appearing* n *times), we define the kernel,* $\ker \omega$*, to be the subset of* $\mathbb{R}^{2n}$*,*

$$\ker \omega := \{v \in \mathbb{R}^{2n} : \omega(v, v_1, \dots, v_{n-1}) = 0, \quad \forall\, v_1, v_2, \dots, v_{n-1} \in \mathbb{R}^{2n}\}.$$

Given a target space $\mathfrak{q} \in Gr_n(\mathbb{R}^{2n})$, we first identify a skew-symmetric n-linear map ω_1 so that $\mathfrak{q} = \ker \omega_1$. For example, if we let $\{q_i\}_{i=1}^n \subset \mathbb{R}^{2n}$ denote a basis for $\mathfrak{q}$, then we can set

$$\omega_1(g_1, \dots, g_n) := \det(g_1 \ \dots \ g_n \ q_1 \dots q_n),$$

or if we interpret the elements $\{g_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$ as 1-forms,

$$g_1 \wedge \cdots \wedge g_n \wedge q_1 \wedge \cdots \wedge q_n = \omega_1(g_1, \dots, g_n) e_1 \wedge \cdots \wedge e_{2n}. \tag{1.4}$$

For notational convenience, we will often write

$$\omega_1(g_1, \dots, g_n) = g_1 \wedge \cdots \wedge g_n \wedge q_1 \wedge \cdots \wedge q_n$$

when, strictly speaking, we mean (1.4).

Next, we let ω_2 denote any skew-symmetric n-linear map for which $\ker \omega_2 \neq \mathfrak{q}$, and we set

$$\mathcal{H}_{\omega_i} := \{\mathfrak{g} \in Gr_n(\mathbb{R}^{2n}) : \mathfrak{g} \cap \ker \omega_i \neq \{0\}\}, \quad i = 1, 2.$$

Then according to Definition 1.3 in [2], the set

$$\mathcal{M} := Gr_n(\mathbb{R}^{2n}) \backslash (\mathcal{H}_{\omega_1} \cap \mathcal{H}_{\omega_2}) \tag{1.5}$$

is a *hyperplane Maslov-Arnold space*.

Definition 1.2. *We say that the flow* $t \mapsto \mathfrak{g}(t)$ *is invariant on* $[a,b]$ *with respect to* ω_1 *and* ω_2 *provided the values*

$$\omega_1(g_1(t), \dots, g_n(t)) \quad \textit{and} \quad \omega_2(g_1(t), \dots, g_n(t))$$

do not simultaneously vanish at any $t \in [a,b]$ *(i.e.,* $\mathfrak{g}(t) \in \mathcal{M}$ *for all* $t \in [a,b]$*). For brevity, we say that the triple* $(\mathfrak{g}(\cdot), \omega_1, \omega_2)$ *is invariant on* $[a,b]$*. (Here, we note that* $\mathfrak{q}$ *is not needed in the triple notation, since* $\mathfrak{q}$ *is determined by* ω_1*.) Likewise, we say that a map* $\mathfrak{g} : [a,b] \times [c,d] \to Gr_n(\mathbb{R}^{2n})$ *is invariant on* $[a,b] \times [c,d]$ *with respect to* ω_1 *and* ω_2 *provided the values*

$$\omega_1(g_1(s,t), \dots, g_n(s,t)) \quad \text{and} \quad \omega_2(g_1(s,t), \dots, g_n(s,t)) \tag{1.6}$$

do not simultaneously vanish at any $(s,t) \in [a,b] \times [c,d]$ *(i.e.,* $\mathfrak{g}(s,t) \in \mathcal{M}$ *for all* $(s,t) \in [a,b] \times [c,d]$*). For brevity, we say that the triple* $(\mathfrak{g}(\cdot,\cdot), \omega_1, \omega_2)$ *is invariant on* $[a,b] \times [c,d]$*. Finally, we will say that a map* $\mathfrak{g} : [a,b] \times [c,d] \to Gr_n(\mathbb{R}^{2n})$ *is invariant on the boundary of* $[a,b] \times [c,d]$ *with respect to* ω_1 *and* ω_2 *provided the values in (1.6) do not simultaneously vanish at any point* (s,t) *on the boundary of* $[a,b] \times [c,d]$*.*

Remark 1.2. *The terminology "invariant" is taken from [2], where it arises naturally as the condition that a path in* $P(\bigwedge^n(\mathbb{R}^{2n}))$ *(i.e., the projective space of all one-dimensional subspaces of the wedge space* $\bigwedge^n(\mathbb{R}^{2n})$*) associated to the flow* $t \mapsto \mathfrak{g}(t)$ *lies entirely in the Maslov-Arnold space introduced in [2]. While this notion of the Maslov-Arnold space is critical to the development of [2], we will only use it indirectly here, and so will omit a precise definition.*

In the event that the flow $t \mapsto \mathfrak{g}(t)$ is invariant on $[a,b]$ with respect to ω_1 and ω_2, the generalized Maslov index of [2] can be computed as the winding number in projective space $\mathbb{R}P^1$ of the map

$$t \mapsto [\omega_1(g_1(t), \dots, g_n(t)) : \omega_2(g_1(t), \dots, g_n(t))] \tag{1.7}$$

through $[0:1]$ (with appropriate conventions taken for counting arrivals and departures; see Section 2 below). If the Maslov-Arnold space is a hyperplane Maslov-Arnold space as in (1.5) then this value is referred to as the *hyperplane index*. Since the currently analysis will be entirely in the hyperplane setting, we will henceforth refer to such counts as hyperplane indices.

Following the convention of [2], we denote the hyperplane index by $\operatorname{Ind}(\cdots)$, though our specific notation is adapted from [25, 26], leading to $\operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$; i.e., $\operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$ is a directed count of the number of times the subspace $\mathfrak{g}(t)$ has non-trivial intersection with $\mathfrak{q}$, counted without multiplicity, as t increases from a to b. The value $\operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$ clearly depends on the choices of ω_1 and ω_2, but we will generally suppress this in the notation, taking it to generally be the case that these choices are clear from context. For ω_1, we will always use (1.4), but there is considerable flexibility in the choice of ω_2. In cases in which two possibilites for ω_2 are considered, we will denote the alternative choice as ω_3, and in order to distinguish the resulting hyperplane indices, we will denote them respectively $\operatorname{Ind}_{\omega_2}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$ and $\operatorname{Ind}_{\omega_3}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$.

For many applications, we would like to compute the hyperplane index associated with a pair of evolving spaces $\mathfrak{g}, \mathfrak{h} : [a,b] \to Gr_n(\mathbb{R}^{2n})$, or more generally (as in the current setting) a pair of evolving spaces $\mathfrak{g} : [a,b] \to Gr_m(\mathbb{R}^n)$ and $\mathfrak{h} : [a,b] \to Gr_{n-m}(\mathbb{R}^n)$, where $m \in \{1, 2, \dots, n-1\}$. Following the approach of Section 3.5 in [19] (as developed for the current setting in [23]), we can proceed by letting $\mathbf{G}(t)$ and $\mathbf{H}(t)$ respectively denote frames for $\mathcal{g}(t)$ and $\mathcal{h}(t)$, and specifying an evolving subspace $\mathcal{f} : [a, b] \to Gr_n(\mathbb{R}^{2n})$ with frame

$$\mathbf{F}(t) := \begin{pmatrix} \mathbf{G}(t) & \mathbf{0}_{n\times(n-m)} \\ \mathbf{0}_{n\times m} & \mathbf{H}(t) \end{pmatrix}. \tag{1.8}$$

Subsequently, we fix a target space $\tilde{\Delta} \in Gr_n(\mathbb{R}^{2n})$ with frame $\tilde{\mathbf{\Delta}} = \binom{-I_n}{I_n}$, and we define the hyperplane index for the pair $\mathcal{g}, \mathcal{h} : [a, b] \to Gr_n(\mathbb{R}^{2n})$ to be

$$\operatorname{Ind}(\mathcal{g}(\cdot), \mathcal{h}(\cdot); [a, b]) := \operatorname{Ind}(\mathcal{f}(\cdot), \tilde{\Delta}; [a, b]), \tag{1.9}$$

where the right-hand side is computed precisely as described above (i.e., as in [2]), depending as always on specified choices of ω_1 and ω_2.

From (1.4), ω_1 is taken to be

$$\omega_1(f_1, f_2, \dots, f_n) := f_1 \wedge \cdots \wedge f_n \wedge \tilde{\delta}_1 \wedge \cdots \wedge \tilde{\delta}_n, \tag{1.10}$$

where the vectors $\{\tilde{\delta}_i\}_{i=1}^n$ comprise the columns of $\tilde{\mathbf{\Delta}} = \binom{-I_n}{I_n}$. In order to take advantage of the additional flexibility with ω_2, we proceed by fixing a (judiciously chosen) invertible $n \times n$ matrix M and setting

$$\omega_2(f_1, f_2, \dots, f_n) := f_1 \wedge \cdots \wedge f_n \wedge \tilde{\sigma}_1 \wedge \cdots \wedge \tilde{\sigma}_n, \tag{1.11}$$

where the vectors $\{\tilde{\sigma}_i\}_{i=1}^n$ comprise the columns of $\tilde{\mathbf{\Sigma}} = \binom{-M}{I_n}$.

Returning to (1.1), we will show in Section 3 that under our Assumptions **(A)** through **(D)** there exists a unique solution $\eta^-(x; \lambda)$ to (1.1) so that

$$\lim_{x\to-\infty} e^{-\mu_-(\lambda)x} \eta^-(x; \lambda) = v^-(\lambda), \tag{1.12}$$

where $\mu_-(\lambda)$ and $v^-(\lambda)$ are as described in Assumption **(C)**. We will let $\mathcal{g}(x; \lambda) \in Gr_1(\mathbb{R}^n)$ denote the path of Grassmannian subspaces with frame $\eta^-(x; \lambda)$. For the space $\mathcal{h}(x; \lambda)$, we begin by observing that under our Assumptions **(A)** through **(D)** we can take a linearly independent collection of solutions to (1.1) $\{y_i^+(x; \lambda)\}_{i=1}^n$, selected so that

$$\lim_{x\to+\infty} e^{-\mu_+(\lambda)x} y_n^+(x; \lambda) = v^+(\lambda), \tag{1.13}$$

where $\mu_+(\lambda)$ and $v^+(\lambda)$ are as described in Assumption **(C)**. With $y_n^+(x; \lambda)$ distinguished in this way, we take $\mathcal{h}(x; \lambda) \in Gr_{n-1}(\mathbb{R}^n)$ to be the space spanned by the collection $\{y_i^+(x; \lambda)\}_{i=1}^{n-1}$. In addition, we characterize this space by designating the wedge product

$$\mathcal{Y}^+(x; \lambda) := y_1^+(x; \lambda) \wedge \cdots \wedge y_{n-1}^+(x; \lambda), \tag{1.14}$$

about which we will establish some general notation in the following remark.

Remark 1.3. *In many cases, we will associate a vector $v \in \mathbb{R}^n$ with a corresponding element of $\bigwedge^1(\mathbb{R}^n)$, $v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$, and likewise we will associate a vector $\mathcal{V} \in \mathbb{R}^n$ with a corresponding element of $\bigwedge^{n-1}(\mathbb{R}^n)$,*

$$\mathcal{V}_1 e_1 \wedge \cdots \wedge e_{n-1} + \mathcal{V}_2 e_1 \wedge \cdots \wedge e_{n-2} \wedge e_n + \cdots + \mathcal{V}_n e_2 \wedge e_3 \wedge \cdots \wedge e_n.$$

Though we will generally use the same notation for both the vector and its associated form, we will keep the two cases separate by consistently using lower case letters for 1-forms and capital calligraphic letters for $(n-1)$-forms. Throughout our analysis, we will often use the relation

$$v \wedge \mathcal{V} = \Big(\sum_{i=1}^{n} (-1)^{i+1} v_i \mathcal{V}_{n+1-i} \Big) e_1 \wedge \cdots \wedge e_n. \tag{1.15}$$

With $\mathcal{Y}^+(x;\lambda)$ defined as in (1.14), it's straightforward to check that

$$\mathcal{Y}^{+\prime}(x;\lambda) = \Big((\operatorname{tr} A(x;\lambda)) I + \tilde{A}(x;\lambda) \Big) \mathcal{Y}^+, \tag{1.16}$$

where

$$\tilde{A}(x;\lambda) = \begin{pmatrix} -a_{nn} & a_{(n-1)n} & -a_{(n-2)n} & \cdots & (-1)^n a_{1n} \\ a_{n(n-1)} & -a_{(n-1)(n-1)} & a_{(n-2)(n-1)} & \cdots & (-1)^{n-1} a_{1(n-1)} \\ -a_{n(n-2)} & a_{(n-1)(n-2)} & -a_{(n-2)(n-2)} & \cdots & (-1)^{n-2} a_{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^n a_{n1} & (-1)^{n-1} a_{(n-1)1} & (-1)^{n-2} a_{(n-2)1} & \cdots & -a_{11} \end{pmatrix}. \tag{1.17}$$

More succinctly, the elements $\{\tilde{a}_{ij}\}_{i,j=1}^n$ of $\tilde{A}$ are related to the elements $\{a_{ij}\}_{i,j=1}^n$ of A by the relation

$$\tilde{a}_{ij} = (-1)^{i+j+1} a_{(n+1-j)(n+1-i)}. \tag{1.18}$$

(See Section 3 below for a straightforward verification.) We set

$$\tilde{\mathcal{Y}}^+(x;\lambda) := e^{-\int_0^x \operatorname{tr} A(\xi;\lambda) d\xi} \mathcal{Y}^+(x;\lambda) \tag{1.19}$$

so that

$$\tilde{\mathcal{Y}}^{+\prime} = \tilde{A}(x;\lambda) \tilde{\mathcal{Y}}^+, \tag{1.20}$$

and we will see in Proposition 3.6 below that

$$\lim_{x \to +\infty} e^{\mu_+(\lambda) x} \tilde{\mathcal{Y}}^+(x;\lambda) = \tilde{\mathcal{V}}^+(\lambda),$$

where $\mu_+(\lambda)$ is as in Assumption **(C)** and $\tilde{\mathcal{V}}^+(\lambda)$ is a right eigenvector (uniquely defined up to a scaling constant) of

$$\tilde{A}_+(\lambda) := \lim_{x \to +\infty} \tilde{A}(x;\lambda),$$

associated to the eigenvalue $-\mu_+(\lambda)$, which is the left-most eigenvalue of $\tilde{A}_+(\lambda)$.

In order to work with ω_2 as specified in (1.11), we let M denote the matrix introduced for ω_2 and set

$$\mathcal{Y}_M^+(x;\lambda) := (M y_1^+(x;\lambda)) \wedge \cdots \wedge (M y_{n-1}^+(x;\lambda)). \tag{1.21}$$

Then $\mathcal{Y}_M^+(x;\lambda)$ satisfies the same relations as $\mathcal{Y}^+(x;\lambda)$, except with $A(x;\lambda)$ replaced everywhere with

$$\mathcal{A}(x;\lambda) := M A(x;\lambda) M^{-1}.$$

In particular, if we set

$$\tilde{\mathcal{Y}}_M^+(x;\lambda) := e^{-\int_0^x \operatorname{tr}\mathcal{A}(\xi;\lambda)d\xi}\mathcal{Y}_M^+(x;\lambda),$$

then

$$\tilde{\mathcal{Y}}_M^{+\prime} = \tilde{\mathcal{A}}(x;\lambda)\tilde{\mathcal{Y}}_M^+,$$

where $\tilde{\mathcal{A}}(x;\lambda)$ is related to $\mathcal{A}(x;\lambda)$ in the same way that $\tilde{A}(x;\lambda)$ is related to $A(x;\lambda)$ (via (1.18)). According to Proposition 3.6 below, we have

$$\lim_{x\to+\infty} e^{\mu_+(\lambda)x}\tilde{\mathcal{Y}}_M^+(x;\lambda) = \tilde{\mathcal{V}}_M^+(\lambda),$$

where $\tilde{\mathcal{V}}_M^+(\lambda)$ a right eigenvector (uniquely defined up to a scaling constant) of

$$\tilde{\mathcal{A}}_+(\lambda) := \lim_{x\to+\infty}\tilde{\mathcal{A}}(x;\lambda), \tag{1.22}$$

associated to the eigenvalue $-\mu_+(\lambda)$.

In our general notation, we now take $\mathbf{G}(x;\lambda) = \eta^-(x;\lambda)$ and

$$\mathbf{H}(x;\lambda) = \begin{pmatrix} y_1^+(x;\lambda) & y_2^+(x;\lambda) & \dots & y_{n-1}^+(x;\lambda)\end{pmatrix},$$

and for some value $c > 0$ to be chosen sufficiently large during the analysis, we set

$$\mathbf{F}^c(x;\lambda) := \begin{pmatrix} \mathbf{G}(x;\lambda) & 0_{n\times(n-1)} \\ 0_{n\times 1} & \mathbf{H}(c;\lambda)\end{pmatrix}.$$

With ω_1 and ω_2 as in (1.10) and (1.11), we correspondingly set

$$\tilde{\omega}_1^c(x;\lambda) := \omega_1(f_1^c(x;\lambda),\dots,f_n^c(x;\lambda)) = \eta^-(x;\lambda)\wedge\mathcal{Y}^+(c;\lambda) \tag{1.23}$$

and

$$\tilde{\omega}_2^c(x;\lambda) := \omega_2(f_1^c(x;\lambda),\dots,f_n^c(x;\lambda)) = \eta^-(x;\lambda)\wedge\mathcal{Y}_M^+(c;\lambda), \tag{1.24}$$

where the vector-functions $\{f_j^c(x;\lambda)\}_{j=1}^n$ comprise the columns of $\mathbf{F}^c(x;\lambda)$, and $\mathcal{Y}^+$ and $\mathcal{Y}_M^+$ are respectively defined in (1.14) and (1.21). Associated with $\tilde{\omega}_1^c(x;\lambda)$ and $\tilde{\omega}_2^c(x;\lambda)$, we define the corresponding normalized functions

$$\tilde{\psi}_i^c(x;\lambda) = \frac{\tilde{\omega}_i^c(x;\lambda)}{|\eta^-(x;\lambda)||\mathcal{Y}^+(c;\lambda)|}, \quad i = 1,2. \tag{1.25}$$

Here, for brevity, the notation of (1.23) and (1.24) takes slight liberties with interpretations of the left and right sides. This convention is discussed in detail in Section 2.

We will be interested separately in limits as c tends to $+\infty$ and as x tends to either $-\infty$ or $+\infty$, prompting the following notational conventions. First, for the limit as c tends to $+\infty$, we observe that we can write

$$\tilde{\psi}_1^c(x;\lambda) = \frac{\eta^-(x;\lambda)\wedge(e^{\mu_+(\lambda)c}\tilde{\mathcal{Y}}^+(c;\lambda))}{|\eta^-(x;\lambda)||e^{\mu_+(\lambda)c}\tilde{\mathcal{Y}}^+(c;\lambda)|}, \tag{1.26}$$

from which it's clear that

$$\tilde{\psi}_1^+(x;\lambda) := \lim_{c\to+\infty} \tilde{\psi}_1^c(x;\lambda) = \frac{\tilde{\omega}_1^+(x;\lambda)}{|\eta^-(x;\lambda)||\tilde{\mathcal{V}}^+(\lambda)|}, \tag{1.27}$$

where we've set

$$\tilde{\omega}_1^+(x;\lambda) := \eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda). \tag{1.28}$$

Likewise, in precisely the same way, we see that

$$\tilde{\psi}_2^+(x;\lambda) := \lim_{c\to+\infty} \tilde{\psi}_2^c(x;\lambda) = \frac{\tilde{\omega}_2^+(x;\lambda)}{|\eta^-(x;\lambda)||\tilde{\mathcal{V}}^+(\lambda)|}, \tag{1.29}$$

where we've set

$$\tilde{\omega}_2^+(x;\lambda) := \eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}_M^+(\lambda). \tag{1.30}$$

Next, for $\tilde{\psi}_1^+(x;\lambda)$, we can write

$$\tilde{\psi}_1^+(x;\lambda) = \frac{e^{-\mu_-(\lambda)x}\eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda)}{|e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)||\tilde{\mathcal{V}}^+(\lambda)|},$$

from which it's clear that

$$\tilde{\psi}_1^{+,-}(\lambda) := \lim_{x\to-\infty} \tilde{\psi}_1^+(x;\lambda) = \frac{v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda)}{|v^-(\lambda)||\tilde{\mathcal{V}}^+(\lambda)|}, \tag{1.31}$$

and similarly

$$\tilde{\psi}_2^{+,-}(\lambda) := \lim_{x\to-\infty} \tilde{\psi}_2^+(x;\lambda) = \frac{v^-(\lambda) \wedge \tilde{\mathcal{V}}_M^+(\lambda)}{|v^-(\lambda)||\tilde{\mathcal{V}}^+(\lambda)|}. \tag{1.32}$$

Under our most general assumptions, the limits of $\tilde{\psi}_1^+(x;\lambda)$ and $\tilde{\psi}_2^+(x;\lambda)$ as x tends to $+\infty$ don't necessarily exist, but in cases for which they do we will designate them respectively as $\tilde{\psi}_1^{+,+}(\lambda)$ and $\tilde{\psi}_2^{+,+}(\lambda)$.

With this notation in place, we are able to state the final set of assumptions that will be needed for our main theorem. These assumptions, which primarily address invariance, are somewhat technical, and so we immediately follow the statement by a lengthy remark addressing both how they should be interpreted and how they can be verified.

(E) Suppose the following conditions hold for $[\lambda_1, \lambda_2] \subset I$, $\lambda_1 < \lambda_2$: (1) for $i = 1, 2$, the limits

$$\tilde{\psi}_1^{+,+}(\lambda_i) := \lim_{x\to+\infty} \tilde{\psi}_1^+(x;\lambda_i) \quad \text{and} \quad \tilde{\psi}_2^{+,+}(\lambda_i) := \lim_{x\to+\infty} \tilde{\psi}_2^+(x;\lambda_i)$$

are well defined; (2) for $i = 1, 2$, the values $\tilde{\psi}_1^+(x;\lambda_i)$ and $\tilde{\psi}_2^+(x;\lambda_i)$ do not simultaneously vanish at any $x \in \mathbb{R}$, and the limit functions $\tilde{\psi}_1^{+,+}(\lambda_i)$ and $\tilde{\psi}_2^{+,+}(\lambda_i)$ do not simultaneously vanish; (3) the values $\tilde{\psi}_1^{+,-}(\lambda)$ and $\tilde{\psi}_2^{+,-}(\lambda)$ do not simultaneously vanish at any $\lambda \in [\lambda_1, \lambda_2]$; and (4) there exists a constant $c_0 > 0$ sufficiently large so that for all $c \geq c_0$ and all $\lambda \in [\lambda_1, \lambda_2]$, the values $\tilde{\psi}_1^c(c;\lambda)$ and $\tilde{\psi}_2^c(c;\lambda)$ do not simultaneously vanish.

Remark 1.4. *We will show in Section 4 that under our Assumption* **(C)**, **(E)***(1) holds for any* λ_i *that is not an eigenvalue of (1.1), while if* λ_i *is an eigenvalue of (1.1), the validity of* **(E)***(1) will be determined by the behavior of its associated eigenfunction in the limit as* x *tends toward* $+\infty$. *(For the applications we have in mind,* λ_1 *will be taken sufficiently negative so that it is not an eigenvalue, and* λ_2 *will be taken to be 0, which will be an eigenvalue.) For* **(E)***(2), we can often show that* λ_1 *can be taken sufficiently negative so that* $\psi_1^+(x;\lambda_1) \neq 0$ *for all* $x \in \mathbb{R}$, *giving half of the condition. On the other hand, we will find that* $\psi_1^+(x;\lambda_2)$ *and* $\psi_2^+(x;\lambda_2)$ *can often be evaluated exactly for all* $x \in \mathbb{R}$, *allowing us to check the second half. For* **(E)***(3), we will see that in many important applications, including the ones we consider in Section 6 of the current analysis, we can show that* $\psi_1^{+,-}(\lambda) \neq 0$ *for all* $\lambda \in [\lambda_1, \lambda_2]$, *which is more than we need. The most challenging assumption to verify is* **(E)***(4), because it's the nature of the method to find indirect information about the spectrum of (1.1) without directly computing the values of* $\psi_1^+(x;\lambda)$ *and* $\psi_2^+(x;\lambda)$ *as* λ *increases from* λ_1 *to* λ_2. *However, in practice we make the following important observation. If (1.1) has no eigenvalues on an interval* $[\lambda_1, \lambda_2]$, *then we can choose* c *sufficiently large so that* $\psi_1^+(c;\lambda) \neq 0$ *for all* $\lambda \in [\lambda_1, \lambda_2]$. *In this way, we have a useful dichotomy: if Assumption* **(E)***(4) fails to hold, then we can conclude that (1.1) certainly has at least one eigenvalue on the interval* $[\lambda_1, \lambda_2]$. *I.e., if an eigenvalue is detected on* $[\lambda_1, \lambda_2]$ *under Assumption* **(E)***(4), and the other parts of Assumptions* **(E)** *are shown to hold, then either there is an eigenvalue on* $[\lambda_1, \lambda_2]$ *because Assumptions* **(E)** *hold, or there is an eigenvalue on* $[\lambda_1, \lambda_2]$ *because Assumption* **(E)***(4) fails to hold. In either case we can rigorously conclude the existence of an eigenvalue on the interval* $[\lambda_1, \lambda_2]$.

We will see in Section 4 that under Assumptions **(A)** through **(E)**, there exists a constant $C > 0$ sufficiently large so that for all $c \geq C$ the sum of hyperplane indices

$$\begin{aligned} &\operatorname{Ind}(\mathscr{g}(-c;\cdot), \mathscr{h}(c;\cdot); [\lambda_1, \lambda_2]) + \operatorname{Ind}(\mathscr{g}(\cdot;\lambda_2), \mathscr{h}(c;\lambda_2); [-c, c]) \\ &\quad - \operatorname{Ind}(\mathscr{g}(c;\cdot), \mathscr{h}(c;\cdot); [\lambda_1, \lambda_2]) - \operatorname{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}(c;\lambda_1); [-c, c]) \end{aligned}$$

remains constant. We will denote this constant $\mathfrak{m}$ and refer to it as the *boundary invariant.*

In what follows, we will define the notation

$$\operatorname{Ind}(\mathscr{g}(\cdot;\lambda), \mathscr{h}^+(\lambda); [-\infty, +\infty]) \tag{1.33}$$

to mean the winding number in projective space $\mathbb{R}P^1$ of the map

$$x \mapsto [\tilde{\psi}_1^+(x;\lambda) : \tilde{\psi}_2^+(x;\lambda)]$$

as x increases from $-\infty$ to $+\infty$, including a possible departure associated with the asymptotic limit on the left and a possible arrival associated with the asymptotic limit on the right. Likewise, we denote by

$$\operatorname{Ind}(\mathscr{g}^-(\cdot), \mathscr{h}^+(\cdot); [\lambda_1, \lambda_2]) \tag{1.34}$$

the winding number in projective space $\mathbb{R}P^1$ of the map

$$\lambda \mapsto [\tilde{\psi}_1^{+,-}(\lambda) : \tilde{\psi}_2^{+,-}(\lambda)]$$

as λ increases from λ_1 to λ_2.

Regarding (1.33), this value is computed by tracking the rotation of a point $p^+(x;\lambda)$ around S^1 as x increases from $-\infty$ to $+\infty$. Under our assumptions, the limits

$$p^{+,-}(\lambda_i) := \lim_{x\to-\infty} p^+(x;\lambda_i) \quad \text{and} \quad p^{+,+}(\lambda_i) := \lim_{x\to+\infty} p^+(x;\lambda_i)$$

both exist. If $p^{+,\pm}(\lambda_i) \neq (-1,0)$, then there is no asymptotic crossing point at the associated side, and the hyperplane index can be computed as usual on that side. In the event that $(-1,0)$ is achieved as one of these asymptotic limits, the situation is slightly more complicated. As a specific case, suppose

$$p^{+,+}(\lambda_1) = (-1,0).$$

It may be the case that as x increases toward $+\infty$ the point $p^+(x;\lambda_1)$ crosses $(-1,0)$ an infinite number of times, so that no true crossing count is valid. Nonetheless, since the limit

$$\lim_{x\to+\infty} p^+(x;\lambda_1)$$

is well defined, we can *define*

$$\operatorname{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}^+(\lambda_1); [0,+\infty])$$

in the following way. (The restriction to one infinite endpoint is simply to allow us to focus on a single side; the case of $-\infty$ is treated similarly.) Given any $\epsilon > 0$, there exists some value L sufficiently large so that

$$|p^+(x;\lambda_1) - (-1,0)| < \epsilon \tag{1.35}$$

for all $x \geq L$. There are three possibilities for the location of $p^+(L;\lambda_1)$: (a) $p^+(L;\lambda_1)$ is a small distance from $(-1,0)$ in the clockwise direction; or (b) $p^+(L;\lambda_1) = (-1,0)$; or (c) $p^+(L;\lambda_1)$ is a small distance from $(-1,0)$ in the counterclockwise direction. We define

$$\operatorname{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}^+(\lambda_1); [0,+\infty]) := \operatorname{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}^+(\lambda_1); [0,L]) + \begin{cases} 1 & \text{case (a)} \\ 0 & \text{cases (b) and (c).} \end{cases}$$

Here, we emphasize that this definition does not depend on the particular choice of L, only on (1.35).

We are now in a position to state the main theorem of the analysis.

