

# Phase-Asymptotic Stability of Transition Front Solutions in Cahn-Hilliard Systems

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## Collaborators

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This is joint work with Bongsuk Kwon, a current Texas A&M post-doc.

The following graduate students worked on numerical calculations related to the project:

Michael Dearman

Andrew Sharp

# Outline of the Talk

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- ▶ Introduction to the main problem
- ▶ Overview of recent results
- ▶ Elements of the Analysis
  - ▶ Spectral issues and the Evans function
  - ▶ Local tracking
  - ▶ Contours and splittings
- ▶ Stability theorem
- ▶ Further work
- ▶ (Time permitting) Notes on the proof
  - ▶ Estimates on the Green's function
  - ▶ Choosing the local shift
  - ▶ Closing the argument

## Cahn-Hilliard Systems

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For  $x \in \mathbb{R}$  and  $u \in \mathbb{R}^m$ , we consider systems of the form

$$u_t = \left( M(u)(-\Gamma u_{xx} + F'(u)) \right)_x$$
$$u(x, 0) = u_0(x).$$

This is a standard model of certain phase separation processes such as spinodal decomposition, where the components of  $u$  characterize  $m$  components of a mixture that contains  $m + 1$  components in all.

Here,  $F \in \mathbb{R}$  is a measure of bulk free energy density,  $M \in \mathbb{R}^{m \times m}$  is a measure of molecular mobility, and  $\Gamma \in \mathbb{R}^{m \times m}$  characterizes interfacial energy. Based on physical considerations, we assume  $M$  and  $\Gamma$  are symmetric and positive definite,  $M$  uniformly so.

We are interested in the phase-asymptotic stability of transition fronts  $\bar{u}(x)$ .

## Spinodal Decomposition

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Spinodal decomposition is a phenomenon in which the rapid cooling (quenching) of a homogeneously mixed alloy with  $m + 1$  components (e.g., iron, chromium, nickel, etc.) causes separation to occur, resolving the mixture into regions of different crystalline structure, separated by steep transition layers.

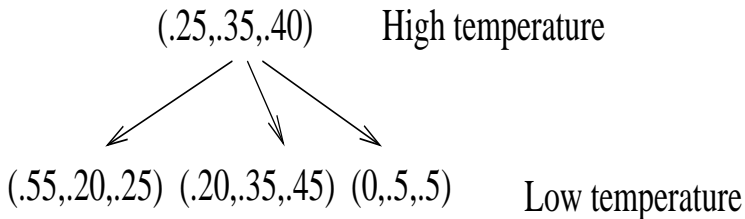


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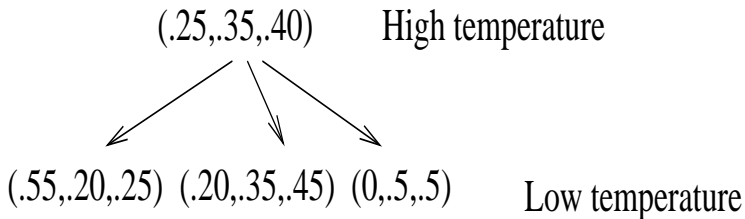


Figure: Possible concentrations.

## The Cahn-Hilliard Energy

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Under certain conditions the energy associated with such a system can be expressed as

$$E(u) := \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle u_x, \Gamma u_x \rangle dx.$$

Here,  $\langle \cdot, \cdot \rangle$  denotes Euclidean inner product.

Physically, we expect  $F$  to have  $m + 1$  distinct minimizers  $\{\xi_j\}_{j=1}^{m+1}$  associated with energy-preferred phases of the mixture. A common example for  $m = 2$  is

$$F(u) = u_1^2 u_2^2 + u_1^2 (1 - u_1 - u_2)^2 + u_2^2 (1 - u_1 - u_2)^2,$$

with minimizers  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ .

## Conserving Mass

---

Cahn-Hilliard systems can be expressed as conservation laws

$$u_t + J_x = 0,$$

with flux

$$J = -M(u)\left(\frac{\delta E}{\delta u}\right)_x,$$

where  $\frac{\delta E}{\delta u} = -\Gamma u_{xx} + F'(u)$  denotes the variational derivative of  $E$ .

This says the system will move from configurations in which a small change in concentration is accompanied by a large change in energy to configurations in which a small change in concentration is accompanied by a small change in energy.

