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Short-time existence theory toward stability for nonlinear parabolic systems

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Abstract. We establish existence of classical solutions for nonlinear parabolic systems in divergence form on $\mathbb{R}^n$, under mild regularity assumptions on coefficients in the problem, and under the assumption of Hölder continuous initial conditions. Our analysis is motivated by the study of stability for stationary and traveling wave solutions arising in such systems. In this setting, large time bounds obtained by pointwise semigroup techniques are often coupled with appropriate short time bounds in order to close an iteration based on Duhamel-type integral equations, and our analysis gives precisely the required short time bounds. This development both clarifies previous applications of this idea (by Zumbrun and Howard) and establishes a general result that covers many additional cases.

1. Introduction

For $u \in \mathbb{R}^N$ and $x \in \mathbb{R}^n$, we consider nonlinear systems

$$\frac{\partial u_i}{\partial t} = \sum_{l=1}^{n} \left\{ \sum_{j=1}^{N} \sum_{|\alpha| \le (2p-1)} A^{ij}_{\alpha,l}(u,x,t) D^{\alpha} u_j \right\}_{x_l}, \tag{1.1}$$

for $i = 1, 2, \dots, N$. Here, p denotes a positive integer, and α is a standard multi-index in x, so that for any function $f(x)$

$$D^{\alpha} f := \frac{\partial^{\alpha_1} \partial^{\alpha_2} \dots \partial^{\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} f.$$

We assume Eq. (1.1) is uniformly parabolic in the following sense:
($\mathcal{P}$) If $\{\lambda_j(\xi; u, x, t)\}_{j=1}^N$ denote the eigenvalues of

$$\sum_{l=1}^{n} \sum_{|\alpha|=(2p-1)} A_{\alpha,l}(u,x,t)(i\xi)^{\alpha}(i\xi_l),$$

then for any compact set $\mathcal{K} \subset \mathbb{R}^N$, and for some values $0 \le \tau < T$,

$$\sup_{|\xi|=1} \mathrm{Re}\lambda_j(\xi; u, x, t) \le -\lambda_0 < 0,$$

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for all $(u, x, t) \in \mathcal{K} \times \mathbb{R}^n \times [\tau, T]$, and $j \in \{1, 2, \dots, N\}$.

Our standing assumptions on the coefficient functions $A^{ij}_{\alpha,l}(u, x, t)$ will be specified in terms of the following definition.

DEFINITION 1.1. *We will say $A^{ij}_{\alpha,l}(u, x, t)$ is Lipschitz–Hölder continuous (exponent γ) uniformly with respect to $\mathcal{U} \subset \mathbb{R}^N \times \mathbb{R}^n \times [\tau, T]$ provided there exists a constant $C = C(\mathcal{U})$ so that*

$$|A^{ij}_{\alpha,l}(u_1, x_1, t_1) - A^{ij}_{\alpha,l}(u_2, x_2, t_2)| \le C\Big\{|u_1 - u_2| + |x_1 - x_2|^\gamma + |t_1 - t_2|^{\frac{\gamma}{2p}}\Big\},$$

for all $(u_1, x_1, t_1), (u_2, x_2, t_2) \in \mathcal{U}$.

We will work with both the weak and strong formulations of (1.1), and correspondingly, we will have two levels of assumptions. Our weak assumptions will be as follows:

(W1) Given any compact set $\mathcal{K} \subset \mathbb{R}^N$, the coefficients $A^{ij}_{\alpha,l}$ are continuous bounded functions in $\Omega_\mathcal{K} := \mathcal{K} \times \mathbb{R}^n \times [\tau, T]$.

(W2) Given any compact set $\mathcal{B} \subset \mathbb{R}^n$, the coefficients $A^{ij}_{\alpha,l}$ are Lipschitz–Hölder continuous (exponent γ) with respect to $(u, x, t) \in \mathcal{K} \times \mathcal{B} \times [\tau, T]$. For all α so that $|\alpha| = 2p - 1$, the $A^{ij}_{\alpha,l}$ are Lipschitz–Hölder continuous (exponent γ) uniformly for $(u, x, t) \in \Omega_\mathcal{K}$.

Our strong assumptions will be as follows:

(S1) In addition to (W1), assume the derivatives $D_u A^{ij}_{\alpha,l}(u, x, t)$ and $D_x A^{ij}_{\alpha,l}(u, x, t)$ both satisfy the assumptions described in (W1) for $A^{ij}_{\alpha,l}$.

(S2) In addition to (W2), assume the derivatives $D_u A^{ij}_{\alpha,l}(u, x, t)$ and $D_x A^{ij}_{\alpha,l}(u, x, t)$ both satisfy the assumptions described in (W2) for $A^{ij}_{\alpha,l}$.

Our analysis is motivated by applications to the study of asymptotic stability for stationary and traveling wave solutions to equations of form (1.1). For example, in [5,9], the authors consider traveling wave solutions $\bar{u}(x - st)$ for viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{1.2}$$

where $u, f \in \mathbb{R}^N$ and $B \in \mathbb{R}^{N \times N}$, and where it is clear since $f(u)$ only appears under differentiation that we can take $f(0) = 0$ without loss of generality. Writing

$$\tilde{A}(u) := \int_0^1 Df(\gamma u)\mathrm{d}\gamma, \tag{1.3}$$

we obtain the relation $f(u) = \tilde{A}(u)u$, and so, (1.2) can be expressed as

$$u_t = \Big(- \tilde{A}(u)u + B(u)u_x \Big)_x, \tag{1.4}$$

or equivalently

$$\frac{\partial u_i}{\partial t} = \left(-\sum_{j=1}^{N} \tilde{A}_{ij}(u)u_j + \sum_{j=1}^{N} B_{ij}(u)u_{j_x} \right)_x .$$

In this way, (1.2) has form (1.1) with $A^{ij}_{0,1} = \tilde{A}_{ij}$ and $A^{ij}_{1,1} = B_{ij}$. Parabolicity is a requirement on the eigenvalues of $-B\xi^2$, and the standard full viscosity assumption made in [5,9] is that the eigenvalues of B all have positive real part (see, e.g., (H1) of [5]). We conclude that

$$\text{Re}\, \lambda_j(\xi; u, x, t) \le -\beta\xi^2,$$

for all $j \in \{1, 2, \ldots, N\}$, where β denotes the smallest real part of any of the eigenvalues of B. Clearly, if we restrict to $|\xi| = 1$, we obtain our parabolicity condition with $\lambda_0 = \beta$. In this case, since the coefficients depend only on u, (W1)–(W2) reduce to the assumption of Hölder continuity on $\tilde{A}(u)$ and $B(u)$ (on compact subsets of $\mathbb{R}^N$) and (S1)–(S2) reduce to Hölder continuity of $D_u\tilde{A}(u)$ and $D_u B(u)$ (on compact subsets of $\mathbb{R}^N$).

Likewise, it is straightforward to verify that multidimensional viscous conservation law systems

$$u_t + \sum_{j=1}^{n} f^j(u)_{x_j} = \sum_{j,k=1}^{n} (B^{jk}(u)u_{x_k})_{x_j} \tag{1.5}$$

for $x \in \mathbb{R}^n$, u, $f^j \in \mathbb{R}^N$, and $B^{jk} \in \mathbb{R}^{N\times N}$ can be expressed in form (1.1) and are parabolic provided

$$\sigma\left(\sum_{j,k=1}^{n} B^{jk}\xi_j\xi_k \right) \ge b_0|\xi|^2.$$

Here, σ denotes spectrum, and our notation signifies that each eigenvalue of the indicated matrix satisfies this condition. Another important family of parabolic equations comprises Cahn–Hilliard systems

$$\frac{\partial u_i}{\partial t} = \nabla \cdot \left\{ \sum_{j=1}^{N} M_{ij}(u)\nabla((-\Gamma\Delta u)_j + F_{u_j}(u)) \right\}, \tag{1.6}$$

which are parabolic provided the product of $N \times N$ matrices $M(u)\Gamma$ is positive definite uniformly in u.

Our main result is the following theorem.

THEOREM 1.1. *Suppose* (1.1) *is uniformly parabolic in the sense of* ($\mathcal{P}$) *that (S1)–(S2) hold and that* $u^\tau(\cdot) \in C^\gamma(\mathbb{R})$ *for some Hölder index* $0 < \gamma < 1$. *Then, there exists a solution to* (1.1)*, denoted u, on some sufficiently small time interval* $[\tau, \tilde{T}]$ *so that* $u(x, \tau) = u^\tau(x)$ *and for any* $\sigma \in (\tau, \tilde{T})$

$$u \in C^{\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [\tau, \tilde{T}]) \cap C^{2p+\gamma, 1+\frac{\gamma}{2p}}(\mathbb{R}^n \times [\sigma, \tilde{T}]).$$

Moreover, u is the unique solution in $C^{\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [\tau, \tilde{T}])$.

REMARK 1.1. *While Theorem 1.1 is interesting in its own right as a sufficiency condition for the short-time existence of strong solutions of* (1.1)*, we have been primarily motivated here by applications to the study of stability for stationary and traveling wave solutions of* (1.1)*. In analyses of viscous conservation laws* [5,9]*, Cahn–Hilliard equations and systems* [2,3]*, and related equations* [4]*, it has been shown that stability can often be established by combining large time bounds obtained by pointwise semigroup techniques with the short-time theory developed here. This procedure is discussed, for example, in* [5] *and* [3]*, where the latter paper bases its discussion directly on the current analysis. We note here that one of the most important elements of this procedure is that for small times, the solution* $u(x,t)$ *of* (1.1) *can be expressed as*

$$u(x,t) = \int_{\mathbb{R}^n} G(x,t;\xi,\tau)u(\xi,\tau)d\xi,$$

where the Green's function G *satisfies estimates developed in* [1]*. This characterization of* u *allows us to easily obtain estimates on derivatives of* u *in terms of* u *itself (at a shifted time) by placing derivatives on the Green's function.*

Alternative approaches and developments in related settings appear, for example, in [6,8], and the substantial list of references discussed in those papers. Aside from a difference of approach (an emphasis on Green's functions here, as opposed to more modern techniques in [6,8]), the current analysis differs from [6,8] and many other investigations in its restriction to Cauchy problems of the form (1.1) on unbounded domains. (Section 4 of [8] is concerned with problems in divergence form, similar to (1.1), on bounded domains). This specialization allows us to obtain a result that (1) is stated in terms convenient for application to stability analyses and (2) requires less regularity on initial conditions than is assumed in any analyses that we are aware of.

Outline of the paper. In Sect. 2, we establish some notational conventions that will be taken throughout the analysis. In Sect. 3, we carry out a linear analysis for a class of linear parabolic systems in weak form, and in Sect. 4, we establish a number of estimates that will be necessary for our nonlinear (Contraction Mapping Theorem) argument. Finally, in Sect. 5, we carry out the CMT argument and establish the full stated regularity of solutions to the strong problem (1.1).

2. Notation

For any $m \times n$ matrix A, we will denote components as A_{ij} or A^{ij}, depending upon convenience. We will use the norm notation

$$|A| := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2 \right)^{\frac{1}{2}}.$$

In calculations in which a new constant appears in each step, we will often take the convention of labeling the constants as C_1, C_2, etc. or (especially in exponents) c_1,

c_2, etc. In many cases, we will begin a calculation by dividing an expression into two summands $I = I_1 + I_2$, and we will continue by further dividing each summand. In this case, we will write $I_1 = J_1 + J_2$ and if necessary $J_1 = K_1 + K_2$, proceeding alphabetically, so that location in the cascade is clear. Once the case of I_1 is finished, we will begin with I_2, starting over with $I_2 = J_1 + J_2$.

Throughout the analysis, we refer to times τ, T and $\tilde{T}$, as discussed in the introduction. Our convention is that τ denotes our initial time, T denotes a possibly large time, and $[\tau, T]$ is an interval over which our equation coefficients satisfy our regularity assumptions. Finally, $\tilde{T} \in (\tau, T)$ denotes a time, so that $\tilde{T} - \tau$ is as small as required by the analysis.

The primary reference for this analysis is Friedman's book [1] in which the statements of results are not numbered by chapter. For clarity here, we will add the relevant chapter to the start of Friedman's numbering, so, for example, Friedman's Theorem 2.1 of Chapter 9 will be designated here as Theorem 9.2.1. In most cases, we will refer to page numbers as well.

3. Friedman's linear theory for the weak formulation

Given any function $\tilde{u}$ in an appropriate function space, we consider the linear problem associated with (1.1)

$$\frac{\partial u_i}{\partial t} = \sum_{l=1}^{n} \left\{ \sum_{j=1}^{N} \sum_{|\alpha| \leq (2p-1)} A^{ij}_{\alpha,l}(\tilde{u}(x,t), x, t) D^{\alpha} u_j \right\}_{x_l}, \tag{3.1}$$

for $i = 1, 2, \ldots, N$. We will write

$$\tilde{A}^{ij}_{\alpha,l}(x,t) := A^{ij}_{\alpha,l}(\tilde{u}(x,t), x, t). \tag{3.2}$$

Our primary goal in this section is to use the parametrix methods of [1] (originally developed by Levi [7]) to analyze a weak form of (3.1). For this analysis, we make the following assumptions on $\tilde{A}^{ij}_{\alpha,l}(x,t)$: for some $T > 0$

(A1) The coefficients $\tilde{A}^{ij}_{\alpha,l}(x,t)$ are continuous bounded functions in $\Omega = \mathbb{R}^n \times [\tau, T]$, and for all α so that $|\alpha| = 2p - 1$, the $\tilde{A}^{ij}_{\alpha,l}(x,t)$ are continuous in t uniformly with respect to (x,t) in Ω.

(A2) The coefficients $\tilde{A}^{ij}_{\alpha,l}(x,t)$ are Hölder continuous (exponent γ) in x uniformly with respect to (x,t) in bounded subsets of Ω, and for all α so that $|\alpha| = 2p - 1$, the $\tilde{A}^{ij}_{\alpha,l}(x,t)$ are Hölder continuous (exponent γ) in x uniformly with respect to (x,t) in Ω.

To begin, we define the function space

$$\mathcal{S} := \{\phi \in C^2(\mathbb{R}^n \times [\tau, T]; \mathbb{R}^n) : \text{spt}(\phi) \subset \mathbb{R}^n \times [\tau, T)\},$$

noting in particular that $\phi \in \mathcal{S} \Rightarrow \phi(x, T) \equiv 0$. We say $u \in C^{2p-1,0}(\mathbb{R}^n \times [\tau, T])$ is a weak solution of (3.1) provided that

$$\int_\tau^T \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial t} u_i \mathrm{d}x\mathrm{d}t = \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i}{\partial x_l} \Big\{ \sum_{j=1}^N \sum_{|\alpha| \le 2p-1} \tilde{A}^{ij}_{\alpha,l}(x,t) D^\alpha u_j \Big\} \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^n} \phi_i(x,\tau) u_i(x,\tau)\mathrm{d}x, \tag{3.3}$$

for each $i \in \{1, 2, , \dots, N\}$ and each function $\phi \in \mathcal{S}$.

Following Friedman's analysis of strongly formulated linear parabolic systems in [1], we will construct a Green's function ($N \times N$ matrix) $G(x, t; \xi, \tau)$ for (3.3). In particular, we construct G so that if $u^\tau(x)$ denotes any function continuous on $\mathbb{R}^n$, then

$$u(x,t) = \int_{\mathbb{R}^n} G(x,t;\xi,\tau) u^\tau(\xi)\mathrm{d}\xi \tag{3.4}$$

satisfies (3.3) with

$$\lim_{t \to \tau^+} u(x,t) = u^\tau(x),$$

for all $x \in \mathbb{R}^n$. We stress at the outset that our approach is constructive, so it is natural to make assumptions on the properties that G is expected to have and to verify them directly from the object we construct. Assuming, then, that G exists, and assuming that we can justify differentiation under the integral sign, we expect G to satisfy the relation

$$\begin{aligned} \int_\tau^T \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial t} \int_{\mathbb{R}^n} \sum_{k=1}^N G_{ik}(x,t;\xi,\tau) u_k^\tau(\xi) \mathrm{d}\xi\mathrm{d}x\mathrm{d}t &= -\int_{\mathbb{R}^n} \phi_i(x,\tau) u_i^\tau(x)\mathrm{d}x \\ &+ \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i}{\partial x_l} \Big\{ \sum_{j=1}^N \sum_{|\alpha| \le 2p-1} \tilde{A}^{ij}_{\alpha,l}(x,t) \\ &\times \int_{\mathbb{R}^n} \sum_{k=1}^N D_x^\alpha G_{jk}(x,t;\xi,\tau) u_k^\tau(\xi)\mathrm{d}\xi \Big\} \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.5}$$

Recalling that formally

$$\int_{\mathbb{R}^n} \phi_i(x,\tau) u_i^\tau(x)\mathrm{d}x = \int_{\mathbb{R}^n} \phi_i(x,\tau) \int_{\mathbb{R}^n} \sum_{k=1}^N G_{ik}(x,\tau;\xi,\tau) u_k^\tau(\xi)\mathrm{d}\xi\mathrm{d}x,$$

we can exchange the order of integration to write

$$\sum_{k=1}^N \int_{\mathbb{R}^n} u_k^\tau(\xi) \Big[\int_\tau^T \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial t} G_{ik}(x,t;\xi,\tau)\mathrm{d}x\mathrm{d}t \Big] \mathrm{d}\xi$$

$$= \sum_{k=1}^{N} \int_{\mathbb{R}^n} u_k^\tau(\xi) \Bigg[- \int_{\mathbb{R}^n} \phi_i(x,\tau) G_{ik}(x,\tau,\xi,\tau) \mathrm{d}x + \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^{n} \frac{\partial \phi_i}{\partial x_l} \left\{ \sum_{j=1}^{N} \sum_{|\alpha| \le 2p-1} \tilde{A}_{\alpha,l}^{ij}(x,t) D_x^\alpha G_{jk}(x,t;\xi,\tau) \right\} \mathrm{d}x \mathrm{d}t \Bigg] \mathrm{d}\xi. \tag{3.6}$$

We will construct G so that

$$\int_\tau^T \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial t} G_{ik}(x,t;\xi,\tau) \mathrm{d}x \mathrm{d}t = -\phi_i(\xi,\tau)\delta_i^k + \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^{n} \frac{\partial \phi_i}{\partial x_l} \left\{ \sum_{j=1}^{N} \sum_{|\alpha| \le 2p-1} \tilde{A}_{\alpha,l}^{ij}(x,t) D_x^\alpha G_{jk}(x,t;\xi,\tau) \right\} \mathrm{d}x \mathrm{d}t, \tag{3.7}$$

where δ_i^k denotes a standard Kronecker delta function.