Theorem 1.1. *For (1.1), let Assumptions* **(A)** *through* **(D)** *hold, and for some fixed interval* $[\lambda_1,\lambda_2] \subset I$, $\lambda_1 < \lambda_2$, *suppose Assumption* **(E)** *holds as well. If* $\mathcal{N}_\#([\lambda_1,\lambda_2])$ *denotes the number of eigenvalues that (1.1) has on the interval* $[\lambda_1,\lambda_2]$, *counted without multiplicity, then*

$$\begin{aligned}\mathcal{N}_\#([\lambda_1,\lambda_2]) \geq \Big| &\operatorname{Ind}(\mathscr{g}(\cdot;\lambda_2), \mathscr{h}^+(\lambda_2); [-\infty,+\infty]) - \operatorname{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}^+(\lambda_1); [-\infty,+\infty]) \\ &+ \operatorname{Ind}(\mathscr{g}^-(\cdot), \mathscr{h}^+(\cdot); [\lambda_1,\lambda_2]) - \mathfrak{m}\Big|.\end{aligned} \tag{1.36}$$

In the remainder of this introduction, we provide some background and context for our analysis and also set out a plan for the paper. For the former, our analysis is motivated by oscillation results for linear Hamiltonian systems, which have their origins in the classical work of Sturm and Morse, respectively [34] and [31]. As discussed at length in [24], numerous authors have contributed to the development and application of such results, and the theory for linear Hamiltonian systems has become well established (see, for example, [10, 22, 24] for development of the general theory, [3, 4, 13, 27, 28, 29] for applications, and [6, 5, 8, 9, 12] for associated numerical calculations.

Such results have all been limited either to linear Hamiltonian systems or systems with underlying Hamiltonian structure, but the recent result [2] provides a tool applicable in fully non-Hamiltonian settings such as those considered here. In [2], the authors employed their generalized Maslov index to obtain an oscillation result for non-Hamiltonian systems on a bounded domain, and the current analysis seems to be the first effort to obtain such oscillation results for a class of non-Hamiltonian systems on $\mathbb{R}$.

The paper is organized as follows. In Section 2, we review elements of the hyperplane index that will be used in our development, and in Section 3 we summarize some general results on solutions of (1.1) that will be necessary for the proof of Theorem 1.1. In Section 4, we prove Theorem 1.1, and in Section 5 we discuss the role of the Evans function in the current setting. Finally, in Section 6 we provide two illustrative applications, first to the generalized KdV equation, and second to the KdV-Burgers equation.

2 Properties of the Hyperplane Index

In this section, we emphasize properties of the hyperplane index that will have a role in our analysis, leaving a full development of the theory to [2]. In particular, a proper discussion of this object requires some items from algebraic topology that are (1) already covered clearly and concisely in [2]; and (2) not critical to the development of our results. Aside from an occasional clarifying comment for interested readers, these items are omitted from the current discussion.

As in the introduction, we let $\mathfrak{g} : [a,b] \to Gr_n(\mathbb{R}^{2n})$ denote a continuous path of Grassmannian subspaces, and we let $\mathfrak{q} \in Gr_n(\mathbb{R}^{2n})$ denote a fixed *target* subspace. We let ω_1 denote a skew-symmetric n-linear map such that $\ker \omega_1 = \mathfrak{q}$, and we let ω_2 denote a second skew-symmetric n-linear map so that the triple $(\mathfrak{g}(\cdot), \omega_1, \omega_2)$ satisfies the invariance property described in Definition 1.2 on the interval $[a,b]$ (i.e., $\mathfrak{g}(t) \in \mathcal{M}$ for all $t \in [a,b]$, where $\mathcal{M}$ is as in (1.5)). Recalling that our notational convention is to fix a choice of frames $\mathbf{G}(t)$ for $\mathfrak{g}(t)$ with columns $\{g_i(t)\}_{i=1}^n$, we set

$$\tilde{\omega}_i(t) := \omega_i(g_1(t), g_2(t), \dots, g_n(t)), \quad i = 1, 2. \tag{2.1}$$

I.e., ω_i will consistently denote a skew-symmetric n-linear map, and $\tilde{\omega}_i$ will consistently denote the evaluation of ω_i along a particular path mapping $[a,b]$ to $Gr_n(\mathbb{R}^{2n})$.

The hyperplane index $\text{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [a,b])$ is then computed as described in (1.7), with appropriate conventions for counting arrivals and departures to and from the point in projective space $[0:1]$ (described below). In practice, we proceed by tracking a point $p(t) \in S^1$, which can be precisely specified as

$$p(t) := \begin{cases} \Big(\frac{\tilde{\omega}_2(t)}{\sqrt{\tilde{\omega}_1(t)^2+\tilde{\omega}_2(t)^2}}, \frac{\tilde{\omega}_1(t)}{\sqrt{\tilde{\omega}_1(t)^2+\tilde{\omega}_2(t)^2}}\Big) & \tilde{\omega}_2(t) \le 0 \\ -\Big(\frac{\tilde{\omega}_2(t)}{\sqrt{\tilde{\omega}_1(t)^2+\tilde{\omega}_2(t)^2}}, \frac{\tilde{\omega}_1(t)}{\sqrt{\tilde{\omega}_1(t)^2+\tilde{\omega}_2(t)^2}}\Big) & \tilde{\omega}_2(t) > 0, \end{cases} \tag{2.2}$$

or equivalently with $\tilde{\omega}_1(t)$ and $\tilde{\omega}_2(t)$ replaced by the scaled variables

$$\tilde{\psi}_1(t) = \frac{\tilde{\omega}_1(t)}{|g_1(t) \wedge g_2(t) \wedge \cdots \wedge g_n(t)|} \quad \tilde{\psi}_2(t) = \frac{\tilde{\omega}_2(t)}{|g_1(t) \wedge g_2(t) \wedge \cdots \wedge g_n(t)|}. \tag{2.3}$$

In the usual way, we think of mapping $\mathbb{R}P^1$ to the left half of the unit circle and then closing to S^1 by equating the points $(0,1)$ and $(0,-1)$. It's clear that t_* is a *crossing point* of the flow (i.e., a point so that $\mathfrak{g}(t_*) \cap \mathfrak{q} \neq \{0\}$) if and only if $p(t_*) = (-1,0)$, so the hyperplane index is computed as a count of the number of times the point $p(t)$ crosses $(-1,0)$. We take crossings in the clockwise direction to be negative and crossings in the counterclockwise direction to be positive. Regarding behavior at the endpoints, if $p(t)$ rotates away from $(-1,0)$ in the clockwise direction as t increases from 0, then the hyperplane index decrements by 1, while if $p(t)$ rotates away from $(-1,0)$ in the counterclockwise direction as t increases from 0, then the hyperplane index does not change. Likewise, if $p(t)$ rotates into $(-1,0)$ in the counterclockwise direction as t increases to 1, then the hyperplane index increments by 1, while if $p(t)$ rotates into $(-1,0)$ in the clockwise direction as t increases to 1, then the hyperplane index does not change. Finally, it's possible that $p(t)$ will arrive at $(-1,0)$ for $t = t_*$ and remain at $(-1,0)$ as t traverses an interval. In these cases, the hyperplane index only increments/decrements upon arrival or departure, and the increments/decrements are determined as for the endpoints (departures determined as with $t = 0$, arrivals determined as with $t = 1$).

Remark 2.1. *In [2], the authors view S^1 as a circle in $\mathbb{C}$, and make the specification*

$$\tilde{p}(t) := \Big(\frac{\tilde{\omega}_1(t) - i\tilde{\omega}_2(t)}{|\tilde{\omega}_1(t) - i\tilde{\omega}_2(t)|}\Big)^2.$$

This choice leads to precisely the same dynamics as those described above, and in particular to the same values of the hyperplane index.

We emphasize, as in the introduction, that in contrast with the Maslov index in the setting of Lagrangian flow, the hyperplane index does not keep track of the dimensions of the intersections.

To set some notation, we let ω_1 and ω_2 be as above, and denote by $\mathcal{P}_{\omega_1,\omega_2}([a,b])$ the collection of all continuous paths $\mathfrak{g} : [a,b] \to \mathcal{M}$, with $\mathcal{M}$ as in (1.5). (I.e., $\mathcal{P}_{\omega_1,\omega_2}([a,b])$ comprises the collection of all continuous paths $\mathfrak{g} : [a,b] \to Gr_n(\mathbb{R}^{2n})$ that are invariant with respect to the skew-symmetric n-linear maps ω_1 and ω_2.) The hyperplane index of [2] has the following properties (see Proposition 3.8 in [2]).

(P1) (Path Additivity) If $\mathfrak{g} \in \mathcal{P}_{\omega_1,\omega_2}([a,b])$ and $\mathfrak{q} = \ker \omega_1$, then for any $\tilde{a}, \tilde{b}, \tilde{c} \in [a,b]$, with $\tilde{a} < \tilde{b} < \tilde{c}$, we have

$$\operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [\tilde{a}, \tilde{c}]) = \operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [\tilde{a}, \tilde{b}]) + \operatorname{Ind}(\mathfrak{g}(\cdot), \mathfrak{q}; [\tilde{b}, \tilde{c}]).$$

(P2) (Homotopy Invariance) If $\mathcal{g}, \mathcal{h} \in \mathcal{P}_{\omega_1,\omega_2}([a,b])$ are homotopic in $\mathcal{M}$ with $\mathcal{g}(a) = \mathcal{h}(a)$ and $\mathcal{g}(b) = \mathcal{h}(b)$ (i.e., if $\mathcal{g}, \mathcal{h}$ are homotopic with fixed endpoints) then

$$\operatorname{Ind}(\mathcal{g}(\cdot), \mathcal{q}; [a,b]) = \operatorname{Ind}(\mathcal{h}(\cdot), \mathcal{q}; [a,b]).$$

2.1 Grassmannian Pairs

In this section, we clarify both the approach and notation from the introduction by providing a general development for computing the hyperplane index for evolving pairs of Grassmannian spaces $\mathcal{g} : [a,b] \to Gr_m(\mathbb{R}^n)$ and $\mathcal{h} : [a,b] \to Gr_{n-m}(\mathbb{R}^n)$, where $m \in \{1, 2, \dots, n-1\}$. In order to facilitate such calculations, we can think of letting $\mathbf{F}(t)$ denote the matrix function specified in (1.8), and taking ω_1 and ω_2 respectively as in (1.10) and (1.11). With these choices in place, we obtain the relation

$$\begin{aligned}\omega_1(f_1, f_2, \dots, f_n) &= \begin{pmatrix} g_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ h_1 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ h_{n-1} \end{pmatrix} \wedge \tilde{\delta}_1 \wedge \cdots \wedge \tilde{\delta}_n \\ &= g_1 \wedge h_1 \wedge \cdots \wedge h_{n-1} \wedge e_{n+1} \wedge \cdots \wedge e_{2n},\end{aligned}$$

and this prompts us to set

$$\tilde{\omega}_1(t) := g_1(t) \wedge h_1(t) \wedge \cdots \wedge h_{n-1}(t). \tag{2.4}$$

Likewise, we'll set

$$\begin{aligned}\omega_2(f_1, \dots, f_n) &= \begin{pmatrix} g_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ h_1 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ h_{n-1} \end{pmatrix} \wedge \tilde{\sigma}_1 \wedge \cdots \wedge \tilde{\sigma}_n \\ &= g_1 \wedge (Mh_1) \wedge \cdots \wedge (Mh_{n-1}) e_{n+1} \cdots \wedge e_{2n},\end{aligned}$$

and also

$$\tilde{\omega}_2(t) := g_1(t) \wedge (Mh_1(t)) \wedge \cdots \wedge (Mh_{n-1}(t)).$$

Having specified ω_1 and ω_2, it's now also convenient to introduce normalized variables

$$\psi_i(f_1, \dots, f_n) := \frac{\omega_i(f_1, \dots, f_n)}{|f_1 \wedge \cdots \wedge f_n|}, \quad i = 1, 2, \tag{2.5}$$

and likewise

$$\begin{aligned}\tilde{\psi}_i(t) &:= \frac{\tilde{\omega}_i(t)}{\left| \begin{pmatrix} g_1(t) \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ h_1(t) \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} 0 \\ h_{n-1}(t) \end{pmatrix} \right|} \\ &= \frac{\tilde{\omega}_i(t)}{|g_1(t)||h_1(t) \wedge \cdots \wedge h_{n-1}(t)|}, \quad i = 1, 2.\end{aligned} \tag{2.6}$$

It's clear from the relationship between $\{\tilde{\omega}_i\}_{i=1}^2$ and $\{\tilde{\psi}_i\}_{i=1}^2$ that if we replace $\{\tilde{\omega}_i\}_{i=1}^2$ with $\{\tilde{\psi}_i\}_{i=1}^2$ in our expression (2.2) for the tracking point $p(t)$, the value of $p(t)$ isn't changed.

2.2 Direction of Rotation

We can employ the approach of Section 4 in [2] to locally analyze the direction associated with a given crossing point. For this, our starting point is the observation that for $p(t)$ near $(-1, 0)$, the location of $p(t)$ can be tracked via the angle

$$\theta(t) = \pi + \tan^{-1} \frac{\tilde{\omega}_1(t)}{\tilde{\omega}_2(t)}, \tag{2.7}$$

with π arising from our convention of placing crossings at $(-1, 0)$. By the monotonicity of $\tan^{-1} x$, the direction of $\theta(t)$ near a value $t = t_*$ for which $\theta(t_*) = \pi$ is determined by the derivative of the ratio $r(t) = \frac{\tilde{\omega}_1(t)}{\tilde{\omega}_2(t)}$, for which $r(t_*) = 0$. Precisely, if $r'(t_*) = \frac{\tilde{\omega}_1'(t_*)}{\tilde{\omega}_2(t_*)} < 0$ then the rotation of $p(t)$ is clockwise at t_*, corresponding with a decrement of the hyperplane index, while if $r'(t_*) > 0$ then the rotation is counterclockwise, corresponding with an increment of the hyperplane index.

2.3 The Boundary Invariant $\mathfrak{m}$

One of the most challenging aspects of working with the hyperplane index is evaluating the boundary invariant $\mathfrak{m}$. One strategy, introduced in [2], is to set

$$\rho(t) = \frac{1}{2}(\tilde{\psi}_1(t)^2 + \tilde{\psi}_2(t)^2), \tag{2.8}$$

and show directly that $\rho(t) \neq 0$ for all $t \in [a, b]$. This has been shown to work in certain cases in both [2] and [23], but in both of those analyses critical use was made of boundedness of the domain of the independent variable.

More generally, a consequence of Lemmas 4.9 and 4.10 in [2] is that under certain fairly general conditions, $\mathfrak{m}$ must be an even integer. Our goal in this section is to slightly relax the assumptions from these lemmas. We begin with the following *exchange principle*, addressing what happens if the skew-symmetric n-linear form ω_2 is exchanged for an alternative choice ω_3.

Lemma 2.1 (Exchange Principle)**.** *Suppose that for some interval $[a, b]$, $a < b$, we have $\mathfrak{g} \in C([a, b], Gr_n(\mathbb{R}^{2n}))$, and that $\{\omega_i\}_{i=1}^3$ are three skew-symmetric n-linear forms on $\mathbb{R}^{2n}$, with $\ker \omega_1 = \mathfrak{q}$. If the triples $(\mathfrak{g}, \omega_1, \omega_2)$ and $(\mathfrak{g}, \omega_1, \omega_3)$ are both invariant on $[a, b]$, and neither of the endpoints $t = a$ and $t = b$ is a crossing point, then the hyperplane indices computed for $(\mathfrak{g}, \omega_1, \omega_2)$ and $(\mathfrak{g}, \omega_1, \omega_3)$ on $[a, b]$ can differ only by an even integer (if at all).*

Proof. Starting with the triple $(\mathfrak{g}, \omega_1, \omega_2)$, a value $t_* \in [a, b]$ is a crossing point if and only if $\tilde{\omega}_1(t_*) = 0$, and by our assumption of invariance, we must correspondingly have $\tilde{\omega}_2(t_*) \neq 0$. As t increases through t_*, $\tilde{\omega}_1(t)$ might change signs, but by continuity $\tilde{\omega}_2(t)$ will not. If $\tilde{\omega}_1(t)$ changes signs, the contribution to the hyperplane index is either $+1$ or -1. On the other hand, if $\tilde{\omega}_1(t)$ fails to change signs, then there is no contribution to the hyperplane index.

Turning to the triple $(\mathfrak{g}, \omega_1, \omega_3)$, precisely the same statements above are true, and in particular we see that a crossing point t_* gives no contribution to the hyperplane index if and only if $\tilde{\omega}_1(t)$ fails to change signs as t increases through t_*. This means that there will be a non-zero contribution to the hyperplane index at t_* for the triple $(\mathcal{g}, \omega_1, \omega_2)$ if and only if there is a non-zero contribution to the hyperplane index at t_* for the triple $(\mathcal{g}, \omega_1, \omega_3)$.

According to these considerations, the hyperplane index for the triple $(\mathcal{g}, \omega_1, \omega_2)$ on $[a, b]$ will have precisely the same number of non-zero crossings as the hyperplane index for the triple $(\mathcal{g}, \omega_1, \omega_3)$ on $[a, b]$. If P_2 and N_2 respectively denote the number of positive and negative crossings for $(\mathcal{g}, \omega_1, \omega_2)$, and P_3 and N_3 respectively denote the number of positive and negative crossings for $(\mathcal{g}, \omega_1, \omega_3)$, then we must have $P_2 + N_2 = P_3 + N_3$. If follows that the difference between the hyperplane indices computed for $(\mathcal{g}, \omega_1, \omega_2)$ and $(\mathcal{g}, \omega_1, \omega_3)$ on $[a, b]$ is

$$P_3 - N_3 - (P_2 - N_2) = (P_2 + N_2 - N_3) - N_3 - P_2 + N_2 = 2N_2 - 2N_3,$$

an even number. $\square$

Lemma 2.2. *Suppose that for some intervals $[a, b]$, $a < b$ and $[c, d]$, $c < d$, $\mathcal{g} \in C([a, b] \times [c, d], Gr_n(\mathbb{R}^{2n}))$, and for some fixed $\mathcal{q} \in Gr_n(\mathbb{R}^{2n})$ let ω_1 denote a skew-symmetric n-linear form on $\mathbb{R}^{2n}$ so that $\ker \omega_1 = \mathcal{q}$. Let ω_2 denote a second skew-symmetric n-linear form on $\mathbb{R}^{2n}$, and suppose that for some point (s_*, t_*) in the interior of $[a, b] \times [c, d]$ we have*

$$\omega_i(g_1(s_*, t_*), g_2(s_*, t_*), \dots, g_n(s_*, t_*)) = 0, \quad i = 1, 2,$$

but that there exists a sufficiently small ball $B \subset \mathbb{R}^2$ centered at (s_*, t_*) so that for all $(s, t) \in B \backslash \{(s_*, t_*)\}$ the values

$$\omega_i(g_1(s, t), g_2(s, t), \dots, g_n(s, t)), \quad i = 1, 2,$$

are not both 0. In short, the triple $(\mathcal{g}, \omega_1, \omega_2)$ loses invariance at an isolated point (s_*, t_*). Then there exists some $\epsilon > 0$ sufficiently small so that for any ball $\mathcal{B} \subset \mathbb{R}^2$ centered at (s_*, t_*) with radius less than ϵ

$$\operatorname{Ind}(\mathcal{g}, \mathcal{q}; \partial \mathcal{B}) \in 2\mathbb{Z},$$

where as with all boundary indices we take $\partial \mathcal{B}$ to be traversed in the counterclockwise direction (though the direction doesn't strictly matter for the result).

Proof. First, let ω_3 denote any third skew-symmetric n-linear form on $\mathbb{R}^{2n}$ so that

$$\omega_3(g_1(s_*, t_*), g_2(s_*, t_*), \dots, g_n(s_*, t_*)) \neq 0.$$

This is always possible by choosing vectors $\{g_i\}_{i=n+1}^{2n}$ so that the collection $\{g_i(s_*, t_*)\}_{i=1}^{n} \cup \{g_i\}_{i=n+1}^{2n}$ comprises a basis for $\mathbb{R}^{2n}$ and taking the kernel of ω_3 to be the space spanned by the collection $\{g_i\}_{i=n+1}^{2n}$ (ensuring that none of the vectors $\{g_i(s_*, t_*)\}_{i=1}^{n}$ is contained in the kernel of ω_3). By continuity of $\mathcal{g}$, we can take $\epsilon > 0$ sufficiently small so that for any ball $\mathcal{B} \subset \mathbb{R}^2$ centered at (s_*, t_*) with radius smaller than ϵ

$$\omega_3(g_1(s, t), g_2(s, t), \dots, g_n(s, t)) \neq 0, \quad \forall\, (s, t) \in \overline{\mathcal{B}},$$

where the overbar denotes closure. I.e., the triple $(\mathcal{g}, \omega_1, \omega_3)$ is invariant on $\overline{\mathcal{B}}$, and so by homotopy invariance

$$\operatorname{Ind}_{\omega_3}(\mathcal{g}, \mathcal{q}, \partial \mathcal{B}) = 0.$$

If $\omega_1(g_1(s,t),\dots,g_n(s,t))$ is identically 0 for $(s,t) \in \partial\mathcal{B}$, then $\mathrm{Ind}_{\omega_i}(\mathscr{g}, \mathscr{q}, \partial\mathcal{B}) = 0$ for $i = 2, 3$ and the claim holds trivially. Otherwise, we can select any $(s_0, t_0) \in \partial\mathcal{B}$ so that $\omega_1(g_1(s_0,t_0),\dots,g_n(s_0,t_0)) \neq 0$ and compute $\mathrm{Ind}_{\omega_i}(\mathscr{g}, \mathscr{q}, \partial\mathcal{B})$, $i = 2, 3$, along $\partial\mathcal{B}$, starting and ending at (s_0, t_0). It follows immediately from the exchange principle that the difference

$$\mathrm{Ind}_{\omega_2}(\mathscr{g}, \mathscr{q}, \partial\mathcal{B}) - \mathrm{Ind}_{\omega_3}(\mathscr{g}, \mathscr{q}, \partial\mathcal{B})$$

is an even number. Since the subtracted index is 0, this gives the claim. □

We can now use Lemma 2.2 to show that under circumstances that hold quite generally the boundary invariant is an even number.

Proposition 2.1. *Suppose that for some intervals $[a,b]$, $a < b$ and $[c,d]$, $c < d$, $\mathscr{g} \in C([a,b] \times [c,d], Gr_n(\mathbb{R}^{2n}))$, and for some fixed $\mathscr{q} \in Gr_n(\mathbb{R}^{2n})$ let ω_1 denote a skew-symmetric n-linear form on $\mathbb{R}^{2n}$ so that $\ker \omega_1 = \mathscr{q}$. Let ω_2 denote a second skew-symmetric n-linear form on $\mathbb{R}^{2n}$, and suppose the triple $(\mathscr{g}, \omega_1, \omega_2)$ is invariant on the boundary of the rectangle $\mathcal{R} := [a,b] \times [c,d]$ and also invariant at all except possibly a finite number of points in the interior of $\mathcal{R}$. Then*

$$\mathrm{Ind}(\mathscr{g}, \mathscr{q}; \partial\mathcal{R}) \in 2\mathbb{Z}.$$

Proof. Let N denote the number of points of invariance in the interior of $\mathcal{R}$, and denote these points $\{(s_i, t_i)\}_{i=1}^N$. Using homotopy invariance, we can compute $\mathrm{Ind}(\mathscr{g}, \mathscr{q}; \partial\mathcal{R})$ by summing the individual hyperplane indices $\mathrm{Ind}(\mathscr{g}, \mathscr{q}; \partial\mathcal{B}_i)$, where $\mathcal{B}_i$ denotes a ball centered at (s_i, t_i) with radius sufficiently small so that $\mathcal{B}_i \subset \mathcal{R}$ and the triple $(\mathscr{g}, \omega_1, \omega_2)$ is invariant on $\partial\mathcal{B}_i$. According to Lemma 2.2, each such index must be an even number, and so the sum must be an even number as well. □

Remark 2.2. *In order to understand why we expect the number of points at which invariance is lost to be finite, we observe that generally the sets*

$$\mathcal{C}_i := \{(s,t) : \tilde{\omega}_i(s,t) = 0\}, \quad i = 1, 2,$$

comprise one-dimensional curves in $[a,b] \times [c,d]$, and points at which invariance is lost are precisely the points at which these curves intersect. In principle, such intersections certainly need not be isolated, but in practice we generally find that they are. For now, the theory is missing a sufficiently general result along these lines, and the boundary invariant must be computed in applications on a case-by-case basis (see Section 6). While a precise value of $\mathfrak{m}$ is certainly optimal, we emphasize that for instability arguments it's often sufficient to identify its parity, since this allows us to determine whether the number of unstable eigenvalues is even or odd.

3 ODE Preliminaries

In this section, we collect several straightforward results associated with solutions to (1.1) and (1.20). As a starting point, the following proposition is adapted from Proposition 1.2 of [33].

Proposition 3.1. *Let Assumptions* **(A)** *through* **(D)** *hold. Then the following statements are true.*

(i) There exists a unique solution $\eta^-(x;\lambda)$ to (1.1) for which the limit

$$\lim_{x\to-\infty} e^{-\mu_-(\lambda)x}\eta^-(x;\lambda) = v^-(\lambda)$$

holds, where $v^-(\lambda)$ is the right eigenvector of $A_-(\lambda)$ described in Assumption **(C)***. Moreover, the convergence is uniform on compact subsets of Ω.*

(ii) There exists a (non-unique) solution $\zeta^+(x;\lambda)$ to (1.1) for which the limit

$$\lim_{x\to+\infty} e^{-\mu_+(\lambda)x}\zeta^+(x;\lambda) = v^+(\lambda)$$

holds, where $v^+(\lambda)$ is the right eigenvector of $A_+(\lambda)$ described in Assumption **(C)***. Moreover, the convergence is uniform on compact subsets of Ω.*

For labeling purposes, we will let $\{y_j^-(x;\lambda)\}_{j=1}^n$ denote a linearly independent collection of solutions to (1.1), indexed so that $\eta^-(x;\lambda) = y_n^-(x;\lambda)$, and we will let $\{y_j^+(x;\lambda)\}_{j=1}^n$ denote a linearly independent collection of solutions to (1.1), indexed so that $\zeta^+(x;\lambda) = y_n^+(x;\lambda)$. In some places, it will be useful to express coordinates of the elements $\{y_j^\pm(x;\lambda)\}_{j=1}^n$ by writing

$$y_j^\pm(x;\lambda) = \begin{pmatrix} y_{1j}^\pm(x;\lambda) & y_{2j}^\pm(x;\lambda) & \cdots & y_{nj}^\pm(x;\lambda)\end{pmatrix}^T, \quad j = 1,2,\dots,n,$$

and we also introduce the matrices

$$Y^\pm(x;\lambda) := \begin{pmatrix} y_1^\pm(x;\lambda) & y_2^\pm(x;\lambda) & \cdots & y_{n-1}^\pm(x;\lambda)\end{pmatrix}.$$

Recalling the specification of $\mathcal{Y}^+(x;\lambda)$ in (1.14), it's straightforward to show that $\mathcal{Y}^+(x;\lambda)$ can be expressed as

$$\begin{aligned}\mathcal{Y}^+(x;\lambda) = d_n^+(x;\lambda)e_1\wedge\cdots\wedge e_{n-1} + d_{n-1}^+(x;\lambda)e_1\wedge\cdots\wedge e_{n-2}\wedge e_n \\ + \cdots + d_1^+(x;\lambda)e_2\wedge\cdots\wedge e_n,\end{aligned} \tag{3.1}$$

where $d_i^+(x;\lambda)$ denotes the determinant of the $(n-1)\times(n-1)$ matrix obtained by eliminating the i^{th} row of Y^+. In this way, we associate $\mathcal{Y}^+(x;\lambda)$ with the vector

$$\mathcal{Y}^+(x;\lambda) = \begin{pmatrix} d_n^+(x;\lambda) & d_{n-1}^+(x;\lambda) & \dots & d_1^+(x;\lambda)\end{pmatrix}^T,$$

with the convention of Remark 1.3.