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$$\text{HEAT EQ: } J = -M(u)(u)_x, \quad \text{HOT} \rightarrow \text{COLD}$$

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## Minimizing the Energy

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Recall that the energy is

$$E(u) := \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle u_x, \Gamma u_x \rangle dx.$$

We expect solutions to evolve toward minimizers of  $E$ . More precisely, it's easy to verify that any solution  $u(x, t)$  in an appropriate function class will satisfy

$$\frac{d}{dt} E(u) = - \int_{-\infty}^{+\infty} \langle M(u) \left( \frac{\delta E}{\delta u} \right)_x, \left( \frac{\delta E}{\delta u} \right)_x \rangle dx \leq 0.$$

One way the system can minimize this is to approach a minimizer of  $F$ , but it's constrained by conservation of mass.

The system can compromise by making transitions from one minimizer of  $F$  to another, but these transitions increase the second term in  $E$ . The resulting *transition fronts* are a balance between these effects.

## Transition Fronts

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We refer to a stationary solution  $\bar{u}(x)$  that connects two minimizers of  $F$  as a transition front. That is,  $\bar{u}(x)$  will satisfy

$$\begin{aligned} \left( M(\bar{u})(-\Gamma \bar{u}_{xx} + F'(\bar{u})) \right)_x &= 0 \\ \lim_{x \rightarrow -\infty} \bar{u}(x) &= u_- = \xi_j \\ \lim_{x \rightarrow +\infty} \bar{u}(x) &= u_+ = \xi_k, \\ \lim_{x \rightarrow \pm\infty} \bar{u}'(x) &= 0, \end{aligned}$$

for some  $j \neq k$ . It's easy to see that  $\bar{u}(x)$  solves

$$-\Gamma \bar{u}_{xx} + F'(\bar{u}) = 0.$$

This is simply the equation

$$\frac{\delta E}{\delta u} = 0.$$

## Example Case

For  $m = 2$ ,  $M(u) \equiv I$ ,  $\Gamma = I$ , and

$$F(u) = u_1^2 u_2^2 + u_1^2 (1 - u_1 - u_2)^2 + u_2^2 (1 - u_1 - u_2)^2.$$

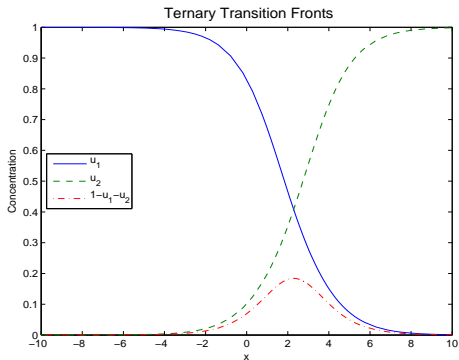


Figure: Ternary Transition Front.

## Existence of Transition Fronts

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The existence of transition fronts for Cahn-Hilliard systems has been established under quite general conditions by:

- ▶ N. D. Alikakos, S. I. Betelu, and X. Chen (2006): for  $m = 2$ , using complex analysis
- ▶ N. D. Alikakos and G. Fusco (2008):  $m \geq 2$ , for  $\Gamma = I$
- ▶ V. Stefanopoulos (2008):  $m \geq 2$ , for  $\Gamma$  positive definite and symmetric

In each of these references, the transition fronts arise as minimizers of the associated energy functional

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle u_x, \Gamma u_x \rangle dx.$$

## Asymptotic Behavior

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Intuitively, we expect that for fairly general initial conditions  $u(x, 0) = u_0(x)$  we will find locally that

$$\lim_{t \rightarrow \infty} u(x, t) = \bar{u}(x)$$

for some transition front  $\bar{u}(x)$ . More precisely, we expect  $u(x, t)$  to approach a combination of transition fronts consistent with mass conservation.

## Asymptotic Stability

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We say  $\bar{u}(x)$  is  $X \rightarrow Y$  stable (for some Banach spaces  $X$  and  $Y$ ) if given any  $\epsilon > 0$  there exists  $\eta > 0$  so that

$$\|u(x, 0) - \bar{u}(x)\|_X < \eta \Rightarrow \|u(x, t) - \bar{u}(x)\|_Y < \epsilon$$

for all  $t \geq 0$ .

We say  $\bar{u}(x)$  is  $X \rightarrow Y$  asymptotically stable if it is stable and there exists  $\eta_0 > 0$  sufficiently small so that

$$\|u(x, 0) - \bar{u}(x)\|_X < \eta_0 \Rightarrow \lim_{t \rightarrow \infty} u(x, t) = \bar{u}(x)$$

in  $Y$ .

## The Shift

Since concentration is conserved, perturbations of a transition front  $\bar{u}(x)$  will not generally approach the front itself, but rather (in the case of stability) a shift of the front. Consider a single component.