Following the general approach of [1], we construct G with the form

$$G(x,t;\xi,\tau) = Z(x-\xi,t;\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y,t;y,\sigma) \Phi^\rho(y,\sigma;\xi,\tau) \mathrm{d}y \mathrm{d}\sigma, \tag{3.8}$$

or in component form

$$G_{ik}(x,t;\xi,\tau) = Z_{ik}(x-\xi,t;\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \sum_{m=1}^{N} Z_{im x_\rho}(x-y,t;y,\sigma) \Phi_{mk}^\rho(y,\sigma;\xi,\tau) \mathrm{d}y \mathrm{d}\sigma, \tag{3.9}$$

where Z and each of the Φ^ρ are $N \times N$ matrices to be specified below. We note for comparison with our reference that [1] addresses strong-form equations,

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{N} \sum_{|\alpha| \le 2p} A_\alpha^{ij}(x,t) D^\alpha u_j, \tag{3.10}$$

and in that setting, the analogous form of G is (the expression for Γ on p. 252 of [1])

$$G(x,t;\xi,\tau) = Z(x-\xi,t;\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^n} Z(x-y,t;y,\sigma) \Phi(y,\sigma;\xi,\tau) \mathrm{d}y \mathrm{d}\sigma.$$

Continuing now with the weak case, the components of $Z(x-\xi,t;y,\tau)$ solve the parametrix equation

$$\frac{\partial Z_{ik}}{\partial t}(x-\xi,t;y,\tau) = \sum_{l=1}^{n} \sum_{j=1}^{N} \sum_{|\alpha| = 2p-1} \tilde{A}_{\alpha,l}^{ij}(y,t) D_x^\alpha \frac{\partial Z_{jk}}{\partial x_l}(x-\xi,t;y,\tau), \tag{3.11}$$

along with the condition that for any $u^\tau \in C(\mathbb{R}^n)$

$$\lim_{t\to\tau+}\int_{\mathbb{R}^n} Z(x-\xi,t;\xi,\tau)u^\tau(\xi)\mathrm{d}\xi = u^\tau(x).$$

Alternatively, we can write (3.11) in vector form for the kth column Z_k

$$\frac{\partial Z_k}{\partial t} = \sum_{l=1}^{n}\sum_{|\alpha|=2p-1}\tilde{A}_{\alpha,l}(y,t)D_x^\alpha \frac{\partial Z_k}{\partial x_l}. \tag{3.12}$$

Since our equation for Z is in the class analyzed by Friedman in [1], we will take the existence of Z and many of its properties directly from that reference.

LEMMA 3.1. *Let assumptions (A1) and (A2) hold, and suppose Z is defined as in* (3.11)*. Then, for each multi-index* $0 \le |\alpha| < \infty$*, there exist constants* c_α *and* C_α *so that*

$$|D_x^\alpha Z_{ik}| \le C_\alpha (t-\tau)^{-\frac{n+|\alpha|}{2p}} e^{-c_\alpha (\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for all $i,k \in \{1,2,\ldots,N\}$*, all* $(x,t) \in \mathbb{R}^n \times (\tau,T]$*, and all* $\xi \in \mathbb{R}^n$*.*

REMARK 3.1. *This lemma is simply a restatement in our context of Theorem 9.2.1 on p. 241 of* [1]*. The proof appears in that reference.*

We now derive integral equations for the matrices Φ^l. To start, we set

$$\begin{aligned} I_{ik} &:= \int_\tau^T\int_{\mathbb{R}^n}\frac{\partial\phi_i}{\partial t}Z_{ik}(x-\xi,t;\xi,\tau)\mathrm{d}x\mathrm{d}x \\ &\quad -\int_\tau^T\int_{\mathbb{R}^n}\sum_{l=1}^{n}\frac{\partial\phi_i}{\partial x_l}\left\{\sum_{j=1}^{N}\sum_{|\alpha|\le 2p-1}\tilde{A}^{ij}_{\alpha,l}(x,t)D_x^\alpha Z_{jk}(x-\xi,t,\xi,\tau)\right\}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Integrating the first summand by parts in t, using (3.11) and rearranging terms, we can write

$$I_{ik} = -\phi_i(\xi,\tau)\delta_i^k - \int_\tau^T\int_{\mathbb{R}^n}\sum_{l=1}^{n}\frac{\partial\phi_i}{\partial x_l}K^l_{ik}(x,t;\xi,\tau)\mathrm{d}x\mathrm{d}t,$$

where

$$\begin{aligned} K^l_{ik}(x,t;\xi,\tau) &:= -\sum_{j=1}^{N}\sum_{|\alpha|=2p-1}\Big(\tilde{A}^{ij}_{\alpha,l}(\xi,t)-\tilde{A}^{ij}_{\alpha,l}(x,t)\Big)D_x^\alpha Z_{jk}(x-\xi,t;\xi,\tau) \\ &\quad +\sum_{j=1}^{N}\sum_{|\alpha|\le 2p-2}\tilde{A}^{ij}_{\alpha,l}(x,t)D_x^\alpha Z_{jk}(x-\xi,t;\xi,\tau). \end{aligned} \tag{3.13}$$

Likewise, set

$$J_{ik} := \int_\tau^T \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial t} \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \sum_{m=1}^N \frac{\partial Z_{im}(x-y,t;y,\sigma)}{\partial x_\rho} \Phi_{mk}^\rho(y,\sigma;\xi,\tau) \mathrm{d}y\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t$$
$$- \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i}{\partial x_l} \Bigg\{ \sum_{j=1}^N \sum_{|\alpha|\le 2p-1} \tilde{A}_{\alpha,l}^{ij}(x,t)$$
$$\times \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \sum_{m=1}^N D_x^\alpha \frac{\partial Z_{jm}(x-y,t;y,\sigma)}{\partial x_\rho} \Phi_{mk}^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \Bigg\} \mathrm{d}x\mathrm{d}t.$$

Integrating by parts again on the first summand, first in x then in t, using (3.11) and rearranging terms, we find

$$J_{ik} = \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i}{\partial x_l} \Phi_{ik}^l(x,t;\xi,\tau)\mathrm{d}x\mathrm{d}t$$
$$- \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i}{\partial x_l} \int_\tau^t \int_{\mathbb{R}^n} \sum_{m=1}^N \sum_{\rho=1}^n \bar{K}_{im}^{l,\rho}(x,t;y,\sigma)\Phi_{mk}^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t,$$

where

$$\bar{K}_{im}^{l,\rho}(x,t;y,\sigma) := - \sum_{j=1}^N \sum_{|\alpha|=2p-1} \left(\tilde{A}_{\alpha,l}^{ij}(\xi,t) - \tilde{A}_{\alpha,l}^{ij}(x,t) \right) D_x^\alpha \frac{\partial Z_{jm}(x-y,t;y,\sigma)}{\partial x_\rho}$$
$$+ \sum_{j=1}^N \sum_{|\alpha|\le 2p-2} \tilde{A}_{\alpha,l}^{ij}(x,t) D_x^\alpha \frac{\partial Z_{jm}(x-\xi,t;\xi,\tau)}{\partial x_\rho}. \tag{3.14}$$

Upon substituting (3.8) into (3.7), we find the relation

$$I_{ik} + J_{ik} = -\phi_i(\xi,\tau)\delta_i^k.$$

Using our expressions for I_{ik} and J_{ik}, we can write this as

$$- \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i(x,t)}{\partial x_l} K_{ik}^l(x,t;\xi,\tau)\mathrm{d}x\mathrm{d}t$$
$$+ \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i(x,t)}{\partial x_l} \Phi_{ik}^l(x,t;\xi,\tau)\mathrm{d}x\mathrm{d}t$$
$$- \int_\tau^T \int_{\mathbb{R}^n} \sum_{l=1}^n \frac{\partial \phi_i(x,t)}{\partial x_l} \int_\tau^t \int_{\mathbb{R}^n} \sum_{m=1}^N \sum_{\rho=1}^n \bar{K}_{im}^{l,\rho}(x,t;y,\sigma)\Phi_{mk}^\rho(y,\sigma;\xi,\tau)\mathrm{d}x\mathrm{d}t = 0.$$

We will construct the matrices Φ^l so that the integrand multipliers of $\frac{\partial \phi_i}{\partial x_l}$ all agree. Writing this result in matrix form, we obtain the collection of matrix integral equations

$$\Phi^l(x,t;\xi,\tau) = K^l(x,t;\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \bar{K}^{l,\rho}(x,t;y,\sigma)\Phi^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma, \quad (3.15)$$

for each $l \in \{1,2,\ldots,n\}$. We observe that for each l, we have the same general form as Friedman's (9.4.7) (from [1]), though our K^l satisfies different estimates than does Friedman's K, and for Friedman K appears twice, in place of both our K^l and our $\bar{K}^{l,\rho}$. For each $l \in \{1,2,\ldots,n\}$, we now proceed by writing Φ^l as an infinite sum

$$\Phi^l(x,t;\xi,\tau) = \sum_{\nu=1}^\infty \Phi^l_\nu(x,t;\xi,\tau), \quad (3.16)$$

where

$$\Phi^l_1(x,t;\xi,\tau) = K^l(x,t;\xi,\tau) \quad (3.17)$$

and for $\nu = 2,3,\ldots,$

$$\Phi^l_\nu(x,t;\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \bar{K}^{l,\rho}(x,t;y,\sigma)\Phi^\rho_{\nu-1}(y,\sigma,\xi,\tau)\mathrm{d}y\mathrm{d}\sigma. \quad (3.18)$$

Following the development on pp. 252–255 in [1] (and for additional details and insight pp. 14–15 of the same reference), we can verify that the sum in (3.16) is uniformly (uniform in x and ξ) a geometric series in $t-\tau$ and so converges for $|t-\tau|<1$.

In preparation for our analysis of the Φ^l, we state a lemma summarizing properties of K^l and $\bar{K}^{l,\rho}$.

LEMMA 3.2. *Let assumptions (A1) and (A2) hold, and suppose K^l and $\bar{K}^{l,\rho}$ are defined, respectively, as in (3.13) and (3.14). Then, there exist constants c and C so that for $0 \le \tau < t \le \tilde{T}$, with $\tilde{T}$ sufficiently small,*

$$|K^l_{ik}(x,t;\xi,\tau)| \le C(t-\tau)^{-1-\frac{n-1-\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

and

$$|\bar{K}^{l,\rho}_{ik}(x,t;\xi,\tau)| \le C(t-\tau)^{-1-\frac{n-\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for all $l,\rho \in \{1,2,\ldots,n\}$, $i,k \in \{1,2,\ldots,N\}$, and $x,\xi \in \mathbb{R}^n$.

Proof Since the K^l and $\bar{K}^{l,\rho}$ are defined in terms of Z and its derivatives, these estimates follow from Lemma 3.1 with one additional observation. Under our assumptions (A1) and (A2), we have the uniform estimate

$$|\tilde{A}^{ij}_{\alpha,l}(\xi,t) - \tilde{A}^{ij}_{\alpha,l}(x,t)| \le \tilde{C}|x-\xi|^\gamma$$

for $|\alpha| = 2p-1$ and some constant $\tilde{C}$. We observe that for $|\alpha| = 2p-1$

$$\left|(\tilde{A}^{ij}_{\alpha,l}(\xi,t) - \tilde{A}^{ij}_{\alpha,l}(x,t))D^{\alpha}_x Z_{jk}(x-\xi,t;\xi,\tau)\right| \leq \tilde{C}C_\alpha|x-\xi|^\gamma(t-\tau)^{-\frac{n+2p-1}{2p}}e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}. \quad (3.19)$$

Now,

$$\frac{|x-\xi|^\gamma}{(t-\tau)^{\frac{\gamma}{2p}}}e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \leq C_1 e^{-\frac{c_\alpha}{2}(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}, \quad (3.20)$$

for some constant C_1, so the right side of (3.19) is

$$\tilde{C}C_\alpha\frac{|x-\xi|^\gamma}{(t-\tau)^{\gamma/2p}}(t-\tau)^{-\frac{n+2p-1-\gamma}{2p}}e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \leq C_2(t-\tau)^{-\frac{n+2p-1-\gamma}{2p}}e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for some constant C_2. Noting that estimates on the other summands in K^l_{ik} are smaller if $t-\tau$ is small, we conclude the claimed estimate.

The proof is similar for $\bar{K}^{l,\rho}$. □

In order to obtain estimates on the Φ^l, we must understand kernel interactions. For this, we will recall Lemma 9.4.7 from p. 253 of [1], which requires the following notation: we set

$$\|x\| := \left(\sum_{i=1}^n |x_i|^q\right)^{1/q} \quad \text{where} \quad q = \frac{2p}{2p-1}, \quad (3.21)$$

and for $\tau < \sigma < t$, we define

$$f_n(x,\xi,y;t,\tau,\sigma) := \left(\frac{\|x-y\|^{2p}}{t-\sigma}\right)^{\frac{1}{2p-1}} + \left(\frac{\|y-\xi\|^{2p}}{\sigma-\tau}\right)^{\frac{1}{2p-1}}.$$

While the norm $\|\cdot\|$ will be convenient for calculations, we will ultimately express our estimates in terms of standard Euclidean norm $|\cdot|$. We note the equivalence

$$\frac{|x|^{2p/(2p-1)}}{2^{p/(2p-1)}} \leq \|x\|^{2p/(2p-1)} \leq |x|^{2p/(2p-1)}.$$

LEMMA 3.3. *Let*

$$I_a := \int_{\mathbb{R}^n}(t-\sigma)^{-\frac{n}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}e^{-af_n(x,\xi,y;t,\tau,\sigma)}dy,$$

where $\tau < \sigma < t$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and a denotes any positive number. For any $0 < \epsilon < 1$, there exists a constant M, depending only on ϵ, a, p and n, so that

$$I_a \leq M(t-\tau)^{-\frac{n}{2p}}e^{-a(1-\epsilon)(\frac{\|x-\xi\|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

REMARK 3.2. *We will repeatedly use Lemma 3.3 in the following form: Given constants c_1 and C_1, there exist constants c_2 and C_2 so that*

$$C_1 \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n}{2p}} (\sigma-\tau)^{-\frac{n}{2p}} e^{-c_1 (\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1 (\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} dy$$
$$\leq C_2 (t-\tau)^{-\frac{n}{2p}} e^{-c_2 (\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

LEMMA 3.4. *Let Assumptions (A1) and (A2) hold and suppose the matrices Φ^l_ν are defined as in (3.17) and (3.18). Then, there exist constants c and C so that for $0 \leq \tau < t \leq \tilde{T}$, with $\tilde{T}$ sufficiently small,*

$$|\Phi^l_\nu (x,t;\xi,\tau)| \leq C(t-\tau)^{-\frac{n+2p-1-\nu\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}$$

for all $l \in \{1, 2, \ldots, n\}$ and $\nu = 1, 2, \ldots,$ and for all $x, \xi \in \mathbb{R}^n$.

REMARK 3.3. *The most important observation in Lemma 3.4 is that to leading order in $t-\tau$ Φ^l is bounded like K^l.*

Proof First, for $\nu = 1$, this is simply Lemma 3.2. For $\nu = 2$,

$$|\Phi^l_2 (x,t;\xi,\tau)| \leq \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n |\bar{K}^{l,\rho}(x,t;y,\sigma) K^\rho (y,\sigma;\xi,\tau)| \mathrm{d}y \mathrm{d}\sigma$$
$$\leq C_1 \int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-1-\frac{n-\gamma}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}}$$
$$\times e^{-c_1 (\frac{|x-y|^{2p}}{t-\tau})^{1/(2p-1)} - c_1 (\frac{|y-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}y \mathrm{d}\sigma.$$

Applying Remark 3.2, we obtain

$$|\Phi^l_2 (x,t;\xi,\tau)| \leq C_2 (t-\tau)^{-\frac{n}{2p}} e^{-c_2 (\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}$$
$$\times \int_\tau^t (t-\sigma)^{-1+\frac{\gamma}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \mathrm{d}\sigma$$
$$\leq C_3 (t-\tau)^{-1-\frac{n-1-2\gamma}{2p}} e^{-c_2 (\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

which is the claim for $\nu = 2$. We observe here that the estimate obtained from the integration over σ is most easily found by dividing the interval of integration into two subintervals, $[\tau, (t-\tau)/2]$ and $[(t-\tau)/2, t]$.

The general step can now be carried out by induction. The main issue regards recovering a constant c that is not reduced during the induction step. (In our calculation, c_2 is smaller than c_1). This is overcome with the observation that the constant arising from $\bar{K}^{l,\rho}$ is always the same. See p. 254 of [1] for details. □

Combining the estimates of Lemmas 3.1 and 3.4, and using representation (3.8), we can obtain estimates on G.

LEMMA 3.5. *Let assumptions (A1) and (A2) hold, and suppose G is defined as in* (3.8). *Then, there exist positive constants c and C so that for any multi-index* $0 \leq |\alpha| < 2p - 1$ *in x, and for* $0 \leq \tau < t \leq \tilde{T}$, *with* $\tilde{T}$ *sufficiently small*

$$|D_x^\alpha G(x,t;\xi,\tau)| \leq C(t-\tau)^{-\frac{n+|\alpha|}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for all $x, \xi \in \mathbb{R}^n$.

Proof Since the estimates on Z are inherited immediately, we need only consider estimates on the integral in (3.8). Assuming that differentiation under the integral sign can be justified, we compute

$$\begin{aligned}
&\left| \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n D_x^\alpha Z_{x_\rho}(x-y,t;y,\sigma)\Phi^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \right| \\
&\leq C_1 \int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n+|\alpha|+1}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \\
&\quad \times e^{-c_1(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y\mathrm{d}\sigma \\
&\leq C_2(t-\tau)^{-\frac{n}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \int_\tau^t (t-\sigma)^{-\frac{|\alpha|+1}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \mathrm{d}\sigma.
\end{aligned}$$

In this last integral, we immediately understand the limitation to $|\alpha| + 1 < 2p$. We obtain an estimate by

$$C_3(t-\tau)^{-\frac{n+|\alpha|}{2p}+\frac{\gamma}{2p}} e^{-c_1(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

which is smaller than the claimed estimate. (We recall that the claimed estimate is determined by Z). Finally, we note that essentially the same argument, with an appeal to the Mean Value Theorem, justifies differentiating under the integral sign. □

REMARK 3.4. *In fact, we can show slightly more. We can establish that* Φ^l *is Hölder continuous and use this fact to justify computing derivatives up to order* $|\alpha| = 2p - 1$, *though we postpone this calculation until we develop the regularity theory of Sect.* 5.3.

One of the most important preliminary observations we will make regards integration of G and its derivatives over $\mathbb{R}^n$. To begin, we observe that if the coefficients $\tilde{A}_{0,l}^{ij}$ (i.e., the coefficients with $|\alpha| = 0$) of (3.1) are all 0 (or simply constant), then the components of u will only appear under differentiation. In this way, we know that if (1.1) is initialized by any constant vector $u(x, 0) \equiv u_0 =$ constant, then it will be solved for all time by the same vector $u(x,t) \equiv u_0$ for all $t \geq 0$. If G denotes a Green's function associated with this equation, we clearly have

$$u_0 = \int_{\mathbb{R}^n} G(x,t;\xi,\tau)u_0 \mathrm{d}\xi, \tag{3.22}$$

for all $u_0 \in \mathbb{R}^n$. It follows that in this case, G integrates to the identity matrix.

Of course we must take care here that the solution $u(x,t) \equiv u_0$ is indeed the solution we obtain through our construction of the Green's function, and uniqueness is not guaranteed by our standard assumptions (A1) and (A2). Precisely, in order to obtain uniqueness, we require the following:

(A3) For each $0 \le |\alpha| \le 2p$, the derivatives $D_x^\beta \tilde{A}^{ij}_{\alpha,l}(x,t)$ for $0 \le |\beta| \le |\alpha|$ are continuous bounded functions in $\Omega = \mathbb{R}^n \times [0,T]$, and they are Hölder continuous with exponent γ uniformly with respect to (x,t) in bounded subsets of Ω.

According to Theorem 9.5.6 on p. 260 of [1], if (A1), (A2) and (A3) all hold, then there exists at most one solution of (3.1) such that for some $k > 0$,

$$\int_0^T \int_{\mathbb{R}^n} |u(x,t)| e^{-k|x|^{2p/(2p-1)}} \mathrm{d}x \mathrm{d}t < \infty.$$

We can proceed by taking sequences of smooth (e.g., mollified) coefficients $\tilde{A}^q_{\alpha,l}$ so that

$$\tilde{A}^q_{\alpha,l}(x,t) \to \tilde{A}_{\alpha,l}(x,t), \quad q \to \infty,$$

pointwise for (x,t) in Ω. The Green's functions associated with these mollified coefficients integrate to identity by uniqueness, and this integral is obtained in the limit for G.