Proposition 3.2. *With $\mathcal{Y}^+$ specified as in (1.14), (1.16) holds.*

Proof. Upon differentiation of (1.14), we obtain the relation

$$\mathcal{Y}^{+\prime}(x;\lambda) = \sum_{j=1}^{n-1} y_1^+\wedge\cdots\wedge A(x;\lambda)y_j^+\wedge\cdots\wedge y_{n-1}^+, \tag{3.2}$$

for which each summand can be understood similarly as in (3.1). Focusing on the first summand

$$(A(x;\lambda)y_1^+) \wedge y_2^+ \wedge \cdots \wedge y_{n-1}^+,$$

we can write

$$A(x;\lambda)y_1^+ = \begin{pmatrix} a_{1j}y_{j1}^+ & a_{2j}y_{j1}^+ & \dots & a_{nj}y_{j1}^+ \end{pmatrix}^T,$$

where for notational brevity summation is assumed over repeated indices. In addition, we introduce the matrix

$$Y_{A1}^+(x;\lambda) := \begin{pmatrix} A(x;\lambda)y_1^+(x;\lambda) & y_2^+(x;\lambda) & \cdots & y_{n-1}^+(x;\lambda) \end{pmatrix},$$

and for each $k \in \{1, 2, \dots, n\}$ we let $d_{A1,k}^+$ denote the determinant of the $(n-1) \times (n-1)$ matrix obtained by eliminating the k^{th} row of Y_{A1}^+. Then, as in (3.1),

$$\begin{aligned} (A(x;\lambda)y_1^+) \wedge \cdots \wedge y_j^+ \wedge \cdots \wedge y_n^+ &= d_{A1,n}^+(x;\lambda) e_1 \wedge \cdots \wedge e_{n-1} \\ &+ d_{A1,n-1}^+(x;\lambda) e_1 \wedge \cdots \wedge e_{n-2} \wedge e_n + \cdots + d_{A1,1}^+(x;\lambda) e_2 \wedge \cdots \wedge e_n. \end{aligned} \tag{3.3}$$

In this way, we have associated $\mathcal{Y}^+$ with the vector $(d_n^+ \; d_{n-1}^+ \; \dots, \; d_1^+)$ and $(Ay_1^+) \wedge y_2^+ \wedge \cdots \wedge y_{n-1}^+$ with the vector $(d_{A1,n}^+ \; d_{A1,n-1}^+ \; \dots, \; d_{A1,1}^+)$, and likewise we can associate the k^{th} summand from (3.2) with a vector $(d_{Ak,n}^+ \; d_{Ak,n-1}^+ \; \dots, \; d_{Ak,1}^+)$. Using these associations, we can write the derivative of the j^{th} component of $\mathcal{Y}^+$ as

$$\mathcal{Y}_j^{+\prime} = \sum_{k=1}^{n-1} d_{Ak,n-(j-1)}^+.$$

Focusing for specificity on the first component $j = 1$, we see that $\mathcal{Y}_1^{+\prime}$ is a sum of n determinants. If we focus still further on terms in this sum associated with a specific entry of the matrix A, then we can readily identify the appearance of that component in our final relation. Using a_{11} as an example case, we can schematically view the terms in $\mathcal{Y}_1^{+\prime}$ associated with a_{11} as arising from the sum

$$\det \begin{pmatrix} a_{11}y_{11}^+ & y_{12}^+ & \cdots & y_{1(n-1)}^+ \\ * & y_{22}^+ & \cdots & y_{2(n-1)}^+ \\ \vdots & \vdots & \dots & \vdots \\ * & y_{(n-1)2}^+ & \cdots & y_{(n-1)(n-1)}^+ \end{pmatrix} + \det \begin{pmatrix} y_{11}^+ & a_{11}y_{12}^+ & \cdots & y_{1(n-1)}^+ \\ y_{21}^+ & * & \cdots & y_{2(n-1)}^+ \\ \vdots & \vdots & \dots & \vdots \\ y_{(n-1)1}^+ & * & \cdots & y_{(n-1)(n-1)}^+ \end{pmatrix} + \dots$$
$$+ \det \begin{pmatrix} y_{11}^+ & y_{12}^+ & \cdots & a_{11}y_{1(n-1)}^+ \\ y_{21}^+ & y_{22}^+ & \cdots & * \\ \vdots & \vdots & \dots & \vdots \\ y_{(n-1)1}^+ & y_{(n-1)2}^+ & \cdots & * \end{pmatrix},$$

where the asterisks indicate terms irrelevant to the calculation (because they don't contain a_{11}). If we now think of expanding each determinant along the column with a_{11}, the combinations of terms including a_{11} are precisely the same as a_{11} multiplied by a determinant expansion along the first row of the matrix

$$\begin{pmatrix} y_{11}^+ & y_{12}^+ & \cdots & y_{1(n-1)}^+ \\ y_{21}^+ & y_{22}^+ & \cdots & y_{2(n-1)}^+ \\ \vdots & \vdots & \cdots & \vdots \\ y_{(n-1)1}^+ & y_{(n-1)2}^+ & \cdots & y_{(n-1)(n-1)}^+ \end{pmatrix}.$$

In summary, the sole multiplier of a_{11} in the expression for $\mathcal{Y}_1^{+\prime}$ is the quantity labeled above as d_n^+, which is precisely the first component of $\mathcal{Y}^+$ (i.e., the component $\mathcal{Y}_1^+$). Proceeding similarly for each element of A and each component of $\mathcal{Y}^+$, we obtain the claim. □

Proposition 3.3. *For any matrix $A \in \mathbb{C}^{n\times n}$, let $\tilde{A}$ be the associated matrix specified in (1.17). Then $\mu \in \sigma(A)$ if and only if $-\mu \in \sigma(\tilde{A})$.*

Proof. If $\mu \in \sigma(A)$ then there exists a left eigenvector of A, which we denote $w \in \mathbb{C}^n$, so that $wA = \mu wA$. If we then specify a new (column) vector $\tilde{\mathcal{V}} \in \mathbb{C}^n$ with components

$$\tilde{\mathcal{V}}_j = (-1)^{n-j} w_{n+1-j}, \quad j = 1, 2, \dots, n, \tag{3.4}$$

then we find by direct calculation that $\tilde{A}\tilde{\mathcal{V}} = -\mu\tilde{\mathcal{V}}$. Reversing the argument gives the converse direction. □

Remark 3.1. *We see from the proof of Proposition 3.3 that if $w^\pm(\lambda)$ is a left eigenvector of $A_\pm(\lambda)$ associated with the simple eigenvalue $\mu_\pm(\lambda)$, with $w^\pm(\lambda)$ chosen to be analytic in λ as in Assumption* **(C)**, *then the corresponding right eigenvector of $\tilde{A}_\pm(\lambda)$ specified via (3.4) will also be analytic. If (3.4) holds, then we correspondingly have*

$$w_k^+ = (-1)^{k-1}\tilde{\mathcal{V}}_{n+1-k}^+. \tag{3.5}$$

In addition, if $v^\pm(\lambda)$ denotes the analytic right eigenvector of $A_\pm(\lambda)$ normalized so that $w^\pm(\lambda)v^\pm(\lambda) = 1$, then we can use (1.15) along with (3.5) to compute

$$\begin{aligned} v^\pm(\lambda) \wedge \tilde{\mathcal{V}}^\pm(\lambda) &= \sum_{i=1}^n (-1)^{i+1} v_i^\pm(\lambda)\tilde{\mathcal{V}}_{n+1-i}^\pm(\lambda) = \sum_{i=1}^n (-1)^{i+1} v_i^\pm(\lambda)(-1)^{i-1} w_i^\pm(\lambda) \\ &= w^\pm(\lambda)v^\pm(\lambda) = 1. \end{aligned}$$

Proposition 3.4. *For any matrix $A \in \mathbb{C}^{n\times n}$, let $\tilde{A}$ be the associated matrix specified in (1.17), and for $u,\mathcal{U} \in \mathbb{R}^n$ interpret u and Au as elements of $\bigwedge^1(\mathbb{R})$ and $\mathcal{U}$ and $\tilde{A}\mathcal{U}$ as elements of $\bigwedge^{n-1}(\mathbb{R})$, as in Remark 1.3. Then*

$$(Au) \wedge \mathcal{U} + u \wedge (\tilde{A}\mathcal{U}) = 0.$$

Proof. First, since Au is interpreted as a 1-form and $\mathcal{U}$ is interpreted as an $(n-1)$-form, the wedge product $(Au) \wedge \mathcal{U}$ is an n-form, which we see from (1.15) is

$$(Au) \wedge \mathcal{U} = \sum_{i,j=1}^n (-1)^{i+1} a_{ij} u_j \mathcal{U}_{n+1-i}.$$

We can compare this with

$$
\begin{aligned}
u \wedge (\tilde{A}\mathcal{U}) &= u_1 \sum_{j=1}^{n} \tilde{a}_{nj}\mathcal{U}_j - u_2 \sum_{j=1}^{n} \tilde{a}_{(n-1)j}\mathcal{U}_j + \cdots + (-1)^{n+1} u_n \sum_{j=1}^{n} \tilde{a}_{1j}\mathcal{U}_j \\
&= \sum_{i,j=1}^{n} (-1)^{i+1} \tilde{a}_{(n+1-i)j} u_i \mathcal{U}_j = \sum_{i,j=1}^{n} (-1)^{i+1} (-1)^{n-i+j} a_{(n+1-j)i} u_i \mathcal{U}_j \\
&= \sum_{i,j=1}^{n} (-1)^{n+1+j} a_{(n+1-j)i} u_i \mathcal{U}_j = \sum_{i,k=1}^{n} (-1)^{k} a_{ki} u_i \mathcal{U}_{n+1-k},
\end{aligned}
$$

where in obtaining the final equality we changed indices from j to $k = n+1-j$. Comparing our expression for $(Au) \wedge \mathcal{U}$ with our expression for $u \wedge (\tilde{A}\mathcal{U})$, we see that the claim is proved. □

With $\eta^-(x;\lambda)$ as specified in (1.12) and $\tilde{\mathcal{Y}}^+(x;\lambda)$ as specified in (1.19), we detect intersections between $\mathrm{Span}\{\eta^-(x;\lambda)\}$ (a one-dimensional subspace of $\mathbb{R}^n$) and $\mathrm{Span}\{y_j^+(x;\lambda)\}_{j=1}^{n-1}$ (an $(n-1)$-dimensional subspace of $\mathbb{R}^n$), with the wedge product

$$
\eta^-(x;\lambda) \wedge \tilde{\mathcal{Y}}^+(x;\lambda).
$$

The following proposition serves as a direct connection between the current analysis and that of [33].

Proposition 3.5. *Suppose $\tilde{\mathcal{Y}}(x;\lambda)$ solves the ODE system $\tilde{\mathcal{Y}}' = \tilde{A}(x;\lambda)\tilde{\mathcal{Y}}$, and let the row vector $z = (z_1\ z_2\ \dots\ z_n)$ be specified with components*

$$
z_j(x;\lambda) := (-1)^{j-1} \tilde{\mathcal{Y}}_{n+1-j}(x;\lambda), \quad j = 1, 2, \dots, n. \tag{3.6}
$$

Then z satisfies the ODE system $z' = -zA(x;\lambda)$. Moreover, if $\tilde{\mathcal{Y}}$ is interpreted as an $(n-1)$-form as in Remark 1.3, and we introduce a column vector y interpreted as a 1-form as in Remark 1.3, then

$$
y \wedge \tilde{\mathcal{Y}}(x;\lambda) = z(x;\lambda) y.
$$

Proof. For the first part of the statement, we can compute directly, writing

$$
\begin{aligned}
z_j' &= (-1)^{j-1} \tilde{\mathcal{Y}}_n' = (-1)^{j-1} \sum_{k=1}^{n} \tilde{a}_{(n+1-j)k} \tilde{\mathcal{Y}}_k \\
&= (-1)^{j-1} \sum_{k=1}^{n} (-1)^{n+k-j} a_{(n+1-k)j} (-1)^{n-k} z_{n+1-k} = -\sum_{k=1}^{n} a_{(n+1-k)j} z_{n+1-k},
\end{aligned}
$$

for $j = 1, 2, \dots, n$, which is precisely $z' = -zA(x;\lambda)$ in component form.

For the second claim, we compute

$$
y \wedge \tilde{\mathcal{Y}} = \sum_{i=1}^{n} (-1)^{i+1} y_i \tilde{\mathcal{Y}}_{n+1-i} = \sum_{i=1}^{n} (-1)^{i+1} y_i (-1)^{i-1} z_i = \sum_{i=1}^{n} \eta_i^- z_i = zy.
$$

□

Remark 3.2. *We see from Proposition 3.5 that as in [33], we could carry out our analysis entirely with appropriate inner products rather than wedge products. Our convention of working with wedge products is motivated by the prospect of extending our analysis to more general settings in which the inner-product formulation isn't viable.*

Proposition 3.5 allows us to adopt Proposition 1.2 from [33] (addressing the variable denoted z here) to a statement about $\tilde{\mathcal{Y}}^+(x;\lambda)$. Precisely, we have the following.

Proposition 3.6. *Let Assumptions* **(A)** *through* **(D)** *hold, and let* $\tilde{\mathcal{V}}^+(\lambda)$ *denote the right eigenvector of* $\tilde{A}_+(\lambda)$ *associated with the eigenvalue* $-\mu_+(\lambda)$*, constructed via (3.5) from* $w^+(\lambda)$ *as in Assumption* **(C)**. *Then there exists a unique solution* $\tilde{\mathcal{Y}}^+(x;\lambda)$ *to (1.20) for which the limit*

$$\lim_{x\to+\infty} e^{\mu_+(\lambda)x}\tilde{\mathcal{Y}}^+(x;\lambda) = \tilde{\mathcal{V}}^+(\lambda)$$

holds, and moreover, the convergence is uniform on compact subsets of Ω.

4 Proof of Theorem 1.1

With $\eta^-(x;\lambda)$ and $\tilde{\mathcal{Y}}^+(x;\lambda)$ as in Propositions 3.1 and 3.6, we let $\mathcal{g}(x;\lambda) \in Gr_1(\mathbb{R}^n)$ denote the path of Grassmannian subspaces with frame $\mathbf{G}(x;\lambda) = \eta^-(x;\lambda)$, and we let $\mathcal{h}(x;\lambda) \in Gr_{n-1}(\mathbb{R}^n)$ denote the path of Grassmannian subspaces with frame

$$\mathbf{H}(x;\lambda) = \begin{pmatrix} y_1^+(x;\lambda) & y_2^+(x;\lambda) & \dots & y_{n-1}^+(x;\lambda) \end{pmatrix}. \tag{4.1}$$

We prove Theorem 1.1 by fixing $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, along with values $c_1 \ll 0$ and $c_2 \gg 0$, and computing the hyperplane index for the pair $\mathcal{g}(x;\lambda)$ and $\mathcal{h}(c_2;\lambda)$ along the following sequence of lines often referred to as the *Maslov box*: (1) fix $x = c_1$ and let λ increase from λ_1 to λ_2 (the *bottom shelf*); (2) fix $\lambda = \lambda_2$ and let x increase from c_1 to c_2 (the *right shelf*); (3) fix $x = c_2$ and let λ decrease from λ_2 to λ_1 (the *top shelf*); and (4) fix $\lambda = \lambda_1$ and let x decrease from c_2 to c_1 (the *left shelf*). See Figure 4.1.

Overview of the Maslov box. Along the top shelf of the Maslov box, we have $x = c_2$, so the hyperplane index

$$\operatorname{Ind}(\mathcal{g}(c_2;\cdot), \mathcal{h}(c_2;\cdot); [\lambda_1, \lambda_2])$$

detects eigenvalues, albeit counted without multiplicity and with no guarantee of monotonicity. If $\mathcal{N}_\#([\lambda_1, \lambda_2])$ denotes the number of eigenvalues that (1.1) has on the interval $[\lambda_1, \lambda_2]$, counted without multiplicity, then in the event of monotonicity the hyperplane index on the top shelf would equal either $\mathcal{N}_\#([\lambda_1, \lambda_2])$ or $-\mathcal{N}_\#([\lambda_1, \lambda_2])$, depending on the direction of the crossings. In the absence of monotonicity, such an equality isn't achieved, and instead we have the inequality,

$$\mathcal{N}_\#([\lambda_1, \lambda_2]) \geq |\operatorname{Ind}(\mathcal{g}(c_2;\cdot), \mathcal{h}(c_2;\cdot); [\lambda_1, \lambda_2])|. \tag{4.2}$$

The hyperplane index along the bottom shelf detects intersections between $\mathcal{g}(c_1;\lambda)$ and $\mathcal{h}(c_2;\lambda)$, and can be denoted

$$\operatorname{Ind}(\mathcal{g}(c_1;\cdot), \mathcal{h}(c_2;\cdot); [\lambda_1, \lambda_2]).$$

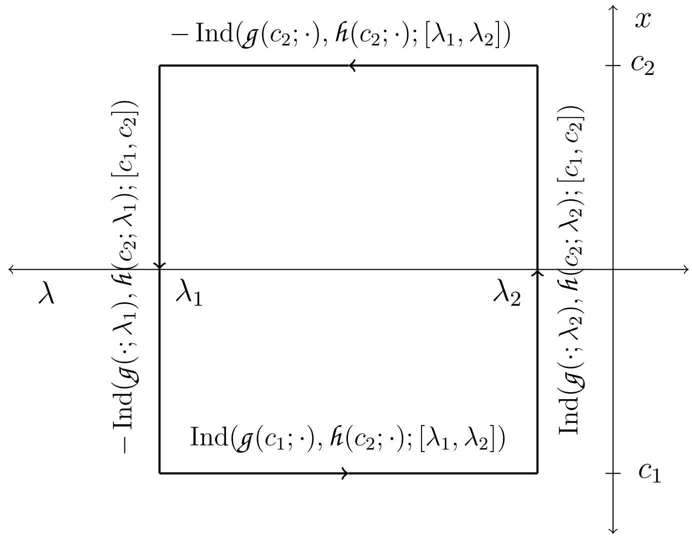

Figure 4.1: The Maslov Box.

Likewise the hyperplane indices along the left and right shelves respectively detect intersections between $g(x;\lambda_i)$ and $h(x;\lambda_i)$, $i=1,2$, as x decreases from c_2 to c_1 (left shelf) and increases from c_1 to c_2 (right shelf). We denote these respectively

$$-\operatorname{Ind}(g(\cdot;\lambda_1), h(c_2;\lambda_1); [c_1,c_2])$$

and

$$\operatorname{Ind}(g(\cdot;\lambda_2), h(c_2;\lambda_2); [c_1,c_2]).$$

From Assumption **(E)**, we can conclude that for c_1 sufficiently negative and c_2 sufficiently positive, we have invariance along each of the four shelves. It follows that we can compute the hyperplane index along the boundary of the Maslov box, and we denote this value $\mathfrak{m}(c_1,c_2)$, writing

$$\begin{aligned}\mathfrak{m}(c_1,c_2) := {}& \operatorname{Ind}(g(c_1;\cdot), h(c_2;\cdot); [\lambda_1,\lambda_2]) + \operatorname{Ind}(g(\cdot;\lambda_2), h(c_2;\lambda_2); [c_1,c_2]) \\ & - \operatorname{Ind}(g(c_2;\cdot), h(c_2;\cdot); [\lambda_1,\lambda_2]) - \operatorname{Ind}(g(\cdot;\lambda_1), h(c_2;\lambda_1); [c_1,c_2]).\end{aligned} \tag{4.3}$$

In order to evaluate the four hyperplane indices in (4.3), we follow the approach outlined in the introduction, beginning with the specification of a third Grassmannian subspace $f^{c_2}(x;\lambda) \in Gr_n(\mathbb{R}^{2n})$ with frame

$$\mathbf{F}^{c_2}(x;\lambda) = \begin{pmatrix} \mathbf{G}(x;\lambda) & 0_{n\times(n-1)} \\ 0_{n\times 1} & \mathbf{H}(c_2;\lambda) \end{pmatrix}.$$

As discussed in Section 2, we set

$$\omega_1^{c_2}(f_1, f_2, \dots, f_n) := f_1(x;\lambda) \wedge \cdots \wedge f_n(x;\lambda) \wedge \tilde{\delta}_1 \wedge \cdots \wedge \tilde{\delta}_n, \tag{4.4}$$

where the vectors $\{\tilde{\delta}_i\}_{i=1}^n$ comprise the columns of $\tilde{\mathbf{\Delta}} = \binom{-I_n}{I_n}$. Likewise, we fix some invertible matrix $M \in \mathbb{R}^{n\times n}$ and set

$$\omega_2^{c_2}(f_1, f_2, \dots, f_n) := f_1(x;\lambda) \wedge \cdots \wedge f_n(x;\lambda) \wedge \tilde{\sigma}_1 \wedge \cdots \wedge \tilde{\sigma}_n, \tag{4.5}$$

where the vectors $\{\tilde{\sigma}_i\}_{i=1}^n$ comprise the columns of $\tilde{\mathbf{\Sigma}} = \binom{-M}{I_n}$.

For the subsequent calculations, we will evaluate ω_1 and ω_2 on the columns of $\mathbf{F}^{c_2}(x;\lambda)$, giving

$$\tilde{\omega}_1^{c_2}(x;\lambda) := \eta^-(x;\lambda) \wedge \mathcal{Y}^+(c_2;\lambda) \tag{4.6}$$

and

$$\tilde{\omega}_2^{c_2}(x;\lambda) := \eta^-(x;\lambda) \wedge \mathcal{Y}_M^+(c_2;\lambda), \tag{4.7}$$

where $\mathcal{Y}^+$ and $\mathcal{Y}_M^+$ are respectively defined in (1.14) and (1.21). (See Section 2.1 for additional details about these wedge products.) Here, $\tilde{\omega}_1^{c_2}(x;\lambda)$ and $\tilde{\omega}_2^{c_2}(x;\lambda)$ are the same as (1.23) and (1.24), except with c replaced by c_2. Likewise, we take $\tilde{\psi}_1^{c_2}(x;\lambda)$, $\tilde{\psi}_1^+(x;\lambda)$, and $\tilde{\psi}_1^{+,-}(\lambda)$ to be as respectively defined in (1.26), (1.27), and (1.31), with analogous definitions for $\tilde{\psi}_2^{c_2}(x;\lambda)$, $\tilde{\psi}_2^+(x;\lambda)$, and $\tilde{\psi}_2^{+,-}(\lambda)$ in (1.25), (1.29), and (1.32). Finally, we let $\tilde{\psi}_1^{c_2,-}(\lambda)$ and $\tilde{\psi}_2^{c_2,-}(\lambda)$ be defined as

$$\tilde{\psi}_i^{c_2,-}(\lambda) := \lim_{x\to-\infty} \tilde{\psi}_i^{c_2}(x;\lambda), \quad i = 1,2.$$

In addition to the specifications above, we will denote by $p^{c_2}(x;\lambda)$ the tracking point p from (2.2) evaluated with $\tilde{\omega}_1$ and $\tilde{\omega}_2$ replaced with $\tilde{\psi}_1^{c_2}(x;\lambda)$ and $\tilde{\psi}_2^{c_2}(x;\lambda)$, and we define $p^+(x;\lambda)$, $p^{+,-}(\lambda)$, and $p^{c_2,-}(\lambda)$ analogously.

We are now in a position to state the following useful lemma.

Lemma 4.1. *Let the assumptions of Theorem 1.1 hold. Given any $\epsilon > 0$, there exists a constant L sufficiently large so that the following hold for all $c_2 \geq L$:*

(1) For $i = 1,2$,

$$|p^{c_2}(x;\lambda_i) - p^+(x;\lambda_i)| < \epsilon$$

for all $x \in \mathbb{R}$.

(2)

$$|p^{c_2,-}(\lambda) - p^{+,-}(\lambda)| < \epsilon$$

for all $\lambda \in [\lambda_1, \lambda_2]$.

Proof. Beginning with (1), we observe from our definitions of $\tilde{\psi}_1^{c_2}(x;\lambda)$ and $\tilde{\psi}_1^+(x;\lambda)$ the relation

$$\tilde{\psi}_1^{c_2}(x;\lambda) - \tilde{\psi}_1^+(x;\lambda) = \Big(\frac{\eta^-(x;\lambda)}{|\eta^-(x;\lambda)|}\Big) \wedge \Big(\frac{e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)}{|e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)|} - \frac{\tilde{\mathcal{V}}^+(\lambda)}{|\tilde{\mathcal{V}}^+(\lambda)|}\Big).$$

Since $\eta^-(x;\lambda)/|\eta^-(x;\lambda)|$ is bounded and

$$\lim_{c_2\to+\infty} \Big(\frac{e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)}{|e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)|} - \frac{\tilde{\mathcal{V}}^+(\lambda)}{|\tilde{\mathcal{V}}^+(\lambda)|}\Big) = 0,$$

we can make the difference $|\tilde{\psi}_1^{c_2}(x;\lambda) - \tilde{\psi}_1^+(x;\lambda)|$ as small as we like by taking c_2 sufficiently large. A similar statement holds for $\tilde{\psi}_2^{c_2}(x;\lambda)$ and $\tilde{\psi}_2^+(x;\lambda)$, and the claim about the difference $|p^{c_2}(x;\lambda_i) - p^+(x;\lambda_i)|$ follows from the continuous dependence of the tracking point on its two inputs, bearing in mind that the points $(0,1)$ and $(0,-1)$ are equated for p. Here, we emphasize that the values $\tilde{\psi}_1^{c_2}(x;\lambda)^2 + \tilde{\psi}_2^{c_2}(x;\lambda)^2$ and $\tilde{\psi}_1^+(x;\lambda)^2 + \tilde{\psi}_2^+(x;\lambda)^2$ are bounded away from zero by virtue of our invariance assumption, so that the pairs $(\tilde{\psi}_1^{c_2}(x;\lambda), \tilde{\psi}_2^{c_2}(x;\lambda))$ and $(\tilde{\psi}_1^{+}(x;\lambda), \tilde{\psi}_2^{+}(x;\lambda))$ are confined to compact subsets of $\mathbb{R}^2$ that do not contain the origin. On such sets, the tracking point p is uniformly continuous in its arguments.

For assertion (2), we write

$$\tilde{\psi}_1^{c_2,-}(\lambda) - \tilde{\psi}_1^{+,-}(\lambda) = \Big(\frac{v^-(\lambda)}{|v^-(\lambda)|}\Big) \wedge \Big(\frac{e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)}{|e^{\mu_+(\lambda)c_2}\tilde{\mathcal{Y}}^+(c_2;\lambda)|} - \frac{\tilde{\mathcal{V}}^+(\lambda)}{|\tilde{\mathcal{V}}^+(\lambda)|}\Big),$$

and the claim follows as for (1). □

Using Lemma 4.1, we can now establish the following lemma, which uses the notation of (1.33) and (1.34).

Lemma 4.2. *Under the assumptions of Theorem 1.1, there exist a positive constant L sufficiently large so that for any constants c_1 and c_2 so that $c_1 \leq -L$ and $c_2 \geq L$ there holds*

$$\begin{aligned}
&\mathrm{Ind}(\mathcal{g}(c_1;\cdot), \mathcal{h}(c_2;\cdot); [\lambda_1,\lambda_2]) + \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_2), \mathcal{h}(c_2;\lambda_2); [c_1,c_2]) - \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_1), \mathcal{h}(c_2;\lambda_1); [c_1,c_2]) \\
&\quad = \mathrm{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}^+(\cdot); [\lambda_1,\lambda_2]) + \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_2), \mathcal{h}^+(\lambda_2); [-\infty,+\infty]) \\
&\qquad - \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_1), \mathcal{h}^+(\lambda_1); [-\infty,+\infty]).
\end{aligned}$$

Proof. We begin by observing that each hyperplane index on the left-hand side of the sought equality is computed by tracking the point $p^{c_2}(x;\lambda)$ around S^1 as the points (x,λ) move along the Maslov box (bottom, right, and left shelves respectively). Likewise, the first hyperplane index on the right-hand side is computed by tracking $p^{+,-}(\lambda)$ around S^1 as λ increases from λ_1 to λ_2, and the latter two hyperplane indices on the right-hand side are computed by tracking (for $i = 1,2$) $p^+(x;\lambda_i)$ around S^1 as x increases from $-\infty$ to $+\infty$. For the proof of Lemma 4.2, our strategy will be to take advantage of Lemma 4.1 to show that these points can be kept close enough so that the indices computed must be equivalent.

We effectively have four cases to consider, based on the asymptotic limits

$$p^{+,+}(\lambda_i) = \lim_{x\to+\infty} p^+(x;\lambda_i), \quad i = 1,2,$$

which necessarily exist under the assumptions of Theorem 1.1. Namely, we can have (1) $p^{+,+}(\lambda_i) \neq (-1,0)$, $i = 1,2$; (2) $p^{+,+}(\lambda_1) \neq (-1,0)$, $p^{+,+}(\lambda_2) = (-1,0)$; (3) $p^{+,+}(\lambda_1) = (-1,0)$, $p^{+,+}(\lambda_2) \neq (-1,0)$; and (4) $p^{+,+}(\lambda_i) = (-1,0)$, $i = 1,2$.

Beginning with Case (1), we first observe that we can take some L sufficiently large so that for all $x \geq L$, we have $p^+(x;\lambda_i) \neq (-1,0)$, $i = 1,2$. We can conclude that for each of $i = 1,2$, and for any $c_2 \geq L$,

$$\mathrm{Ind}(\mathcal{g}(\cdot;\lambda_i), \mathcal{h}^+(\lambda_i); [c_2,+\infty]) = 0; \tag{4.8}$$

In addition, for Case (1) neither λ_1 nor λ_2 can be an eigenvalue, so we must have $p^{c_2}(c_2;\lambda_i) \neq (-1,0)$, $i = 1,2$.

We now think of starting the evolution of $p^{c_2}(x;\lambda)$ and $p^+(x;\lambda)$ at the point (c_2,λ_1) (top left corner of the Maslov box). According to Lemma 4.1, given any $\epsilon > 0$ we can choose L sufficiently large (possibly larger than before) so that for all $c_2 \geq L$

$$|p^{c_2}(x;\lambda_1) - p^+(x;\lambda_1)| < \epsilon \tag{4.9}$$

for all $x \in \mathbb{R}$. Likewise, according to Lemma 4.1, we can choose L sufficiently large so that for all $c_2 \geq L$

$$|p^{c_2,-}(\lambda) - p^{+,-}(\lambda)| < \epsilon \tag{4.10}$$

for all $\lambda \in [\lambda_1, \lambda_2]$. Last, we can complete a U-shaped contour by choosing L sufficiently large so that for all $c_2 \geq L$

$$|p^{c_2}(x; \lambda_2) - p^{+}(x; \lambda_2)| < \epsilon \tag{4.11}$$

for all $x \in \mathbb{R}$.