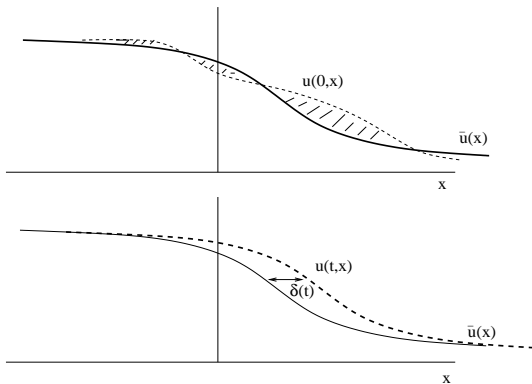


Figure: The shifted wave.

# Phase-Asymptotic Stability

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We define our perturbation as

$$v(x, t) := u(x + \delta(t), t) - \bar{u}(x).$$

We say  $\bar{u}(x)$  is  $X \rightarrow Y$  phase-stable if there exists a shift function  $\delta(t)$  so that for any  $\epsilon > 0$  there exists  $\eta > 0$  so that

$$\|v(x, 0)\|_X < \eta \Rightarrow \|v(x, t)\|_Y < \epsilon$$

for all  $t \geq 0$ .

We say  $\bar{u}(x)$  is  $X \rightarrow Y$  phase-asymptotically stable if  $\bar{u}(x)$  is  $X \rightarrow Y$  phase-stable and there exists a shift function  $\delta(t)$  and a value  $\eta_0 > 0$  so that

$$\|v(x, 0)\|_X \leq \eta_0 \Rightarrow \lim_{t \rightarrow \infty} \|v(x, t)\|_Y = 0.$$

## Goal

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We establish  $L^1 \cap L^\infty \rightarrow L^p$  phase-asymptotic stability for all  $p > 1$ .

We obtain  $L^1 \cap L^\infty \rightarrow L^1$  phase-stability.

## Overview of Recent Results

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For  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , general Cahn-Hilliard systems have the form

$$u_{j_t} = \nabla \cdot \left\{ \sum_{i=1}^m M_{ji}(u) \nabla \left( (-\Gamma \Delta u)_i + F_{u_i}(u) \right) \right\}.$$

1. For  $m = 1$ ,  $n \geq 1$ , phase-asymptotic stability of transition fronts has been established for general physical equations:
  - ▶  $n = 1$ : [H. Comm. Math. Phys. 2007]
  - ▶  $n \geq 2$ : [H. Physica D 2007]
2. For  $m \geq 2$ ,  $n = 1$ , phase-asymptotic stability has been established for many physical systems:
  - ▶ Spectral analysis, [H. and Bongsuk Kwon, Discrete and Continuous Dynamical Systems A, to appear 2012]
  - ▶ Nonlinear analysis, [H. and Bongsuk Kwon, Two submitted preprints 2011]
3. The goal of this program is to establish phase-asymptotic stability for the general case.

## Linearization

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We define our perturbation as

$$v(x, t) := u(x + \delta(t), t) - \bar{u}(x),$$

where  $\delta(t)$  tracks the shift between  $u(x, t)$  and  $\bar{u}(x)$ . The perturbation equation is

$$v_t = \left( M(\bar{u})(-\Gamma v_{xx} + F''(\bar{u})v)_x \right)_x + \dot{\delta}(t)(\bar{u}_x + v_x) + Q_x,$$

where

$$|Q| \leq C \left[ e^{-\eta|x|} |v|^2 + |v| |v_x| + |v| |v_{xxx}| \right].$$

This is advantageous (over  $|v|^2$ ) because we expect  $v_x$  and  $v_{xxx}$  to decay faster than  $v$  as  $|x| + t \rightarrow \infty$ . On the other hand, we expect  $v_x$  and  $v_{xxx}$  to blow up respectively like  $t^{-1/4}$  and  $t^{-3/4}$  as  $t \rightarrow 0$ .

# The Associated Eigenvalue Problem

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The associated linear equation is

$$v_t = Lv := \left( M(\bar{u})(-\Gamma v_{xx} + F''(\bar{u})v)_x \right)_x.$$

If we look for solutions of the form  $v(x, t) = e^{\lambda t}\phi(x)$ , we obtain the eigenvalue problem

$$L\phi = \lambda\phi.$$

The resolvent for this problem is

$$R(\lambda; L) := (\lambda I - L)^{-1},$$

and we say  $\lambda \in \mathbb{C}$  is in the resolvent set of  $L$  if  $R(\lambda; L)$  is a bounded linear operator.

## Spectrum of $L$ : $\sigma(L) = \sigma_{pt} \cup \sigma_{ess}$

---

We consider two (not necessarily disjoint) sets of spectrum. The point spectrum is

$$\sigma_{pt} := \{\lambda \in \mathbb{C} : L\phi = \lambda\phi \text{ for some } \phi \in H^2, \phi \neq 0\}.$$

We refer to elements of  $\sigma_{pt}$  as eigenvalues. By essential spectrum  $\sigma_{ess}$ , we mean any value  $\lambda \in \mathbb{C}$  that is not in the resolvent set and is not an isolated eigenvalue with finite multiplicity.