Finally, we can guarantee that the Green's function associated with the weak formulation satisfies the same property by noting that for the problem with mollified coefficients, the Green's function for the weak formulation will be the same as for the strong formulation by construction. Again, integration to identity is obtained in the limit.

So far, our discussion has centered around the case $\tilde{A}^{ij}_{0,l} \equiv 0$. At this point, we take advantage of our constructive approach to verify that the general case is a slight perturbation of this more restrictive case. First, we can write

$$\begin{aligned} K^l &= P^l + Q^l \\ \bar{K}^{l,\rho} &= \bar{P}^{l,\rho} + \bar{Q}^{l,\rho}, \end{aligned}$$

where Q^l and $\bar{Q}^{l,\rho}$ denote the portions of K^l and $\bar{K}^{l,\rho}$, respectively, that involve $\tilde{A}_{0,l}$. Precisely, from (3.13) and (3.14),

$$\begin{aligned} Q^l_{ik}(x,t;\xi,\tau) &= \sum_{j=1}^N \tilde{A}^{ij}_{0,l}(x,t) Z_{jk}(x-\xi,t;\xi,\tau); \quad \text{i.e., } Q^l = \tilde{A}_{0,l} Z \\ \bar{Q}^{l,\rho}_{im}(x,t;\xi,\tau) &= \sum_{j=1}^N \tilde{A}^{ij}_{0,l}(x,t) \frac{\partial Z_{jm}(x-\xi,t;\xi,\tau)}{\partial x_\rho}; \quad \text{i.e., } \bar{Q}^{l,\rho} = \tilde{A}_{0,l} \frac{\partial Z}{\partial x_\rho}. \end{aligned} \tag{3.23}$$

We also write

$$\Phi^l_\nu = \Phi^{l,1}_\nu + \Phi^{l,0}_\nu,$$

where $\Phi_\nu^{l,0}$ comprises all terms in Φ_ν^l [from (3.17) and (3.18)] that include $\tilde{A}_{0,l}$. Finally,

$$\Phi^l = \sum_{\nu=1}^{\infty} \Phi_\nu^l = \sum_{\nu=1}^{\infty} (\Phi_\nu^{l,1} + \Phi_\nu^{l,0}) =: \Phi^{l,1} + \Phi^{l,0}. \tag{3.24}$$

In this way, we can construct our Green's function (3.8) as

$$\begin{aligned} G(x,t;\xi,\tau) &= Z(x-\xi,t;\xi,\tau) \\ &\quad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y,t;y,\sigma) \\ &\quad \times \Big(\Phi^{\rho,1}(y,\sigma;\xi,\tau) + \Phi^{\rho,0}(y,\sigma;\xi,\tau)\Big) \mathrm{d}y\mathrm{d}\sigma. \end{aligned} \tag{3.25}$$

Noting that the Green's function for the case $\tilde{A}_{0,l}^{ij}(x,t) \equiv 0$ is precisely

$$\begin{aligned} G(x,t;\xi,\tau) &= Z(x-\xi,t;\xi,\tau) \\ &\quad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y,t;y,\sigma)\Phi^{\rho,1}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma, \end{aligned}$$

we have from (3.22) the useful relation

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big\{ Z(x-\xi,t;\xi,\tau) \\ &\quad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y,t;y,\sigma)\Phi^{\rho,1}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \Big\} \mathrm{d}\xi = I, \end{aligned} \tag{3.26}$$

for all (x,t) in Ω.

In order to estimate the remaining part of G, we first require an estimate on $\Phi^{l,0}$. We begin by noting that $\Phi_1^{l,0} = Q^l$, and generally (directly from (3.18))

$$\begin{aligned} \Phi_\nu^l(x,t;\xi,\tau) &= \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Big(\bar{P}^{l,\rho}(x,t;y,\sigma) + \bar{Q}^{l,\rho}(x,t;y,\sigma) \Big) \\ &\quad \times \Phi_{\nu-1}^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma. \end{aligned} \tag{3.27}$$

For $\nu = 2$,

$$\begin{aligned} \Phi_2^l(x,t;\xi,\tau) &= \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Big(\bar{P}^{l,\rho} + \bar{Q}^{l,\rho} \Big)(x,t;y,\sigma) \\ &\quad \times \Big(P^\rho + Q^\rho \Big)(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma. \end{aligned} \tag{3.28}$$

We clearly have three terms that involve Q^l and/or $\bar{Q}^{l,\rho}$. Each can be analyzed in the same way, so we focus on the choice

$$I := \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \bar{P}^{l,\rho}(x,t;y,\sigma) Q^\rho(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma.$$

Recalling definition (3.23) and the estimates of Lemmas 3.1 and 3.2, we have

$$|I| \le C_1 \int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-1-\frac{n-\gamma}{2p}} (\sigma-\tau)^{-\frac{n}{2p}} e^{-c_1(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}} \times e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y\mathrm{d}\sigma$$
$$\le C_2(t-\tau)^{-\frac{n}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \int_\tau^t (t-\sigma)^{-1+\frac{\gamma}{2p}} \mathrm{d}\sigma$$
$$\le C_3(t-\tau)^{-\frac{n-\gamma}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

In this calculation, we have used Remark 3.2. Proceeding similarly for the other two terms in $\Phi_2^{l,0}$, we conclude

$$|\Phi_2^{l,0}(x,t;\xi,\tau)| \le C(t-\tau)^{-\frac{n-\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}$$

for some constants c and C.

We see that $\Phi_2^{l,0}$ does not blow up as fast for $t \to \tau$ as does $\Phi_1^{l,0}$, and we can see from (3.27) that the rate of blow-up (or decay for ν sufficiently large) on $\Phi_\nu^{l,0}$ will generally be improved over the rate associated with $\Phi_{\nu-1}^{l,0}$ by a factor of $(t-\tau)^{\frac{\gamma}{2p}}$. In this way, we recognize that the leading order term for t sufficiently close to τ is $\Phi_1^{l,0} = Q^l$. We conclude the estimate

$$|\Phi^{l,0}(x,t;\xi,\tau)| \le C(t-\tau)^{-\frac{n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}. \tag{3.29}$$

We now state a lemma that will be fundamental to our analysis.

LEMMA 3.6. *Let assumptions (A1) and (A2) hold, and suppose G is defined as in* (3.8). *Then, there exists an $N \times N$ matrix function $R(x,t;\tau)$ and a constant C so that for any multi-index $0 \le |\alpha| < 2p-1$ in x, and for $0 \le \tau < t \le \tilde{T}$, with $\tilde{T}$ sufficiently small*

$$\int_{\mathbb{R}^n} G(x,t;\xi,\tau)d\xi = I + R(x,t;\tau),$$

and

$$|D_x^\alpha R(x,t;\tau)| \le C(t-\tau)^{1-\frac{1+|\alpha|}{2p}}, \tag{3.30}$$

for all $x \in \mathbb{R}^n$.

Proof Following the calculations leading up to Lemma 3.6, we see that all that remains is to establish the claimed estimates on

$$R(x,t;\tau) := \int_{\mathbb{R}^n} \left\{ \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n Z_{x_\rho}(x-y,t;y,\sigma)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \right\} \mathrm{d}\xi.$$

Formally, for each α described in the theorem's statement, we can write

$$D_x^\alpha R(x,t;\tau) = \int_{\mathbb{R}^n} \left\{ \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n D_x^\alpha Z_{x_\rho}(x-y,t;y,\sigma)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \right\}\mathrm{d}\xi,$$

from which we obtain the estimate

$$\begin{aligned}
|D_x^\alpha R(x,t;\tau)| &\le C_1 \int_{\mathbb{R}^n} \left\{ \int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n+|\alpha|+1}{2p}} (\sigma-\tau)^{-\frac{n}{2p}} \right. \\
&\qquad \left. \times e^{-c_1(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y\mathrm{d}\sigma \right\} \mathrm{d}\xi \\
&\le C_2 \int_{\mathbb{R}^n} (t-\tau)^{-\frac{n}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \int_\tau^t (t-\sigma)^{-\frac{|\alpha|+1}{2p}} \mathrm{d}\sigma\mathrm{d}\xi \\
&\le C_3 \int_{\mathbb{R}^n} (t-\tau)^{1-\frac{n+|\alpha|+1}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\xi \\
&\le C_4 (t-\tau)^{1-\frac{|\alpha|+1}{2p}}.
\end{aligned}$$

□

LEMMA 3.7. *Let assumptions (A1) and (A2) hold, and suppose G is defined as in (3.8). Then, there exist constant C and $\tilde{C}$ so that the following estimates hold for $0 \le \tau < t \le \tilde{T}$, with $\tilde{T}$ sufficiently small:*

(I) For any $x_1, x_2 \in \mathbb{R}^n$, $0 \le \tau < t \le \tilde{T}$,

$$\left| \int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau) \Big) d\xi \right| \le C(t-\tau)^{1-\frac{1+\gamma}{2p}} |x_1 - x_2|^\gamma;$$

(II) For any $x_1, x_2 \in \mathbb{R}^n$, $0 \le \tau < t \le \tilde{T}$, and for any $f \in C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$

$$\left| \int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau) \Big) f(\xi) d\xi \right| \le \tilde{C} |x_1 - x_2|^\gamma;$$

(III) For any $x \in \mathbb{R}^n$ and $0 \le \tau < t_1 < t_2 \le \tilde{T}$

$$\left| \int_{\mathbb{R}^n} \Big(G(x,t_1;\xi,\tau) - G(x,t_2;\xi,\tau) \Big) d\xi \right| \le C(t_2-\tau)^{1-\frac{1+\gamma}{2p}} (t_2 - t_1)^{\frac{\gamma}{2p}};$$

(IV) For any $x \in \mathbb{R}^n$ and $0 \le \tau < t_1 < t_2 \le \tilde{T}$, and for any $f \in C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$

$$\left| \int_{\mathbb{R}^n} \Big(G(x,t_1;\xi,\tau) - G(x,t_2;\xi,\tau) \Big) f(\xi) d\xi \right| \le \tilde{C} (t_2 - t_1)^{\frac{\gamma}{2p}};$$

The constants C and $\tilde{C}$ both depend on the bounds of the PDE coefficients and the Hölder constant for f. The constant C additionally depends on the Hölder constant associated with the coefficients $\tilde{A}_{\alpha,l}$, while the constant $\tilde{C}$ does not.

REMARK 3.5. *The final note regarding constants will be extremely important in our proof of the main theorem.*

Proof of Case (I). If we combine (3.25) with (3.26), we find

$$\int_{\mathbb{R}^n}\Big(G(x_1,t;\xi,\tau)-G(x_2,t;\xi,\tau)\Big)\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(x_1,x_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi,$$

where

$$\Delta Z_{x_\rho}(x_1,x_2):=Z_{x_\rho}(x_1-y,t;y,\sigma)-Z_{x_\rho}(x_2-y,t;y,\sigma).$$

(Here, and in similar instances below, we suppress dependence on certain variables for notational brevity).

At this point, we divide the analysis into two cases: (1) $|x_1-x_2|\le(t-\tau)^{\frac{1}{2p}}$; and (2) $|x_1-x_2|>(t-\tau)^{\frac{1}{2p}}$.

Case (1) $|x_1-x_2|\le(t-\tau)^{\frac{1}{2p}}$. For Case (1), we write

$$\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(x_1,x_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^n}\int_\tau^{t-\frac{1}{2}|x_1-x_2|^{2p}}\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(x_1,x_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$+\int_{\mathbb{R}^n}\int_{t-\frac{1}{2}|x_1-x_2|^{2p}}^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(x_1,x_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$=:I_1+I_2.$$

For I_1, we can apply the Mean Value Theorem to the components of Z_{x_ρ}. We have

$$Z^{ij}_{x_\rho}(x_1-y,t;\xi,\tau)-Z^{ij}_{x_\rho}(x_2-y,t;\xi,\tau)=D_xZ^{ij}_{x_\rho}(x^*-y,t;\xi,\tau)\cdot(x_1-x_2),\quad(3.31)$$

for some $x^*=x^*(x_1,x_2,y,\xi;t,\tau)$ (depending also on i and j) on the line between x_1 and x_2. According to Lemma 3.1

$$|D_xZ^{ij}_{x_\rho}(x^*-y,t;\xi,\tau)|\le C(t-\tau)^{-\frac{n+2}{2p}}e^{-c(\frac{|x^*-y|^{2p}}{(t-\tau)})^{1/(2p-1)}}.$$

We write

$$x_2-y=x_2-x^*+x^*-y\Rightarrow|x_2-y|\le|x_2-x^*|+|x^*-y|,$$

which implies

$$\Big(\frac{|x_2-y|^{2p}}{t-\tau}\Big)^{\frac{1}{2p-1}}\le 2^{\frac{2p}{2p-1}}\bigg\{\Big(\frac{|x_2-x^*|^{2p}}{t-\tau}\Big)^{\frac{1}{2p-1}}+\Big(\frac{|x^*-y|^{2p}}{t-\tau}\Big)^{\frac{1}{2p-1}}\bigg\}.$$

Upon rearranging this and raising expressions as exponents of e, we find

$$e^{-(\frac{|x^*-y|^{2p}}{t-\tau})^{1/(2p-1)}} \le e^{-2^{-\frac{2p}{2p-1}}(\frac{|x_2-y|^{2p}}{t-\tau})^{1/(2p-1)}} e^{(\frac{|x_2-x^*|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

Now, x^* is on the line between x_1 and x_2 so

$$|x_2 - x^*| \le |x_2 - x_1|.$$

For I_1, we have $\tau \le \sigma \le t - \frac{1}{2}|x_1 - x_2|^{2p}$, so that

$$t - \sigma \ge \frac{1}{2}|x_1 - x_2|^{2p} \ge \frac{1}{2}|x_2 - x^*|^{2p}.$$

It follows that

$$e^{(\frac{|x_2-x^*|^{2p}}{t-\sigma})^{1/(2p-1)}} \le e^{2^{1/(2p-1)}}.$$

We see that there exist constants C_1 and c_1, which can be expressed explicitly from the preceding considerations, so that

$$e^{-c(\frac{|x^*-y|^{2p}}{t-\tau})^{1/(2p-1)}} \le C_1 e^{-c_1(\frac{|x_2-y|^{2p}}{t-\tau})^{1/(2p-1)}}. \tag{3.32}$$

Combining these observations, we can compute (using Lemma 3.3)

$$\begin{aligned}|I_1| \le\ & C_2|x_2 - x_1| \int_{\mathbb{R}^n} \int_\tau^{t-\frac{1}{2}|x_1-x_2|^{2p}} \int_{\mathbb{R}^n} \Big\{ (t-\sigma)^{-\frac{n+2}{2p}} (\sigma - \tau)^{-\frac{n}{2p}} \\ & \times e^{-c_1(\frac{|x_2-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_2(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \Big\} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ \le\ & C_3|x_2 - x_1|(t-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_\tau^{t-\frac{1}{2}|x_1-x_2|^{2p}} (t-\sigma)^{-\frac{1}{p}} e^{-c_3(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi.\end{aligned}$$

Recalling again that $|x_1 - x_2| \le [2(t-\sigma)]^{1/2p}$, we have, for $\sigma \in [\tau, t - \frac{1}{2}|x_1 - x_2|^{2p}]$, the inequality

$$|x_1 - x_2|(t-\sigma)^{-\frac{1}{2p}} \le 2^{\frac{1-\gamma}{2p}} |x_2 - x_1|^\gamma (t-\sigma)^{-\frac{\gamma}{2p}}.$$

We have, then,

$$\begin{aligned}|I_1| \le\ & C_4|x_2 - x_1|^\gamma (t-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_\tau^{t-\frac{1}{2}|x_1-x_2|^{2p}} (t-\sigma)^{-\frac{1+\gamma}{2p}} e^{-c_3(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi \\ \le\ & C_5|x_2 - x_1|^\gamma (t-\tau)^{1-\frac{n+1+\gamma}{2p}} \int_{\mathbb{R}^n} e^{-c_3(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\xi \\ \le\ & C_6|x_2 - x_1|^\gamma (t-\tau)^{1-\frac{1+\gamma}{2p}}.\end{aligned}$$

For I_2, we use the more rudimentary estimate

$$|Z_{x_\rho}(x_1 - y, t; y, \sigma) - Z_{x_\rho}(x_2 - y, t; y, \sigma)| \\ \leq C(t-\sigma)^{-\frac{n+1}{2p}} \left\{ e^{-c(\frac{|x_1-y|^{2p}}{t-\sigma})^{1/(2p-1)}} + e^{-c(\frac{|x_2-y|^{2p}}{t-\sigma})^{1/(2p-1)}} \right\}. \quad (3.33)$$

Estimates on I_2 can be divided into two terms, one associated with each summand on the right-hand side of this last inequality. For notational convenience, we will express these as $I_1 = J_1 + J_2$. For J_1, we have

$$|J_1| \leq C_1 \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_1-x_2|^{2p}}^{t} \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n+1}{2p}} (\sigma-\tau)^{-\frac{n}{2p}} \\ \times e^{-c_1(\frac{|x_1-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ \leq C_2(t-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_1-x_2|^{2p}}^{t} (t-\sigma)^{-\frac{1}{2p}} e^{-c_2(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi. \quad (3.34)$$

At this point, we observe the integral

$$\int_{t-\frac{1}{2}|x_1-x_2|^{2p}}^{t} (t-\sigma)^{-\frac{1}{2p}} \mathrm{d}\sigma = -\frac{1}{1-\frac{1}{2p}} (t-\sigma)^{1-\frac{1}{2p}} \Big|_{t-\frac{1}{2}|x_1-x_2|^{2p}}^{t} \\ = -\frac{1}{1-\frac{1}{2p}} \left(\frac{1}{2}|x_1-x_2|^{2p} \right)^{(1-\frac{1}{2p})} = -\frac{1}{1-\frac{1}{2p}} |x_1 - x_2 f|^{2p-1}.$$

We can conclude the estimate

$$|J_1| \leq C_2 |x_1 - x_2|^{2p-1}.$$

Finally, recalling that we remain in Case (1), we have

$$|x_1 - x_2|^{2p-1} = |x_1 - x_2|^\gamma |x_1 - x_2|^{2p-1-\gamma} \leq |x_1 - x_2|^\gamma (t-\tau)^{\frac{2p-1-\gamma}{2p}},$$

which gives the claimed estimate. The analysis of J_2 is almost identical.

Case (2) $|x_1 - x_2| > (t-\tau)^{\frac{1}{2p}}$. For Case (2), we again use (3.33), which again leads to two terms, which we designate $I_2 = J_1 + J_2$. (We recall our convention that even when we have expressed I_1 as a sum $J_1 + J_2$, we write $I_2 = J_1 + J_2$ with a new choice of J_1 and J_2). Proceeding as in (3.34), we compute

$$|J_1| \leq C_1(t-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{\tau}^{t} (t-\sigma)^{-\frac{1}{2p}} e^{-c_2(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi \\ \leq C_2(t-\tau)^{1-\frac{1}{2p}} \leq C_2(t-\tau)^{1-\frac{1+\gamma}{2p}} |x_1 - x_2|^\gamma,$$

where in obtaining this final inequality, we have observed that in Case (2) $(t-\tau)^{\frac{\gamma}{2p}} < |x_1 - x_2|^\gamma$.

This establishes Case (I) of the lemma.