At this point, we can choose ϵ sufficiently small so that the winding numbers associated with the points $p^{c_2}(x;\lambda)$ and $p^+(x;\lambda)$ must be the same on the following U-shaped contour: (1) fix $\lambda = \lambda_1$, and let x decrease from c_2 to $-\infty$ (asymptotic limit sense); (2) for the asymptotic limit points $p^{c_2,-}(\lambda)$ and $p^{+,-}(\lambda)$, let λ increase from λ_1 to λ_2; and (3) fix $\lambda = \lambda_2$ and let x increase from $-\infty$ to c_2. In total, we obtain the index relation

$$\begin{aligned} &\text{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}(c_2; \cdot); [\lambda_1, \lambda_2]) + \text{Ind}(\mathcal{g}(\cdot; \lambda_2), \mathcal{h}(c_2; \lambda_2); [-\infty, c_2]) \\ &\quad - \text{Ind}(\mathcal{g}(\cdot; \lambda_1), \mathcal{h}(c_2; \lambda_1); [-\infty, c_2]) \\ &= \text{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}^+(\cdot); [\lambda_1, \lambda_2]) + \text{Ind}(\mathcal{g}(\cdot; \lambda_2), \mathcal{h}^+(\lambda_2); [-\infty, c_2]) \\ &\quad - \text{Ind}(\mathcal{g}(\cdot; \lambda_1), \mathcal{h}^+(\lambda_1); [-\infty, c_2]). \end{aligned} \tag{4.12}$$

For the indices on the right-hand side of this relation, we can use (4.8) to obtain precisely the expressions stated on the right-hand side in Lemma 4.2. For the indices on the left-hand side of this relation, we observe that by virtue of our invariance assumption **(E)**(3) for the bottom shelf we can take L sufficiently large so that for all $c_1 \leq -L$ and $c_2 \geq L$, the scaled variables $\tilde{\psi}_1^{c_2}(x;\lambda)$ and $\tilde{\psi}_2^{c_2}(x;\lambda)$ do not simultaneously vanish at any point of the asymptotic rectangle $[-\infty, c_1] \times [\lambda_1, \lambda_2]$. Accordingly, we can use homotopy invariance in the Maslov-Arnold space to see that

$$\begin{aligned} &\text{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}(c_2; \cdot); [\lambda_1, \lambda_2]) + \text{Ind}(\mathcal{g}(\cdot; \lambda_2), \mathcal{h}(c_2; \lambda_2); [-\infty, c_1)) \\ &\quad - \text{Ind}(\mathcal{g}(\cdot; \lambda_1), \mathcal{h}(c_2; \lambda_1); [-\infty, c_1]) = \text{Ind}(\mathcal{g}(c_1; \cdot), \mathcal{h}(c_2; \cdot); [\lambda_1, \lambda_2]). \end{aligned}$$

If we think of solving this last relation for $\text{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}(c_2; \cdot); [\lambda_1, \lambda_2])$ and substituting the result into the left-hand side of (4.12), we see that the left-hand side of (4.12) becomes

$$\text{Ind}(\mathcal{g}(c_1; \cdot), \mathcal{h}(c_2; \cdot); [\lambda_1, \lambda_2]) + \text{Ind}(\mathcal{g}(\cdot; \lambda_2), \mathcal{h}(c_2; \lambda_2); [c_1, c_2]) - \text{Ind}(\mathcal{g}(\cdot; \lambda_1), \mathcal{h}(c_2; \lambda_1); [c_1, c_2]),$$

which is precisely the left-hand side claimed in Lemma 4.2.

Turning to Case (2), the critical difference is that we now have $p^{c_2}(c_2; \lambda_2) = (-1, 0)$, and additionally

$$p^{+,+}(\lambda_2) = \lim_{x \to +\infty} p^+(x; \lambda_2) = (-1, 0). \tag{4.13}$$

Similarly as for Case (1), we would like to think of tracking relevant points around S^1 as points (x, λ) move along the lower U-shaped contour, but for Case (2) we have three subcases for the location of $p^+(c_2; \lambda_2)$: (i) $p^+(c_2; \lambda_2)$ is rotated slightly away from $(-1, 0)$ in the clockwise direction; or (ii) $p^+(c_2; \lambda_2) = (-1, 0)$; or $p^+(c_2; \lambda_2)$ is rotated slightly away from $(-1,0)$ in the counterclockwise direction. It follows immediately from (4.13) and our specification of hyperplane indices computed on unbounded domains that

$$\operatorname{Ind}(\mathscr{g}(\cdot;\lambda_2),\mathscr{h}^+(\lambda_2);[-\infty,+\infty]) = \operatorname{Ind}(\mathscr{g}(\cdot;\lambda_2),\mathscr{h}^+(\lambda_2);[-\infty,c_2]) + \begin{cases} +1 & \text{case (i)} \\ 0 & \text{cases (ii) or (iii).} \end{cases} \tag{4.14}$$

For all cases, given any $\epsilon > 0$, we can choose L large enough so that (4.9), (4.10), and (4.11) all hold for all $c_2 \geq L$. It follows that the quantities on the left and right sides of (4.12) can differ only by either 0 or 1. We claim that the discrepancies are precisely as follows: the quantity obtained by subtracting the right-hand side of (4.12) from the left must be +1 for Case (i) and 0 for Cases (ii) and (iii). To see this, we need only consider the possible ways in which the points $p^{c_2}(c_2;\lambda_2)$ and $p^+(c_2;\lambda_2)$ can arrive respectively at $p^{c_2}(c_2;\lambda_2) = (-1,0)$ and the Case (i) location of $p^+(c_2;\lambda_2)$. Ignoring transient crossings (i.e., crossings along with return crossings), either $p^{c_2}(c_2;\lambda_2)$ has arrived at $(-1,0)$ one more time moving in the counterclockwise direction than $p^+(c_2;\lambda_2)$, or $p^+(c_2;\lambda_2)$ has crossed $(-1,0)$ once more in the clockwise direction than $p^+(c_2;\lambda_2)$. In either case, the discrepancy is +1. The reasoning is similar for Cases (ii) and (iii), and we see that the discrepancies are precisely the values on the right-hand side of (4.14). In this way, we see that the possible discrepancy between the left and right sides of (4.12) are precisely addressed in Case (2) by the inclusion of $+\infty$ in the index

$$\operatorname{Ind}(\mathscr{g}(\cdot;\lambda_2),\mathscr{h}^+(\lambda_2);[-\infty,+\infty]),$$

and the equality of Lemma 4.2 is seen to hold.

Cases (3) and (4) can be handled similarly. □

One thing clear from Lemma 4.2 is that the left-hand side of the stated relation takes the same value for all $c_1 \leq -L$ and $c_2 \geq L$ (because the right-hand side is independent of c_1 and c_2). In order to show from (4.3) that $\mathfrak{m}(c_1,c_2)$ is actually independent of c_1 and c_2, we last need to establish that the top-shelf index

$$\operatorname{Ind}(\mathscr{g}(c_2;\cdot),\mathscr{h}(c_2;\cdot);[\lambda_1,\lambda_2])$$

is independent of c_2 (for c_2 taken sufficiently large). To see this, we first recall that crossing points along the top shelf correspond precisely with eigenvalues of (1.1) and so certainly do not depend on c_2. This means we only need to verify that the directions we associate with these crossings must be independent of c_2.

If the eigenvalues in $[\lambda_1,\lambda_2]$ aren't discrete, we set $\mathcal{N}_\#([\lambda_1,\lambda_2]) = \infty$, in which case the claim of Theorem 1.1 holds by convention, so we can restrict our analysis to the case of discrete spectrum. For this, we suppose $\lambda_0 \in (\lambda_1,\lambda_2)$ is an isolated eigenvalue and we consider a Maslov box sufficiently small so that λ_0 is the only eigenvalue it contains. (See Figure 4.2.) In particular, we compute the hyperplane index detecting intersections between $\mathscr{g}(x;\lambda)$ and $\mathscr{h}(x;\lambda)$ as this small box is traversed. According to our Assumption **(E)**(4), we have invariance along this Maslov box and its interior, so we can use homotopy invariance in the associated Maslov-Arnold space to conclude that

$$\begin{aligned}&\operatorname{Ind}(\mathscr{g}(c_2;\cdot),\mathscr{h}(c_2;\cdot);[\lambda_0-h,\lambda_0+h]) \\ &\qquad - \operatorname{Ind}(\mathscr{g}(c_2+\Delta c_2;\cdot),\mathscr{h}(c_2+\Delta c_2;\cdot);[\lambda_0-h,\lambda_0+h]) = 0.\end{aligned}$$

Since only a single crossing occurs for each of these hyperplane indices, it must be the case that these crossings are in the same direction. This discussion has been for interior values $\lambda_0 \in (\lambda_1, \lambda_2)$, but the endpoints $\lambda_0 = \lambda_1$ and $\lambda_0 = \lambda_2$ can be treated similarly if λ_1 or λ_2 is an eigenvalue, with boxes either extending to the right of λ_1 or to the left of λ_2.

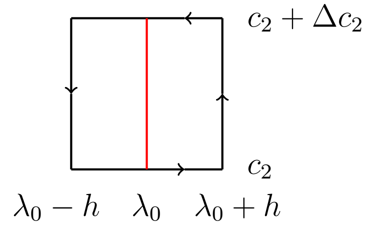

Figure 4.2: The Maslov Box at λ_0.

These considerations allow us to associate a value $\mathfrak{m}$ with any interval $[\lambda_1, \lambda_2] \subset I$ for which the assumptions of Theorem 1.1 apply. In particular, by rearranging (4.3), we can write

$$\begin{aligned}\operatorname{Ind}(\mathscr{g}(c_2; \cdot), \mathscr{h}(c_2; \cdot); [\lambda_1, \lambda_2]) &= \operatorname{Ind}(\mathscr{g}(\cdot; \lambda_2), \mathscr{h}(c_2; \lambda_2); [c_1, c_2]) \\ &\quad - \operatorname{Ind}(\mathscr{g}(\cdot; \lambda_1), \mathscr{h}(c_2; \lambda_1); [c_1, c_2]) + \operatorname{Ind}(\mathscr{g}(c_1; \cdot), \mathscr{h}(c_2; \cdot); [\lambda_1, \lambda_2]) - \mathfrak{m}.\end{aligned}$$

Upon substitution of the relation from Lemma 4.2, we obtain

$$\begin{aligned}\operatorname{Ind}(\mathscr{g}(c_2; \cdot), \mathscr{h}(c_2; \cdot); [\lambda_1, \lambda_2]) &= \operatorname{Ind}(\mathscr{g}(\cdot; \lambda_2), \mathscr{h}^+(\lambda_2); [-\infty, +\infty]) \\ &\quad - \operatorname{Ind}(\mathscr{g}(\cdot; \lambda_1), \mathscr{h}^+(\lambda_1); [-\infty, +\infty]) + \operatorname{Ind}(\mathscr{g}^-(\cdot), \mathscr{h}^+(\cdot); [\lambda_1, \lambda_2]) - \mathfrak{m}.\end{aligned} \tag{4.15}$$

Last, using (4.2), we obtain the claimed inequality

$$\begin{aligned}\mathcal{N}_{\#}([\lambda_1, \lambda_2]) \geq \Big| &\operatorname{Ind}(\mathscr{g}(\cdot; \lambda_2), \mathscr{h}^+(\lambda_2); [-\infty, +\infty]) \\ &- \operatorname{Ind}(\mathscr{g}(\cdot; \lambda_1), \mathscr{h}^+(\lambda_1); [-\infty, +\infty]) + \operatorname{Ind}(\mathscr{g}^-(\cdot), \mathscr{h}^+(\cdot); [\lambda_1, \lambda_2]) - \mathfrak{m}\Big|.\end{aligned}$$

This completes the proof of Theorem 1.1.

4.1 The Bottom and Left Shelves

In this section, we provide additional information about calculations associated with the bottom and left shelves, emphasizing cases in which one or both of these values can be shown to be 0.

4.1.1 The Bottom Shelf

In Theorem 1.1, the hyperplane index associated with the bottom shelf detects intersections between the spaces $\mathscr{g}^-(\lambda)$ and $\mathscr{h}^+(\lambda)$ as λ increases from λ_1 to λ_2. These intersections can be detected as zeros of the wedge product

$$v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda),$$

where we recall from Proposition 3.1 that $v^-(\lambda)$ denotes a right eigenvector of $A_-(\lambda)$ associated with the eigenvalue $\mu_-(\lambda)$, and we recall from Proposition 3.6 that $\tilde{\mathcal{V}}^+(\lambda)$ denotes a right eigenvector of $\tilde{A}_+(\lambda)$ associated with the eigenvalue $-\mu_+(\lambda)$. In general, this allows us to explicitly compute the hyperplane index along the bottom shelf by working with the scaled variables $\tilde{\psi}_1^{+,-}(\lambda)$ and $\tilde{\psi}_2^{+,-}(\lambda)$, defined respectively in (1.31) and (1.32). In some cases, including the applications we consider in Section 6, we can show that for each $\lambda \in [\lambda_1, \lambda_2]$, we have $v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) \neq 0$, from which we can conclude that there are no crossing points along the bottom shelf. In such cases, we naturally have invariance along the bottom shelf, and additionally

$$\mathrm{Ind}(\mathscr{g}^-(\cdot), \mathscr{h}^+(\cdot); [\lambda_1, \lambda_2]) = 0.$$

4.1.2 The Left Shelf

In most applications, we expect to take λ_1 sufficiently negative so that it is not an eigenvalue of (1.1), and in such cases it's straightforward to verify that the limit

$$\tilde{\psi}^{+,+}(\lambda_1) = \lim_{x \to +\infty} \tilde{\psi}^+(x; \lambda_1)$$

is well defined. Precisely, we can prove the following lemma.

Lemma 4.3. *Let Assumptions* **(A)** *through* **(D)** *hold, and fix any $\lambda_0 \in [\lambda_1, \lambda_2]$ that is not an eigenvalue of (1.1). Then there exists a constant $c_+(\lambda_0) \neq 0$ so that*

$$\lim_{x \to +\infty} e^{-\mu_+(\lambda_0)x} \eta^-(x; \lambda_0) = c_+(\lambda_0) v^+(\lambda_0),$$

where $v^+(\lambda_0)$ is the eigenvector of $A_+(\lambda_0)$ associated with the eigenvalue $\mu_+(\lambda_0)$. It follows that

$$\tilde{\psi}_1^{+,+}(\lambda_0) := \lim_{x \to +\infty} \tilde{\psi}_1^+(x; \lambda_0) = \frac{v^+(\lambda_0) \wedge \tilde{\mathcal{V}}^+(\lambda_0)}{|v^+(\lambda_0)||\tilde{\mathcal{V}}^+(\lambda_0)|}$$
$$\tilde{\psi}_2^{+,+}(\lambda_0) := \lim_{x \to +\infty} \tilde{\psi}_2^+(x; \lambda_0) = \frac{v^+(\lambda_0) \wedge \tilde{\mathcal{V}}_M^+(\lambda_0)}{|v^+(\lambda_0)||\tilde{\mathcal{V}}^+(\lambda_0)|}.$$

Proof. As in the discussion of (1.13), we let $\{y_j^+(x; \lambda_0)\}_{j=1}^n$ denote a basis of linearly independent solutions of (1.1) indexed so that (1.13) holds. Then there exist constants $\{c_j(\lambda_0)\}_{j=1}^n$ so that

$$\eta^-(x; \lambda_0) = \sum_{j=1}^n c_j(\lambda_0) y_j^+(x; \lambda_0), \quad i = 1, 2, \tag{4.16}$$

where we must have $c_n(\lambda_0) \neq 0$ or λ_0 would be an eigenvalue of (1.1) (in the sense described in the introduction). According to Assumption **(C)**, along with our labeling convention, the solutions $\{y_j^+(x; \lambda)\}_{j=1}^{n-1}$ are all $\mathbf{o}(e^{\mu_+(\lambda_0)x})$ as $x \to +\infty$, so

$$\begin{aligned}\lim_{x \to +\infty} e^{-\mu_+(\lambda_0)x} \eta^-(x; \lambda_0) &= \sum_{j=1}^n c_j(\lambda_0) \lim_{x \to +\infty} e^{-\mu_+(\lambda_0)x} y_j^+(x; \lambda_0) \\ &= c_n(\lambda_0) \lim_{x \to +\infty} e^{-\mu_+(\lambda_0)x} y_n^+(x; \lambda_0) = c_n(\lambda_0) v^+(\lambda).\end{aligned}$$

The first claim of our lemma follows from denoting $c_n(\lambda_0)$ by $c_+(\lambda_0)$.

For the second claim, we can write

$$\tilde{\psi}_1^+(x;\lambda_0) = \frac{(e^{-\mu_+(\lambda_0)x}\eta^-(x;\lambda_0)) \wedge \tilde{\mathcal{V}}^+(\lambda)}{|e^{-\mu_+(\lambda_0)x}\eta^-(x;\lambda_0)||\tilde{\mathcal{V}}^+(\lambda)|},$$

from which the claim regarding $\tilde{\psi}_1^{+,+}(\lambda_0)$ is clear upon taking $x \to +\infty$. The claim about $\tilde{\psi}_2^{+,+}(\lambda_0)$ follows similarly. □

In some cases, including the applications we consider in Section 6, we can take λ_1 sufficiently negative so that there are no crossings along the left shelf. To understand conditions under which this occurs, we recall that in the left-shelf computation

$$\text{Ind}(\mathscr{g}(\cdot;\lambda_1), \mathscr{h}^+(\lambda_1); [-\infty, +\infty]),$$

we detect intersections between the evolving subspaces $\mathscr{g}(x;\lambda_1)$ and the fixed target space $\mathscr{h}^+(\lambda_1)$. If we can show that

$$\eta^-(x;\lambda_1) \wedge \tilde{\mathcal{V}}^+(\lambda_1) \neq 0$$

for all $x \in \mathbb{R}$, and additionally that no intersection is obtained in the limit as $x \to \pm\infty$, then we can conclude that there are no crossings on the left shelf. For this discussion, we will lean heavily on the development of [33], especially the statement and proof of Proposition 1.17 from that reference.

Proposition 4.1. *Let Assumptions* **(A)** *through* **(D)** *hold, with the interval I unbounded on the left. For the matrices $A_\pm(\lambda)$ specified in Assumption* **(B)***, suppose $A_-(\lambda) = A_+(\lambda)$, and that there exists a diagonalizing matrix $V_-(\lambda)$, so that for each $\lambda \in I$ the matrix*

$$D_-(\lambda) := V_-(\lambda)^{-1}A_-(\lambda)V_-(\lambda)$$

is diagonal with the eigenvalues of $A_-(\lambda)$ on its diagonal, and in particular with $\mu_-(\lambda)$ in the first column of the first row. In addition, we set

$$F(x;\lambda) := V_-(\lambda)^{-1}(A(x;\lambda) - A_-(\lambda))V_-(\lambda),$$

and assume the following three items:

(i) There exists a positive constant λ_∞ and a corresponding constant C_∞ so that for all $\lambda \leq -\lambda_\infty$ we have the inequality

$$\int_{-\infty}^{+\infty} |F(x;\lambda)|dx \leq C_\infty.$$

(ii) The limit

$$\lim_{x_0 \to -\infty} \int_{-\infty}^{x_0} |F(x;\lambda)|dx = 0$$

converges uniformly for all $\lambda \leq -\lambda_\infty$.

(iii) If e_1 denotes the usual first Euclidian basis element, then

$$\lim_{\lambda \to -\infty} \int_{-\infty}^{+\infty} |F(x;\lambda)e_1| dx = 0.$$

Under these assumptions, we can conclude

$$V_-(\lambda)^{-1}(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) = e_1 + \mathbf{o}(1), \quad \lambda \to -\infty,$$

and likewise if $z^+(x;\lambda)$ is the solution to $z^{+\prime} = -z^+ A(x;\lambda)$ associated to $\tilde{\mathcal{Y}}^+(x;\lambda)$ via (3.6), then

$$e^{\mu_+(\lambda)x} z^+(x;\lambda) V_+(\lambda) e_1 = 1 + \mathbf{o}(1), \quad \lambda \to -\infty,$$

where in both cases the order term is uniform for $x \in \mathbb{R}$. In addition, the vector $z^+(0;\lambda)V_+(\lambda)$ remains bounded as λ tends toward $-\infty$.

Proof. We begin by looking for solutions to (1.1) of the form

$$y(x;\lambda) = e^{\mu_-(\lambda)x} V_-(\lambda)(e_1 + v(x;\lambda))$$

for which we find by direct calculation that

$$v'(x;\lambda) = V_-(\lambda)^{-1}(A_-(\lambda) - \mu_-(\lambda)I)V_-(\lambda)(e_1 + v(x;\lambda)) + F(x;\lambda)(e_1 + v(x;\lambda)).$$

For notational convenience, we set

$$B_-(\lambda) := V_-(\lambda)^{-1}(A_-(\lambda) - \mu_-(\lambda)I)V_-(\lambda),$$

and we observe that according to our convention with $V_-(\lambda)$ we have $B_-(\lambda)e_1 = 0$. This allows us to express $v'(x;\lambda)$ in the more compact form

$$v' = B_-(\lambda)v + F(x;\lambda)(v + e_1). \tag{4.17}$$

Our immediate goal is to use this last relation to obtain a convenient expression for $v(x;\lambda)$, but first we determine the asymptotic behavior of $v(x;\lambda)$ as x tends toward $-\infty$. Our particular interest is the function v associated with $\eta^-(x;\lambda)$, which we take to be defined by the relation

$$\eta^-(x;\lambda) = e^{\mu_-(\lambda)x} V_-(\lambda)(v(x;\lambda) + e_1). \tag{4.18}$$

Solving for $v(x;\lambda)$, we find

$$v(x;\lambda) = -e_1 + e^{-\mu_-(\lambda)x} V_-(\lambda)^{-1} \eta^-(x;\lambda).$$

According to Proposition 3.1, we have the convergence

$$\lim_{x \to -\infty} e^{-\mu_-(\lambda)x} \eta^-(x;\lambda) = v^-(\lambda),$$

where $v^-(\lambda)$ is the first column of $V_-(\lambda)$ (the eigenvector of $A_-(\lambda)$ corresponding with eigenvalue $\mu_-(\lambda)$). It follows that $V_-(\lambda)^{-1}v^-(\lambda) = e_1$, and consequently $v(x;\lambda) \to 0$ as $x \to -\infty$.

Returning now to (4.17), we note that we can express the equation as

$$(e^{-B_-(\lambda)x}v)' = e^{-B_-(\lambda)x}F(x;\lambda)(v+e_1).$$

Since $\mu_-(\lambda)$ is the largest eigenvalue of $A_-(\lambda)$, the eigenvalues of $B_-(\lambda)$ must all be non-positive. Observing that v is bounded on any interval $[x_0, x]$, $x_0 < x$, we see that we can integrate on an arbitrary such interval to obtain the relation

$$e^{-B_-(\lambda)x}v(x;\lambda) - e^{-B_-(\lambda)x_0}v(x_0;\lambda) = \int_{x_0}^{x} e^{-B_-(\lambda)\xi}F(\xi;\lambda)(v(\xi;\lambda)+e_1)d\xi. \tag{4.19}$$

The exponential $e^{-B_-(\lambda)x_0}$ remains bounded as $x_0 \to -\infty$, and $v(x_0) \to 0$ as $x_0 \to -\infty$. Additionally since $F(\xi;\lambda)$ is assumed to be integrable on $\mathbb{R}$ (Assumption (i)), we can take a limit with $x_0 \to -\infty$ on both sides of this last expression to see that

$$e^{-B_-(\lambda)x}v(x;\lambda) = \int_{-\infty}^{x} e^{-B_-(\lambda)\xi}F(\xi;\lambda)(v(\xi;\lambda)+e_1)d\xi,$$

or equivalently

$$v(x;\lambda) = \int_{-\infty}^{x} e^{B_-(\lambda)(x-\xi)}F(\xi;\lambda)(v(\xi;\lambda)+e_1)d\xi.$$

By construction, $B_-(\lambda)$ is a diagonal matrix, and it is easily seen that the matrix norm of $\exp(B_-(\lambda)(x-\xi))$ is 1. This allows us to write

$$\begin{aligned}
|v(x;\lambda)| &\leq \int_{-\infty}^{x} |e^{B_-(\lambda)(x-\xi)}||F(\xi;\lambda)(v(\xi;\lambda)+e_1)|d\xi \\
&= \int_{-\infty}^{x} |F(\xi;\lambda)(v(\xi;\lambda)+e_1)|d\xi \\
&\leq \int_{-\infty}^{x} |F(\xi;\lambda)v(\xi;\lambda)|d\xi + \int_{-\infty}^{x} |F(\xi;\lambda)e_1|d\xi \\
&\leq \sup_{\xi\leq x}|v(\xi;\lambda)| \int_{-\infty}^{x} |F(\xi;\lambda)|d\xi + \int_{-\infty}^{x} |F(\xi;\lambda)e_1|d\xi.
\end{aligned}$$

Using Assumption (ii), we can take x_0 sufficiently negative so that for all $x \leq x_0$,

$$|v(x;\lambda)| \leq \frac{1}{2}\sup_{\xi\leq x_0}|v(\xi;\lambda)| + \int_{-\infty}^{x_0} |F(\xi;\lambda)e_1|d\xi.$$

We can now take a supremum on both sides over $x \leq x_0$ to see that

$$\frac{1}{2}\sup_{\xi\leq x_0}|v(\xi;\lambda)| \leq \int_{-\infty}^{x_0} |F(\xi;\lambda)e_1|d\xi.$$

Using Assumption (iii), we see that the right-hand side of this last expression tends to 0 as $\lambda \to -\infty$, so for some fixed $x_0 \ll 0$, we can assert that

$$\lim_{\lambda\to-\infty}\sup_{\xi\leq x_0}|v(\xi;\lambda)| = 0.$$

We now fix x_0 as such a value and return to (4.19) to see that

$$|v(x;\lambda)| \le \Big(|v(x_0;\lambda)| + \int_{x_0}^{x} |F(\xi;\lambda)e_1| d\xi\Big) + \int_{x_0}^{x} |F(\xi;\lambda)||v(\xi;\lambda)| d\xi.$$

If we apply Grönwall's inequality to this integral relation, we obtain the inequality

$$\begin{aligned} |v(x;\lambda)| &\le \Big(|v(x_0;\lambda)| + \int_{x_0}^{x} |F(\xi;\lambda)e_1| d\xi\Big) e^{\int_{x_0}^{x} |F(\xi;\lambda)| d\xi} \\ &\le \Big(|v(x_0;\lambda)| + \int_{x_0}^{\infty} |F(\xi;\lambda)e_1| d\xi\Big) e^{\int_{x_0}^{\infty} |F(\xi;\lambda)| d\xi}. \end{aligned} \tag{4.20}$$

Here, the quantity in large parentheses tends to 0 as $\lambda \to -\infty$, so we can choose $\lambda_\infty \gg 0$ sufficiently large so that $v(x;\lambda)$ is small for all $x \in \mathbb{R}$, $\lambda \le -\lambda_\infty$.

Last, upon multiplication of (4.18) on the left by $V_-(\lambda)^{-1} e^{-\mu_-(\lambda)x}$, we obtain the relation

$$V_-(\lambda)^{-1}(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) = e_1 + v(x;\lambda).$$

We have seen that $v(x;\lambda) = \mathbf{o}(1)$, $\lambda \to -\infty$ uniformly for $x \in \mathbb{R}$, giving the first claim.

The second claim is proven similarly, combining the second part of the proof of Proposition 1.17 from [33] with the first part of the current proof. Here, we primarily just indicate how the final observation on boundedness of $z^+(0;\lambda)V_+(\lambda)$ is established. The proof in this case begins by setting

$$w(x;\lambda) := -e_1^T + e^{\mu_+(\lambda)x} z^+(x;\lambda)V_+(\lambda).$$

Then, proceeding as in the first part of this proof, we can establish that there exists a constant C sufficiently large so that for all λ sufficiently negative we have the bound

$$|w(x;\lambda)| \le C, \quad \forall\, x \in \mathbb{R}.$$

In particular, $w(0;\lambda)$ satisfies this estimate, and we have the relation

$$z^+(0;\lambda)V_+(\lambda) = e_1^T + w(0;\lambda),$$

verifying that the left-hand side is bounded as $\lambda \to -\infty$. □

Remark 4.1. *The only place in the proof of Proposition 4.1 in which we absolutely require the assumption that $A_- = A_+$ is in obtaining the second inequality in (4.20). Nonetheless, it is critical at that point, and for the general case of $A_- \neq A_+$, verifying the absence of crossings on the left shelf is more delicate. (See the appendix of this paper for one example.)*

We can now use Proposition 4.1 to establish the following result on crossings along the left shelf.

Proposition 4.2. *Suppose the assumptions of Proposition 4.1 hold. Then there exists a value $\lambda_\infty > 0$ sufficiently large so that for all $\lambda_1 \le -\lambda_\infty$ there holds*

$$\ell(x;\lambda_1) \cap \ell^+(\lambda_1) = \{0\}$$

for all $x \in \mathbb{R}$, and in particular,

$$\text{Ind}(\ell(\cdot;\lambda_1), \ell^+(\lambda_1), [-\infty, +\infty]) = 0.$$

Proof. First, crossings along the left shelf correspond precisely with zeros of

$$(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) \wedge \tilde{\mathcal{V}}^+(\lambda),$$

which by virtue of Proposition 3.5 can be expressed as

$$w^+(\lambda)\cdot(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)),$$

where $w^+(\lambda)$ is related to $\tilde{\mathcal{V}}^+(\lambda)$ as in 3.5.