Proof of Case (II). For Case (II), we divide the analysis into the same two subcases as we used in Case (I).

Case (1) $|x_1 - x_2| \leq (t-\tau)^{\frac{1}{2p}}$. For Case (1), we write

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau)\Big) f(\xi)\mathrm{d}\xi\\&= \int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau)\Big)\Big(f(\xi) - f(x_2)\Big)\mathrm{d}\xi\\&\quad + \int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau)\Big) f(x_2)\mathrm{d}\xi = I_1 + I_2.\end{aligned}$$

For I_1, we apply the Mean Value Theorem to G, similarly as applied to Z_{x_ρ} in (3.31) to obtain the estimate

$$|G_{ij}(x_1,t;\xi,\tau) - G_{ij}(x_2,t;\xi,\tau)| \leq C|x_1 - x_2|(t-\tau)^{-\frac{n+1}{2p}} e^{-c(\frac{|x^*-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for some positive constants c and C, and for $x^* = x^*(x_1, x_2, t, \xi, \tau)$ (depending also on i and j) on the line between x_1 and x_2. Using (3.32) with ξ replacing y, we obtain the inequality

$$e^{-c(\frac{|x^*-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \leq C_1 e^{-c_1(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

In this way, we can write

$$\begin{aligned}|I_1| &\leq C_2|x_1 - x_2|(t-\tau)^{-\frac{n+1}{2p}} \int_{\mathbb{R}^n} e^{-c_1(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} |\xi - x_2|^\gamma \mathrm{d}\xi\\&\leq C_3|x_1 - x_2|(t-\tau)^{-\frac{1-\gamma}{2p}} \leq C_3|x_1 - x_2|^\gamma,\end{aligned}$$

where in obtaining the penultimate inequality, we have used the idea of (3.20), while in obtaining the final inequality we have simply used the inequality defining Case (1). We emphasize that C_3, denoted $\tilde{C}$ in the statement of our lemma, depends on the Hölder constant associated with f, but not on the Hölder constants associated with the PDE coefficients $\tilde{A}^{ij}_{\alpha,l}$.

In this case, we obtain a much smaller term from I_2 by directly applying the result of Case (I) from the lemma.

Case (2) $|x_1 - x_2| < (t-\tau)^{\frac{1}{2p}}$. For Case (2), we write

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(G(x_1,t;\xi,\tau) - G(x_2,t;\xi,\tau)\Big) f(\xi)\mathrm{d}\xi\\&= \int_{\mathbb{R}^n} G(x_1,t;\xi,\tau)(f(\xi) - f(x_1))\mathrm{d}\xi - \int_{\mathbb{R}^n} G(x_2,t;\xi,\tau)(f(\xi) - f(x_2))\mathrm{d}\xi\\&\quad + \int_{\mathbb{R}^n} G(x_1,t;\xi,\tau)(f(x_1) - f(x_2))\mathrm{d}\xi\end{aligned}$$

$$+\int_{\mathbb{R}^n}\Big(G(x_1,t;\xi,\tau)-G(x_2,t;\xi,\tau)\Big)f(x_2)\mathrm{d}\xi$$
$$=: I_1+I_2+I_3+I_4.$$

For I_1, we have from Lemma 3.5

$$|I_1|\le C_1(t-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n} e^{-c_1(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}|\xi-x_1|^\gamma\mathrm{d}\xi \le C_2(t-\tau)^{\frac{\gamma}{2p}}\le C_2|x_1-x_2|^\gamma,$$

where we have used (3.20) and the inequality defining Case (2). The analysis of I_2 is clearly the same as that of I_1, resulting in the same estimate. For I_3, we have

$$|I_3|\le C_1(t-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n} e^{-c_1(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}|x_2-x_1|^\gamma\mathrm{d}\xi\le C_2|x_1-x_2|^\gamma.$$

Finally, using the estimate from Case (I), we see that I_4 is much smaller.

Proof of Case (III). If we combine (3.25) with (3.26), we find

$$\int_{\mathbb{R}^n}\Big(G(x,t_1;\xi,\tau)-G(x,t_2;\xi,\tau)\Big)\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^n}\int_\tau^{t_1}\int_{\mathbb{R}^n}\sum_{\rho=1}^n Z_{x_\rho}(x-y,t_1;y,\sigma)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$-\int_{\mathbb{R}^n}\int_\tau^{t_2}\int_{\mathbb{R}^n}\sum_{\rho=1}^n Z_{x_\rho}(x-y,t_2;y,\sigma)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$

It will be convenient to rearrange the right-hand side of this last relation as

$$\int_{\mathbb{R}^n}\int_\tau^{t_1}\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(t_1,t_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$-\int_{\mathbb{R}^n}\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\sum_{\rho=1}^n Z_{x_\rho}(x-y,t_2;y,\sigma)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi =: I_1+I_2, \tag{3.35}$$

where

$$\Delta Z_{x_\rho}(t_1,t_2):=Z_{x_\rho}(x-y,t_1;y,\sigma)-Z_{x_\rho}(x-y,t_2;y,\sigma).$$

At this point, we divide the analysis into two cases, in precisely the same spirit as our analyses of (I) and (II): (1) $t_2-t_1\le t_1-\tau$; and (2) $t_2-t_1>t_1-\tau$.

Case (1) $t_2-t_1\le t_1-\tau$. For Case (1), we observe that $\tau\le t_1-\frac{1}{2}(t_2-t_1)\le t_1$, allowing us to write

$$I_1=\int_{\mathbb{R}^n}\int_\tau^{t_1-\frac{1}{2}(t_2-t_1)}\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(t_1,t_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$+\int_{\mathbb{R}^n}\int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1}\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(t_1,t_2)\Phi^{\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$=: J_1+J_2.$$

For J_1, we apply the Mean Value Theorem to Z_{x_ρ} in t, noting that estimates on t-derivatives of Z can be obtained from Lemma 3.1 and the defining relation (3.11). Proceeding similarly as in (3.31), we find that for each component Z^{ij}

$$\begin{aligned}\left|Z^{ij}_{x_\rho}(x-y,t_1;y,\sigma)-Z^{ij}_{x_\rho}(x-y,t_2;y,\sigma)\right|\\ \le C_1(t^*-\sigma)^{-1-\frac{n+1}{2p}}e^{-c_1(\frac{|x-y|^{2p}}{t^*-\sigma})^{1/(2p-1)}}(t_2-t_1),\end{aligned} \tag{3.36}$$

for some constants c_1 and C_1 and some value $t^*=t^*(t_1,t_2,x,y,\sigma)$ between t_1 and t_2. For J_1, we have

$$t_1-\sigma\le t^*-\sigma<3(t_1-\sigma),$$

so that for new constants c_2 and C_2, we have

$$\begin{aligned}\left|Z^{ij}_{x_\rho}(x-y,t_1;y,\sigma)-Z^{ij}_{x_\rho}(x-y,t_2;y,\sigma)\right|\\ \le C_2(t_1-\sigma)^{-1-\frac{n+1}{2p}}e^{-c_2(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}}(t_2-t_1).\end{aligned}$$

We obtain the inequality

$$\begin{aligned}|J_1|&\le C_3(t_2-t_1)\int_{\mathbb{R}^n}\int_\tau^{t_1-\frac{1}{2}(t_2-t_1)}\int_{\mathbb{R}^n}(t_1-\sigma)^{-1-\frac{n+1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}\\ &\quad\times e^{-c_2(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}}e^{-c_2(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\ &\le C_4(t_2-t_1)(t_1-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_\tau^{t_1-\frac{1}{2}(t_2-t_1)}(t_1-\sigma)^{-1-\frac{1}{2p}}e^{-c_3(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.\end{aligned}$$

Carrying out the integration over σ explicitly, and recalling that in Case (1) $(t_1-\tau)^{-1/2p}\le(t_2-t_1)^{-1/2p}$, we obtain the estimate

$$\begin{aligned}|J_1|&\le C_5(t_2-t_1)^{1-\frac{1}{2p}}(t_1-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}e^{-c_3(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\xi\\ &\le C_6(t_2-t_1)^{1-\frac{1}{2p}}\le C_6(t_2-t_1)^{\frac{\gamma}{2p}}(t_1-\tau)^{1-\frac{1+\gamma}{2p}}.\end{aligned}$$

For J_2, we use the idea of (3.33) to obtain an estimate by two terms, which we denote $J_2=K_1+K_2$. For the first,

$$\begin{aligned}|K_1|&\le C_1\int_{\mathbb{R}^n}\int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1}\int_{\mathbb{R}^n}(t_1-\sigma)^{-\frac{n+1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}\\ &\quad\times e^{-c_2(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}}e^{-c_2(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\ &\le C_2(t_1-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1}(t_1-\sigma)^{-\frac{1}{2p}}e^{-c_3(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.\end{aligned}$$

Carrying out the integration over σ explicitly, we obtain an estimate by

$$|K_1| \le C_3(t_1-\tau)^{-\frac{n}{2p}}(t_2-t_1)^{1-\frac{1}{2p}}\int_{\mathbb{R}^n} e^{-c_3(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\xi$$
$$\le C_4(t_2-t_1)^{1-\frac{1}{2p}} \le C_4(t_2-t_1)^{\frac{\gamma}{2p}}(t_1-\tau)^{1-\frac{1+\gamma}{2p}}.$$

The term for K_2 can be analyzed similarly, completing the analysis of J_2, which completes the analysis of I_1 [from (3.35)].

For I_2, we use the estimates of Lemma 3.1 to write

$$|I_2| \le C_1\int_{\mathbb{R}^n}\int_{t_1}^{t_2}\int_{\mathbb{R}^n}(t_2-\sigma)^{-\frac{n+1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}$$
$$\times e^{-c_1(\frac{|x-y|^{2p}}{t_2-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$\le C_2(t_2-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_{t_1}^{t_2}(t_2-\sigma)^{-\frac{1}{2p}}e^{-c_2(\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.$$

Once again carrying out the integration over σ explicitly, we estimate

$$|I_2| \le C_3(t_2-\tau)^{-\frac{n}{2p}}(t_2-t_1)^{1-\frac{1}{2p}}\int_{\mathbb{R}^n} e^{-c_2(\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}}\mathrm{d}\xi$$
$$\le C_4(t_2-t_1)^{1-\frac{1}{2p}} \le C_4(t_2-t_1)^{\frac{\gamma}{2p}}(t_2-\tau)^{1-\frac{1+\gamma}{2p}}.$$

Case (2) $t_2-t_1 > t_1-\tau$. For Case (2), we use the idea of (3.33) (with different values of t instead of different values of x), and we express the resulting two terms as $I_1 = J_1 + J_2$. For J_1, we write

$$|J_1| \le C_1\int_{\mathbb{R}^n}\int_{\tau}^{t_1}\int_{\mathbb{R}^n}(t_1-\sigma)^{-\frac{n+1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}$$
$$\times e^{-c_1(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$\le C_2(t_1-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_{\tau}^{t_1}(t_1-\sigma)^{-\frac{1}{2p}}e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.$$

Carrying out the integration over σ explicitly, we estimate

$$|J_1| \le C_3(t_1-\tau)^{1-\frac{n+1}{2p}}\int_{\mathbb{R}^n} e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}}\mathrm{d}\xi$$
$$\le C_4(t_1-\tau)^{1-\frac{1}{2p}} \le C_4(t_2-t_1)^{\frac{\gamma}{2p}}(t_1-\tau)^{1-\frac{1+\gamma}{2p}}.$$

The analysis of J_2 is similar.

Finally, for I_2, we have

$$|I_2| \le C_1\int_{\mathbb{R}^n}\int_{t_1}^{t_2}\int_{\mathbb{R}^n}(t_2-\sigma)^{-\frac{n+1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}$$
$$\times e^{-c_1(\frac{|x-y|^{2p}}{t_2-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$\le C_2(t_2-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_{t_1}^{t_2}(t_2-\sigma)^{-\frac{1}{2p}}e^{-c_2(\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.$$

In this case, we obtain

$$|I_2| \le C_3(t_2-\tau)^{-\frac{n}{2p}}(t_2-t_1)^{1-\frac{1}{2p}} \int_{\mathbb{R}^n} e^{-c_2(\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}} \mathrm{d}\xi$$
$$\le C_4(t_2-t_1)^{1-\frac{1}{2p}} \le C_4(t_2-t_1)^{\frac{\gamma}{2p}}(t_2-\tau)^{1-\frac{1+\gamma}{2p}}.$$

Proof of Case (IV). For Case (IV), we divide the analysis into the same two subcases as we used in Case (III).

Case (1) $t_2 - t_1 \le t_1 - \tau$. For Case (1), we write

$$\int_{\mathbb{R}^n} \Big(G(x,t_1;\xi,\tau) - G(x,t_2;\xi,\tau) \Big) f(\xi)\mathrm{d}\xi$$
$$= \int_{\mathbb{R}^n} \Big(G(x,t_1;\xi,\tau) - G(x,t_2;\xi,\tau) \Big)\Big(f(\xi) - f(x) \Big)\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^n} \Big(G(x,t_1;\xi,\tau) - G(x,t_2;\xi,\tau) \Big) f(x)\mathrm{d}\xi = I_1 + I_2.$$

For I_1, we would like to apply the Mean Value Theorem as in our analysis of Case (II), but we must keep in mind that when working with the weak formulation, G is not necessarily differentiable in t. Using (3.8), we can write

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} \Delta Z(t_1,t_2)(f(\xi)-f(x))\mathrm{d}\xi \\ &\quad + \int_{\mathbb{R}^n}\int_\tau^{t_1}\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(t_1,t_2)\Phi^\rho(y,\sigma;\xi,\tau)(f(\xi)-f(x))\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\ &\quad - \int_{\mathbb{R}^n}\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\sum_{\rho=1}^n Z_{x_\rho}(x-y,t_2;y,\sigma)\Phi^\rho(y,\sigma;\xi,\tau)(f(\xi)-f(x))\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\ &=: J_1 + J_2 + J_3, \end{aligned} \tag{3.37}$$

where

$$\Delta Z(t_1,t_2) := Z(x-\xi,t_1;\xi,\tau) - Z(x-\xi,t_2;\xi,\tau)$$
$$\Delta Z_{x_\rho}(t_1,t_2) := Z_{x_\rho}(x-y,t_1;y,\sigma) - Z_{x_\rho}(x-y,t_2;y,\sigma).$$

For J_1, we apply the Mean Value Theorem to the components of Z to obtain estimates of the form

$$\begin{aligned} &\Big| Z^{ij}(x-\xi,t_1;\xi,\tau) - Z^{ij}(x-\xi,t_2;\xi,\tau) \Big| \\ &\quad \le C_1(t^*-\tau)^{-1-\frac{n}{2p}} e^{-c_1(\frac{|x-\xi|^{2p}}{t^*-\tau})^{1/(2p-1)}}(t_2-t_1), \end{aligned} \tag{3.38}$$

where $t^* = t^*(t_1,t_2,x,\xi,\tau)$ (depending also on i and j) is a value between t_1 and t_2. In Case (1), $(t_2-t_1) \le (t^*-\tau)$ and

$$\frac{1}{2}(t_2-\tau) \le (t^*-\tau) \le 2(t_2-\tau).$$

Combining these inequalities, we can conclude

$$|J_1| \leq C_2(t_2-t_1)^{\frac{\gamma}{2p}} \int_{\mathbb{R}^n} (t_2-\tau)^{-\frac{\gamma}{2p}-\frac{n}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}} |\xi - x|^{\gamma} \mathrm{d}\xi$$
$$\leq C_3(t_2-t_1)^{\frac{\gamma}{2p}}.$$

For J_2, we write

$$J_2 = \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Delta Z_{x_\rho}(t_1,t_2) \Phi^{\rho}(y,\sigma;\xi,\tau)(f(\xi)-f(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi$$
$$\int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Delta Z_{x_\rho}(t_1,t_2) \Phi^{\rho}(y,\sigma;\xi,\tau)(f(\xi)-f(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi$$
$$=: K_1 + K_2. \tag{3.39}$$

For K_1 since $\sigma < t_1 - \frac{1}{2}(t_2-t_1)$, we can use the Mean Value Theorem again. In this case,

$$\frac{1}{2}(t_2-t_1) < t^* - \sigma < 3(t_1-\sigma),$$

and we obtain an inequality

$$|K_1| \leq C_1(t_2-t_1)^{\frac{\gamma}{2p}} \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} \int_{\mathbb{R}^n} \Big\{ (t_1-\sigma)^{-\frac{\gamma}{2p}-\frac{n+1}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}}$$
$$\times e^{-c_1(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi-x|^{\gamma} \Big\} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi$$
$$\leq C_2(t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} \Big\{ (t_1-\sigma)^{-\frac{\gamma}{2p}-\frac{1}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}}$$
$$\times e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} |\xi-x|^{\gamma} \Big\} d\sigma \mathrm{d}\xi.$$

Carrying out the remaining two integrals, we find

$$|K_1| \leq C_3(t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} |\xi-x|^{\gamma} \mathrm{d}\xi$$
$$\leq C_4(t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{\frac{\gamma}{2p}}.$$

For $t_1 - \tau$ small, this gives the claimed estimate with an arbitrarily small choice of constant $\tilde{C}$. (This last point is important, because C_4 depends on the Hölder constant for the PDE coefficients, and $\tilde{C}$ does not).

For K_2, we proceed by writing

$$\begin{aligned} K_2 \,=\, & \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y, t_1; y, \sigma) \Phi^\rho(y, \sigma; \xi, \tau) \\ & \times (f(\xi) - f(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ & - \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_2} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{x_\rho}(x-y, t_2; y, \sigma) \Phi^\rho(y, \sigma; \xi, \tau) \\ & \times (f(\xi) - f(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ =: \, & L_1 + L_2. \end{aligned}$$

For L_1, we estimate

$$\begin{aligned} |L_1| \le & \, C_1 \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} \Big\{ (t_1-\sigma)^{-\frac{n+1}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \\ & \times e^{-c_1(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \Big\} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ \le & \, C_2 (t_1-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} (t_1-\sigma)^{-\frac{1}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \\ & \times e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} |\xi - x|^\gamma d\sigma \mathrm{d}\xi. \end{aligned}$$

Recalling that in this case $(\sigma - \tau) > \frac{1}{2}(t_1 - \tau)$, we obtain

$$\begin{aligned} |L_1| \le & \, C_3 (t_1-\tau)^{-1+\frac{1+\gamma-n}{2p}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} (t_1-\sigma)^{-\frac{1}{2p}} \\ & \times e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} |\xi - x|^\gamma d\sigma \mathrm{d}\xi \\ \le & \, C_4 (t_1-\tau)^{-1+\frac{1+\gamma-n}{2p}} (t_2-t_1)^{1-\frac{1}{2p}} \int_{\mathbb{R}^n} e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \mathrm{d}\xi \\ \le & \, C_5 (t_1-\tau)^{-1+\frac{1+2\gamma}{2p}} (t_2-t_1)^{1-\frac{1}{2p}}. \end{aligned}$$

In this case $t_2 - t_1 < t_1 - \tau$, and using this, we conclude

$$|L_1| \le C_5 (t_1-\tau)^{\frac{\gamma}{2p}} (t_2-t_1)^{\frac{\gamma}{2p}},$$

which is sufficient.

The expression L_2 can be analyzed similarly, and this finishes the analysis of K_2 and thus of J_2 [from (3.37)].