According to Proposition 4.2, we have the asymptotic relation

$$V_-(\lambda)^{-1}(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) = e_1 + \mathbf{o}(1), \quad \lambda \to -\infty,$$

uniformly for $x \in \mathbb{R}$, and according to the normalization in Assumption **(C)**, along with our assumption that $A_+(\lambda)$ is diagonalizable, we can write

$$w^+(\lambda)V_+(\lambda) = e_1^T.$$

Under the assumptions of Proposition 4.1, we have $V_-(\lambda) = V_+(\lambda)$, allowing us to compute

$$\begin{aligned} w^+(\lambda)\cdot(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) &= e_1^T V_-(\lambda)^{-1}(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) \\ &= e_1^T(e_1 + \mathbf{o}(1)) = 1 + \mathbf{o}(1). \end{aligned}$$

We see immediately that for λ sufficiently negative we must have

$$(e^{-\mu_-(\lambda)x}\eta^-(x;\lambda)) \wedge \tilde{\mathcal{V}}^+(\lambda) > 0,$$

for all $x \in \mathbb{R}$, indicating that there are no crossings along the left shelf. □

5 The Evans Function

One of our goals is to place information gained from the Evans function into the broader geometrical framework of the current analysis. The Evans function in this setting has already been elegantly developed in [33], so we proceed primarily by translating the results obtained there into the current setting.

Generally, the Evans function serves as a characteristic function for eigenvalue problems such as (1.1) (standard references include [1, 15, 16, 17, 18, 20], along with [33]). For (1.1) under Assumptions **(A)** through **(D)**, it's natural to specify the Evans function as the wedge product

$$D(\lambda) := \eta^-(x;\lambda) \wedge \tilde{\mathcal{Y}}^+(x;\lambda), \tag{5.1}$$

where Proposition 3.4 allows us to verify that the right-hand side is independent of x (i.e., its derivative with respect to x is 0).

Remark 5.1. *According to Proposition 3.5, our specification (5.1) is equivalent to the specification from [33], which with z as in Proposition 3.5 can be expressed here as*

$$D_{PW}(\lambda) = z(0;\lambda)\cdot\eta^-(0;\lambda).$$

This correspondence between $D(\lambda)$ and $D_{PW}(\lambda)$ allows us to adapt results from [33] directly to the current setting, though we include some details of the proofs for completeness.

Proposition 5.1. *Let Assumptions* **(A)** *through* **(D)** *hold, and let the Evans function* $D(\lambda)$ *be specified as in (5.1). Then for any* $\lambda \in I$*,*

$$\begin{aligned} D'(\lambda) &= \Big(\frac{v^{-\prime}(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda)}{v^-(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda)} + \frac{v^+(\lambda) \wedge \tilde{\mathcal{V}}^{+\prime}(\lambda)}{v^+(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda)}\Big) D(\lambda) \\ &+ \int_{-\infty}^0 \Big((A_\lambda(x;\lambda) - \mu_-'(\lambda) I)\eta^-(x;\lambda)\Big) \wedge \tilde{\mathcal{Y}}^+(x;\lambda) dx \\ &+ \int_0^{+\infty} \Big((A_\lambda(x;\lambda) - \mu_+'(\lambda) I)\eta^-(x;\lambda)\Big) \wedge \tilde{\mathcal{Y}}^+(x;\lambda) dx. \end{aligned}$$

Here $v^-(\lambda)$ *is an eigenvector of* $A_-(\lambda)$ *corresponding to the eigenvalue* $\mu_-(\lambda)$*;* $\tilde{\mathcal{V}}^-(\lambda)$ *is an eigenvector of* $\tilde{A}_-(\lambda)$ *corresponding to the eigenvalue* $-\mu_-(\lambda)$*;* $v^+(\lambda)$ *is an eigenvector of* $A_+(\lambda)$ *corresponding to the eigenvalue* $\mu_+(\lambda)$*; and* $\tilde{\mathcal{V}}^+(\lambda)$ *is an eigenvector of* $\tilde{A}_+(\lambda)$ *corresponding to the eigenvalue* $-\mu_+(\lambda)$*.*

Proof. Our Proposition 5.1 is effectively a restatement of Theorem 1.11 from [33] in the current setting, and we only briefly sketch the proof, also adapted to our setting. First, for $x < 0$, it's useful to write

$$D(\lambda) = u^-(x;\lambda) \wedge \tilde{\mathcal{U}}^-(x;\lambda),$$

where we're introducing the notation

$$\begin{aligned} u^-(x;\lambda) &:= e^{-\mu_-(\lambda)x} \eta^-(x;\lambda) \\ \tilde{\mathcal{U}}^-(x;\lambda) &= e^{+\mu_-(\lambda)x} \tilde{\mathcal{Y}}^+(x;\lambda). \end{aligned}$$

Recalling that $\eta^{-\prime} = A\eta^-$ and $\tilde{\mathcal{Y}}^{+\prime} = \tilde{A}\tilde{\mathcal{Y}}^+$, we can write

$$\begin{aligned} u^{-\prime} &= -\mu_- u^- + Au \\ \tilde{\mathcal{U}}^{-\prime} &= \mu_- \tilde{\mathcal{U}}^- + \tilde{A}\tilde{\mathcal{U}}^-. \end{aligned}$$

Taking a λ-derivative of these expressions, we find

$$\begin{aligned} u_\lambda^{-\prime} &= -\mu_-' u^- - \mu_- u_\lambda^- + A_\lambda u^- + A u_\lambda^- \\ \tilde{\mathcal{U}}_\lambda^{-\prime} &= \mu_-' \tilde{\mathcal{U}}^- + \mu_- \tilde{\mathcal{U}}_\lambda^- + \tilde{A}_\lambda \tilde{\mathcal{U}}^- + \tilde{A}\tilde{\mathcal{U}}_\lambda^-. \end{aligned}$$

We now differentiate $u_\lambda^- \wedge \tilde{\mathcal{U}}^-$ in x,

$$\begin{aligned} \frac{d}{dx}(u_\lambda^- \wedge \tilde{\mathcal{U}}^-) &= \Big((A_\lambda - \mu_-' I)u^-\Big) \wedge \tilde{\mathcal{U}}^- + (A u_\lambda^-) \wedge \tilde{\mathcal{U}}^- + u_\lambda^- \wedge (\tilde{A}\tilde{\mathcal{U}}^-) \\ &= \Big((A_\lambda - \mu_-' I)u^-\Big) \wedge \tilde{\mathcal{U}}^-, \end{aligned}$$

where the second equality follows from Proposition 3.4. For $R > 0$, we can now integrate on $(-R, 0)$ to obtain the relation

$$\begin{aligned} & u_\lambda^-(0;\lambda) \wedge \tilde{\mathcal{U}}^-(0;\lambda) - u_\lambda^-(-R;\lambda) \wedge \tilde{\mathcal{U}}^-(-R;\lambda) \\ &\quad = \int_{-R}^0 \Big((A_\lambda(x;\lambda) - \mu_-'(\lambda) I)u^-(x;\lambda)\Big) \wedge \tilde{\mathcal{U}}^-(x;\lambda) dx. \end{aligned} \tag{5.2}$$

Our next goal will be to take a limit of this expression as R tends to $+\infty$, and for this we first need to look carefully at the wedge product $u_\lambda^-(-R;\lambda) \wedge \tilde{\mathcal{U}}^-(-R;\lambda)$. Beginning with $u_\lambda^-(-R;\lambda)$, we recall from Proposition 3.1 (and the definition of $u^-(-R;\lambda)$) that

$$\lim_{R\to+\infty} u^-(-R;\lambda) = v^-(\lambda),$$

where $v^-(\lambda)$ is analytic on Ω and the convergence is uniform on compact subsets of Ω. It follows that the limit of the derivatives with respect to λ converges to derivatives of the limits,

$$\lim_{R\to+\infty} u_\lambda^-(-R;\lambda) = v^{-\prime}(\lambda).$$

Next, for $\tilde{\mathcal{U}}^-(-R;\lambda)$ we have the complication that we don't have a convenient expression for $\tilde{\mathcal{Y}}^+(-R;\lambda)$ for large values of R. Nonetheless, recalling that $\tilde{\mathcal{Y}}^+(x;\lambda)$ can be viewed as a solution to the ODE $\tilde{\mathcal{Y}}^{+\prime} = \tilde{A}(x;\lambda)\tilde{\mathcal{Y}}^+$, we can characterize $\tilde{\mathcal{Y}}^+(-R;\lambda)$ via a basis of solutions to this ODE constructed for $x \ll 0$. Precisely, under the assumptions of Proposition 3.6 we can construct a basis $\{\tilde{\mathcal{Y}}_j^-(x;\lambda)\}_{j=1}^n$ for the solutions of $\tilde{\mathcal{Y}}' = \tilde{A}(x;\lambda)\tilde{\mathcal{Y}}$, indexed so that

$$\lim_{R\to+\infty} e^{-\mu_-(\lambda)R}\tilde{\mathcal{Y}}_1^-(-R;\lambda) = \tilde{\mathcal{V}}^-(\lambda),$$

where $\tilde{\mathcal{V}}^-(\lambda)$ is an eigenvector of $\tilde{A}_-(\lambda)$ associated with the eigenvalue $-\mu_-(\lambda)$ (which is the most negative eigenvalue of $\tilde{A}_-(\lambda)$), and additionally

$$\lim_{R\to+\infty} e^{-\mu_-(\lambda)R}\tilde{\mathcal{Y}}_j^-(-R;\lambda) = 0, \quad j = 2,3,\dots,n.$$

If we now write

$$\tilde{\mathcal{Y}}^+(-R;\lambda) = \sum_{j=1}^n \tilde{c}_j(\lambda)\tilde{\mathcal{Y}}_j^-(-R;\lambda),$$

for some expansion coefficients $\{\tilde{c}_j(\lambda)\}_{j=1}^n$ then

$$\tilde{\mathcal{U}}^-(-R;\lambda) = \sum_{j=1}^n \tilde{c}_j(\lambda)e^{-\mu_-(\lambda)R}\tilde{\mathcal{Y}}_j^-(-R;\lambda),$$

and

$$\lim_{R\to+\infty} \tilde{\mathcal{U}}^-(-R;\lambda) = \tilde{c}_1(\lambda)\tilde{\mathcal{V}}^-(\lambda).$$

In order to better understand the nature of the expansion coefficient $\tilde{c}_1(\lambda)$, we recall that $D(\lambda)$ is independent of x, allowing us to write

$$D(\lambda) = \lim_{R\to+\infty} u^-(-R;\lambda) \wedge \tilde{\mathcal{U}}^-(-R;\lambda) = \tilde{c}_1(\lambda)v^-(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda).$$

We see that

$$\tilde{c}_1(\lambda) = \frac{D(\lambda)}{v^-(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda)},$$

and consequently

$$\lim_{R\to+\infty} u_\lambda^-(-R;\lambda)\wedge\tilde{\mathcal{U}}^-(-R;\lambda)=\frac{v^{-\prime}(\lambda)\wedge\tilde{\mathcal{V}}^-(\lambda)}{v^-(\lambda)\wedge\tilde{\mathcal{V}}^-(\lambda)}D(\lambda).$$

Combining this last relation with (5.2), in which we take the limit as $R\to-\infty$, we obtain the relation

$$\begin{aligned} u_\lambda^-(0;\lambda)\wedge\tilde{\mathcal{U}}^-(0;\lambda)&=\frac{v^{-\prime}(\lambda)\wedge\tilde{\mathcal{V}}^-(\lambda)}{v^-(\lambda)\wedge\tilde{\mathcal{V}}^-(\lambda)}D(\lambda)\\ &+\int_{-\infty}^0\Big((A_\lambda(x;\lambda)-\mu_-'(\lambda)I)u^-(x;\lambda)\Big)\wedge\tilde{\mathcal{U}}^-(x;\lambda)dx. \end{aligned}\tag{5.3}$$

By a similar calculation, if we start with

$$D(\lambda)=u^+(x;\lambda)\wedge\tilde{\mathcal{U}}^+(x;\lambda),$$

where

$$\begin{aligned} u^+(x;\lambda)&:=e^{-\mu_+(\lambda)x}\eta^-(x;\lambda)\\ \tilde{\mathcal{U}}^+(x;\lambda)&=e^{+\mu_+(\lambda)x}\tilde{\mathcal{Y}}^+(x;\lambda). \end{aligned}$$

we find the relation

$$\frac{d}{dx}(u^+(x;\lambda)\wedge\tilde{\mathcal{U}}_\lambda^+(x;\lambda))=u^+(x;\lambda)\wedge((\tilde{A}_\lambda(x;\lambda)+\mu_+'(\lambda)I)\tilde{\mathcal{U}}^+(x;\lambda)).$$

Upon integrating on $(0,R)$ and using Proposition 3.4, we obtain the relation

$$\begin{aligned} &u_\lambda^+(R;\lambda)\wedge\tilde{\mathcal{U}}^+(R;\lambda)-u_\lambda^+(0;\lambda)\wedge\tilde{\mathcal{U}}^+(0;\lambda)\\ &\quad=\int_0^R\Big((-A_\lambda(x;\lambda)+\mu_+'(\lambda)I)u^+(x;\lambda)\Big)\wedge\tilde{\mathcal{U}}^+(x;\lambda)dx. \end{aligned}\tag{5.4}$$

Proceeding similarly as before, we obtain the relation

$$\lim_{R\to+\infty} u_\lambda^+(R;\lambda)\wedge\tilde{\mathcal{U}}^+(R;\lambda)=\frac{v^+(\lambda)\wedge\tilde{\mathcal{V}}^{+\prime}(\lambda)}{v^+(\lambda)\wedge\tilde{\mathcal{V}}^+(\lambda)}D(\lambda),$$

and from (5.4) (by taking a limit as R tends to $+\infty$)

$$\begin{aligned} u_\lambda^+(0;\lambda)\wedge\tilde{\mathcal{U}}^+(0;\lambda)&=\frac{v^+(\lambda)\wedge\tilde{\mathcal{V}}^{+\prime}(\lambda)}{v^+(\lambda)\wedge\tilde{\mathcal{V}}^+(\lambda)}D(\lambda)\\ &+\int_0^{+\infty}\Big((A_\lambda(x;\lambda)-\mu_+'(\lambda)I)u^+(x;\lambda)\Big)\wedge\tilde{\mathcal{U}}^+(x;\lambda)dx. \end{aligned}\tag{5.5}$$

At this point, let's note the relation

$$\begin{aligned} u^-(0;\lambda)&=u^+(0;\lambda)=\eta^-(0;\lambda)\\ \tilde{\mathcal{U}}^-(0;\lambda)&=\tilde{\mathcal{U}}^+(0;\lambda)=\tilde{\mathcal{Y}}^+(0;\lambda). \end{aligned}$$

Starting with

$$D(\lambda) = u^-(0;\lambda) \wedge \tilde{\mathcal{U}}^-(0;\lambda),$$

these relations allow us to write

$$\begin{aligned} D'(\lambda) &= u^-_\lambda(0;\lambda) \wedge \tilde{\mathcal{U}}^-(0;\lambda) + u^-(0;\lambda) \wedge \tilde{\mathcal{U}}^-_\lambda(0;\lambda) \\ &= u^-_\lambda(0;\lambda) \wedge \tilde{\mathcal{U}}^-(0;\lambda) + u^+(0;\lambda) \wedge \tilde{\mathcal{U}}^+_\lambda(0;\lambda). \end{aligned}$$

Combining this final relation with (5.3) and (5.5), we obtain the assertion of Proposition 5.1. □

In the following proposition, we address a special case that will be important for our applications.

Proposition 5.2. *Let Assumptions* **(A)** *through* **(D)** *hold, with additionally* $A_{\lambda\lambda}(x;0) = 0$ *(trivially true if* $A(x;\lambda)$ *depends linearly on* λ*), and let the Evans function* $D(\lambda)$ *be specified as in (5.1). If* $D(0) = 0$*, then*

$$D'(0) = \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx.$$

Moreover, if additionally $D'(0) = 0$*, then*

$$D''(0) = \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta^-_\lambda(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx + \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+_\lambda(x;0)dx.$$

Proof. First, from Proposition 5.1, we have the relation

$$\begin{aligned} D'(0) = &\int_{-\infty}^{0} \Big((A_\lambda(x;0) - \mu'_-(0)I)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx \\ &+ \int_{0}^{+\infty} \Big((A_\lambda(x;0) - \mu'_+(0)I)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx. \end{aligned}$$

The wedge product $\eta^-(x;0) \wedge \tilde{\mathcal{Y}}^+(x;0)$ is just $D(0)$, and so is identically 0, allowing us to reduce this to

$$D'(0) = \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx,$$

as claimed.

Next, we assume $D(0) = 0$ and $D'(0) = 0$, and we compute $D''(0)$. We proceed by differentiating the expression for $D'(\lambda)$ in Proposition 5.1. First, since $D(0) = 0$ and $D'(0) = 0$, we see that the λ-derivative of

$$\Big(\frac{v^{-\prime}(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda)}{v^-(\lambda) \wedge \tilde{\mathcal{V}}^-(\lambda)} + \frac{v^+(\lambda) \wedge \tilde{\mathcal{V}}^{+\prime}(\lambda)}{v^+(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda)}\Big)D(\lambda),$$

evaluated at $\lambda = 0$ must be 0. The remaining terms in $D'(\lambda)$ can be expressed as

$$\begin{aligned} &\int_{-\infty}^{0} \Big(A_\lambda(x;\lambda)\eta^-(x;\lambda)\Big) \wedge \tilde{\mathcal{Y}}^+(x;\lambda) - \mu'_-(\lambda)D(\lambda)dx \\ &\quad + \int_{0}^{+\infty} \Big(A_\lambda(x;\lambda)\eta^-(x;\lambda)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0) - \mu'_+(\lambda)D(\lambda)dx. \end{aligned}$$

Upon differentiation in λ and evaluation at $\lambda = 0$ (and using the assumptions $A_{\lambda\lambda}(x;0) = 0$, $D(0) = 0$ and $D'(0) = 0$), we are left with

$$D''(0) = \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta_\lambda^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0)dx + \int_{-\infty}^{+\infty} \Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}_\lambda^+(x;0)dx,$$

as claimed. □

In order to take full advantage of the relations for $D'(0)$ and $D''(0)$ from Proposition 5.2, it's useful to be clear about the connection between $\tilde{\omega}_1(x;\lambda)$ and $D(\lambda)$. For this, we note from Propositions 3.1 and 3.6 and the specifications

$$D(\lambda) = \eta^-(x;\lambda) \wedge \tilde{\mathcal{Y}}^+(x;\lambda)$$

and

$$\tilde{\omega}_1^+(x;\lambda) = \eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda),$$

that given any closed interval $K \subset I$, and any $\epsilon > 0$, we can take $x > 0$ sufficiently large so that

$$|D(\lambda) - e^{-\mu_+(\lambda)x}\tilde{\omega}_1^+(x;\lambda)| < \epsilon \tag{5.6}$$

for all $\lambda \in K$. Since $D(\lambda)$ is analytic at $\lambda = 0$, its sign for λ sufficiently close to 0 is determined by the first non-zero value $D(0)$, $D'(0)$, $D''(0)$ etc. via the relation

$$D(\lambda) = D(0) + D'(0)\lambda + \frac{1}{2}D''(0)\lambda^2 + \dots.$$

In order to be clear about this process, let's take the specific case $D(0) = 0$ and $D'(0) > 0$, and let's suppose (as will be the case in our applications) that by taking x sufficiently large we can fix the sign of $\tilde{\psi}_2^+(x;0)$, say $\tilde{\psi}_2^+(x;0) > 0$. Under these conditions, there exists a value $\delta > 0$ sufficiently small so that $D(\lambda) < 0$ for all $\lambda \in (-\delta, 0)$, and in particular $D(-\delta/2) < 0$. It follows that we can take x sufficiently large so that $\tilde{\psi}_1(x;-\delta/2) < 0$ (because we must have $\tilde{\omega}_1^+(x;\lambda) < 0$ by (5.6)). Since $\tilde{\psi}_2(x;\lambda)$ depends smoothly on λ, we can additionally choose δ small enough so that $\tilde{\psi}_2(x;-\delta/2) > 0$, which places $p^+(x;-\delta/2)$ in the second quadrant. We can conclude that

$$\mathrm{Ind}(\mathcal{g}(x;\cdot), \mathcal{h}(x;\cdot); [-\delta/2, 0]) = +1.$$

Returning to (4.15) from the proof of Theorem 1.1, we can write (with $\lambda_2 = 0$)

$$\begin{aligned} &\mathrm{Ind}(\mathcal{g}(c_2;\cdot), \mathcal{h}(c_2;\cdot); [\lambda_1, -\delta/2]) + \mathrm{Ind}(\mathcal{g}(x;\cdot), \mathcal{h}(x;\cdot); [-\delta/2, 0]) \\ &\quad = \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_2), \mathcal{h}^+(\lambda_2); [-\infty, +\infty]) - \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_1), \mathcal{h}^+(\lambda_1); [-\infty, +\infty]) \\ &\quad + \mathrm{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}^+(\cdot); [\lambda_1, \lambda_2]) - \mathfrak{m}, \end{aligned} \tag{5.7}$$

from which we now obtain a lower bound on $\mathcal{N}_\#([\lambda_1, \lambda_2))$ rather than on $\mathcal{N}_\#([\lambda_1, \lambda_2])$. In particular, in this case, we obtain the relation

$$\begin{aligned} \mathcal{N}_\#([\lambda_1, \lambda_2)) + 1 \geq \Big| &\mathrm{Ind}(\mathcal{g}(\cdot;\lambda_2), \mathcal{h}^+(\lambda_2); [-\infty, +\infty]) - \mathrm{Ind}(\mathcal{g}(\cdot;\lambda_1), \mathcal{h}^+(\lambda_1); [-\infty, +\infty]) \\ &+ \mathrm{Ind}(\mathcal{g}^-(\cdot), \mathcal{h}^+(\cdot); [\lambda_1, \lambda_2]) - \mathfrak{m}\Big|. \end{aligned} \tag{5.8}$$

Additional information about the spectrum of (1.1) can be found by also computing the limit of $D(\lambda)$ as λ tends toward $-\infty$. For this, we will make use of Proposition 4.1, so we make the same assumptions as stated there.

Proposition 5.3. *Let the Assumptions of Proposition 4.1 hold, and let $D(\lambda)$ be specified as in (5.1). Then*

$$\lim_{\lambda \to -\infty} D(\lambda) = 1.$$

Proof. First, we recall that $v^-(\lambda)$ denotes an eigenvector of $A_-(\lambda)$ associated with the eigenvalue $\mu_-(\lambda)$, and $\tilde{\mathcal{V}}^+(\lambda)$ denotes an eigenvector of $\tilde{A}_+(\lambda)$ associated with the eigenvalue $-\mu_+(\lambda)$. The normalization in place via Assumption **(C)** and the relation between $\tilde{\mathcal{V}}^+(\lambda)$ and $w^+(\lambda)$ (described in Remark 3.1) is

$$v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) = 1. \tag{5.9}$$

According to Proposition 4.1, we have the relations

$$V_-(\lambda)^{-1}\eta^-(0;\lambda) = e_1 + \mathbf{o}(1), \quad \lambda \to -\infty,$$

and

$$z^+(0;\lambda)V_+(\lambda)e_1 = 1 + \mathbf{o}(1), \quad \lambda \to -\infty,$$

where under the Assumptions of Proposition 4.1 we have $V_-(\lambda) = V_+(\lambda)$. Using Remark 5.1, we can now compute

$$\begin{aligned} D(\lambda) &= \eta^-(0;\lambda) \wedge \tilde{\mathcal{Y}}^+(0;\lambda) = z^+(0;\lambda) \cdot \eta^-(0;\lambda) = z^+(0;\lambda)V_+(\lambda)V_-(\lambda)^{-1}\eta^-(0;\lambda) \\ &= z^+(0;\lambda)V_+(\lambda)(e_1 + \mathbf{o}(1)) = z^+(0;\lambda)V_+(\lambda)e_1 + z^+(0;\lambda)V_+(\lambda)\mathbf{o}(1) = 1 + \mathbf{o}(1), \end{aligned}$$

where we have used the boundedness of $z^+(0;\lambda)V_+(\lambda)$ (from the final assertion of Proposition 4.1). □

6 Applications

In this section, we illustrate the implementation of Theorem 1.1 with two applications.

6.1 The Generalized KdV Equation

As our first application, we consider traveling waves $\bar{u}(x - st)$ occurring as solutions to the generalized Korteweg-De Vries equation

$$u_t + f(u)_x = -u_{xxx}, \quad f(u) = \frac{u^{p+1}}{p+1}, \quad p \geq 1. \tag{6.1}$$

For these calculations, it will be convenient to shift to a moving coordinate system for which $\bar{u}(x)$ is a stationary solution to

$$u_t + (f(u) - su)_x = -u_{xxx}. \tag{6.2}$$

It follows that $\bar{u}(x)$ is a solution to the ODE

$$\bar{u}^p \bar{u}' - s\bar{u}' = -\bar{u}''', \tag{6.3}$$

for which we can readily check that for $s > 0$

$$\bar{u}(x) = \alpha \operatorname{sech}^{2/p}(\gamma x), \quad \alpha = (\frac{1}{2} s(p+1)(p+2))^{1/p}, \quad \gamma = \frac{p\sqrt{s}}{2} \tag{6.4}$$

is an exact solution. (These solutions are taken from [33].) Our goal is to use the preceding analysis to analyze the spectral stability/instability of the wave (6.4) as a solution to (6.2).

If we linearize (6.2) about the wave $\bar{u}(x)$, we obtain the associated eigenvalue problem

$$L\phi = -\phi''' - (a(x)\phi)' = \lambda\phi, \quad a(x) = \bar{u}(x)^p - s. \tag{6.5}$$

As a starting point for the analysis, we recall from [21, 30] that the essential spectrum of L can be determined from the asymptotic equation

$$-\phi''' - a_\pm \phi' = \lambda\phi, \tag{6.6}$$

where

$$a_\pm = \lim_{x \to \pm\infty} a(x) = -s = -4\frac{\gamma^2}{p^2}.$$

(For this application, $a_- = a_+$, so the subscript $\pm$ is intended to signify that both sides are analyzed at once.) Precisely, the essential spectrum of L comprises the values λ for which $\phi(x;k) = e^{ikx}$ is a solution to (6.6) for some $k \in \mathbb{R}$, namely

$$\sigma_{\text{ess}}(L) = \{\lambda \in \mathbb{C} : \lambda = ik^3 - ia_\pm k, \ k \in \mathbb{R}\},$$

which is the imaginary axis.

In the usual way, we express (6.5) as a first-order system by introducing a vector function $y = (y_1 \ y_2 \ y_3)^T$ with $y_1 = \phi$, $y_2 = \phi'$, and $y_3 = \phi''$, yielding (1.1) with

$$A(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda - a'(x) & -a(x) & 0 \end{pmatrix}, \tag{6.7}$$

and correspondingly

$$A_\pm(\lambda) := \lim_{x \to \pm\infty} A(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & -a_\pm & 0 \end{pmatrix}.$$

For each fixed $\lambda \in \mathbb{C}$, the eigenvalues of $A_\pm(\lambda)$ are roots μ of the cubic polynomial

$$h(\mu;\lambda) = \mu^3 + a_\pm \mu + \lambda. \tag{6.8}$$

For $\lambda = 0$, the roots are easily found to be

$$\mu_1(0) = -2\frac{\gamma}{p}, \quad \mu_2(0) = 0, \quad \mu_3(0) = 2\frac{\gamma}{p},$$

where we've recalled that $a_\pm = -4\frac{\gamma^2}{p^2}$. As λ decreases from 0, the graph of $h(\mu;\lambda)$ will keep the same form but decrease so that the roots $\mu_1(\lambda)$ and $\mu_2(\lambda)$ will initially move toward one another while the root $\mu_3(\lambda)$ increases. As λ continues to decrease, the roots $\mu_1(\lambda)$ and $\mu_2(\lambda)$ will coalesce at some value $\lambda = \lambda_c$ into a complex conjugate pair. The precise value of λ_c isn't critical to our analysis, but one easily finds it to be

$$\lambda_c = -2(s/3)^{3/2}. \tag{6.9}$$

For λ above this coalescence value (i.e., for $\lambda \in (\lambda_c, 0]$) we can associate an eigenvector $v_j(\lambda)$ with each $\mu_j(\lambda)$, $j \in \{1,2,3\}$,

$$v_j(\lambda) = (1 \ \mu_j(\lambda) \ \mu_j(\lambda)^2)^T, \quad j = 1,2,3. \tag{6.10}$$

Correspondingly, it's straightforward to identify three linearly independent solutions of (1.1) with $A(x;\lambda)$ as in (6.7),

$$y_j^\pm(x;\lambda) = e^{\mu_j(\lambda)x}(v_j(\lambda) + E_j^\pm(x;\lambda)),$$

where $E_j^\pm(x;\lambda) = \mathbf{O}(e^{-\alpha|x|})$ uniformly in λ on compact subsets of I for some fixed constant $\alpha > 0$. (See, e.g., [35].) The eigenvector $v_3(\lambda)$ has the same form for all $\lambda \le 0$, and up to a choice of scaling is $v^-(\lambda)$ from Proposition 3.1. Likewise, up to a choice of scaling, $\eta^-(x;\lambda)$ is $y_3^-(x;\lambda)$. On the other hand, once λ decreases to the coalescence threshold, it becomes problematic to separate the solutions associated with $\mu_1(\lambda)$ and $\mu_2(\lambda)$, and one of the advantages of the wedge-product formulation is that no such separation is necessary.

Turning to consideration of $\tilde{A}(x;\lambda)$, we see from (1.17) that for this application

$$\tilde{A}(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ -a(x) & 0 & 1 \\ \lambda + a'(x) & 0 & 0 \end{pmatrix},$$

and correspondingly

$$\tilde{A}_\pm(\lambda) = \lim_{x\to\pm\infty} \tilde{A}(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ -a_\pm & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}. \tag{6.11}$$

According to Proposition 3.3, $\tilde{A}_\pm(\lambda)$ has eigenvalues $\tilde{\mu}_j(\lambda) = -\mu_{3-j+1}(\lambda)$, $j = 1,2,3$, and we readily find that the associated eigenvectors have the form $\tilde{\mathcal{V}}_j = (1 \ \tilde{\mu}_j \ \lambda/\tilde{\mu}_j)^T$. In particular, $\tilde{\mu}_1(\lambda) = -\mu_3(\lambda)$, and up to a choice of scaling $\tilde{\mathcal{V}}_1(\lambda)$ is $\tilde{\mathcal{V}}^+(\lambda)$ from Proposition 3.6.