For J_3, we estimate directly

$$|J_3| \le C_1 \int_{\mathbb{R}^n} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big\{ (t_2-\sigma)^{-\frac{n+1}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \times e^{-c_1 (\frac{|x-y|^{2p}}{t_2-\sigma})^{1/(2p-1)}} e^{-c_1 (\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \Big\} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi$$
$$\le C_2 (t_2-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t_1}^{t_2} \Big\{ (t_2-\sigma)^{-\frac{1}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \times e^{-c_2 (\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \Big\} \mathrm{d}\sigma \mathrm{d}\xi.$$

We find

$$|J_3| \le C_3 (t_2-\tau)^{-\frac{n}{2p}} (t_2-t_1)^{\frac{\gamma}{2p}} \int_{\mathbb{R}^n} e^{-c_2 (\frac{|x-\xi|^{2p}}{t_2-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \mathrm{d}\xi$$
$$\le C_4 (t_2-\tau)^{\frac{\gamma}{2p}} (t_2-t_1)^{\frac{\gamma}{2p}},$$

which is sufficient for $t_2 - \tau$ sufficiently small.

This concludes the analysis of I_1. Since I_2 can be understood from Case (III), the argument is complete. □

4. Estimates for the contraction argument

In this section, we gather some important preliminary observations that will be used in our contraction mapping argument. Given some function $u^\tau \in C^\gamma(\mathbb{R}^n)$, for some Hölder index $0 < \gamma < 1$, we will work with the metric space

$$\mathcal{S} := \{u \in C^{\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [\tau, \tilde{T}]) : u(x,\tau) = u^\tau(x), \|u\|_{C^{\gamma,\frac{\gamma}{2p}}} \le K\}, \tag{4.1}$$

for some constant $K > 0$ and some sufficiently small time $\tilde{T} > 0$. Here,

$$\|u\|_{C^{\gamma,\frac{\gamma}{2p}}} := \sup_{\substack{x\in\mathbb{R}^n \\ t\in[\tau,\tilde{T}]}} |u(x,t)| + \sup_{\substack{x_1,x_2\in\mathbb{R}^n, x_1\ne x_2 \\ t\in[\tau,\tilde{T}]}} \frac{|u(x_1,t)-u(x_2,t)|}{|x_1-x_2|^\gamma} + \sup_{\substack{x\in\mathbb{R}^n \\ t_1,t_2\in[\tau,\tilde{T}], t_1\ne t_2}} \frac{|u(x,t_1)-u(x,t_2)|}{|t_1-t_2|^{\frac{\gamma}{2p}}}. \tag{4.2}$$

We recall that given any $\tilde{u} \in \mathcal{S}$, we can define the associated linear problem (3.1), and we denote by $Z^{\tilde{u}}$ the parametrix associated with this problem, and by $\Phi^{\tilde{u},\rho,1}$ and $\Phi^{\tilde{u},\rho,0}$ the respective $\Phi^{\rho,1}$ and $\Phi^{\rho,0}$ [as defined in (3.24)]. In what follows, we drop the tilde notation for convenience.

We will set

$$\begin{aligned}\Delta Z(u,v) &:= Z^u(x-\xi,t;\xi,\tau) - Z^v(x-\xi,t;\xi,\tau)\\ \Delta Z_t(u,v) &:= Z^u_t(x-\xi,t;\xi,\tau) - Z^v_t(x-\xi,t;\xi,\tau)\\ \Delta\Phi^l(u,v) &:= \Phi^{u,l}(x,t;\xi,\tau) - \Phi^{v,l}(x,t;\xi,\tau)\\ \Delta\Phi^{l,0}(u,v) &:= \Phi^{u,l,0}(x,t;\xi,\tau) - \Phi^{v,l,0}(x,t;\xi,\tau).\end{aligned} \tag{4.3}$$

LEMMA 4.1. *Suppose ($\mathcal{P}$) and (W1)–(W2) hold, $u, v \in \mathcal{S}$, and Z^u, Z^v satisfy* (3.11) *with, respectively, $\tilde{A}^{ij}_{\alpha,l}(x,t) = A^{ij}_{\alpha,l}(u(x,t),x,t)$ and $\tilde{A}^{ij}_{\alpha,l}(x,t) = A^{ij}_{\alpha,l}(v(x,t),x,t)$. Then, for $0<\tau<t\le\tilde{T}$, with $\tilde{T}$ sufficiently small, and for any multi-index α, there exist constants c, c_α and C, C_α so that*

(I)

$$\left|D^\alpha_x \Delta Z(u,v)\right| \le C_\alpha \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{\frac{\gamma-n-|\alpha|}{2p}} e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}$$

$$\left|D^\alpha_x \Delta Z_t(u,v)\right| \le C_\alpha \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-1+\frac{\gamma-n-|\alpha|}{2p}} e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

(II)

$$\left|\Delta\Phi^l(u,v)\right| \le C \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-1+\frac{1+\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}$$

$$\left|\Delta\Phi^{l,0}(u,v)\right| \le C \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for each $l \in \{1,2,\ldots,n\}$.

Remark on the proof of Part (I). The proof of Part (I) closely follows Friedman's proof of Lemma 9.3.3 in [1], and we omit most of the details. In obtaining our formulation, we use one additional fact,

$$\sup_{x\in\mathbb{R}} |u(x,t)-v(x,t)| \le \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{\frac{\gamma}{2p}}. \tag{4.4}$$

This is clear since, by definition

$$\left|(u(x,t_1)-v(x,t_1)) - (u(x,t_2)-v(x,t_2))\right| \le \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{\frac{\gamma}{2p}},$$

for all $t_1, t_2 \in [\tau,\tilde{T}]$, $t_1 \ne t_2$. The claim is immediate upon taking $t_2 = t$ and $t_1 = \tau$ (keeping in mind that $u, v \in \mathcal{S} \Rightarrow u(x,\tau) = v(x,\tau) = u^\tau(x)$).

Proof of Part (II). We begin by writing

$$\Phi^{u,l} = \sum_{\nu=1}^{\infty} \Phi^{u,l}_\nu,$$

where $\Phi_1^{u,l} = K^{u,l}$ and for $\nu = 2, 3, \dots$

$$\Phi_\nu^{u,l}(x,t;\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \bar{K}^{u,l,\rho}(x,t;y,\sigma)\Phi_{\nu-1}^{u,\rho}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma;$$

i.e., (3.17) and (3.18) in our current notation. Here, we denote by $K^{u,l}$ and $\bar{K}^{u,l,\rho}$, respectively, the expressions for K^l and $\bar{K}^{l,\rho}$ [from (3.13) and (3.14)] associated with u.

Noting that

$$\Phi^{u,l} - \Phi^{v,l} = \sum_{\nu=1}^\infty \left(\Phi_\nu^{u,l} - \Phi_\nu^{v,l}\right), \tag{4.5}$$

we consider the differences $\Phi_\nu^{u,l} - \Phi_\nu^{v,l}$, beginning with $\nu = 1$. In this case, we have

$$\begin{aligned}
&\left(\Phi_1^{u,l}(x,t;\xi,\tau) - \Phi_1^{v,l}(x,t;\xi,\tau)\right)_{ik} = \left(K^{u,l}(x,t;\xi,\tau) - K^{v,l}(x,t;\xi,\tau)\right)_{ik}\\
&= -\sum_{j=1}^N \sum_{|\alpha|=2p-1} \left\{\left(A_{\alpha,l}^{ij}(u,\xi,t) - A_{\alpha,l}^{ij}(u,x,t)\right)D_x^\alpha Z_{jk}^u\right.\\
&\quad \left.-\left(A_{\alpha,l}^{ij}(v,\xi,t) - A_{\alpha,l}^{ij}(v,x,t)\right)D_x^\alpha Z_{jk}^v\right\}\\
&\quad +\sum_{j=1}^N \sum_{|\alpha|\le 2p-2} \left\{A_{\alpha,l}^{ij}(u,x,t)D_x^\alpha Z_{jk}^u - A_{\alpha,l}^{ij}(v,x,t)D_x^\alpha Z_{jk}^v\right\} =: I_1 + I_2,
\end{aligned}$$

where for notational brevity we have omitted the dependence of Z on $(x,t;\xi,\tau)$, and where u and v always depend on (x,t) or (ξ,t), consistent with the remaining dependence of $A_{\alpha,l}^{ij}$. We can rearrange I_1 as

$$\begin{aligned}
I_1 &= -\sum_{j=1}^N \sum_{|\alpha|=2p-1} \left\{\left(A_{\alpha,l}^{ij}(u,\xi,t) - A_{\alpha,l}^{ij}(u,x,t)\right)\left(D_x^\alpha Z_{jk}^u - D_x^\alpha Z_{jk}^v\right)\right\}\\
&\quad -\sum_{j=1}^N \sum_{|\alpha|=2p-1} \left\{\left(A_{\alpha,l}^{ij}(u,\xi,t) - A_{\alpha,l}^{ij}(v,\xi,t)\right) - \left(A_{\alpha,l}^{ij}(u,x,t)\right.\right.\\
&\quad \left.\left.-A_{\alpha,l}^{ij}(v,x,t)\right)\right\} D_x^\alpha Z_{jk}^v\\
&=: J_1 + J_2.
\end{aligned}$$

For J_1, we note, using (W2) and the fact that $u \in \mathcal{S}$,

$$\left|A_{\alpha,l}^{ij}(u,\xi,t) - A_{\alpha,l}^{ij}(u,x,t)\right| \le C_1\left(|u(\xi,t) - u(x,t)| + |\xi - x|^\gamma\right) \le C_2|\xi - x|^\gamma.$$

Using in addition Part I of this lemma and the idea of (3.20), we find

$$|J_1| \leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-1+\frac{1+2\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

For J_2, we note

$$\begin{aligned}\left|A^{ij}_{\alpha,l}(u,\xi,t)-A^{ij}_{\alpha,l}(v,\xi,t)\right| &\leq C|u(\xi,t)-v(x,t)| \\ &\leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{\frac{\gamma}{2p}}, \end{aligned} \tag{4.6}$$

and conclude

$$|J_2| \leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-1+\frac{1+\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

(If we assume greater regularity on the coefficients $A^{ij}_{\alpha,l}$ [i.e., we assume (S1)–(S2)], we can recover the estimate for J_1; in fact, this is the primary reason for taking this choice of arrangement).

Proceeding similarly for I_2, we can take advantage of the lower-order derivative to obtain an estimate that is smaller than the estimate on I_1 for $t-\tau$ sufficiently small. This concludes the analysis for $\nu=1$.

According to our definitions, we have obtained an estimate on the difference $|K^{u,l}-K^{v,l}|$, and by almost precisely the same calculation, we can verify

$$\begin{aligned}&\left|\bar{K}^{u,l,\rho}(x,t;\xi,\tau)-\bar{K}^{u,l,\rho}(x,t;\xi,\tau)\right| \\ &\quad\leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-1+\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.\end{aligned} \tag{4.7}$$

For $\nu=2,3,\ldots$, we proceed by writing

$$\begin{aligned}&\Phi^{u,l}_{\nu}(x,t;\xi,\tau)-\Phi^{v,l}_{\nu}(x,t;\xi,\tau)\\ &=\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n\Big\{\bar{K}^{u,l,\rho}(x,t;y,\sigma)\Phi^{u,\rho}_{\nu-1}(y,\sigma;\xi,\tau)\\ &\quad-\bar{K}^{v,l,\rho}(x,t;y,\sigma)\Phi^{v,\rho}_{\nu-1}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma\\ &=\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n\Big(\bar{K}^{u,l,\rho}(x,t;y,\sigma)-\bar{K}^{v,l,\rho}(x,t;y,\sigma)\Big)\Phi^{u,\rho}_{\nu-1}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\\ &\quad+\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n\bar{K}^{v,l,\rho}(x,t;y,\sigma)\Big(\Phi^{u,\rho}_{\nu-1}(y,\sigma;\xi,\tau)-\Phi^{v,\rho}_{\nu-1}(y,\sigma;\xi,\tau)\Big)\mathrm{d}y\mathrm{d}\sigma\\ &=: I_1+I_2^{\nu-1},\end{aligned}$$

the superscript on I_2 serving as an index rather than a power. We can estimate I_1 as in the case $\nu=1$ and obtain the same estimate we found for J_1 in that case. For $I_2^{\nu-1}$, we can use the estimate for $\nu = 1$ to show that I_2^1 satisfies the same bound as J_1 (from the case $\nu = 1$). In this way, the difference $\Phi_2^{u,l} - \Phi_2^{v,l}$ is smaller than the difference $\Phi_1^{u,l} - \Phi_1^{v,l}$, and the general case follows by induction.

For the final estimate of Lemma 4.1, we begin by observing $\Phi_1^{u,l,0} = Q^{u,l}$. Computing directly, we find

$$\begin{aligned}
&\Phi_1^{u,l,0}(x,t;\xi,\tau) - \Phi_1^{v,l,0}(x,t;\xi,\tau)\\
&\quad = A_{0,l}(u,x,t)Z^u(x-\xi,t;\xi,\tau) - A_{0,l}(v,x,t)Z^v(x-\xi,t;\xi,\tau)\\
&\quad = \Big(A_{0,l}(u,x,t) - A_{0,l}(v,x,t)\Big)Z^u(x-\xi,t;\xi,\tau)\\
&\qquad + A_{0,l}(v,x,t)\Big(Z^u(x-\xi,t;\xi,\tau) - Z^v(x-\xi,t;\xi,\tau)\Big) =: I_1 + I_2. \qquad (4.8)
\end{aligned}$$

Proceeding with (4.6) and Part I of the lemma, we conclude

$$|I_1| + |I_2| \leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

Likewise, for $\nu = 2, 3, \dots,$ we write

$$\begin{aligned}
&\Phi^{u,l,0}(x,t;\xi,\tau) - \Phi^{v,l,0}(x,t;\xi,\tau)\\
&\quad = \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{\bar{P}^{u,l,\rho}(x,t;y,\sigma)\Phi_{\nu-1}^{u,\rho,0}(y,\sigma;\xi,\tau)\\
&\qquad - \bar{P}^{v,l,\rho}(x,t;y,\sigma)\Phi_{\nu-1}^{v,\rho,0}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma\\
&\qquad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{\bar{Q}^{u,l,\rho}(x,t;y,\sigma)\Phi_{\nu-1}^{u,\rho,1}(y,\sigma;\xi,\tau)\\
&\qquad - \bar{Q}^{v,l,\rho}(x,t;y,\sigma)\Phi_{\nu-1}^{v,\rho,1}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma\\
&\quad =: I_1 + I_2.
\end{aligned}$$

For I_1, we rearrange terms as

$$\begin{aligned}
I_1 &= \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{\bar{P}^{u,l,\rho}(x,t;y,\sigma) - \bar{P}^{v,l,\rho}(x,t;y,\sigma)\Big\}\Phi_{\nu-1}^{u,\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\\
&\quad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \bar{P}^{v,l,\rho}(x,t;y,\sigma)\Big\{\Phi_{\nu-1}^{u,\rho,0}(y,\sigma;\xi,\tau) - \Phi_{\nu-1}^{v,\rho,0}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma\\
&=: J_1 + J_2^{\nu-1}.
\end{aligned}$$

For J_1, we use (3.29) and note that the difference $\bar{P}^{u,l,\rho} - \bar{P}^{v,l,\rho}$ satisfies the same estimates as the difference $\bar{K}^{u,l,\rho} - \bar{K}^{v,l,\rho}$ [i.e., (4.7)] to find

$$|J_1| \leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

For J_2, we start with the case $\nu = 2$ for which we can use (4.8) to obtain an estimate smaller than the one on J_1 by factor $(t-\tau)^{\frac{\gamma}{2p}}$. Once we establish the full estimate on $I_1 + I_2$, we will be able to obtain the general statement by induction.

For I_2, we write

$$\begin{aligned} I_2 &= \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \left\{ \bar{Q}^{u,l,\rho}(x,t;y,\sigma) - \bar{Q}^{v,l,\rho}(x,t;y,\sigma) \right\} \Phi_{\nu-1}^{u,\rho,1}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma \\ &\quad + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \bar{Q}^{v,l,\rho}(x,t;y,\sigma) \left\{ \Phi_{\nu-1}^{u,\rho,1}(y,\sigma;\xi,\tau) - \Phi_{\nu-1}^{v,\rho,1}(y,\sigma;\xi,\tau) \right\} \mathrm{d}y\mathrm{d}\sigma \\ &=: J_1 + J_2. \end{aligned}$$

We can estimate the difference $\bar{Q}^{u,l,\rho} - \bar{Q}^{v,l,\rho}$ in a manner very similar to our previous calculations, and combining this estimate with (3.4) and the first estimate of Part (II), we find the estimate

$$|J_1| + |J_2| \le C \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} (t-\tau)^{\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}.$$

This completes the proof of Lemma 4.1. □

5. Nonlinear analysis

Given a function $\tilde{u}(x,t)$, let $G^{\tilde{u}}(x,t;\xi,\tau)$ denote the Green's function associated with (3.1), as constructed in Sect. 3. Fix some function $u^\tau \in C^\gamma(\mathbb{R}^n)$ and define the nonlinear map (dropping the tilde for notational brevity)

$$\mathcal{T}u := \int_{\mathbb{R}^n} G^u(x,t;\xi,\tau) u^\tau(\xi)\mathrm{d}\xi. \tag{5.1}$$

Our goal in this section is to verify that $\mathcal{T}$ is an invariant contraction map on the metric space $\mathcal{S}$ defined in (4.1).

We note at the outset that if the coefficients of (3.1) are defined by $u \in \mathcal{S}$, then by virtue of (W1)–(W2) we can conclude (A1)–(A2) will hold. In this way, we can employ all the lemmas established in Sect. 3. In particular, the Hölder constants associated with the coefficients of (3.1) will depend on K.

5.1. Invariance

We begin by showing that $u \in \mathcal{S} \Rightarrow \mathcal{T}u \in \mathcal{S}$. First, we see from Lemma 3.6 that by continuously extending $\mathcal{T}u$ in the limit as $t \to \tau^+$, we have $(\mathcal{T}u)(x,\tau) = u^\tau(x)$.

In order to see that $\|\mathcal{T}u\|_{C^{\gamma,\frac{\gamma}{2p}}} < K$, we need to consider the three summands of (4.2) applied to $\mathcal{T}u$. First,

$$\mathcal{T}u(x,t) := \int_{\mathbb{R}^n} G^u(x,t;\xi,\tau)u^\tau(x)\mathrm{d}\xi + \int_{\mathbb{R}^n} G^u(x,t;\xi,\tau)(u^\tau(\xi) - u^\tau(x))\mathrm{d}\xi.$$

Using Lemmas 3.5 and 3.6, we find that for $t-\tau$ sufficiently small

$$|\mathcal{T}(x,t)| \leq |u^\tau(x)| + C(t-\tau)^{\frac{\gamma}{2p}},$$

for some constant C. By taking K sufficiently large, we can ensure

$$\sup_{\substack{x\in\mathbb{R}^n \\ t\in[\tau,\tilde{T}]}} |(\mathcal{T}u)(x,t)| < \frac{K}{3}.$$

Next, we have directly from Lemma 3.7 Part (II) that

$$|(\mathcal{T}u)(x_1,t) - (\mathcal{T}u)(x_2,t)| \leq \tilde{C}|x_1-x_2|^\gamma,$$

where $\tilde{C}$ does not depend on the Hölder constant associated with the coefficients in (3.1) and consequently does not depend on K. Accordingly, we can choose K large enough so that

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^n, x_1\neq x_2 \\ t\in[\tau,\tilde{T}]}} \frac{|(\mathcal{T}u)(x_1,t) - (\mathcal{T}u)(x_2,t)|}{|x_1-x_2|^\gamma} < \frac{K}{3}.$$

Finally, using Lemma 3.7 Part (IV), we find that K can be chosen sufficiently large so that

$$\sup_{\substack{x\in\mathbb{R}^n \\ t_1,t_2\in[\tau,\tilde{T}], t_1\neq t_2}} \frac{|(\mathcal{T}u)(x,t_1) - (\mathcal{T}u)(x,t_2)|}{|t_1-t_2|^{\frac{\gamma}{2p}}} < \frac{K}{3}.$$

Combining these inequalities, we clearly have

$$\|\mathcal{T}u\|_{C^{\gamma,\frac{\gamma}{2p}}} < K,$$

and so $\mathcal{T}u \in \mathcal{S}$.