Since this application is in the setting of Proposition 4.2, we can conclude immediately that we can take λ_1 sufficiently negative so that the hyperplane index associated with the left shelf gives no contribution. In addition, we see from (5.9) in the proof of Proposition 5.3 that $v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) = 1$ for all $\lambda \le 0$, ensuring that there are no crossing points along the bottom shelf (as discussed in Section 4.1.1). We note that in order to obtain this normalization, we can use $v^-(\lambda) = \kappa(\lambda)v_3(\lambda)$ and $\tilde{\mathcal{V}}^+(\lambda) = \kappa(\lambda)\tilde{\mathcal{V}}_1(\lambda)$, where

$$\kappa(\lambda) = 1/\sqrt{-\lambda/\mu_3(\lambda) + 2\mu_3(\lambda)^2}. \tag{6.12}$$

Next, we turn to the evaluation of $D(0)$, $D'(0)$, and $D''(0)$, which we adapt from [33] with only minor changes required for the current wedge-based formulation. To start, we observe that since $y_3^-(x;0)$ is the only solution from our basis $\{y_j^-(x;0)\}_{j=1}^3$ that decays as x tends to $-\infty$ it must be the case that there exists some constant k_- so that (keeping in mind that $\eta^-(x;\lambda)$ is just a rescaling of $y_3^-(x;\lambda)$)

$$\eta^-(x;0) = k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix}. \tag{6.13}$$

According to Proposition 3.1, we can write

$$\lim_{x\to-\infty} e^{-\mu_3(0)x} k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix} = v^-(0).$$

Recalling that the first component of $v^-(\lambda)$ has been chosen by convention to be positive and that $\bar{u}'(x) > 0$ for all $x < 0$, we conclude that $k_- > 0$.

In addition, we need to understand the nature of $\tilde{\mathcal{Y}}^+(x;0)$, which solves the ODE

$$\tilde{\mathcal{Y}}^{+\prime} = \tilde{A}(x;0)\tilde{\mathcal{Y}}^+.$$

In particular, we see that if we write $\tilde{\mathcal{Y}}^+ = (\tilde{\mathcal{Y}}_1^+ \; \tilde{\mathcal{Y}}_2^+ \; \tilde{\mathcal{Y}}_3^+)^T$, then $\tilde{\mathcal{Y}}_1^{+\prime} = \tilde{\mathcal{Y}}_2^+$, so that

$$\tilde{\mathcal{Y}}_1^{+\prime\prime} = \tilde{\mathcal{Y}}_2^{+\prime} = -a(x)\tilde{\mathcal{Y}}_1^+ + \tilde{\mathcal{Y}}_3^+. \tag{6.14}$$

Differentiating once more, we see that

$$\tilde{\mathcal{Y}}_1^{+\prime\prime\prime} = -(a(x)\tilde{\mathcal{Y}}_1^+)' + \tilde{\mathcal{Y}}_3^{+\prime} = -(a(x)\tilde{\mathcal{Y}}_1^+)' + a'(x)\tilde{\mathcal{Y}}_1^+ = -a(x)\tilde{\mathcal{Y}}_1^{+\prime}. \tag{6.15}$$

Comparing this equation with (6.3), and recalling the definition of $a(x)$ in (6.5), we see that $\tilde{\mathcal{Y}}_1^+(x;0)$ solves the same equation as $\bar{u}(x)$, which is the only solution of this equation that lies left in $\mathbb{R}$. It follows that there exists a constant k_+ so that

$$\tilde{\mathcal{Y}}^+(x;0) = \begin{pmatrix} \tilde{\mathcal{Y}}_1^+(x;0) \\ \tilde{\mathcal{Y}}_2^+(x;0) \\ \tilde{\mathcal{Y}}_3^+(x;0) \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{Y}}_1^+(x;0) \\ \tilde{\mathcal{Y}}_1^{+\prime}(x;0) \\ \tilde{\mathcal{Y}}_1^{+\prime\prime}(x;0) + a(x)\tilde{\mathcal{Y}}_1^+(x;0) \end{pmatrix} = k_+ \begin{pmatrix} \bar{u}(x) \\ \bar{u}'(x) \\ \bar{u}''(x) + a(x)\bar{u}(x) \end{pmatrix}. \tag{6.16}$$

Recalling the asymptotic relation

$$\lim_{x\to+\infty} e^{\mu_+(0)x}\tilde{\mathcal{Y}}^+(x;0) = \tilde{\mathcal{V}}^+(0),$$

and noting that $\bar{u}(x) > 0$ for all $x > 0$, we see that $k_+ > 0$.

We can now directly compute

$$\begin{aligned} D(0) = \eta^-(x;0) \wedge \tilde{\mathcal{Y}}^+(x;0) &= k_- k_+ \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} \bar{u}(x) \\ \bar{u}'(x) \\ \bar{u}''(x) + a(x)\bar{u}(x) \end{pmatrix} \\ &= k_- k_+ \bar{u}(x)(\bar{u}'''(x) + a(x)\bar{u}'(x)) = 0. \end{aligned}$$

Using

$$A_\lambda(x;0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

along with the relations above for $\eta^-(x;0)$ and $\tilde{\mathcal{Y}}^+(x;0)$, we find that

$$\Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0) = -k_-k_+\bar{u}'(x)\bar{u}(x),$$

from which we see that (using the expression for $D'(0)$ in Proposition 5.2)

$$D'(0) = -k_-k_+ \int_{-\infty}^{+\infty} \frac{d}{dx}\frac{\bar{u}(x)^2}{2} dx = 0.$$

In order to evaluate the integrals in the expression for $D''(0)$ in Proposition 5.2, we need to understand the functions $\eta_\lambda^-(x;0)$ and $\tilde{\mathcal{Y}}_\lambda^+(x;0)$. Starting with $\eta_\lambda^-(x;0)$, we recall that the first component of $\eta^-(x;\lambda)$, $\phi(x;\lambda) := \eta_1^-(x;\lambda)$, solves the eigenvalue problem

$$-\phi''' - (a(x)\phi)' = \lambda\phi.$$

Upon differentiation in λ, we see that ϕ_λ solves the equation

$$-\phi_\lambda''' - (a(x)\phi_\lambda)' = \phi + \lambda\phi_\lambda.$$

From (6.13) we can write $\phi(x;0) = k_-\bar{u}'(x)$, so that with $\lambda = 0$ we obtain the equation

$$-\phi_\lambda'''(x;0) - (a(x)\phi_\lambda(x;0))' = k_-\bar{u}'(x). \tag{6.17}$$

For comparison, we note that by differentiating (6.3) in s we obtain the relation

$$p\bar{u}^{p-1}\bar{u}_s + \bar{u}^p\bar{u}_s' - \bar{u}' - s\bar{u}_s' = -\bar{u}_s''', \tag{6.18}$$

which we can re-write as

$$-\bar{u}_s''' - (a(x)\bar{u}_s)' = -\bar{u}'. \tag{6.19}$$

We see that the function $-k_-\bar{u}_s(x;s)$ solves the inhomogeneous ODE (6.17). We've already observed that the only left-decaying solution to the associated homogeneous equation is $\bar{u}'(x)$, so we must have

$$\phi_\lambda(x;0) = \beta_-\bar{u}'(x;s) - k_-\bar{u}_s(x;s)$$

for some constant $\beta_- \in \mathbb{R}$.

Next, if we repeat the calculations leading to (6.15) with $\lambda \neq 0$, we obtain the relation

$$\tilde{\mathcal{Y}}_1^{+\,'''} + a(x)\tilde{\mathcal{Y}}_1^{+\,'} = \lambda\tilde{\mathcal{Y}}_1^+.$$

Upon differentiation in λ, we find that $\varphi(x;\lambda) := \partial_\lambda\tilde{\mathcal{Y}}_1^+(x;\lambda)$ satisfies the equation

$$\varphi''' + a(x)\varphi' = \lambda\varphi + \tilde{\mathcal{Y}}_1^+.$$

Evaluating now at $\lambda = 0$ and recalling (6.16), we see that

$$\varphi'''(x;0) + a(x)\varphi'(x;0) = k_+\bar{u}(x;s). \tag{6.20}$$

For comparison, we integrate (6.19) (and change signs) to see that

$$\bar{u}_s'' + a(x)\bar{u}_s = \bar{u}, \tag{6.21}$$

for which the constant of integration is seen to be 0 since $\bar{u}(x;s)$ tends to 0 along with its derivatives as x tends to $\pm\infty$. If we now introduce an integrated variable

$$\mathcal{U}(x;s) := \int_{-\infty}^{x} \bar{u}_s(\xi;s)d\xi,$$

we can express (6.21) as

$$\mathcal{U}''' + a(x)\mathcal{U}' = \bar{u}.$$

In this way, we see that a particular solution of (6.20) is $k_+\mathcal{U}(x;s)$, and since $\bar{u}(x;s)$ is the only solution (up to constant multiplication) of the associated homogeneous equation, we can write

$$(\partial_\lambda \tilde{\mathcal{Y}}_1^+)(x;0) = \varphi(x;0) = \beta_+\bar{u}(x;s) + k_+\mathcal{U}(x;s)$$

for some constant β_+.

We are now in a position to readily evaluate the expression for $D''(0)$ in Proposition 5.2. For the first summand, we compute

$$\begin{aligned}
-k_+ \int_{-\infty}^{+\infty} &\begin{pmatrix} 0 \\ 0 \\ \beta_-\bar{u}'(x;s) - k_-\bar{u}_s(x;s) \end{pmatrix} \wedge \begin{pmatrix} \bar{u}(x;s) \\ \bar{u}'(x;s) \\ \bar{u}''(x;s) + a(x)\bar{u}(x;s) \end{pmatrix} dx \\
&= -k_+ \int_{-\infty}^{+\infty} (\beta_-\bar{u}'(x;s) - k_-\bar{u}_s(x;s))\bar{u}(x;s)dx = k_-k_+ \int_{-\infty}^{+\infty} \bar{u}(x;s)\bar{u}_s(x;s)dx,
\end{aligned}$$

where we've observed that $\bar{u}(x;s)\bar{u}'(x;s)$ integrates to 0. Turning to the second summand in the expression for $D''(0)$ in Proposition 5.2, we compute

$$\begin{aligned}
-k_- \int_{-\infty}^{+\infty} &\begin{pmatrix} 0 \\ 0 \\ \bar{u}'(x;s) \end{pmatrix} \wedge \begin{pmatrix} \beta_+\bar{u}(x;s) + k_+\mathcal{U}(x;s) \\ * \\ * \end{pmatrix} dx \\
&= -k_- \int_{-\infty}^{+\infty} (\beta_+\bar{u}(x;s) + k_+\mathcal{U}(x;s))\bar{u}'(x;s)dx = -k_-k_+ \int_{-\infty}^{+\infty} \mathcal{U}(x;s)\bar{u}'(x;s)dx,
\end{aligned}$$

where the asterisks indicate terms that don't have a role in the calculation. If we integrate this last integral by parts and observe that there is no contribution from the boundary, we see that it becomes precisely the above expression

$$+k_-k_+ \int_{-\infty}^{+\infty} \bar{u}(x)\bar{u}_s(x)dx.$$

In total, we can write

$$D''(0) = k_- k_+ \frac{d}{ds} \int_{-\infty}^{+\infty} \bar{u}(x;s)^2 dx.$$

Using the specification in (6.4), we see that

$$\int_{-\infty}^{+\infty} \bar{u}(x;s)^2 dx = \alpha(s)^2 \int_{-\infty}^{+\infty} \operatorname{sech}^{4/p}(\gamma(s)x) dx = c_p \frac{\alpha(s)^2}{\gamma(s)},$$

where $c_p := \int_{-\infty}^{+\infty} \operatorname{sech}^{4/p} x dx$, and in obtaining this expression we've used a change of variables $y = \gamma(s)x$. We've seen above that $k_- > 0$ and $k_+ > 0$, and it's clear that $c_p > 0$, so

$$\operatorname{sgn} D''(0) = \operatorname{sgn} \frac{d}{ds} \frac{\alpha(s)^2}{\gamma(s)}.$$

In order to determine the sign on the right-hand side, it's convenient to write

$$\ln \frac{\alpha(s)^2}{\gamma(s)} = \frac{2}{p} \ln(\frac{1}{2} s(p+1)(p+2)) - \ln(\frac{1}{2} p \sqrt{s}) = \frac{2}{p} \ln s - \ln \sqrt{s} + C,$$

where C simply notes additional terms constant in s. Differentiating in s, we now see that

$$(\frac{\alpha(s)^2}{\gamma(s)})^{-1} \frac{d}{ds} \frac{\alpha(s)^2}{\gamma(s)} = \frac{4-p}{2ps}.$$

We conclude that

$$\operatorname{sgn} D''(0) = \begin{cases} +1 & 1 \leq p < 4 \\ -1 & 4 < p. \end{cases} \tag{6.22}$$

We are now in a position to compute the hyperplane index along the right shelf. Our starting point for this calculation is the map

$$\begin{aligned} \tilde{\omega}_1^+(x;0) = \eta^-(x;0) \wedge \tilde{\mathcal{V}}^+(0) &= k_- \kappa(0) \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -\mu_3(0) \\ 0 \end{pmatrix} \\ &= \frac{k_- p}{2\sqrt{2}\gamma} \Big\{ 2\frac{\gamma}{p} \bar{u}''(x) + \bar{u}'''(x) \Big\}, \end{aligned}$$

where we've used the relations $\mu_3(0) = 2\gamma/p$ and $\kappa(0) = p/(2\sqrt{2}\gamma)$.

We see from this that in order to identify crossing points along the right shelf, we need to look for roots of the relation $\frac{2\gamma}{p} \bar{u}''(x) + \bar{u}'''(x)$. Using (6.4), we can compute

$$\bar{u}'(x) = \frac{2\alpha\gamma}{p} \operatorname{sech}^{\frac{2}{p}-1}(\gamma x)(-\operatorname{sech}(\gamma x) \tanh(\gamma x)) = -\frac{2\gamma}{p} \bar{u}(x) \tanh(\gamma x) \tag{6.23}$$

In addition, by integrating (6.3), we obtain the relation

$$\bar{u}'' = s\bar{u} - \frac{\bar{u}^{p+1}}{p+1},$$

and from (6.3) itself we can write

$$\bar{u}''' = (s - \bar{u}^p)\bar{u}' = -\frac{2\gamma}{p}\bar{u}(s - \bar{u}^p)\tanh(\gamma x).$$

Upon combining these relations, we see that

$$\frac{2\gamma}{p}\bar{u}''(x) + \bar{u}'''(x) = \frac{2\gamma}{p}\bar{u}(x)\Big(s - s\tanh(\gamma x) + \bar{u}^p\tanh(\gamma x) - \frac{\bar{u}^p}{p+1}\Big). \tag{6.24}$$

Using now the identity

$$\bar{u}(x)^p = \alpha^p \operatorname{sech}^2(\gamma x) = \alpha^p(1 - \tanh^2(\gamma x)),$$

we can express the quantity in large parentheses on the right-hand side of (6.24) in terms of the variable $z = \tanh(\gamma x) \in (-1, 1)$. Precisely, we can write this expression as

$$\Psi(z) = (1 - z)\Big\{s + \alpha^p z(1 + z) - \frac{\alpha^p(1 + z)}{p + 1}\Big\}.$$

As expected, since $\lambda = 0$ is an eigenvalue, we have a crossing point in the limit as $x \to +\infty$ (corresponding with $z \to 1$), and otherwise there is a one-to-one relationship between zeros of the polynomial $\Psi(z)$ and crossing points for $\tilde{\omega}_1^+(x; 0)$ for $x \in \mathbb{R}$. The quantity in curved brackets in (6.1) is a quadratic in z, which can be expressed as

$$\Psi_2(z) = \alpha^p z^2 + \frac{1}{2}ps(p + 2)z - \frac{ps}{2}. \tag{6.25}$$

Writing out α^p, we see that we're looking for roots of the quadratic expression

$$\frac{1}{2}s(p + 1)(p + 2)z^2 + \frac{1}{2}ps(p + 2)z - \frac{ps}{2} = 0 \implies (p + 1)(p + 2)z^2 + p(p + 2)z - p = 0.$$

Checking the right-hand side at the values $z = -1, 0, 1$, we respectively obtain the values 2, $-p$ and $2(p + 1)^2$, from which we see that $\Psi_2(z)$ has two real roots z_1 and z_2 on the interval $(-1, 1)$, and moreover that $z_1 < 0 < z_2$. Correspondingly, $\tilde{\omega}_1^+(x; 0)$ has two real roots x_1 and x_2 on $\mathbb{R}$, with $x_1 < 0 < x_2$.

In order to understand possible contributions to the hyperplane index from the limits $x \to \pm\infty$, we work with the scaled map

$$\tilde{\psi}_1^+(x; 0) = \frac{\tilde{\omega}_1^+(x; 0)}{|\eta^-(x; 0)||\tilde{\mathcal{V}}^+(0)|},$$

for which we've seen in (1.31) has the well-defined asymptotic limit

$$\tilde{\psi}_1^{+,-}(0) = \lim_{x \to -\infty} \tilde{\psi}_1^+(x; 0) = \frac{v^-(0) \wedge \tilde{\mathcal{V}}^+(0)}{|v^-(0)||\tilde{\mathcal{V}}^+(0)|} =: c_0,$$

where according to our scaling convention c_0 is the positive constant

$$c_0 = \frac{1}{|v^-(0)||\tilde{\mathcal{V}}^+(0)|}.$$

In this case, the wave $\bar{u}(x)$ is symmetric about $x = 0$, allowing us to see that (due to (6.13))

$$\lim_{x\to+\infty} \frac{\eta^-(x;0)}{|\eta^-(x;0)|} = -\frac{v^+(0)}{|v^+(0)|},$$

where

$$v^+(0) = \kappa(0) \begin{pmatrix} 1 \\ -\mu_3(0) \\ \mu_3(0)^2 \end{pmatrix} = \frac{p}{2\sqrt{2}\gamma} \begin{pmatrix} 1 \\ -2\gamma/p \\ 4\gamma^2/p^2 \end{pmatrix}.$$

We see that

$$\begin{aligned} \tilde{\psi}_1^{+,+}(0) &= \lim_{x\to+\infty} \tilde{\psi}_1^+(x;0) = -\frac{v^+(0) \wedge \tilde{\mathcal{V}}^+(0)}{|v^+(0)||\tilde{\mathcal{V}}^+(0)|} \\ &= -\frac{\kappa(0)^2}{|v^+(0)||\tilde{\mathcal{V}}^+(0)|} \begin{pmatrix} 1 \\ -2\gamma/p \\ 4\gamma^2/p^2 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -2\gamma/p \\ 0 \end{pmatrix} = 0. \end{aligned}$$

The graph of the function $\tilde{\psi}_1^+(x;0)$ is depicted in Figure 6.1.

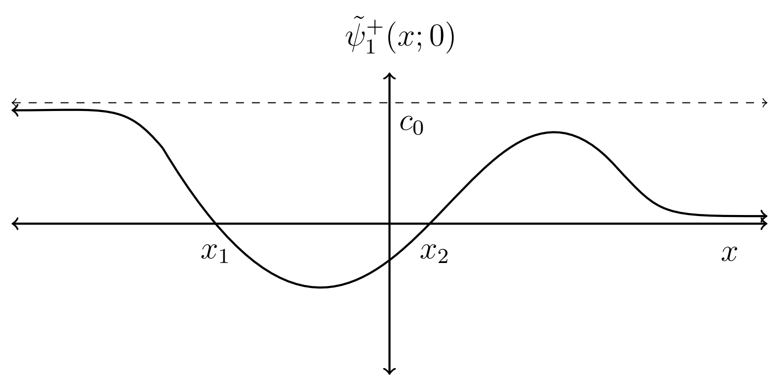

Figure 6.1: The function $\tilde{\psi}_1^+(x;0)$ in the case of the generalized KdV equation.

In order to compute the hyperplane index along the right shelf, we also need to specify $\tilde{\omega}_2^+(x;0)$, and for this we have considerable flexibility, particularly in our choice of the constant matrix M. For reasons that will become clear just below, we will find it convenient to take

$$M = \begin{pmatrix} 0 & 1 & 1/\sqrt{s} \\ 1 & 0 & 0 \\ 1 & 0 & -1/s \end{pmatrix}. \tag{6.26}$$

We now set

$$\tilde{\omega}_2^+(x;0) = \eta_-(x;0) \wedge \tilde{\mathcal{V}}_M(0),$$

where $\tilde{\mathcal{V}}_M(0)$ is an eigenvector of the asymptotic matrix $\tilde{\mathcal{A}}^+(0)$ (from (1.22)) associated with the eigenvalue $-\mu_3(0)$, namely

$$\tilde{\mathcal{V}}_M^+(0) = \begin{pmatrix} -s/\mu_3(0) + \sqrt{s} \\ -\mu_3(0) + \sqrt{s} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that

$$\tilde{\omega}_2^+(x;0) = k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = k_- \bar{u}'(x) = -\frac{2k_-\gamma}{p} \bar{u}(x) \tanh(\gamma x).$$

As with $\tilde{\omega}_1^+(x;0)$, in order to accommodate the limits $x \to \pm\infty$, we will work with the scaled map

$$\tilde{\psi}_2^+(x;0) = \frac{\tilde{\omega}_2^+(x;0)}{|\eta^-(x;0)||\tilde{\mathcal{V}}^+(0)|}, \tag{6.27}$$

for which we have the limits

$$\begin{aligned} \tilde{\psi}_2^{+,-}(0) &= \lim_{x\to-\infty} \tilde{\psi}_2^+(x;0) = \frac{v^-(0) \wedge \tilde{\mathcal{V}}_M^+(0)}{|v^-(0)||\tilde{\mathcal{V}}^+(0)|} \\ &= \frac{p}{2\sqrt{2}\gamma} \begin{pmatrix} 1 \\ 2\gamma/p \\ 4\gamma^2/p^2 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{p}{2\sqrt{2}\gamma} > 0, \end{aligned}$$

and likewise

$$\begin{aligned} \tilde{\psi}_2^{+,+}(0) &= \lim_{x\to+\infty} \tilde{\psi}_2^+(x;0) = -\frac{v^+(0) \wedge \tilde{\mathcal{V}}_M^+(0)}{|v^+(0)||\tilde{\mathcal{V}}^+(0)|} \\ &= -\frac{p}{2\sqrt{2}\gamma} \begin{pmatrix} 1 \\ -2\gamma/p \\ 4\gamma^2/p^2 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{p}{2\sqrt{2}\gamma} < 0. \end{aligned}$$

In total, we see that $\tilde{\psi}_2^+(x;0)$ is as depicted in Figure 6.2.

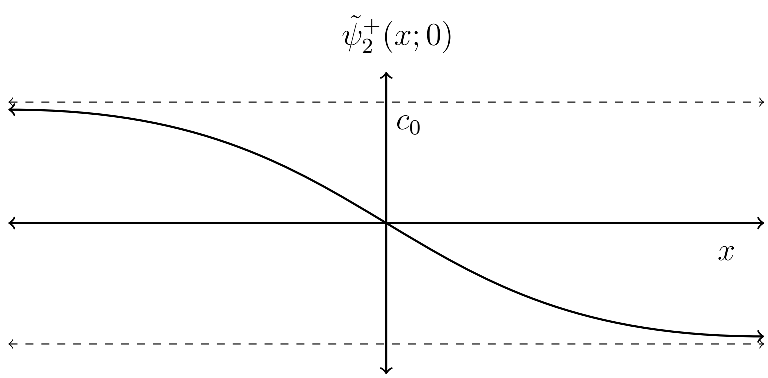

Figure 6.2: The function $\tilde{\psi}_2^+(x;0)$ in the case of the generalized KdV equation.

Before computing the hyperplane index on the right shelf, we verify that we have invariance. First, the only roots of $\tilde{\psi}_1^+(x;0)$ are the values designated as x_1 and x_2 above, and these satisfy $x_1 < 0 < x_2$. By contrast, the only real root of $\tilde{\psi}_2^+(x;0)$ is $x = 0$, so there is no value $x \in \mathbb{R}$ for which both $\tilde{\psi}_1^+(x;0)$ and $\tilde{\psi}_2^+(x;0)$ vanishe. In addition, since $\tilde{\psi}_2^{+,\pm}(0) \neq 0$, invariance is not lost at the asymptotic endstates.

In order to compute the hyperplane index along the right shelf, we need only identify all conjugate points on the interval $[-\infty, +\infty]$ (allowing $\pm\infty$ to serve asymptotically as crossing points, though $-\infty$ has already been ruled out by our analysis of the bottom shelf) and assign a direction to each. For any finite crossing point x_*, this direction is determined by the ratio

$$\frac{\tilde{\psi}_1^{+\prime}(x_*; 0)}{\tilde{\psi}_2^{+}(x_*; 0)},$$

while for the crossing point at $+\infty$ we will work directly with the rotation described by $p^+(x; 0)$ as x tends toward $+\infty$. For the former, we see immediately from Figure 6.1 that $\tilde{\psi}_1^{+\prime}(x_1; 0) < 0$ and $\tilde{\psi}_1^{+\prime}(x_2; 0) > 0$, while from Figure 6.2 we see that $\tilde{\psi}_2^{+}(x_1; 0) > 0$ and $\tilde{\psi}_2^{+}(x_2; 0) < 0$. We can conclude that a direction of -1 should be assigned to each of these crossings.

For the asymptotic crossing as $x \to +\infty$, we observe from Figure 6.1 that for all $x > 0$ sufficiently large we have $\tilde{\psi}_1^{+}(x; 0) > 0$, while from Figure 6.2 we see that for all $x > 0$, $\tilde{\psi}_2^{+}(x; 0) < 0$. This pairing places the point $p^+(x; 0)$ in the second quadrant, from which we see that the approach of $p^+(x; 0)$ to the point $(-1, 0)$ will be in the positive (i.e., in the counterclockwise) direction, giving a contribution of $+1$ to the hyperplane index. Combining this with the negative crossings at x_1 and x_2, we conclude that in this case, the hyperplane index along the right shelf satisfies the relation

$$\mathrm{Ind}(\mathscr{g}(\cdot; 0), \mathscr{h}^+(0); [-\infty, +\infty]) = -1.$$

According to Theorem 1.1, we can conclude that *if* we have invariance along the top shelf then the count $\mathcal{N}_\#((-\infty, 0])$ of non-positive eigenvalues of (6.5) satisfies the relation

$$\mathcal{N}_\#((-\infty, 0]) \geq \Big| \mathrm{Ind}(\mathscr{g}(\cdot; 0), \mathscr{h}^+(0); [-\infty, +\infty]) - \mathfrak{m} \Big| = |-1 - \mathfrak{m}|. \tag{6.28}$$

In order to count the number of strictly negative eigenvalues, we would like to replace $\mathcal{N}_\#((-\infty, 0])$ with $\mathcal{N}_\#((-\infty, 0))$, and we have seen in Section 5 how this can be accomplished via the Evans function. For the current example, we have the two cases specified in (6.22). First, for $1 \leq p < 4$, we have $D''(0) > 0$, from which we see that for $\lambda < 0$ sufficiently close to 0 we must have $D(\lambda) > 0$ and consequently (via (5.6)) $\tilde{\psi}_1^{+}(x; \lambda) > 0$ for x sufficiently large. In addition, we've seen that for the current application $\tilde{\psi}_2^{+}(x; 0) < 0$ for $x > 0$. This pairing $\tilde{\psi}_1^{+}(x; \lambda) > 0$ and $\tilde{\psi}_2^{+}(x; 0) < 0$ places $p^+(x; \lambda)$ in the second quadrant, indicating that as λ increases to 0 $p(x; \lambda)$ rotates into $(-1, 0)$ in the positive (i.e., the counterclockwise) direction, thus incrementing the hyperplane index by $+1$. As in the discussion of (5.8), this allows us to refine (6.28) to the statement

$$\mathcal{N}_\#((-\infty, 0)) + 1 \geq |1 + \mathfrak{m}|.$$

Since we may have $\mathfrak{m} = 0$, this relation is consistent with spectral stability (i.e., consistent with the count $\mathcal{N}_\#((-\infty, 0)) = 0$). Although we cannot conclude spectral stability from this calculation, we note that spectral stability is known to hold in this case. (See p. 50 of [33] for discussion and references.)

On the other hand, for $p > 4$, we have $D''(0) < 0$, with everything else as before, and the same considerations described just above determine that in this case the relation (6.28) can be refined to

$$\mathcal{N}_{\#}((-\infty, 0)) \geq |1 + \mathfrak{m}|. \tag{6.29}$$

If the boundary invariant $\mathfrak{m}$ is an even number, as suggested by Lemma 2.2 (see also Remark 6.1 just below), then we can conclude that there is an unstable eigenvalue in this case, and so $\bar{u}(x)$ must be spectrally unstable.