5.2. Contraction

The contraction argument consists of establishing three inequalities, associated with the summands in our $C^{\gamma,\frac{\gamma}{2p}}$ norm. We carry these out in the next three subsections.

5.2.1. Supremum inequality

In this section, we verify that there exists a value $0<\theta<1$ so that

$$\|\mathcal{T}u - \mathcal{T}v\|_{C^{\gamma,\frac{\gamma}{2p}}} < \theta\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}, \tag{5.2}$$

for all $u, v \in \mathcal{S}$.

We begin by writing, for $u, v \in \mathcal{S}$,

$$\mathcal{T}u - \mathcal{T}v = \int_{\mathbb{R}^n} \Big(G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau)\Big)u^\tau(\xi)\mathrm{d}\xi$$
$$= \int_{\mathbb{R}^n} \Big(G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau)\Big)u^\tau(x)\mathrm{d}\xi$$
$$+ \int_{\mathbb{R}^n} \Big(G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau)\Big)(u^\tau(\xi) - u^\tau(x))\mathrm{d}\xi =: I_1 + I_2.$$

Using (3.26), we find that

$$I_1 = u^\tau(x)\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Big\{Z^u_{x_\rho}(x-y,t;y,\sigma)\Phi^{u,\rho,0}(y,\sigma;\xi,\tau)$$
$$-Z^v_{x_\rho}(x-y,t;y,\sigma)\Phi^{v,\rho,0}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$= u^\tau(x)\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n \Delta Z_{x_\rho}(u,v)\Phi^{u,\rho,0}(y,\sigma;\xi,\tau)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$+u^\tau(x)\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n Z^v_{x_\rho}(x-y,t;y,\sigma)\Delta\Phi^{\rho,0}(u,v)\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi =: J_1 + J_2,$$

where

$$\Delta Z_{x_\rho}(u,v) := Z^u_{x_\rho}(x-y,t;y,\sigma) - Z^v_{x_\rho}(x-y,t;y,\sigma)$$
$$\Delta\Phi^{\rho,0}(u,v) := \Phi^{u,\rho,0}(y,\sigma;\xi,\tau) - \Phi^{v,\rho,0}(y,\sigma;\xi,\tau). \tag{5.3}$$

Using (3.29) and Part (I) of Lemma 4.1, we compute

$$|J_1| \le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}\int_{\mathbb{R}^n}\int_\tau^t\int_{\mathbb{R}^n}(t-\sigma)^{\frac{\gamma-n-1}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}$$
$$\times e^{-c_1(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi$$
$$\le C_2\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_\tau^t(t-\sigma)^{\frac{\gamma-1}{2p}}e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi$$
$$\le C_3\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{1+\frac{\gamma-1}{2p}}.$$

Likewise, for J_2 we combine (3.1) with the first estimate in Part (II) of Lemma 4.1 to obtain precisely the same estimate we found for J_1.

For I_2, we write

$$G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau) = Z^u(x-\xi,t;\xi,\tau) - Z^v(x-\xi,t;\xi,\tau)$$
$$+\int_\tau^t\int_{\mathbb{R}^n}\sum_{\rho=1}^n\Big\{Z^u_{x_\rho}(x-y,t;y,\sigma)\Phi^{u,\rho}(y,\sigma;\xi,\tau)$$
$$-Z^v_{x_\rho}(x-y,t;y,\sigma)\Phi^{v,\rho}(y,\sigma;\xi,\tau)\Big\}\mathrm{d}y\mathrm{d}\sigma.$$

Rearranging terms similarly as in our analysis of I_1 and using Lemma 4.1, we can verify the estimate

$$\Big|G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau)\Big| \\ \leq C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{\frac{\gamma-n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}. \tag{5.4}$$

Integrating, we find

$$|I_2| \leq C(t-\tau)^{\frac{\gamma}{p}}\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}.$$

This establishes the first part of (5.2)

$$\sup_{\substack{x\in\mathbb{R}^n \\ t\in[\tau,\tilde{T}]}} |\mathcal{T}u - \mathcal{T}v| \leq \theta\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}, \tag{5.5}$$

for $\tilde{T}$ sufficiently small.

REMARK 5.1. *We observe for future reference that we have established here that for any $f \in C^\gamma(\mathbb{R}^n)$ (and for $t-\tau$ sufficiently small), we have the estimate*

$$\Big|\int_{\mathbb{R}^n}\Big(G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau)\Big)f(\xi)d\xi\Big| \leq C(t-\tau)^{\frac{\gamma}{2p}}\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}.$$

5.2.2. Δx Inequality

Next, we establish the inequality

$$\sup_{\substack{x_1,x_2\in\mathbb{R}^n, x_1\neq x_2 \\ t\in[\tau,\tilde{T}]}} \frac{\Big|\Big(\mathcal{T}u(x_1,t) - \mathcal{T}v(x_1,t)\Big) - \Big(\mathcal{T}u(x_2,t) - \mathcal{T}v(x_2,t)\Big)\Big|}{|x_1-x_2|^\gamma} \leq \theta\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}. \tag{5.6}$$

We divide this analysis into two cases, $|x_1-x_2| \leq (t-\tau)^{1/(2p)}$ (denoted Case X1) and $|x_1-x_2| > (t-\tau)^{1/(2p)}$ (denoted Case X2).
Case X1. $|x_1-x_2| \leq (t-\tau)^{1/(2p)}$. We begin by writing

$$\Big(\mathcal{T}u(x_1,t) - \mathcal{T}v(x_1,t)\Big) - \Big(\mathcal{T}u(x_2,t) - \mathcal{T}v(x_2,t)\Big) \\ = \int_{\mathbb{R}^n}\Big\{\Big(G^u(x_1,t;\xi,\tau) - G^v(x_1,t;\xi,\tau)\Big) \\ -\Big(G^u(x_2,t;\xi,\tau) - G^v(x_2,t;\xi,\tau)\Big)\Big\}u^\tau(\xi)\mathrm{d}\xi$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \left\{ \Big(G^u(x_1, t; \xi, \tau) - G^v(x_1, t; \xi, \tau) \Big) \right. \\
&\quad \left. - \Big(G^u(x_2, t; \xi, \tau) - G^v(x_2, t; \xi, \tau) \Big) \right\} u^\tau(x_2) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^n} \left\{ \Big(G^u(x_1, t; \xi, \tau) - G^v(x_1, t; \xi, \tau) \Big) \right. \\
&\quad \left. - \Big(G^u(x_2, t; \xi, \tau) - G^v(x_2, t; \xi, \tau) \Big) \right\} \Big(u^\tau(\xi) - u^\tau(x_2) \Big) \mathrm{d}\xi =: I_1 + I_2.
\end{aligned}
\tag{5.7}
$$

For I_1, we can use (3.26) to see that

$$
I_1 = u^\tau(x_2) \int_{\mathbb{R}^n} \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big(\Delta Z_{x_\rho} \Phi^{\rho,0}(x_1; u, v) - \Delta Z_{x_\rho} \Phi^{\rho,0}(x_2; u, v) \Big) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi,
\tag{5.8}
$$

where

$$
\begin{aligned}
\Delta Z_{x_\rho} \Phi^{\rho,0}(x; u, v) := &\; Z^u_{x_\rho}(x - y, t; y, \sigma) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \\
&- Z^v_{x_\rho}(x - y, t; y, \sigma) \Phi^{v,\rho,0}(y, \sigma; \xi, \tau).
\end{aligned}
\tag{5.9}
$$

In the current case (i.e., for $|x_1 - x_2| \leq (t - \tau)^{1/(2p)}$), we can divide the interval $[\tau, t]$ into a union of two subintervals $[\tau, t - \frac{1}{2}|x_2 - x_1|^{2p}]$ and $[t - \frac{1}{2}|x_2 - x_1|^{2p}, t]$. We have

$$
\begin{aligned}
I_1 &= u^\tau(x_2) \int_{\mathbb{R}^n} \int_\tau^{t - \frac{1}{2}|x_2 - x_1|^{2p}} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big(\Delta Z_{x_\rho} \Phi^{\rho,0}(x_1; u, v) \\
&\quad - \Delta Z_{x_\rho} \Phi^{\rho,0}(x_2; u, v) \Big) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\
&\quad + u^\tau(x_2) \int_{\mathbb{R}^n} \int_{t - \frac{1}{2}|x_2 - x_1|^{2p}}^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big(\Delta Z_{x_\rho} \Phi^{\rho,0}(x_1; u, v) \\
&\quad - \Delta Z_{x_\rho} \Phi^{\rho,0}(x_2; u, v) \Big) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\
&=: J_1 + J_2.
\end{aligned}
$$

For J_1, there is no problem applying the Mean Value Theorem to each summand in the integrand. We obtain expressions of the form

$$
\begin{aligned}
&\left\{ \Delta Z_{x_\rho} \Phi^{\rho,0}(x_1; u, v) - \Delta Z_{x_\rho} \Phi^{\rho,0}(x_2; u, v) \right\}_{ij} \\
&\quad = D_x \left\{ \Delta Z_{x_\rho} \Phi^{\rho,0}(x^*; u, v) \right\}_{ij} \cdot (x_1 - x_2),
\end{aligned}
\tag{5.10}
$$

for some vector $x^* = x^*_{ij}(x_1, x_2, t, y, \sigma, \xi, \tau)$ (depending also on i and j) on the line between x_1 and x_2. Proceeding now as in previous calculations, we write

$$\begin{aligned}
&\Big\{\Delta Z_{x_\rho}\Phi^{\rho,0}(x^*; u, v)\Big\}_{ij}\\
&= \Big\{\Big(D_x Z^u{}_{x_\rho}(x^*-y, t; y, \sigma) - D_x Z^v{}_{x_\rho}(x^*-y, t; y, \sigma)\Big)\Phi^{u,\rho,0}(y, \sigma; \xi, \tau)\Big\}_{ij}\\
&\quad + \Big\{D_x Z^v{}_{x_\rho}(x^*-y, t; y, \sigma)\Big(\Phi^{u,\rho,0}(y, \sigma; \xi, \tau) - \Phi^{v,\rho,0}_{qj}(y, \sigma; \xi, \tau)\Big)\Big\}_{ij}\\
&=: K_1^* + K_2^*.
\end{aligned} \tag{5.11}$$

For K_1^*, we observe from Lemma 4.1 and using (3.29)

$$\begin{aligned}
|K_1^*| &\le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\sigma)^{-\frac{n+2-\gamma}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}\\
&\quad\times e^{-c_1(\frac{|x^*-y|^{2p}}{t-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}.
\end{aligned} \tag{5.12}$$

Likewise, for K_2 we have

$$\begin{aligned}
|K_2^*| &\le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\sigma)^{-\frac{n+2}{2p}}(\sigma-\tau)^{-\frac{n-\gamma}{2p}}\\
&\quad\times e^{-c_1(\frac{|x^*-y|^{2p}}{t-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}.
\end{aligned} \tag{5.13}$$

Each of these terms corresponds with a summand in J_1, and we denote the full expression for J_1 as $J_1 = K_1 + K_2$, where K_1 comprises terms like K_1^* and K_2 comprises terms like K_2^*.

Using the argument following (3.31), we find

$$\begin{aligned}
|K_1| &\le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}\int_{\mathbb{R}^n}\int_\tau^{t-\frac{1}{2}|x_2-x_1|^{2p}}\int_{\mathbb{R}^n}(t-\sigma)^{-\frac{n+2-\gamma}{2p}}(\sigma-\tau)^{-\frac{n}{2p}}\\
&\quad\times e^{-c_1(\frac{|x^*-y|^{2p}}{t-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}|x_2-x_1|\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\
&\le C_2\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-\frac{n}{2p}}\\
&\quad\times\int_{\mathbb{R}^n}\int_\tau^{t-\frac{1}{2}|x_2-x_1|^{2p}}(t-\sigma)^{-\frac{2-\gamma}{2p}}|x_2-x_1|e^{-c_2(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}\mathrm{d}\sigma\mathrm{d}\xi.
\end{aligned}$$

In this case $|x_2-x_1| \le [\frac{1}{2}(t-\sigma)]^{\frac{1}{2p}}$, and we obtain the estimate

$$|K_1| \le C_3|x_2-x_1|^\gamma\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{1-\frac{1}{2p}},$$

which for $t-\tau$ small is much better than we require.

A similar argument leads to the same estimate on K_2.

For J_2, we cannot apply the Mean Value Theorem, because the associated higher-order derivatives are not integrable up to t. Instead, we proceed directly, writing $J_2 = K_1 + K_2$, where

$$K_1 = \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Big(Z^u_{x_\rho}(x_1 - y, t; y, \sigma) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \\ - Z^v_{x_\rho}(x_1 - y, t; y, \sigma) \Phi^{v,\rho,0}(y, \sigma; \xi, \tau) \Big) \mathrm{d}y \sigma \mathrm{d}\xi. \tag{5.14}$$

We rearrange the summands in the integrand into two terms

$$\Big(Z^u_{x_\rho}(x_1 - y, t; y, \sigma) - Z^v_{x_\rho}(x_1 - y, t; y, \sigma) \Big) \Phi^{v,\rho,0}(y, \sigma; \xi, \tau) \\ + Z^v_{x_\rho}(x_1 - y, t; y, \sigma) \Big(\Phi^{u,\rho,0}(y, \sigma; \xi, \tau) - \Phi^{v,\rho,0}(y, \sigma; \xi, \tau) \Big) =: L_1 + L_2. \tag{5.15}$$

Employing now (3.29) and Lemma 4.1, we compute

$$|L_1| \leq C_1 \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} \int_{\mathbb{R}^n} (t - \sigma)^{-\frac{n+1-\gamma}{2p}} (\sigma - \tau)^{-\frac{n}{2p}} \\ \times e^{-c_1 (\frac{|x_1-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1 (\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ \leq C_2 \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} (t - \tau)^{-\frac{n}{2p}} \\ \times \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} (t - \sigma)^{-\frac{1-\gamma}{2p}} e^{-c_2 (\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi.$$

Bearing in mind the limits of σ integration, we obtain the estimate

$$|L_1| \leq C_3 \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} |x_1 - x_2|^{2p-(1-\gamma)} \leq C_4 \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} (t - \tau)^{1-\frac{1}{2p}} |x_1 - x_2|^{\gamma},$$

where in obtaining this last inequality, we have used the inequality defining the current case ($|x_2 - x_1| \leq (1 - \tau)^{\frac{1}{2p}}$). The quantity denoted L_2 can be analyzed similarly, and this completes the analysis of K_1. Likewise, we can analyze K_2 similarly as K_1, since all that changes is that x_1 is replaced by x_2. This complete the analysis of J_2 and hence of I_1 [from (5.7)].

Turning now to I_2, we have

$$I_2 = \int_{\mathbb{R}^n} \Big\{ \Delta Z(x_1; u, v) - \Delta Z(x_2; u, v) \Big\} (u^\tau(\xi) - u^\tau(x_2)) \mathrm{d}\xi \\ + \int_{\mathbb{R}^n} \int_{\tau}^{t} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Big\{ \Delta Z_{x_\rho} \Phi^\rho(x_1; u, v) - \Delta Z_{x_\rho} \Phi^\rho(x_2; u, v) \Big\} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ =: J_1 + J_2,$$

where ΔZ and $\Delta Z_{x_\rho}\Phi^\rho$ are defined similarly as in (5.9).

For J_1, we can apply the Mean Value Theorem similarly as in (5.10) to obtain an estimate of the form

$$
\begin{aligned}
|J_1| &\le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}|x_2-x_1|\int_{\mathbb{R}^n}(t-\tau)^{-\frac{n+1-\gamma}{2p}}e^{-c_1(\frac{|x^*-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}|\xi-x_2|^\gamma \mathrm{d}\xi\\
&\le C_2\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}|x_2-x_1|(t-\tau)^{-\frac{1-2\gamma}{2p}}\\
&\le C_3\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}|x_2-x_1|^\gamma(t-\tau)^{\frac{\gamma}{2p}}.
\end{aligned}
$$

For J_2, we write

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}^n}\int_\tau^{t-\frac{1}{2}|x_2-x_1|^{2p}}\int_{\mathbb{R}^n}\Big\{\Delta Z_{x_\rho}\Phi^\rho(x_1;u,v)-\Delta Z_{x_\rho}\Phi^\rho(x_2;u,v)\Big\}\\
&\quad\times(u^\tau(\xi)-u^\tau(x_2))\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\
&\quad+\int_{\mathbb{R}^n}\int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t}\int_{\mathbb{R}^n}\Big\{\Delta Z_{x_\rho}\Phi^\rho(x_1;u,v)-\Delta Z_{x_\rho}\Phi^\rho(x_2;u,v)\Big\}\\
&\quad\times(u^\tau(\xi)-u^\tau(x_2))\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\
&=: K_1+K_2.
\end{aligned}
$$

For K_1, we apply the Mean Value Theorem precisely as in (5.10) and (5.11), except with $\Phi^{u,\rho,0}$ and $\Phi^{v,\rho,0}$, respectively, replaced by $\Phi^{u,\rho}$ and $\Phi^{v,\rho}$. We express the rearrangement of (5.11) as $K_1 = L_1 + L_2$, and from Lemmas 4.1 and 3.4, and using the argument following (3.31) to accommodate the value of x^*, we obtain the estimate

$$
\begin{aligned}
|L_1| &\le C_1\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}\int_{\mathbb{R}^n}\int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t}\int_{\mathbb{R}^n}(t-\sigma)^{-\frac{n+2-\gamma}{2p}}(\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}}\\
&\quad\times e^{-c_1(\frac{|x_1-y|^{2p}}{t-\sigma})^{1/(2p-1)}}e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}|x_1-x_2||\xi-x_2|^\gamma\mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi\\
&\le C_2\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-\frac{n}{2p}}\int_{\mathbb{R}^n}\int_{t-\frac{1}{2}|x_2-x_1|^{2p}}(t-\sigma)^{-\frac{2-\gamma}{2p}}(\sigma-\tau)^{-1+\frac{1+\gamma}{2p}}\\
&\quad\times e^{-c_2(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}|x_1-x_2||\xi-x_2|^\gamma\mathrm{d}\sigma\mathrm{d}\xi.
\end{aligned}
$$

On this interval $|x_1-x_2|\le[2(t-\sigma)]^{\frac{1}{2p}}$, and we can write

$$
\begin{aligned}
|L_1| &\le C_3\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-\frac{n}{2p}}|x_1-x_2|^\gamma\\
&\quad\times\int_{\mathbb{R}^n}\int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t}(t-\sigma)^{-\frac{1}{2p}}(\sigma-\tau)^{-1+\frac{1+\gamma}{2p}}e^{-c_2(\frac{|x_2-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}|\xi-x_2|\mathrm{d}\sigma\mathrm{d}\xi\\
&\le C_4\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{-\frac{n-\gamma}{2p}}|x_1-x_2|^\gamma\int_{\mathbb{R}^n}e^{-c_2(\frac{|x_2-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}|\xi-x_2|\mathrm{d}\xi\\
&\le C_5\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}(t-\tau)^{\frac{\gamma}{p}}|x_1-x_2|^\gamma.
\end{aligned}
$$

The analysis of L_2 is almost identical to that of L_1 and gives the same estimate. This completes the analysis of K_1, and we turn to K_2, for which we avoid the Mean Value Theorem. We rearrange terms similarly as in our expressions $J_1 = K_1 + K_2$ leading into (5.14) and express the right-hand side as $K_2 = L_1(x_1) + L_2(x_2)$. We then further separate these terms as in (5.15), starting with $L_1 = M_1 + M_2$. For M_1, we employ the estimates of Lemmas 4.1 and 3.4 to obtain the estimate

$$\begin{aligned}|M_1| &\le C_1 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n+1-\gamma}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \\ &\quad \times e^{-c_1(\frac{|x_1-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi - x_2|^{\gamma} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\ &\le C_2 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} (t-\sigma)^{-\frac{1-\gamma}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \\ &\quad \times e^{-c_2(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} |\xi - x_2|^{\gamma} \mathrm{d}\sigma \mathrm{d}\xi.\end{aligned}$$

Over this interval, $\sigma - \tau > \frac{1}{2}(t-\tau)$, and so, we can compute

$$\begin{aligned}|M_1| &\le C_3 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-1+\frac{1+\gamma-n}{2p}} \int_{\mathbb{R}^n} \int_{t-\frac{1}{2}|x_2-x_1|^{2p}}^{t} (t-\sigma)^{-\frac{1-\gamma}{2p}} \\ &\quad \times e^{-c_3(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} |\xi - x_2|^{\gamma} \mathrm{d}\sigma \mathrm{d}\xi \\ &\le C_4 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-1+\frac{1+\gamma-n}{2p}} \int_{\mathbb{R}^n} |x_1-x_2|^{2p-1+\gamma} \\ &\quad \times e^{-c_4(\frac{|x_1-\xi|^{2p}}{t-\tau})^{1/(2p-1)}} |\xi - x_2|^{\gamma} \mathrm{d}\xi.\end{aligned}$$

At this point, we use the triangle inequality $|\xi - x_2| \le |\xi - x_1| + |x_1 - x_2|$. For the summand $|\xi - x_1|$, we obtain an estimate by

$$C_5 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{-1+\frac{1+2\gamma}{2p}} |x_1-x_2|^{2p-1+\gamma} \le C_6 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t-\tau)^{\frac{\gamma}{p}} |x_1-x_2|^{\gamma}.$$

The summand $|x_1 - x_2|$ in our triangle inequality leads to the same estimate, completing the analysis of M_1. The analysis of M_2 is similar, leading to the same estimate, and this concludes the analysis of L_1. The analysis of L_2 is similar to that of L_1, concluding the analysis of K_2, which in turn concludes the analysis of J_2. Finally, this concludes the analysis of I_2 [from (5.7)], and we have concluded the result for Case $X1$.