In summary, we expect $\bar{u}(x)$ to be stable for $1 \leq p < 4$ and unstable for $p > 4$. This is precisely the conclusion of [33], obtained there in the following way. For $1 \leq p < 4$, we've seen that $D(\lambda) > 0$ for $\lambda < 0$ sufficiently close to 0, and we also know from Proposition 5.3 that $D(\lambda) \to +1$ as $\lambda \to -\infty$. This arrangement is consistent with an absence of real roots of $D(\lambda)$ on $(-\infty, 0)$, and so consistent with the case of spectral stability. (Of course, an even number of eigenvalues is possible, so no positive conclusion can be reached based on this calculation.) Likewise, if $p > 4$, then $D''(0) < 0$ and so $D(\lambda) < 0$ for $\lambda < 0$ sufficiently close to 0. Since we still have the limit $D(\lambda) \to 1$ as $\lambda \to \infty$, this arrangement guarantees that $D(\lambda)$ has a least one real root on the interval $(-\infty, 0)$, so we certainly have spectral instability.

Remark 6.1. *As noted in Lemma 2.2, under fairly general conditions we have that $\mathfrak{m}$ is an even number. For the current application, we can verify this rigorously by combining the analysis of [33] described just above with our analysis of the remaining shelves of the Maslov box.*

At this point, we have carried out the full analysis required to reach our conclusions, but in order to illustrate how the method is working, we provide numerically generated depictions of the Maslov box for two cases, one stable and one unstable. For these calculations, we will work with

$$\tilde{\omega}_1^+(x;\lambda) = \eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda),$$

where we recall that

$$\tilde{\mathcal{V}}^+(\lambda) = \kappa(\lambda) \begin{pmatrix} 1 \\ -\mu_3(\lambda) \\ -\lambda/\mu_3(\lambda) \end{pmatrix},$$

with $\kappa(\lambda)$ serving as the scaling constant specified in (6.12). We have, then,

$$\tilde{\omega}_1^+(x;\lambda) = \kappa(\lambda)\Big\{ -\frac{\lambda}{\mu_3(\lambda)} \eta_1^-(x;\lambda) + \mu_3(\lambda)\eta_2^-(x;\lambda) + \eta_3^-(x;\lambda)\Big\}. \tag{6.30}$$

Likewise, we set

$$\tilde{\omega}_2^+(x;0) = \eta^-(x;\lambda) \wedge \tilde{\mathcal{V}}_M(\lambda),$$

where $\tilde{\mathcal{V}}_M(\lambda)$ is an eigenvector of the asymptotic matrix $\tilde{\mathcal{A}}_+(\lambda)$ (from (1.22)) associated with the eigenvalue $-\mu_3(\lambda)$, namely

$$\tilde{\mathcal{V}}_M^+(\lambda) = \begin{pmatrix} -s/\mu_3(\lambda) + \sqrt{s} \\ -\mu_3(\lambda) + \sqrt{s} \\ 1 \end{pmatrix},$$

where no specific normalization is required. It follows that

$$\tilde{\omega}_2^+(x;\lambda) = k_-\Big\{\eta_1^-(x;\lambda) - \eta_2^-(x;\lambda)(-\mu_3(\lambda) + \sqrt{s}) + \eta_3^-(x;\lambda)(-s/\mu_3(\lambda) + \sqrt{s})\Big\}. \quad (6.31)$$

Using (6.30) and (6.31), we can now generate spectral curves throughout a fixed Maslov box by numerically generating $\eta^-(x;\lambda)$ throughout the box. As an example of the stable case, we will carry this out for $p = 7/2$ and $s = 1/2$, using $[-7,0]\times[-5,5]$ as the (truncated) Maslov box. (See Figure 6.3.) We've seen in our analytic calculation that each crossing on the right shelf gives a contribution of -1 to the hyperplane index. In addition, we've seen that there is an additional contribution of $+1$ obtained in the limit as $x \to +\infty$, but this is never picked up on any box truncated in the x-direction. Finally, by using the Evans function, we were able to show that for c sufficiently large the tracking point $p^+(c;\lambda)$ rotates in the clockwise direction as λ decreases from 0. We see that as x increases toward c, $p^+(x;0)$ rotates toward $(-1,0)$ in the counterclockwise direction without ever arriving at $(-1,0)$, and then when $x = c$ and λ is decreased from 0, the point $p^+(c;\lambda)$ rotates back in the clockwise direction so that the corner point at $x = c$ and $\lambda = 0$ does not increment the hyperplane index. In total, we can conclude that in this case $\mathfrak{m} = -2$. In order to have such a value for $\mathfrak{m}$, there must be at least one point at which invariance is lost in the open box $(-7,0)\times(-5,5)$, and since one condition for loss of invariance is $\tilde{\omega}_1^+(x;\lambda) = 0$, this point must occur along the spectral curve (depicted in red in Figure 6.3). Numerically searching along this curve for zeros of $\tilde{\omega}_2^+(x;\lambda)$, we find that invariance seems to be lost at about $x = .348$ and $\lambda = -1.706$ (working with increments .001 in both x and λ). At first glance, it may seem that the turnaround point of the spectral curve at $(\lambda, x) \cong (-3.4, 0)$ is a likely candidate for the point at which invariance is lost, but this depends entirely on the choice of $\tilde{\omega}_2^+(x;\lambda)$. In particular, as we've seen in the proof of Lemma 2.2, the point of invariance can always be changed by changing the choice of $\tilde{\omega}_2^+(x;\lambda)$.

For the unstable case, we'll again take $s = 1/2$, this time with $p = 9/2$. Proceeding similarly as in the previous case, we numerically generate the spectral curves for the Maslov box $[-7,0]\times[-5,5]$. (See Figure 6.4.) In this case, we see a compressed spectral curve in the upper right corner of the Maslov box, and in order to clarify the behavior there, we include a second Maslov box on $[-1,0]\times[-5,5]$. (See Figure 6.5.) As in the previous case with $p = 7/2$, the crossings along the right shelf each contribute -1 to the hyperplane index. As x increases to a sufficiently large value, the tracking point $p^+(x;0)$ moves toward $(-1,0)$ in the counterclockwise direction. In addition, we have seen from our analysis of the Evans function that for c sufficiently large, $p^+(c;\lambda)$ rotates away from $(-1,0)$ in the counterclockwise direction as λ decreases from 0. In this way, we see that for c sufficiently large, the hyperplane index increments by $+1$ as $p^+(x;\lambda)$ traverses the corner at $x = c$ and $\lambda = 0$. Finally, there is a contribution of -1 from the eigenvalue at $\lambda = -.0959$ (with an increment in the calculation of .0001). As with the case with $p = 7/2$, we can conclude that $\mathfrak{m} = -2$. Once again, we see that invariance is lost for at least one point, and computing numerically we approximate this point as $x = -.286$ and $\lambda = -4.563$ (with a step size of .001 in both x and λ).

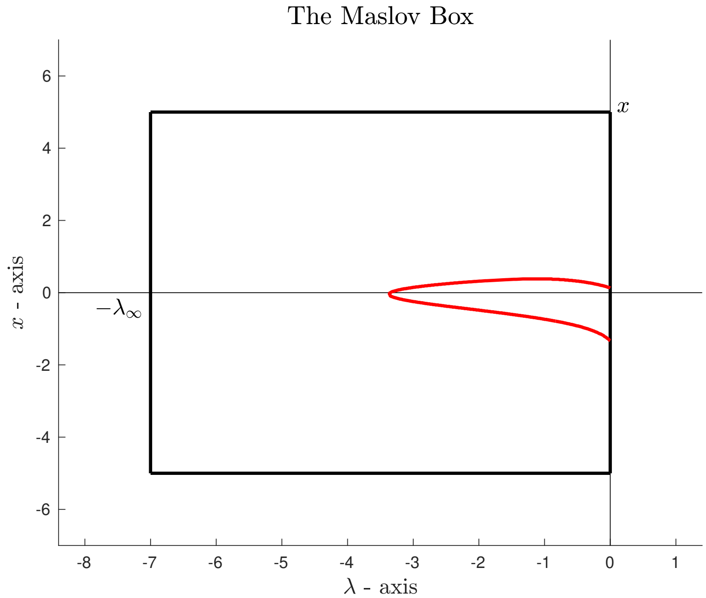

Figure 6.3: Maslov Box associated with (6.3) for $s = 1/2$ and $p = 7/2$.

6.2 The Korteweg-de Vries-Burgers Equation

For our second application, we consider the KdV-Burgers equation

$$u_t + uu_x = u_{xx} + \nu u_{xxx}, \tag{6.32}$$

where $\nu \in \mathbb{R}$ is fixed. It's known (see, e.g., [32]) that there exist stationary solutions $\bar{u}(x)$ for (6.32), which satisfy the asymptotic conditions

$$\lim_{x\to\pm\infty} \bar{u}(x) = \mp 1. \tag{6.33}$$

Such solutions satisfy the ODE

$$\nu\bar{u}''' + \bar{u}'' - \bar{u}\bar{u}' = 0, \tag{6.34}$$

which we can integrate to

$$\nu\bar{u}'' + \bar{u}' - \frac{1}{2}(\bar{u}^2 - 1) = 0. \tag{6.35}$$

For $|\nu| \le 1/4$ these solutions are known to be monotonic, while for $|\nu| > 1/4$ they are known to oscillate as x tends to $+\infty$. Such a wave, generated numerically, is depicted in Figure 6.6 for $\nu = 10$. For specificity, we will take $\nu > 0$ throughout our calculations, noting that the case $\nu < 0$ can be addressed similarly.

Remark 6.2. *In Theorem 1.1 of [32], the authors consider a more general equation than (6.35), and with a different scaling, namely*

$$-c\phi + \frac{1}{p+1}\phi^{p+1} + \phi'' = \alpha\phi', \tag{6.36}$$

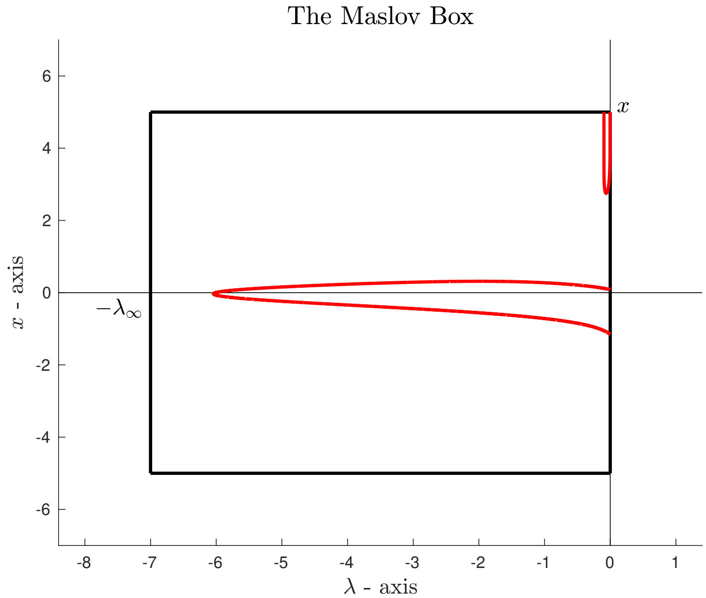

Figure 6.4: Maslov Box associated with (6.3) for $s = 1/2$ and $p = 9/2$.

where $p \geq 1$ and c and α are taken to be positive constants. Our equation (6.35) can be obtained from (6.36) in the case $\nu > 0$ by setting $c = 1$, $\alpha = 1/\sqrt{\nu}$, and

$$\bar{u}(x) = -\phi(-x/\sqrt{\nu}) + 1,$$

and for $\nu < 0$ by setting $c = 1$, $\alpha = 1/\sqrt{-\nu}$, and

$$\bar{u}(x) = \phi(x/\sqrt{-\nu}) - 1.$$

If we linearize (6.32) about the wave $\bar{u}(x)$, we obtain the eigenvalue problem

$$L\phi = -\nu\phi''' - \phi'' + (\bar{u}(x)\phi)' = \lambda\phi, \tag{6.37}$$

where the sign has again been chosen so that eigenvalues with a negative real part signify spectral instability. According to (6.33), the associated asymptotic problems are

$$-\nu\phi''' - \phi'' \pm \phi' = \lambda\phi. \tag{6.38}$$

According to [21, 30], we can understand the essential spectrum of L by looking for solutions to (6.38) of the form $\phi(x;k) = e^{ikx}$. We find

$$\sigma_{\text{ess}}(L) = \{\lambda \in \mathbb{C} : \lambda = i\nu k^3 + k^2 \pm ik,\, k \in \mathbb{R}\},$$

from which it's clear that the essential spectrum of L is confined to the right complex half-plane along with $\lambda = 0$.

In order to express (6.37) in our standard form (1.1), we set $y_1 = \phi$, $y_2 = \phi'$, and $y_3 = \nu\phi''$, so that

$$y' = A(x;\lambda)y, \quad A(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/\nu \\ \bar{u}'(x) - \lambda & \bar{u}(x) & -1/\nu \end{pmatrix}, \tag{6.39}$$

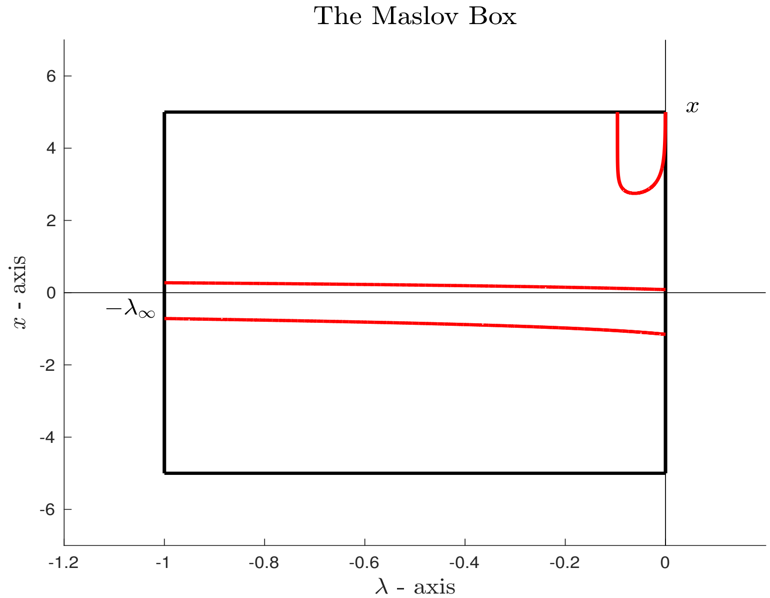

Figure 6.5: Additional detail for the Maslov Box associated with (6.3) for $s = 1/2$ and $p = 9/2$.

with the corresponding asymptotic matrices

$$A_\pm(\lambda) := \lim_{x \to \pm\infty} A(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1/\nu \\ -\lambda & \mp 1 & -1/\nu \end{pmatrix}.$$

The eigenvalues of $A_\pm(\lambda)$ are easily seen to be roots μ of the function

$$h_\pm(\mu;\lambda) = \mu^3 + \frac{1}{\nu}\mu^2 \pm \frac{1}{\nu}\mu + \frac{\lambda}{\nu}. \tag{6.40}$$

For $\lambda = 0$, the roots of $h_\pm(\mu; 0)$ are readily computed,

$$\mu = 0, \quad \frac{1}{2\nu}(-1 - \sqrt{1 \mp 4\nu}), \quad \frac{1}{2\nu}(-1 + \sqrt{1 \mp 4\nu}).$$

Recalling that we're taking $\nu > 0$, we see that for $h_-(\mu; 0)$ these roots are naturally ordered as

$$\mu_1^-(0) = \frac{1}{2\nu}(-1 - \sqrt{1 + 4\nu}) < 0, \quad \mu_2^-(0) = 0, \quad \mu_3^-(0) = \frac{1}{2\nu}(-1 + \sqrt{1 + 4\nu}) > 0.$$

As λ decreases from 0, the graph of the cubic function $h_-(\mu; 0)$ will lower so that the values $\mu_1^-(\lambda)$ and $\mu_2^-(\lambda)$ will approach one another, while $\mu_3^-(\lambda)$ will increase. As λ continues to decrease, the roots $\mu_1^-(\lambda)$ and $\mu_2^-(\lambda)$ will coalesce at some value $\lambda = \lambda_c$ into a complex conjugate pair. For λ above this coalescence value (i.e., for $\lambda \in (\lambda_c, 0]$) we can associate an eigenvector $v_j^-(\lambda)$ with each $\mu_j^-(\lambda)$, $j \in \{1, 2, 3\}$,

$$v_j^-(\lambda) = (1 \ \mu_j^-(\lambda) \ \nu\mu_j^-(\lambda)^2)^T, \quad j = 1, 2, 3. \tag{6.41}$$

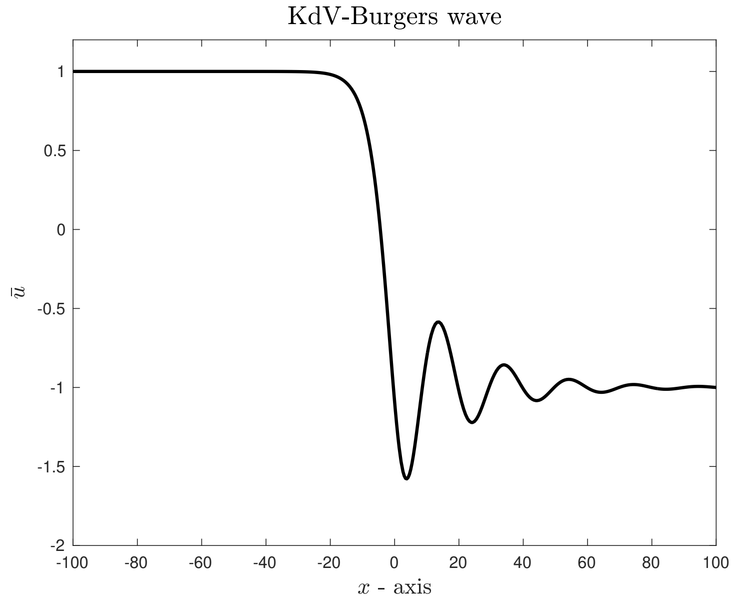

Figure 6.6: Stationary solution $\bar{u}(x)$ for (6.32) with $\nu = 10$.

The eigenvector $v_3^-(\lambda)$ has the same form for all $\lambda \le 0$, and up to a scaling factor is the eigenvector $v^-(\lambda)$ from Proposition 3.1.

Proceeding similarly for $h_+(\mu;\lambda)$, we first observe that in this case $\text{Re}\sqrt{1-4\nu} < 1$, so we have the ordering

$$\mu_1^+(0) = \frac{1}{2\nu}(-1-\sqrt{1-4\nu}), \quad \mu_2^+(0) = \frac{1}{2\nu}(-1+\sqrt{1-4\nu}), \quad \mu_3^+(0) = 0, \tag{6.42}$$

where $\mu_1^+(0)$ and $\mu_2^+(0)$ both have negative real part. If $\nu > 1/4$, then $\mu_1^+(0)$ and $\mu_2^+(0)$ will comprise a complex conjugate pair, while if $0 < \nu < 1/4$, then $\mu_1^+(\lambda)$ and $\mu_2^+(\lambda)$ will coalesce into such a pair as λ decreases. In either case, the value $\mu_3^+(\lambda)$ will remain real and increasing as λ decreases from 0. (The borderline case $\nu = 1/4$ requires additional work and won't be considered in the current analysis.)

We next consider $\tilde{A}(x;\lambda)$ (from (1.17)), which in this case is

$$\tilde{A}(x;\lambda) = \begin{pmatrix} 1/\nu & 1/\nu & 0 \\ \bar{u}(x) & 0 & 1 \\ \lambda - \bar{u}'(x) & 0 & 0 \end{pmatrix}, \tag{6.43}$$

with the corresponding asymptotic matrix

$$\tilde{A}_+(\lambda) = \lim_{x\to+\infty} \tilde{A}(x;\lambda) = \begin{pmatrix} 1/\nu & 1/\nu & 0 \\ -1 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}. \tag{6.44}$$

The unique left-most eigenvalue of $\tilde{A}_+(\lambda)$ is $\tilde{\mu}_1^+(\lambda) = -\mu_3^+(\lambda)$, with associated eigenvector

$$\tilde{\mathcal{V}}_1^+(\lambda) = \begin{pmatrix} 1 \\ -\nu\mu_3^+(\lambda) - 1 \\ \nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1 \end{pmatrix}, \tag{6.45}$$

which up to a scaling factor is $\tilde{\mathcal{V}}^+(\lambda)$ from Remark 3.1.

At this point, we have the pieces in place to verify that for any choice of $\lambda_\infty > 0$ the hyperplane index along the bottom shelf for the interval $[-\lambda_\infty, 0]$ gives no contribution. Namely, using the development of Section 4.1, we can verify this assertion if we can show that for all $\lambda \le 0$

$$v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) \neq 0. \tag{6.46}$$

(In contrast with the previous case, since $A_-(\lambda) \neq A_+(\lambda)$ here, relation (6.46) isn't immediate.) Using our expressions just above for $v^-(\lambda)$ and $\tilde{\mathcal{V}}^+(\lambda)$, with $\kappa(\lambda)$ and $\tilde{\kappa}(\lambda)$ serving respectively as scaling factors for $v^-(\lambda)$ and $\tilde{\mathcal{V}}^+(\lambda)$, we can compute this wedge product to be

$$\begin{aligned} v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) &= \kappa(\lambda)\tilde{\kappa}(\lambda) \begin{pmatrix} 1 \\ \mu_3^-(\lambda) \\ \nu\mu_3^-(\lambda)^2 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -\nu\mu_3^+(\lambda) - 1 \\ \nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1 \end{pmatrix} \\ &= \kappa(\lambda)\tilde{\kappa}(\lambda) \Big\{ \nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1 - \mu_3^-(\lambda)(-\nu\mu_3^+(\lambda) - 1) + \nu\mu_3^-(\lambda)^2 \Big\} \\ &= \kappa(\lambda)\tilde{\kappa}(\lambda) \Big\{ \nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1 + \nu\mu_3^-(\lambda)\mu_3^+(\lambda) + \mu_3^-(\lambda) + \nu\mu_3^-(\lambda)^2 \Big\}. \end{aligned}$$

Recalling that $\mu_3^-(\lambda) > 0$ for all $\lambda \le 0$ and $\mu_3^+(\lambda) \ge 0$ for all $\lambda \le 0$, we see that $v^-(\lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) > 0$ for all $\lambda \le 0$, verifying (6.46).

Before moving on, we observe that since $A_-(\lambda) \neq A_+(\lambda)$ in this case, we cannot conclude immediately from Proposition 4.2 that there are no crossings along the left shelf. Nonetheless, such a conclusion can be drawn from an argument based on energy methods. Since that argument has a different flavor than the current considerations, it has been placed in an appendix.

Next, we turn to the evaluation of $D(0)$ and $D'(0)$ (it will turn out that $D''(0)$ isn't required). To this end, we first observe that since $A_-(\lambda)$ only has one positive eigenvalue, the solution $\eta^-(x;\lambda)$ described in Proposition 3.1 is (up to a multiplicative constant) the only solution of (6.39) that decays as x tends to $-\infty$. In this way, we see that that there exists some constant k_- so that

$$\eta^-(x;0) = k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}'''(x) \end{pmatrix}. \tag{6.47}$$

According to Proposition 3.1, we can write

$$\lim_{x \to -\infty} e^{-\mu_3^-(0)x} k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}'''(x) \end{pmatrix} = \lim_{x \to -\infty} e^{-\mu_3^-(0)x} \eta^-(x;0) = v^-(\lambda).$$

Recalling that the first component of $v^-(\lambda)$ is positive and $\bar{u}'(x) < 0$ for $x \ll 0$, we conclude that $k_- < 0$.

In addition, we need to understand the nature of $\tilde{\mathcal{Y}}^+(x;0)$, which solves the ODE

$$\tilde{\mathcal{Y}}^{+\prime} = \tilde{A}(x;0)\tilde{\mathcal{Y}}^+, \tag{6.48}$$

where $\tilde{A}(x;\lambda)$ as as in (6.44). Proceeding similarly as in (6.14) and (6.15), we find that the first component of $\tilde{\mathcal{Y}}^+(x;0)$ satisfies the equation

$$\nu\tilde{\mathcal{Y}}_1^{+\prime\prime\prime} - \tilde{\mathcal{Y}}_1^{+\prime\prime} - \bar{u}(x)\tilde{\mathcal{Y}}_1^{+\prime} = 0. \tag{6.49}$$

It's clear that one solution of this equation is $\tilde{\mathcal{Y}}_1^+(x;0) \equiv 1$, and upon substitution of this component into the full system (6.48) we see that one family of solutions is

$$\tilde{\mathcal{Y}}^+(x;0) = k_+ \begin{pmatrix} 1 \\ -1 \\ -\bar{u}(x) \end{pmatrix},$$

for some constant k_+. Moreover, since all other (linearly independent) solutions to (6.48) grow at exponential rate as $x \to +\infty$, this must indeed be the solution $\tilde{\mathcal{Y}}^+(x;0)$ we're seeking. Recalling from Proposition 3.6 that

$$\lim_{x\to\infty} e^{-\mu_3^+(0)x}\tilde{\mathcal{Y}}^+(x;0) = \tilde{\mathcal{V}}^+(0),$$

we see that we must have $k_+ > 0$. We can now directly compute

$$\begin{aligned} D(0) = \eta^-(x;0) \wedge \tilde{\mathcal{Y}}^+(x;0) &= k_-k_+ \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ -\bar{u}(x) \end{pmatrix} \\ &= k_-k_+\Big\{\nu\bar{u}'''(x) + \bar{u}''(x) - \bar{u}(x)\bar{u}'(x)\Big\} = 0, \end{aligned}$$

where the final equality holds by the specification of $\bar{u}(x)$ as a stationary solution to (6.32) (i.e., from (6.35)).

Using

$$A_\lambda(x;0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

along with the relations above for $\eta^-(x;0)$ and $\tilde{\mathcal{Y}}^+(x;0)$, we find that

$$\Big(A_\lambda(x;0)\eta^-(x;0)\Big) \wedge \tilde{\mathcal{Y}}^+(x;0) = -k_-k_+\bar{u}'(x),$$

from which we conclude (using the expression for $D'(0)$ in Proposition 5.2)

$$D'(0) = -k_-k_+ \int_{-\infty}^{+\infty} \bar{u}'(x)dx = -k_-k_+(u_+ - u_-) = 2k_-k_+ < 0.$$

We now compute the hyperplane index along the right shelf. First, we will detect crossing points with the function

$$\tilde{\omega}_1^+(x;0) = \eta^-(x;0) \wedge \tilde{\mathcal{V}}^+(0) = k_-\tilde{\kappa}(0) \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -\nu\mu_3^+(0) - 1 \\ \nu\mu_3^+(0)^2 + \mu_3^+(0) + 1 \end{pmatrix}$$

$$= k_-\tilde{\kappa}(0) \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = k_-\tilde{\kappa}(0)\Big(\nu\bar{u}'''(x) + \bar{u}''(x) + \bar{u}'(x)\Big).$$

We recall that $\nu\bar{u}'''(x) + \bar{u}''(x) = \bar{u}(x)\bar{u}'(x)$, so that

$$\tilde{\omega}_1^+(x;0) = k_-\tilde{\kappa}(0)\bar{u}'(x)(\bar{u}(x) + 1). \tag{6.50}$$

For $0 < \nu < 1/4$, in which case $\bar{u}(x)$ decreases monotonically from 1 to -1, there are no crossing points $x_* \in \mathbb{R}$ (though there is an asymptotic crossing point at $+\infty$). More interesting, for $\nu > 1/4$, oscillations lead to crossings at each critical point of $\bar{u}(x)$ and also at each value $x_* \in \mathbb{R}$ so that $\bar{u}(x_*) = -1$.

In order to compute the hyperplane index along the right shelf, we also need to specify $\tilde{\omega}_2^+(x;0)$, and for this we have considerable flexibility, particularly in our choice of the constant matrix M. For reasons that will become clear just below, we will find it convenient to take

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.51}$$

We now set

$$\tilde{\omega}_2^+(x;0) = \eta_-(x;0) \wedge \tilde{\mathcal{V}}_M(0),$$

where $\tilde{\mathcal{V}}_M(0)$ is an eigenvector of the asymptotic matrix $\tilde{\mathcal{A}}_+(0)$ associated with the eigenvalue $-\mu_3(0)$, where $\tilde{\mathcal{A}}_+(0)$ is computed via (1.22)) from the matrix

$$\mathcal{A}_+(0) = MA_+(0)M^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 - 1/\nu \end{pmatrix}.$$

We find

$$\tilde{\mathcal{V}}_M^+(0) = -\begin{pmatrix} -\mu_3^+(0) \\ -\mu_3^+(0) - 1/\nu \\ \mu_3^+(0)^2 - (1 - 1/\nu)\mu_3^+(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\nu \\ 0 \end{pmatrix},$$

from which it follows that

$$\tilde{\omega}_2^+(x;0) = k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \bar{u}'''(x) \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1/\nu \\ 0 \end{pmatrix} = -(k_-/\nu)\bar{u}''(x). \tag{6.52}$$

For the ensuing discussion, it will be necessary to understand $\bar{u}''(x)$ for arbitrarily large values of x. For this, we first observe that for the eigenvalues $\mu_1^+(0)$ and $\mu_2^+(0)$ from (6.42) the associated eigenvectors can be chosen to be

$$v_i^+(0) = \begin{pmatrix} 1 \\ \mu_i^+(0) \\ \nu\mu_i^+(0)^2 \end{pmatrix}, \quad i = 1, 2. \tag{6.53}$$

For $0 < \nu < 1/4$, the eigenvalues $\mu_1^+(0)$ and $\mu_2^+(0)$ are real and distinct, satisfying $\mu_1^+(0) < \mu_2^+(0) < 0$, and we can readily construct individual solutions

$$y_i^+(x;0) = e^{\mu_i^+(0)x}(v_i^+(0) + E_i^+(x;0)), \quad i = 1, 2,$$

where $E_i^+(x;0)$ decays at exponential rate in x as x tends to $+\infty$. Since $\eta^-(x;0)$ solves (6.39) and decays at exponential rate as x tends to $+\infty$, and additionally since $\mu_3^+(0) = 0$, there must exist constants C_1 and C_2 so that

$$\eta^-(x;0) = k_- \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \\ \nu\bar{u}''(x) \end{pmatrix} = C_1 e^{\mu_1^+(0)x}(v_1^+(0)+E_1^+(x;0))+C_2 e^{\mu_2^+(0)x}(v_2^+(0)+E_2^+(x;0)). \tag{6.54}$$

In addition, upon integration of the first component on (x, ∞), we obtain the relation

$$k_-(\bar{u}(x)+1) = C_1 e^{\mu_1^+(0)x}(\frac{1}{\mu_1^+(0)} + \mathbf{O}(e^{-\alpha|x|})) + C_2 e^{\mu_2^+(0)x}(\frac{1}{\mu_2^+(0)} + \mathbf{O}(e^{-\alpha|x|})), \tag{6.55}$$

for some fixed $\alpha > 0$. Combining these observations, we can conclude that for $C_2 = 0$ we have the limits

$$\lim_{x\to+\infty} \frac{\bar{u}''(x)}{\bar{u}'(x)} = \mu_1^+(0); \quad \lim_{x\to+\infty} \frac{\bar{u}'(x)}{\bar{u}(x)+1} = \mu_1^+(0), \tag{6.56}$$

while for $C_2 \neq 0$ the same relations hold with $\mu_1^+(0)$ replaced by $\mu_2^+(0)$.