Case X2. $|x_1 - x_2| > (t-\tau)^{1/(2p)}$. In this case, we use the simple inequality

$$\begin{aligned}&\Big|\Big(\mathcal{T}u(x_1,t) - \mathcal{T}v(x_1,t)\Big) - \Big(\mathcal{T}u(x_2,t) - \mathcal{T}v(x_2,t)\Big)\Big| \\ &\le \Big|\mathcal{T}u(x_1,t) - \mathcal{T}v(x_1,t)\Big| + \Big|\mathcal{T}u(x_2,t) - \mathcal{T}v(x_2,t)\Big| \\ &=: I_1 + I_2.\end{aligned}$$

According to Remark 5.1,

$$|I_1| \leq C(t-\tau)^{\frac{\gamma}{p}} \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}.$$

In this case $(t-\tau)^{\frac{\gamma}{2p}} \leq |x_1 - x_2|^\gamma$, and we immediately obtain the inequality

$$|I_1| \leq C(t-\tau)^{\frac{\gamma}{2p}} |x_1 - x_2|^\gamma \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}.$$

Clearly, I_2 can be analyzed similarly, and this completes the analysis of $X2$. Combining cases $X1$ and $X2$, we have established the claimed inequality (5.6).

5.2.3. Δt Inequality

Finally, we establish the inequality

$$\sup_{\substack{t_1,t_2\in[\tau,\tilde{T}],t_1\neq t_2\\ x\in\mathbb{R}^n}} \frac{\left|\Big(\mathcal{T}u(x,t_1)-\mathcal{T}v(x,t_1)\Big)-\Big(\mathcal{T}u(x,t_2)-\mathcal{T}v(x,t_2)\Big)\right|}{|t_1-t_2|^{\frac{\gamma}{2p}}} \leq \theta \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}}, \tag{5.16}$$

for some $\tilde{T}$ sufficiently small and some $0 < \theta < 1$. Without loss of generality, we will take $t_1 \leq t_2$.

Case T1. $t_2 - t_1 < t_1 - \tau$. We begin by writing

$$\begin{aligned}
&\Big(\mathcal{T}u(x,t_1) - \mathcal{T}v(x,t_1)\Big) - \Big(\mathcal{T}u(x,t_2) - \mathcal{T}v(x,t_2)\Big)\\
&= \int_{\mathbb{R}^n} \Big\{\Delta G(t_1; u, v) - \Delta G(t_2; u, v)\Big\} u^\tau(x) d\xi\\
&\quad + \int_{\mathbb{R}^n} \Big\{\Delta G(t_1; u, v) - \Delta G(t_2; u, v)\Big\} (u^\tau(\xi) - u^\tau(x)) d\xi\\
&=: I_1 + I_2,
\end{aligned} \tag{5.17}$$

where

$$\Delta G(t; u, v) := G^u(x,t;\xi,\tau) - G^v(x,t;\xi,\tau). \tag{5.18}$$

Beginning with I_1, we conclude from (3.26)

$$\begin{aligned}
I_1 = \int_{\mathbb{R}^n} \Big\{ &\int_\tau^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^{\rho,0}(t_1; u, v) dy d\sigma\\
&- \int_\tau^{t_2} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v) dy d\sigma \Big\} u^\tau(\xi) d\xi
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \left\{ \int_\tau^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^{\rho,0}(t_1; u, v) - \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v) \right] \mathrm{d}y \mathrm{d}\sigma \right\} u^\tau(x) \mathrm{d}\xi$$

$$- \int_{\mathbb{R}^n} \left\{ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v) \mathrm{d}y \mathrm{d}\sigma \right\} u^\tau(\xi) \mathrm{d}\xi =: J_1 + J_2, \qquad (5.19)$$

where

$$\Delta Z_{x_\rho} \Phi^{\rho,0}(t; u, v) := Z^u_{x_\rho}(x - y, t; y, \sigma) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau)$$
$$- Z^v_{x_\rho}(x - y, t; y, \sigma) \Phi^{v,\rho,0}(y, \sigma; \xi, \tau).$$

For J_1, we write

$$\begin{aligned} J_1 = & \int_{\mathbb{R}^n} \left\{ \int_\tau^{t_1 - \frac{1}{2}(t_2 - t_1)} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \left[\Delta Z_{x_\rho} \Phi^{\rho,0}(t_1; u, v) \right.\right. \\ & \left.\left. - \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v) \right] \mathrm{d}y \mathrm{d}\sigma \right\} u^\tau(x) \mathrm{d}\xi \\ & + \int_{\mathbb{R}^n} \left\{ \int_{t_1 - \frac{1}{2}(t_2 - t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \left[\Delta Z_{x_\rho} \Phi^{\rho,0}(t_1; u, v) \right.\right. \\ & \left.\left. - \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v) \right] \mathrm{d}y \mathrm{d}\sigma \right\} u^\tau(x) \mathrm{d}\xi \\ =: & K_1 + K_2. \end{aligned}$$

For K_1, we can apply the Mean Value Theorem in t to the difference

$$\Delta Z_{x_\rho} \Phi^{\rho,0}(t_1; u, v) - \Delta Z_{x_\rho} \Phi^{\rho,0}(t_2; u, v).$$

More precisely, similarly as in (5.10), we can express the ij entry of this matrix as

$$\left\{ {Z^u}_{tx_\rho}(x - y, t^*; y, \sigma) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \right.$$
$$\left. - {Z^v}_{tx_\rho}(x - y, t^*; y, \sigma) \Phi^{v,\rho,0}(y, \sigma; \xi, \tau) \right\}_{ij} (t_1 - t_2), \qquad (5.20)$$

for some value $t^* = t^*(t_1, t_2, x, y, \xi, \sigma, \tau)$ (also depending on i and j) between t_1 and t_2. As usual, we now rearrange this last expression into convenient differences

$$\left\{ \left({Z^u}_{tx_\rho}(x - y, t^*; y, \sigma) - {Z^v}_{tx_\rho}(x - y, t^*; y, \sigma) \right) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \right\}_{ij} (t_1 - t_2)$$
$$+ \left\{ {Z^v}_{tx_\rho}(x - y, t^*; y, \sigma) \left(\Phi^{v,\rho,0}(y, \sigma; \xi, \tau) - \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \right) \right\}_{ij} (t_1 - t_2). \qquad (5.21)$$

We respectively associate these last two summands with terms we will denote $L_1 + L_2$. Using Lemma 4.1 and the estimate (3.29), we find (suppressing dependence of t^* on q)

$$|L_1| \le C_1 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{\tau}^{t_1 - \frac{1}{2}(t_2 - t_1)} \int_{\mathbb{R}^n} (t^* - \sigma)^{-1-\frac{n+1-\gamma}{2p}} (\sigma - \tau)^{-\frac{n}{2p}} \times e^{-c_1(\frac{|x_1 - y|^{2p}}{t^* - \sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma - \tau})^{1/(2p-1)}} (t_2 - t_1) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi.$$

Over this interval of integration $t^* - \sigma \ge \frac{1}{2}(t_2 - t_1)$, and so we have the inequality

$$(t^* - \sigma)^{-1-\frac{n+1-\gamma}{2p}} (t_2 - t_1) \le (t^* - \sigma)^{-\frac{n+1}{2p}} (t_2 - t_1)^{\frac{\gamma}{2p}}.$$

At the same time,

$$t_1 - \sigma \le t^* - \sigma \le 3(t_1 - \sigma),$$

so in all appearances $t^* - \sigma$ can be replaced with $t_1 - \sigma$ (with new constants). Combining these observations and carrying out the integration over y, we obtain the inequality

$$\begin{aligned} |L_1| &\le C_2 (t_2 - t_1)^{\frac{\gamma}{2p}} \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_1 - \tau)^{-\frac{n}{2p}} \\ &\quad \times \int_{\mathbb{R}^n} \int_{\tau}^{t_1 - \frac{1}{2}(t_2 - t_1)} (t_1 - \sigma)^{-\frac{1}{2p}} e^{-c_2(\frac{|x_1 - \xi|^{2p}}{t_1 - \sigma})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi \\ &\le C_3 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2 - t_1)^{\frac{\gamma}{2p}} (t_1 - \tau)^{1-\frac{1}{2p}}, \end{aligned}$$

which is much smaller than our claim. The analysis of L_2 is similar, leading to the same estimate, and this completes the analysis of K_1.

For K_2, we avoid the Mean Value Theorem, analyzing instead each expression

$$\Delta Z_{x_\rho} \Phi^{\rho,0}(t; u, v)$$

individually. We express the resulting expression as $K_2 = L_1 + L_2$, and in both cases, we use the rearrangement

$$\begin{aligned} &\Big(Z^u_{x_\rho}(x - y, t_j; y, \sigma) - Z^v_{x_\rho}(x - y, t_j; y, \sigma) \Big) \Phi^{u,\rho,0}(y, \sigma; \xi, \tau) \\ &\quad + Z^v_{x_\rho}(x - y, t_j; y, \sigma) \Big(\Phi^{u,\rho,0}(y, \sigma; \xi, \tau) - \Phi^{v,\rho,0}(y, \sigma; \xi, \tau) \Big). \end{aligned} \tag{5.22}$$

For L_1, we express the associated integrals as $L_1 = M_1 + M_2$, and for M_1, we obtain the inequality

$$
\begin{aligned}
|M_1| \le{}& C_1 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} (t_1 - \sigma)^{-\frac{n+1-\gamma}{2p}} (\sigma - \tau)^{-\frac{n}{2p}} \\
&\times e^{-c_1(\frac{|x_1-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\
\le{}& C_2 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_1 - \tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} (t_1 - \sigma)^{-\frac{1-\gamma}{2p}} \\
&\times e^{-c_2(\frac{|x_1-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} \mathrm{d}\sigma \mathrm{d}\xi.
\end{aligned}
$$

Carrying out the integration over σ, we obtain the estimate

$$
\begin{aligned}
|M_1| \le{}& C_3 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_1 - \tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} (t_2 - t_1)^{1-\frac{1-\gamma}{2p}} e^{-c_2(\frac{|x_1-\xi|^{2p}}{t_1-\tau})^{1/(2p-1)}} \mathrm{d}\xi \\
\le{}& C_3 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2 - t_1)^{1-\frac{1-\gamma}{2p}} \le C_4 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2 - t_1)^{\frac{\gamma}{2p}} (t_1 - \tau)^{1-\frac{1}{2p}},
\end{aligned}
$$

where in obtaining this last inequality we have observed that we are in the case $t_2 - t_1 < t_1 - \tau$.

The analysis of M_2 is similar and leads to the same estimate, and this completes the analysis of L_1. For L_2, the only difference is that t_1 is replaced by t_2, but in this case

$$
t_1 - \tau \le t_2 - \tau \le 2(t_1 - \tau),
$$

and so we obtain the same estimate with different constants. This completes the analysis of K_2, which in turn completes the analysis of J_1 [from (5.19)].

For J_2, we use (5.22) (with t_2) for the integrand and express the resulting summands as $J_2 = K_1 + K_2$. Proceeding similarly as in the analysis of M_1 just above, we obtain the same estimate as there. At this point, we have verified

$$
|I_1| \le C_4 \|u - v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2 - t_1)^{\frac{\gamma}{2p}} (t_1 - \tau)^{1-\frac{1}{2p}}.
$$

For I_2, we write

$$
\begin{aligned}
I_2 ={}& \int_{\mathbb{R}^n} \Big(\Delta Z(t_1; u, v) - \Delta Z(t_2; u, v) \Big) (u^\tau(\xi) - u^\tau(x)) \mathrm{d}\xi \\
&+ \int_{\mathbb{R}^n} \int_\tau^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^\rho(t_1; u, v)(u^\tau(\xi) - u^\tau(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\
&- \int_{\mathbb{R}^n} \int_\tau^{t_2} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho} \Phi^\rho(t_2; u, v)(u^\tau(\xi) - u^\tau(x)) \mathrm{d}y \mathrm{d}\sigma \mathrm{d}\xi \\
=:{}& J_1 + J_2. \qquad (5.23)
\end{aligned}
$$

For J_1, we apply the Mean Value Theorem in the form

$$\Big(\Delta Z(t_1; u, v) - \Delta Z(t_2; u, v)\Big)_{ij}$$
$$= \Big\{ {Z^u}_t(x-y, t^*; \xi, \tau) - {Z^v}_t(x-y, t^*; \xi, \tau) \Big\}_{ij} (t_1 - t_2),$$

for $t^* = t^*(t_1, t_2, x, y, \xi, \tau)$ (depending also on i and j) between t_1 and t_2. Using Lemma 4.1, we estimate

$$|J_1^*| \leq C_1(t_2 - t_1)\|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} \int_{\mathbb{R}^n} (t^* - \tau)^{-1-\frac{n-\gamma}{2p}} e^{-c_1(\frac{|x-y|^{2p}}{t^*-\tau})^{1/(2p-1)}} |x - \xi|^\gamma d\xi,$$

where J_1 comprises terms of the form J_1^*. Keeping in mind that we are in the Case $T1$, we have the inequality

$$t_1 - \tau \leq t^* - \tau \leq 2(t_1 - \tau),$$

which allows us to replace $t^* - \tau$ with $t_1 - \tau$ up to a constant. Upon making this substitution and integrating, we obtain

$$|J_1^*| \leq C_2(t_2 - t_1)(t_1 - \tau)^{-1+2\frac{\gamma}{2p}} \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}} \leq C_3(t_2 - t_1)^{\frac{\gamma}{2p}} (t_1 - \tau)^{\frac{\gamma}{2p}} \|u - v\|_{C^{\gamma, \frac{\gamma}{2p}}},$$

where in obtaining this last inequality, we have used the inequality defining Case $T1$. This completes the analysis of J_1 [from (5.23)].

For J_2, it is useful to write

$$\begin{aligned} J_2 = &\int_{\mathbb{R}^n} \int_\tau^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{ \Delta Z_{x_\rho} \Phi^\rho(t_1; u, v) - \Delta Z_{x_\rho} \Phi^\rho(t_2; u, v) \Big\} \\ &\times (u^\tau(\xi) - u^\tau(x)) dy d\sigma d\xi \\ &- \int_{\mathbb{R}^n} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Delta Z_{x_\rho}(t_2; u, v) \Phi^\rho (u^\tau(\xi) - u^\tau(x)) dy d\sigma d\xi \\ =: &\, K_1 + K_2. \end{aligned} \tag{5.24}$$

For K_1, we further subdivide the intervals of integration, writing

$$\begin{aligned} K_1 = &\int_{\mathbb{R}^n} \int_\tau^{t_1 - \frac{1}{2}(t_2 - t_1)} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{ \Delta Z_{x_\rho} \Phi^\rho \\ &\times (t_1; u, v) - \Delta Z_{x_\rho} \Phi^\rho(t_2; u, v) \Big\} (u^\tau(\xi) - u^\tau(x)) dy d\sigma d\xi \\ &+ \int_{\mathbb{R}^n} \int_{t_1 - \frac{1}{2}(t_2 - t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^n \Big\{ \Delta Z_{x_\rho} \Phi^\rho(t_1; u, v) - \Delta Z_{x_\rho} \Phi^\rho(t_2; u, v) \Big\} \\ &\times (u^\tau(\xi) - u^\tau(x)) dy d\sigma d\xi \\ =: &\, L_1 + L_2. \end{aligned} \tag{5.25}$$

For L_1, we apply the Mean Value Theorem as in (5.20), and rearrange the result as in (5.21) into $L_1 = M_1 + M_2$. For the resulting M_1, we obtain terms of the form

$$|M_1^*| \leq C_1 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} \int_{\mathbb{R}^n} (t^*-\sigma)^{-1-\frac{n+1-\gamma}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \times e^{-c_1(\frac{|x-y|^{2p}}{t^*-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} (t_2-t_1)|\xi-x|^{\gamma} \mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi.$$

In this case, we have the inequality

$$t_1 - \sigma \leq t^* - \sigma \leq 3(t_1-\sigma),$$

so that $t^*-\sigma$ is interchangeable with $t_1-\sigma$ up to a change of constants. In addition, $t_2 - t_1 \leq 2(t_1-\sigma)$, and we can write

$$\begin{aligned}
|M_1^*| &\leq C_2 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{\frac{\gamma}{2p}} \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} \\
&\quad \times \int_{\mathbb{R}^n} (t_1-\sigma)^{-\frac{n+1}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \\
&\quad \times e^{-c_2(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}} e^{-c_2(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi-x|^{\gamma} \mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\
&\leq C_3 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{-\frac{n}{2p}} \\
&\quad \times \int_{\mathbb{R}^n} \int_{\tau}^{t_1-\frac{1}{2}(t_2-t_1)} (t_1-\sigma)^{-\frac{1}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \\
&\quad \times e^{-c_3(\frac{|x-\xi|^{2p}}{t_1-\sigma})^{1/(2p-1)}} |\xi-x|^{\gamma} \mathrm{d}\sigma\mathrm{d}\xi.
\end{aligned}$$

Integrating in both σ and ξ, we conclude

$$|L_1^*| \leq C_4 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{\frac{\gamma}{2p}}.$$

Since M_1 comprises terms of form M_1^*, this completes the analysis of M_1. The analysis of M_2 is similar, and so, we have concluded the estimate for L_1 [from (5.25)].

For L_2, we avoid the Mean Value Theorem, proceeding instead by writing

$$\begin{aligned}
L_2 &= \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Delta Z_{x_\rho} \Phi^{\rho}(t_1; u, v)(u^{\tau}(\xi) - u^{\tau}(x)) \mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\
&\quad - \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} \Delta Z_{x_\rho} \Phi^{\rho}(t_2; u, v)(u^{\tau}(\xi) - u^{\tau}(x)) \mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\
&=: M_1 + M_2.
\end{aligned}$$

For M_1, we rearrange $\Delta Z_{x_\rho} \Phi^{\rho}$ as

$$\Delta Z_{x_\rho} \Phi^{\rho}(t_1; u, v) := \Big\{ Z^u_{x_\rho}(x-\xi, t_1; \xi, \tau) - Z^v_{x_\rho}(x-\xi, t_1; \xi, \tau) \Big\} \Phi^{u,\rho}(y, \sigma; \xi, \tau) + Z^v_{x_\rho}(x-\xi, t_1; \xi, \tau) \Big\{ \Phi^{u,\rho}(y, \sigma; \xi, \tau) - \Phi^{v,\rho}(y, \sigma; \xi, \tau) \Big\},$$

and we use this arrangement to write $M_1 = N_1 + N_2$.

For N_1, we employ the estimates of Lemmas 3.4 and 4.1 to obtain an estimate by

$$\begin{aligned}|N_1| &\le C_1 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} (t_1-\sigma)^{-\frac{n+1-\gamma}{2p}} (\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} \\ &\quad \times e^{-c_1(\frac{|x-y|^{2p}}{t_1-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} |\xi - x|^\gamma \mathrm{d}y\mathrm{d}\sigma\mathrm{d}\xi \\ &\le C_2 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_1-\tau)^{-\frac{n}{2p}} \int_{\mathbb{R}^n} \int_{t_1-\frac{1}{2}(t_2-t_1)}^{t_1} (t_1-\sigma)^{-\frac{1-\gamma}{2p}} (\sigma-\tau)^{-1+\frac{1+\gamma}{2p}} \\ &\quad \times e^{-c_2(\frac{|x-\xi|^{2p}}{t_1-\sigma})^{1/(2p-1)}} |\xi - x|^\gamma \mathrm{d}\sigma\mathrm{d}\xi.\end{aligned}$$

Over this interval of integration in σ, we have the inequality $\sigma - \tau \ge \frac{1}{2}(t_1-\tau)$, and consequently, the term $(\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}}$ can be replaced with $(t_1-\tau)^{-1-\frac{n-1-\gamma}{2p}}$ (with a change of constant). Upon making this replacement and integrating in both σ and ξ, we obtain the estimate

$$\begin{aligned}|N_1| &\le C_3 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{1-\frac{1-\gamma}{2p}} (t_1-\tau)^{-1+\frac{1+\gamma}{2p}} \\ &\le C_4 \|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_2-t_1)^{\frac{\gamma}{2p}} (t_1-\tau)^{\frac{\gamma}{2p}},\end{aligned}$$

where in obtaining this last inequality, we have used the inequality defining case $T1$. This completes the analysis of N_1. The analysis of N_2 is similar, and this completes the analysis of M_1. The analysis of M_2 is similar to the analysis of M_1 (with t_2 replacing t_1), and we obtain the same estimate (keeping in mind $t_2 - \tau \le 2(t_1 - \tau)$ in this case). This completes the analysis of L_2, which in turn completes the analysis of K_1 [from (5.24)].

For K_2, we can proceed almost exactly as we did with M_1 and M_2, except that the limits on the integration over σ change. We obtain the same estimate we found above for M_1. This concludes the analysis of K_2, which concludes the analysis for J_2 and in turn I_2 [from (5.17)]. This finishes Case $T1$.

Case T2. $(t_2-t_1) > (t_1-\tau)$. For this case, we will not need to apply the Mean Value Theorem, and the analysis will be much easier. In particular, we simply estimate

$$\begin{aligned}&\Big|\Big(\mathcal{T}u(x,t_1) - \mathcal{T}v(x,t_1)\Big) - \Big(\mathcal{T}u(x,t_2) - \mathcal{T}v(x,t_2)\Big)\Big| \\ &\le \Big|\mathcal{T}u(x,t_1) - \mathcal{T}v(x,t_1)\Big| + \Big|\mathcal{T}u(x,t_2) - \mathcal{T}v(x,t_2)\Big| =: |I_1| + |I_2|.\end{aligned}$$

We now analyze each of these summands on the right-hand side by the general method we used in the strand (I_2-J_2-K_1-L_2) of Case $T1$. We obtain the estimates

$$|I_j| \le C\|u-v\|_{C^{\gamma,\frac{\gamma}{2p}}} (t_j-\tau)^{\frac{\gamma}{p}}.$$

In this case $t_1 - \tau \le (t_2 - t_1)$, and likewise $t_2 - \tau \le 2(t_2 - t_1)$, and this immediately gives the claimed estimate (5.16) for $\tilde{T} - \tau$ sufficiently small.

5.3. Regularity

Our estimates from Sects. 5.1 and 5.2 are sufficient by virtue of the Contraction Mapping Theorem to ensure the existence of a unique solution to the weak formulation of (1.1). We stress that this construction has been carried out in the context of our weak assumptions (W1)–(W2), along of course with uniform parabolicity ($\mathcal{P}$). We summarize our work so far in the following theorem.

THEOREM 5.1. *Suppose* (1.1) *is uniformly parabolic in the sense of* ($\mathcal{P}$) *that (W1)–(W2) hold and that for some value* $\tau \in [0, T)$, $u^\tau(\cdot) \in C^\gamma(\mathbb{R})$ *for some Hölder index* $0 < \gamma < 1$. *Then, there exists a value* $\tilde{T} \in (\tau, T)$, *with* $\tilde{T} - \tau$ *possibly small, so that for any* $\sigma \in (\tau, \tilde{T})$ *there exists a weak solution to* (1.1)

$$u \in C^{\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [0, \tilde{T}]) \cap C^{2p-1+\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [\sigma, \tilde{T}]).$$

Moreover, u is the unique weak solution of (1.1) *in* $C^{\gamma, \frac{\gamma}{2p}}(\mathbb{R}^n \times [0, \tilde{T}])$.

In this section, we verify that under the stronger conditions (S1) and (S2), u is actually a classical solution to (1.1).

We recall that by construction, we can write our weak solution as

$$u(x, t) = \int_{\mathbb{R}^n} G(x, t; \xi, \tau) u^\tau(\xi) \mathrm{d}\xi, \tag{5.26}$$

for a function G (previously denoted G^u) that can be expressed as

$$G(x, t; \xi, \tau) = Z(x - \xi, t; \xi, \tau) + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n Z_{x_\rho}(x - y, t; y, \sigma) \Phi^\rho(y, \sigma; \xi, \tau) \mathrm{d}y \mathrm{d}\sigma.$$

Using the estimates of Lemmas 3.1 and 3.4, we readily verify that G is $2p-1$ times differentiable in x with estimate

$$|D_x^\alpha G(x, t; \xi, \tau)| \leq C(t - \tau)^{-\frac{n+|\alpha|}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for all $|\alpha| \leq 2p - 1$. The difficulty arises when we try to put a $2p$ x derivatives or one t derivative on G, in which case we must take considerable care with integrability over $[\tau, t]$.

Under assumptions (S1) and (S2), we can differentiate Z with respect to y, and in particular, we can express the useful relation

$$\frac{d}{dy_\rho} Z(x - y, t; y, \sigma) = -Z_{x_\rho}(x - y, t; y, \sigma) + Z_{y_\rho}(x - y, t; y, \sigma), \tag{5.27}$$

where Z_{y_ρ} denotes differentiation with respect to y_ρ *only as it appears in the third place holder*. This allows us to express G (after integrating by parts) as

$$G(x, t; \xi, \tau) = Z(x - \xi, t; \xi, \tau) + \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n Z_{y_\rho}(x - y, t; y, \sigma) \Phi^\rho(y, \sigma; \xi, \tau) \mathrm{d}y \mathrm{d}\sigma$$
$$+ \int_\tau^t \int_{\mathbb{R}^n} \sum_{\rho=1}^n Z(x - y, t; y, \sigma) \Phi^\rho_{y_\rho}(y, \sigma; \xi, \tau) \mathrm{d}y \mathrm{d}\sigma. \tag{5.28}$$

Of these three summands, Z is already understood, and the third summand is effectively the same term that arises in Friedman's strong analysis. More precisely, our $\Phi^{\rho}_{y_\rho}$ blows up at the same rate in $\sigma - \tau$ as Friedman's Φ (cf (9.4.7) on p. 252 of [1]). We focus, then, on the second integrand, which we denote as $V(x, t; \xi, \tau)$. In addition, we write

$$J(x, t, \sigma; \xi, \tau) := \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} Z_{y_\rho}(x - y, t; y, \sigma)\Phi^{\rho}(y, \sigma; \xi, \tau)\mathrm{d}y,$$

so that

$$V(x, t; \xi, \tau) = \int_{\tau}^{t} J(x, t, \sigma; \xi, \tau)\mathrm{d}\sigma.$$

We begin by verifying that V is $2p$ times differentiable in x. To begin, we observe that for $t > \sigma$, we can write

$$D_x^{\alpha} J(x, t, \sigma; \xi, \tau) = \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} D_x^{\alpha} Z_{y_\rho}(x - y, t; y, \sigma)\Phi^{\rho}(y, \sigma; \xi, \tau)\mathrm{d}y.$$

According to (9.3.11) on p. 249 of [1], we have, for any multi-index α, the estimate

$$|D_x^{\alpha} Z_{y_\rho}(x - y, t; y, \sigma)| \leq C(t - \sigma)^{-\frac{n+|\alpha|}{2p}} e^{-c(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}}, \tag{5.29}$$

where we note in particular that the y-differentiation (for y only in the third position) does not increase the blow-up as t approaches σ. In this way, we can start with the naive estimate

$$\begin{aligned} |D_x^{\alpha} J(x, t, \sigma; \xi, \tau)| &\leq C_1 \int_{\mathbb{R}^n} (t - \sigma)^{-\frac{n+|\alpha|}{2p}} (\sigma - \tau)^{-1-\frac{n-1-\gamma}{2p}} \\ &\quad \times e^{-c_1(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}} e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} \mathrm{d}y \\ &\leq C_2 (t - \tau)^{-\frac{n}{2p}} (t - \sigma)^{-\frac{|\alpha|}{2p}} (\sigma - \tau)^{-1+\frac{1+\gamma}{2p}} \\ &\quad \times e^{-c_2(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}}. \end{aligned} \tag{5.30}$$

Here, we can pause to observe the fundamental problem that for $|\alpha| = 2p$, this estimate is not integrable in σ up to $\sigma = t$. In order to remedy this, we obtain an alternative estimate by writing

$$\begin{aligned} D_x^{\alpha} J(x, t, \sigma; \xi, \tau) &= \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} D_x^{\alpha} Z_{y_\rho}(x - y, t; y, \sigma)\Phi^{\rho}(x, \sigma; \xi, \tau)\mathrm{d}y \\ &+ \int_{\mathbb{R}^n} \sum_{\rho=1}^{n} D_x^{\alpha} Z_{y_\rho}(x-y, t; y, \sigma)\Big\{\Phi^{\rho}(x, \sigma; \xi, \tau) - \Phi^{\rho}(y, \sigma; \xi, \tau)\Big\}\mathrm{d}y = I_1 + I_2. \end{aligned}$$

Beginning with I_1, we note that, for each summand, Φ^ρ can be pulled out of the integration. We recall that for any fixed $z \in \mathbb{R}^n$, the function $Z(x-y,t;z,\sigma)$ is a Green's function for the PDE

$$\frac{\partial Z_{ik}}{\partial t}(x-\xi,t;z,\tau) = \sum_{l=1}^{n}\sum_{j=1}^{N}\sum_{|\alpha|=2p-1} \tilde{A}^{ij}_{\alpha,l}(z,t) D_x^\alpha \frac{\partial Z_{jk}}{\partial x_l}(x-\xi,t;z,\tau). \quad (5.31)$$

Since constant vectors in $\mathbb{R}^n$ are clearly solutions to this system, we must have the identities

$$\int_{\mathbb{R}^n} Z(x-y,t;z,\sigma)\mathrm{d}y \equiv I \Rightarrow \int_{\mathbb{R}^n} Z_{z_\rho}(x-y,t;z,\sigma)\mathrm{d}y \equiv 0. \quad (5.32)$$

In this way, we can write

$$\int_{\mathbb{R}^n} D_x^\alpha Z_{z_\rho}(x-y,t;y,\sigma)\mathrm{d}y$$
$$= \int_{\mathbb{R}^n} D_x^\alpha \Big(Z_{z_\rho}(x-y,t;y,\sigma) - D_x^\alpha Z_{z_\rho}(x-y,t;x,\sigma)\Big)\mathrm{d}y,$$

bearing in mind that the differentiation D_x^α is only with respect to x as it appears in the first placeholder. According to Lemma 9.3.4 of [1], we have

$$\Big| D_x^\alpha Z_{y_\rho}(x-y,t;y,\sigma) - D_x^\alpha Z_{y_\rho}(x-y,t;x,\sigma)\Big|$$
$$\le C(t-\sigma)^{-\frac{n+|\alpha|}{2p}} |y-x|^\gamma e^{-c(\frac{|x-y|^{2p}}{t-\sigma})^{1/(2p-1)}},$$

for some positive constants c and C. We see that

$$\left| \int_{\mathbb{R}^n} D_x^\alpha Z_{z_\rho}(x-y,t;y,\sigma)\mathrm{d}y \right| \le C_1(t-\sigma)^{\frac{\gamma-|\alpha|}{2p}}.$$

In this way,

$$|I_1| \le C_2(t-\sigma)^{\frac{\gamma-|\alpha|}{2p}}(\sigma-\tau)^{-1-\frac{n-1-\gamma}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}, \quad (5.33)$$

for some positive constants c_2 and C_2.

For I_2, we observe that precisely the same analysis that leads to (9.4.17) on p. 255 of [1] leads to the inequality

$$\Big|\Phi^\rho(y,\sigma;\xi,\tau) - \Phi^\rho(x,\sigma;\xi,\tau)\Big|$$
$$\le C_1|x-y|^\beta(\sigma-\tau)^{-1-\frac{n+\beta-1-\gamma}{2p}}\left\{ e^{-c_1(\frac{|y-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}} + e^{-c_1(\frac{|x-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}\right\},$$

for any $0 < \beta < \gamma$ and some positive constants c_1 and C_1. Upon integrating, we immediately see that

$$|I_2| \le C_2(t-\tau)^{-\frac{n}{2p}}(t-\sigma)^{-1+\frac{\beta}{2p}}(\sigma-\tau)^{-1+\frac{\gamma+1-\beta}{2p}} e^{-c_2(\frac{|x-\xi|^{2p}}{\sigma-\tau})^{1/(2p-1)}}, \quad (5.34)$$

for some constants c_2 and C_2.

Now, we evaluate

$$\int_\tau^t \Big|D_x^\alpha J(x,t,\sigma;\xi,\tau)\Big| \mathrm{d}\sigma = \int_\tau^{\frac{\tau+t}{2}} \Big|D_x^\alpha J(x,t,\sigma;\xi,\tau)\Big| \mathrm{d}\sigma + \int_{\frac{\tau+t}{2}}^t \Big|D_x^\alpha J(x,t,\sigma;\xi,\tau)\Big| \mathrm{d}\sigma.$$

For the integration on $[\tau, \frac{\tau+t}{2}]$, we use (5.30), while for integration over $[\frac{\tau+t}{2}, t]$, we use (5.33) and (5.34). We find

$$\int_\tau^t \Big|D_x^\alpha J(x,t,\sigma;\xi,\tau)\Big| \mathrm{d}\sigma \le C(t-\tau)^{-1-\frac{n-1-\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for some positive constants c and C.

We have verified that the second summand of $G(x,t;\xi,\tau)$ [in (5.28)] is $2p$ times differentiable in x. By analyzing the third summand in a similar way, we conclude that for each multi-index $|\alpha| \le 2p$, we have the estimate

$$|D_x^\alpha G(x,t;\xi,\tau)| \le C_\alpha (t-\tau)^{-\frac{n+|\alpha|}{2p}} e^{-c_\alpha(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for some positive constants c_α and C_α. (We note for clarity that the estimates on G are ultimately determined by those on Z). This is precisely the same estimate we found in Lemma 3.5, extended to the broader range $|\alpha| \le 2p$.

Turning now to differentiation with respect to t, we begin by writing (for $h > 0$ small)

$$V(x,t+h;\xi,\tau) - V(x,t;\xi,\tau) = \int_t^{t+h} J(x,t+h,\sigma;\xi,\tau) d\sigma + \int_\tau^t \Big(J(x,t+h,\sigma;\xi,\tau) - J(x,t,\sigma;\xi,\tau)\Big) \mathrm{d}\sigma =: I_1 + I_2.$$

For I_1, we can analyze $J(x,t+h,\sigma;\xi,\tau)$ similarly as we did $D_x^\alpha J$ (in fact, with less effort), and we find

$$|J(x,t,t^-;\xi,\tau)| \le C(t-\tau)^{-1-\frac{n-1-\gamma}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

where by $J(x,t,t^-;\xi,\tau)$ we mean

$$\lim_{\sigma\to t^-} J(x,t,\sigma;\xi,\tau).$$

For I_2, we use the Mean Value Theorem to write

$$I_2 = h\int_\tau^t \frac{\partial J}{\partial t}(x,t^*,\sigma;\xi,\tau) \mathrm{d}\sigma,$$

for some $t^* = t^*(t, h, x, \xi, \sigma, \tau)$ between t and $t+h$. We observe from the definition of J and the observation from (3.11) that time derivatives of Z can be exchanged for $2p$ space derivatives that this integrand can be estimated as in our analysis of D_x^α differentiation. We find

$$\lim_{h\to 0^+} |\frac{I_2}{h}| \le C(t-\tau)^{-1-\frac{n-1-\gamma}{2p}}.$$

Noting that a very similar argument works for $h < 0$, we conclude the estimate

$$|G_t(x, t; \xi, \tau)| \le C(t-\tau)^{-1-\frac{n}{2p}} e^{-c(\frac{|x-\xi|^{2p}}{t-\tau})^{1/(2p-1)}},$$

for some positive constants c and C.

We can now differentiate u in (5.26) directly (bringing derivatives under the integral sign), and we see that $u \in C^{2p,1}(\mathbb{R}^n \times [\sigma, \tilde{T}]$ for any $\sigma \in (\tau, \tilde{T}]$. We conclude that u is in fact a strong solution of our original equation (1.1).

Finally, we obtain the additional Hölder regularity $C^{2p+\gamma,1+\frac{\gamma}{2p}}(\mathbb{R}^n \times [\sigma, \tilde{T}]$ for all $\tau < \sigma \le \tilde{T}$ by an argument similar to the proof of Lemma 4.1, augmented by the observations used in this section to obtain higher-order regularity.

This concludes the proof of Theorem 1.1. □

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