We are now in a position to compute the hyperplane index on the right shelf for $0 < \nu < 1/4$. We have already seen that there are no crossing points $x_* \in \mathbb{R}$, so we only need to understand the nature of the asymptotic crossing point as x tends to $+\infty$. For this, we work with the scaled map

$$\tilde{\psi}_1^+(x;0) = \frac{\tilde{\omega}_1^+(x;0)}{|\eta^-(x;0)||\tilde{\mathcal{V}}^+(0)|},$$

which according to (1.31) has the well-defined asymptotic limit

$$\tilde{\psi}_1^{+,-}(0) = \lim_{x\to-\infty} \tilde{\psi}_1^+(x;0) = \frac{v^-(0) \wedge \tilde{\mathcal{V}}^+(0)}{|v^-(0)||\tilde{\mathcal{V}}^+(0)|} > 0,$$

where the final inequality follows from our verification of (6.46). From (6.54), we see that

$$\lim_{x\to+\infty} \frac{\eta^-(x;\lambda)}{|\eta^-(x;\lambda)|} = \begin{cases} v_2^+(0)/|v_2^+(0)| & C_2 \neq 0 \\ v_1^+(0)/|v_1^+(0)| & C_2 = 0. \end{cases}$$

In either case,

$$v_i^+(0) \wedge \tilde{\mathcal{V}}^+(0) = \tilde{\kappa}(0) \begin{pmatrix} 1 \\ \mu_i^+(0) \\ \nu\mu_i^+(0)^2 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \tilde{\kappa}(0)(\mu\mu_i^+(0)^2 + \mu_i^+(0) + 1) = 0,$$

where the final equality holds because $\mu_1^+(0)$ and $\mu_2^+(0)$ are the non-zero roots of $h_+(\mu;0)$. In this way, we see that a crossing must be associated with the asymptotic limit as x tends to $+\infty$. In order to understand the sign associated with this crossing, we consider directly the signs of $\tilde{\psi}_1^+(x;0)$ and $\tilde{\psi}_2^+(x;0)$ as x tends to $+\infty$.

First, since $k_- < 0$ and $\bar{u}'(x) < 0$ for all $x \in \mathbb{R}$, we see from (6.50) that in this case $\tilde{\psi}_1^+(x;0) > 0$ for all $x \in \mathbb{R}$. In addition, we see from (6.55) and (6.56) that $\bar{u}''(x) > 0$ for x sufficiently large, so from (6.52) we see that $\tilde{\psi}_2^+(x;0) > 0$ for all x sufficiently large. With $\tilde{\psi}_1^+(x;0)$ and $\tilde{\psi}_2^+(x;0)$ both positive, $p^+(x;0)$ lies in the third quadrant and so approaches $(-1,0)$ in the clockwise direction as x tends to $+\infty$. According to our convention, the hyperplane index does not increment in this case, and we can conclude that for $0 < \nu < 1/4$, we have

$$\mathrm{Ind}(\mathcal{g}(\cdot;0), \mathcal{h}^+(0); [-\infty, +\infty]) = 0.$$

According to Theorem 1.1, we can conclude that *if* we have invariance along the top shelf in this case then the count $\mathcal{N}_\#((-\infty,0])$ of non-positive eigenvalues of (6.37) satisfies the inequality

$$\mathcal{N}_\#((-\infty,0]) \geq |\mathfrak{m}|.$$

As in Section 6.1, we can use information about the Evans function to obtain an estimate for $\mathcal{N}_\#((-\infty,0))$ rather than $\mathcal{N}_\#((-\infty,0])$. We've seen that for (6.37) we have $D(0) = 0$ and $D'(0) < 0$, from which we can conclude that for x sufficiently large we must have $\tilde{\psi}_1^+(x;\lambda) > 0$ for all $\lambda < 0$ sufficiently close to 0. In addition, we've seen that for the current application $\tilde{\psi}_2^+(x;0) > 0$ for $x \gg 0$. This pairing $\tilde{\psi}_1^+(x;\lambda) > 0$ and $\tilde{\psi}_2^+(x;0) > 0$ places $p^+(x;\lambda)$ in the third quadrant, signifying that as λ increases to 0 $p^+(x;\lambda)$ rotates into $(-1,0)$ in the clockwise direction, and the hyperplane index is not incremented. This allows us to refine (6.28) to the statement

$$\mathcal{N}_\#((-\infty,0)) \geq |\mathfrak{m}|.$$

Since we may have $\mathfrak{m} = 0$, this relation is consistent with stability (though does not imply stability).

We now turn to the interesting case $\eta > 1/4$, for which oscillations in the wave $\bar{u}(x)$ suggest the possible onset of instability. Our primary interest in this example is determining why, from the current geometric point of view, such instability doesn't occur. As usual, we begin by identifying all conjugate points on $\mathbb{R}$. We see from (6.50) that these are values $x_* \in \mathbb{R}$ for which either $\bar{u}'(x_*) = 0$ or $\bar{u}(x_*) = -1$. The first such point, which we will denote x_1, occurs when $\bar{u}(x)$ first crosses the horizontal line at -1. If we order further crossings as the sequence $x_1 < x_2 < x_3 < ...$, we see from Figure 6.1 that $\bar{u}'(x_2) = 0$, $\bar{u}(x_3) = -1$, $\bar{u}'(x_4) = 0$, and so on, with an infinite number of crossings in total.

Remark 6.3. *For $\nu > 1/4$, the wave $\bar{u}(x)$ corresponds with a connection in the $(\bar{u}, \bar{u}')$ phase plane from a saddle point at $(1,0)$ to a stable spiral at $(-1,0)$, ensuring the qualitative properties described here. See Theorem 1.1 in [32] for details.*

In order to assign directions to these crossing points, we express (6.35) as

$$\nu\bar{u}''(x) + \bar{u}'(x) = \frac{1}{2}(\bar{u}(x)^2 - 1). \tag{6.57}$$

For x_1, we see that

$$\nu \bar{u}''(x_1) = -\bar{u}'(x_1) < 0,$$

and subsequently the sign alternates for x_3, x_5 etc. (i.e., $\nu \bar{u}''(x_3) = -\bar{u}'(x_3) > 0$, $\nu \bar{u}''(x_5) = -\bar{u}'(x_5) < 0$ and so on). Likewise, for x_2 we see that

$$\nu \bar{u}''(x_2) = \frac{1}{2}(\bar{u}(x_2)^2 - 1) > 0,$$

with the sign again alternating for x_2, x_4, etc. Combining these observations, we conclude that

$$\operatorname{sgn} \tilde{\omega}_2^+(x_i; 0) = \operatorname{sgn} \bar{u}''(x) = \begin{cases} +1 & i = 2, 3, 6, 7, \dots \\ -1 & i = 1, 4, 5, 8, 9, \dots . \end{cases}$$

Likewise,

$$\tilde{\omega}_1^{+\prime}(x; 0) = \frac{k_-}{\sqrt{3}} \Big(\bar{u}''(x)(1 + \bar{u}(x)) + \bar{u}'(x)^2 \Big),$$

and so

$$\tilde{\omega}_1^{+\prime}(x_i; 0) = \frac{k_-}{\sqrt{3}} \bar{u}'(x_i)^2 < 0, \quad i = 1, 3, 5, \dots,$$

with also

$$\tilde{\omega}_1^{+\prime}(x_i; 0) = \frac{k_-}{\sqrt{3}} \bar{u}''(x_i)(1 + \bar{u}(x_i)) > 0, \quad i = 2, 4, 6, \dots,$$

Putting these observations together, we see that

$$\operatorname{sgn} \frac{\tilde{\omega}_1^{+\prime}(x_i; 0)}{\tilde{\omega}_2^+(x_i; 0)} = \begin{cases} +1 & i = 1, 2, 5, 6, \dots \\ -1 & i = 3, 4, 7, 8, \dots . \end{cases}$$

We can conclude that as x increases the value of

$$\operatorname{Ind}(\mathscr{g}(\cdot; 0), \mathscr{h}^+(0); (-\infty, x])$$

cycles among the values $\{0, 1, 2\}$, starting with 0 (for $x < x_1$). The full hyperplane index doesn't exist, but the cancellation in this calculation suggests that every spectral curve that enters the Maslov box through the right shelf also exits through the right shelf. These considerations suggest that even for $\nu > 1/4$ the wave $\bar{u}(x)$ might be spectrally stable, and indeed the additional numerical calculations carried out below bear this out.

As with the application considered in Section 6.1, we finish this section with a numerical evaluation of the spectral curves associated with (6.37). For this calculation, we will work with the functions

$$\begin{aligned} \tilde{\omega}_1^+(x; \lambda) &= \eta^-(x; \lambda) \wedge \tilde{\mathcal{V}}^+(\lambda) \\ &= \tilde{\kappa}(\lambda) \Big\{ (\nu \mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1) \eta_1^-(x; \lambda) + (\nu \mu_3^+(\lambda) + 1) \eta_2^-(x; \lambda) + \eta_3^-(x; \lambda) \Big\} \end{aligned}$$

and

$$\begin{aligned} \tilde{\omega}_2^+(x; \lambda) = \eta^-(x; \lambda) \wedge \tilde{\mathcal{V}}_M^+(\lambda) = -\Big\{ &(\mu_3^+(\lambda)^2 - (1 - 1/\nu) \mu_3^+(\lambda)) \eta_1^-(x; \lambda) \\ &+ (\mu_3^+(\lambda) + 1/\nu) \eta_2^-(x; \lambda) - \mu_3^+(\lambda) \eta_3^-(x; \lambda) \Big\}, \end{aligned}$$

where $\tilde{\mathcal{V}}^+(\lambda)$ is as above and

$$\tilde{\mathcal{V}}_M^+(\lambda) = \begin{pmatrix} \mu_3^+(\lambda) \\ \mu_3^+(\lambda) + 1/\nu \\ -\mu_3^+(\lambda)^2 + (1 - 1/\nu)\mu_3^+(\lambda) \end{pmatrix},$$

is an eigenvector associated with the eigenvalue $-\mu_3^+(\lambda)$ for the matrix $\tilde{\mathcal{A}}_+(\lambda)$ defined via (1.17) from

$$\mathcal{A}_+(\lambda) = MA_+(\lambda)M^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -\lambda & 0 & 0 \\ 1-\lambda & -1 & 1 - 1/\nu \end{pmatrix}.$$

Using $\tilde{\omega}_1^+(x;\lambda)$ and $\tilde{\omega}_2^+(x;\lambda)$, we can now generate the spectral curves on any truncated Maslov box by numerically computing $\eta^-(x;\lambda)$. As expected, if this is done for any $\nu \in (0, 1/4)$, no spectral curves enter the Maslov box, and so there are no spectral curves to depict. In addition, as x increases to some sufficiently large value c, the tracking point $p^+(x;0)$ rotates toward $(-1, 0)$ in the clockwise direction (without reaching it), and as λ decreases from 0 $p^+(c;\lambda)$ rotates away from $(-1, 0)$ in the counterclockwise direction. In this way, we see that for c sufficiently large, there is no contribution to the Maslov box for any $\nu \in (0, 1/4)$. As a specific case, calculations were carried out for $\nu = 1/8$, and it was confirmed that for the Maslov box $[-5, 0] \times [-20, 20]$ there are no crossings on the boundary, so $\mathfrak{m} = 0$. This includes no crossings on the top shelf, indicating no unstable eigenvalues.

The cases $\nu > 1/4$ are more interesting. We have already seen in our analytic calculations that for all $\nu > 1/4$, there are in fact an infinite number of crossings on the right shelf at a sequence of values $\{x_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} x_i = \infty$. Correspondingly, we expect to find an infinite number of spectral curves entering and exiting the Maslov box along the right shelf. Since our numerical calculations will be truncated, they won't confirm this expectation, but they provide evidence that the spectral curves are precisely as expected over the window of investigation. We will carry out calculations for two specific cases, $\nu = 2$ and $\nu = 5$. These seem to give an indication of how the picture varies as ν decreases below 2 and increases above 5.

First, for $\nu = 2$, the spectral curves entering through the right shelf are seen numerically to be contained in the vertical strip associated with the λ interval $[-.02, 0]$. In Figure 6.7, spectral curves in the Maslov box $[-.02, 0] \times [-22, 22]$ are depicted. In order to see that the crossings continue, we also provide a second figure depicting the Maslov box $[-.0002, 0] \times [-22, 22]$ in which the upper spectral curves from Figure 6.7 are more fully resolved, and two additional spectral curve becomes apparent. See Figure 6.8.

As a point of comparison, we also numerically generate spectral curves for the case $\nu = 5$. As ν increases, the oscillations of the stationary solution $\bar{u}(x;\nu)$ spread out, so we expect the spectral curves to be farther apart. This is indeed the case, as show in Figure 6.9. As ν continues to increase, we expect the spectral curves to be spaced farther apart and to go farther into the Maslov box.

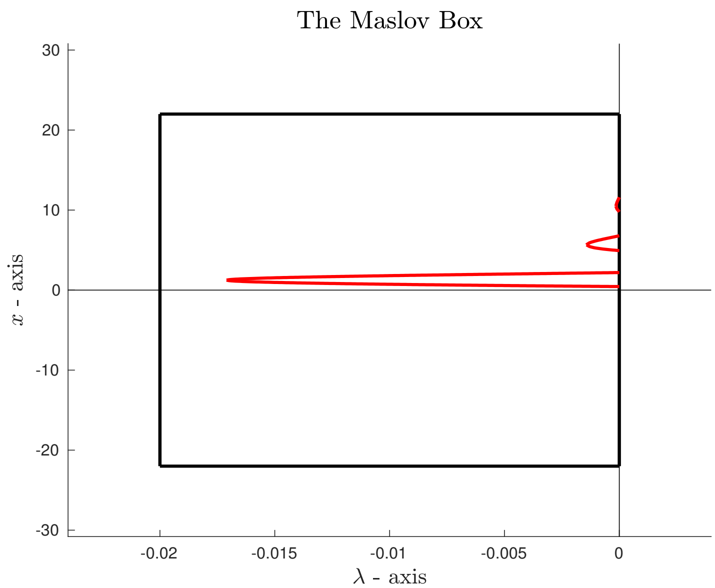

Figure 6.7: Maslov Box associated with (6.32) for $\nu = 2$.

A Appendix

In this appendix, we include a verification of the claim in Section 6.2 that in our analysis of the KdV-Burgers equation, there are no intersections along the left shelf. First, in order to have an intersection along the left shelf, there must be some value $s \in \mathbb{R}$ so that

$$\eta^-(s;\lambda_1) \wedge \tilde{\mathcal{V}}^+(\lambda_1) = 0.$$

I.e., λ_1 must be an eigenvalue for the half-line problem

$$\begin{aligned} -\nu\phi''' - \phi'' + (\bar{u}(x)\phi)' = \lambda\phi, \quad x \in (-\infty, s) \\ -\frac{\lambda}{\mu_3^+(\lambda)}\phi(s) - (\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)})\phi'(s) + \nu\phi''(s) = 0, \end{aligned} \tag{A.1}$$

where in formulating this boundary condition we have used $\tilde{\mathcal{V}}^+(\lambda)$ from (6.45) to write

$$\begin{pmatrix} \phi(s) \\ \phi'(s) \\ \nu\phi''(s) \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -\nu\mu_3^+(\lambda) - 1 \\ \nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1 \end{pmatrix}$$
$$= (\nu\mu_3^+(\lambda)^2 + \mu_3^+(\lambda) + 1)\phi(s) + (\nu\mu_3^+(\lambda) + 1)\phi'(s) + \nu\phi''(s) = 0,$$

then rearranged terms using with the eigenvalue relation

$$\nu\mu_3^+(\lambda)^3 + \mu_3^+(\lambda)^2 + \mu_3^+(\lambda) = -\lambda$$

(i.e., $h_+(\mu_3^+;\lambda) = 0$) with h_+ as in (6.40).

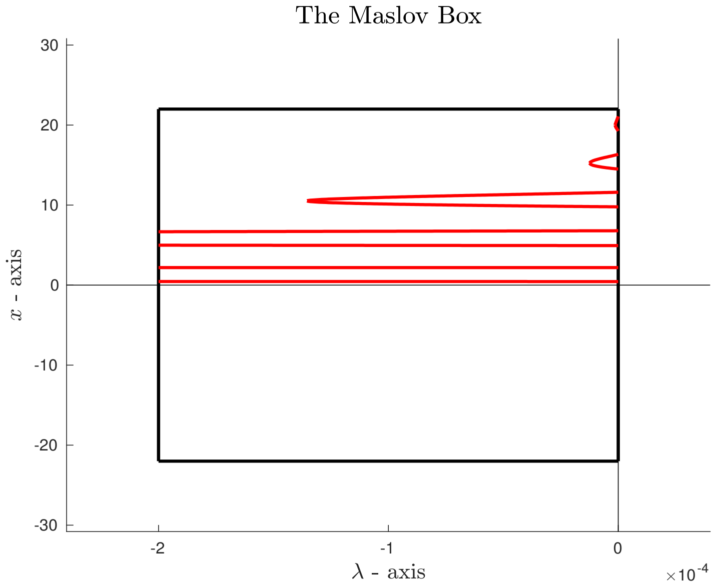

Figure 6.8: Maslov Box associated with (6.32) for $\nu = 2$, including additional spectral curves.

We proceed via a standard energy argument, which begins with the assumption that for some $\lambda \leq 0$ there exists a solution $\phi(x)$ to (A.1) so that $\phi(x) \to 0$ as $x \to -\infty$, necessarily at exponential rate under our assumptions. We multiply (A.1) by ϕ and integrate over $(-\infty, s)$ to obtain the relation

$$-\nu \int_{-\infty}^{s} \phi''' \phi dx - \int_{-\infty}^{s} \phi'' \phi dx + \int_{-\infty}^{s} (\bar{u}(x)\phi)' \phi dx = \lambda \int_{-\infty}^{s} \phi^2 dx. \tag{A.2}$$

For the first summand on the left-hand side of (A.2), we can integrate by parts to write

$$\begin{aligned} -\nu \int_{-\infty}^{s} \phi''' \phi dx &= -\nu \phi''(s)\phi(s) + \nu \int_{-\infty}^{s} \phi'' \phi' dx \\ &= -\nu \phi''(s)\phi(s) + \frac{\nu}{2} \phi'(s)^2. \end{aligned} \tag{A.3}$$

Likewise, for the second summand on the left-hand side of (A.2), we can integrate by parts to write

$$-\int_{-\infty}^{s} \phi'' \phi dx = -\phi'(s)\phi(s) + \int_{-\infty}^{s} \phi'^2 dx, \tag{A.4}$$

and for the third we similarly obtain the relation

$$\int_{-\infty}^{s} (\bar{u}(x)\phi)' \phi dx = \bar{u}(s)\phi(s)^2 - \int_{-\infty}^{s} \bar{u}(x) \phi \phi' dx. \tag{A.5}$$

Next, we obtain lower bounds on the right-hand sides of the expressions (A.3), (A.4), and (A.5). Starting with (A.5), we'll set

$$C := \sup_{x \in \mathbb{R}} |\bar{u}(x)|,$$

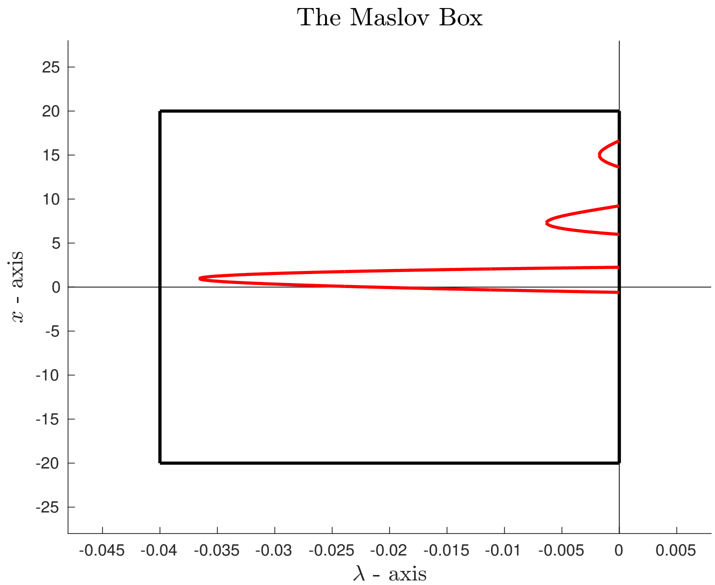

Figure 6.9: Maslov Box associated with (6.32) for $\nu = 5$.

which is bounded due to continuity of $\bar{u}(x)$ and the endstate conditions. In addition, for a value $\epsilon > 0$ to be chosen sufficiently small below, we note the standard inequality

$$|\phi||\phi'| \leq \frac{1}{2\epsilon}|\phi|^2 + \frac{\epsilon}{2}|\phi'|^2. \tag{A.6}$$

This allows us to express the lower bound

$$-\int_{-\infty}^{s} \bar{u}(x)\phi\phi' dx \geq -\frac{C}{2\epsilon}\int_{-\infty}^{s} \phi^2 dx - \frac{C\epsilon}{2}\int_{-\infty}^{s} \phi'^2 dx.$$

Next, in order to estimate $\bar{u}(s)\phi(s)^2$, we observe that $\phi(s)^2$ can be expressed as

$$\phi(s)^2 = \int_{-\infty}^{s} \frac{d}{dx}\phi^2 dx = \int_{-\infty}^{s} 2\phi\phi' dx,$$

which we can combine with (A.6) to see that

$$\phi(s)^2 \leq \frac{1}{\epsilon}\int_{-\infty}^{s} \phi^2 dx + \epsilon\int_{-\infty}^{s} \phi'^2 dx. \tag{A.7}$$

Upon combining these observations, we obtain the inequality

$$\int_{-\infty}^{s} (\bar{u}(x)\phi)'\phi dx \geq -\frac{3C}{2\epsilon}\int_{-\infty}^{s} \phi^2 dx - \frac{3C\epsilon}{2}\int_{-\infty}^{s} \phi'^2 dx.$$

Next, for the right-hand side of (A.4), we can use (A.6) once again (with a new constant δ in place of ϵ) to write

$$-\phi'(s)\phi(s) + \int_{-\infty}^{s} \phi'^2 dx \geq -\frac{1}{2\delta}|\phi|^2 - \frac{\delta}{2}|\phi'|^2 + \int_{-\infty}^{s} \phi'^2 dx,$$

and subsequently we can use (A.7) to obtain the inequality

$$-\int_{-\infty}^{s} \phi''\phi dx \geq -\frac{1}{2\epsilon\delta}\int_{-\infty}^{s} \phi^2 dx - \frac{\epsilon}{2\delta}\int_{-\infty}^{s} \phi'^2 dx - \frac{\delta}{2}|\phi'|^2 + \int_{-\infty}^{s} \phi'^2 dx.$$

Last, for (A.3), we use the boundary condition in (A.1) to write

$$\nu\phi''(s) = \frac{\lambda}{\mu_3^+(\lambda)}\phi(s) + (\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)})\phi'(s),$$

from which we see that

$$-\nu\phi''(s)\phi(s) = -\frac{\lambda}{\mu_3^+(\lambda)}\phi(s)^2 - (\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)})\phi'(s)\phi(s).$$

The sign of the first summand on the right-hand side is beneficial, but the second summand on the right-hand side will take some work to control. First, similarly as with (A.6), given any $\eta > 0$, we have the inequality

$$|(\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)})\phi'(s)\phi(s)| \leq \frac{1}{2\eta}|(\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)})|^2\phi(s)^2 + \frac{\eta}{2}\phi'(s)^2.$$

Recalling that $\mu_3^+(\lambda)$ solves the polynomial equation

$$\nu\mu^3 + \mu^2 + \mu + \lambda = 0,$$

we see that the behavior of $\mu_3^+(\lambda)$ for $\lambda \ll 0$ can be characterized by the limit

$$\lim_{\lambda\to-\infty} \frac{\nu\mu_3^+(\lambda)^3}{\lambda} = -1.$$

In particular,

$$\lim_{\lambda\to-\infty} \frac{|\lambda|^2/|\mu_3^+(\lambda)|^4}{|\lambda|/|\mu_3^+(\lambda)|} = \lim_{\lambda\to-\infty} \frac{|\lambda|}{|\mu_3^+(\lambda)|^3} = \nu.$$

Given any $\kappa > 0$, we can find $\Lambda \gg 0$ sufficiently large so that

$$\frac{|\lambda|^2}{|\mu_3^+(\lambda)|^4} \leq (\nu + \kappa)\frac{|\lambda|}{|\mu_3^+(\lambda)|}, \quad \forall\, \lambda < -\Lambda.$$

In addition, since $\lambda/\mu_3^+(\lambda)^2 < -1/\mu_3^+(\lambda) < 0$, we have the simple inequality

$$|\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)}| \leq |\frac{\lambda}{\mu_3^+(\lambda)^2}|,$$

and this allows us to express the estimate

$$\frac{1}{2\eta}|\frac{\lambda}{\mu_3^+(\lambda)^2} + \frac{1}{\mu_3^+(\lambda)}|^2 \leq \frac{1}{2\eta}|\frac{\lambda}{\mu_3^+(\lambda)^2}|^2 \leq \frac{1}{2\eta}(\nu + \kappa)\frac{|\lambda|}{|\mu_3^+(\lambda)|}.$$

We now see that

$$-\nu\phi''(s)\phi(s) \geq -\frac{\lambda}{\mu_3^+(\lambda)}\phi(s)^2 - \frac{1}{2\eta}(\nu+\kappa)\frac{|\lambda|}{|\mu_3^+(\lambda)|}\phi(s)^2 - \frac{\eta}{2}\phi'(s)^2.$$

At this point, we choose $\eta = \frac{3}{4}\nu$ and $\kappa = \frac{1}{4}\nu$, for which the inequality becomes

$$-\nu\phi''(s)\phi(s) \geq -\frac{\lambda}{\mu_3^+(\lambda)}\phi(s)^2 - \frac{5}{6}\frac{|\lambda|}{|\mu_3^+(\lambda)|}\phi(s)^2 - \frac{3}{8}\nu\phi'(s)^2 \geq -\frac{3}{8}\nu\phi'(s)^2.$$

If we now put all of these inequalities together, we obtain the lower bound

$$\begin{aligned}\lambda\int_{-\infty}^{s}\phi^2 dx \geq -\frac{3}{8}\nu\phi'(s)^2 + \frac{\nu}{2}\phi'(s)^2 - \frac{1}{2\epsilon\delta}\int_{-\infty}^{s}\phi^2 dx - \frac{\epsilon}{2\delta}\int_{-\infty}^{s}\phi'^2 dx \\ - \frac{\delta}{2}\phi'(s)^2 + \int_{-\infty}^{s}\phi'^2 dx - \frac{3C}{2\epsilon}\int_{-\infty}^{s}\phi^2 dx - \frac{3C\epsilon}{2}\int_{-\infty}^{s}\phi'^2 dx.\end{aligned}$$

Finally, we will complete the calculation by making judicious choices for ϵ and δ based on the values of ν and C. For this we'll require

$$\begin{aligned}\frac{3}{8}\nu + \frac{\epsilon}{2} &\leq \frac{\nu}{2} \\ \frac{\epsilon}{2\delta} + \frac{3C\epsilon}{2} &\leq 1.\end{aligned}$$

To be concrete, we take the specific values

$$\epsilon = \min\{\frac{1}{4}\nu, \frac{1}{3C}\} \quad \text{and} \quad \delta = \epsilon.$$

It follows that

$$\lambda\int_{-\infty}^{s}\phi^2 dx \geq -\Big(\frac{1}{2\epsilon\delta} + \frac{3C}{2\epsilon}\Big)\int_{-\infty}^{s}\phi^2 dx,$$

from which we see that any eigenvalue $\lambda \in \mathbb{R}$ of (A.1) must satisfy the inequality

$$\lambda \geq -\Big(\frac{1}{2\epsilon\delta} + \frac{3C}{2\epsilon}\Big),$$

where ϵ and δ are the fixed constants chosen above. By choosing λ_1 below this threshold, we can ensure that there are no crossings on the left shelf.

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author email: phoward@tamu.edu