Roughly, eigenvalues characterize local (transitional) behavior of the transition front, while the essential spectrum characterizes endstate behavior.

# The Classical Semigroup Framework

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We can write our full perturbation equation as

$$v_t = Lv + \dot{\delta}(t)(\bar{u}_x + v_x) + Q_x.$$

For an initial perturbation  $v(x, 0)$ , we can express  $v(x, t)$  in semigroup formalism as

$$v(x, t) = e^{Lt} v_0 + \int_0^t e^{L(t-s)} \left[ \dot{\delta}(s) \bar{u}'(y) + \dot{\delta}(s) v_y(y, s) + Q_y \right] ds,$$

where by Laplace transform

$$e^{Lt} := \frac{1}{2\pi i} \int_{\Omega} e^{\lambda t} R(\lambda; L) d\lambda.$$

Here,  $\Omega$  denotes a contour in the resolvent set of  $L$ , entirely to the right of  $\sigma(L)$ , so that  $\arg \lambda \rightarrow \pm\theta$  as  $|\lambda| \rightarrow \infty$  for some  $\theta \in (\frac{\pi}{2}, \pi)$ . In addition we can move  $\Omega$  as allowed by Cauchy's Theorem.

## Spectrum and Contour

Clearly, if  $\sigma(L) \subset (-\infty, -\kappa]$  for some  $\kappa > 0$  then  $e^{Lt}$  will decay at exponential rate in  $t$ .

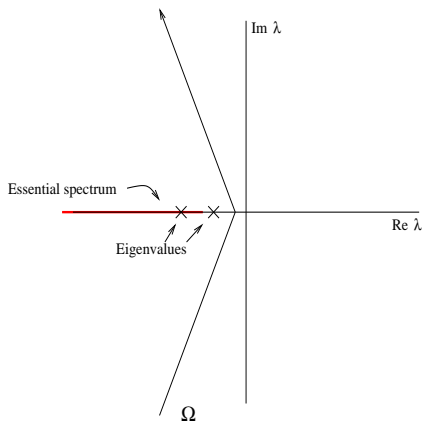


Figure: Example of a clearly stable spectrum.



## Spectrum and Contour

---

We can accommodate the eigenvalue at  $\lambda = 0$  by separating out a term that does not decay in time.

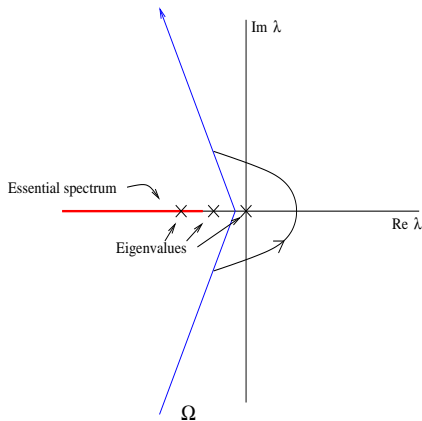


Figure: Accommodating the neutral eigenvalue.

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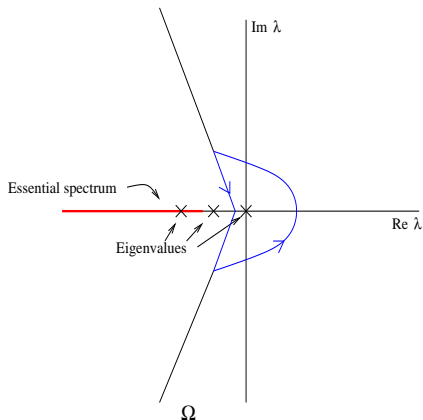


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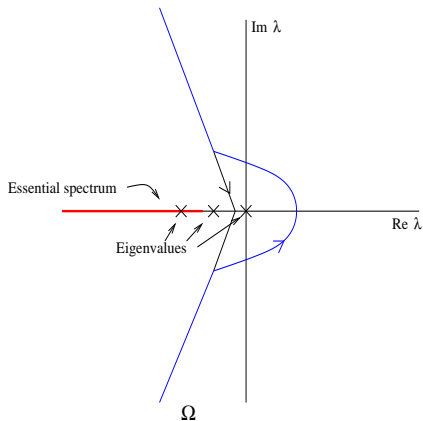


Figure: Accommodating the neutral eigenvalue.

# Spectrum and Contour

For Cahn-Hilliard systems  $\sigma_{ess} = (-\infty, 0]$ .

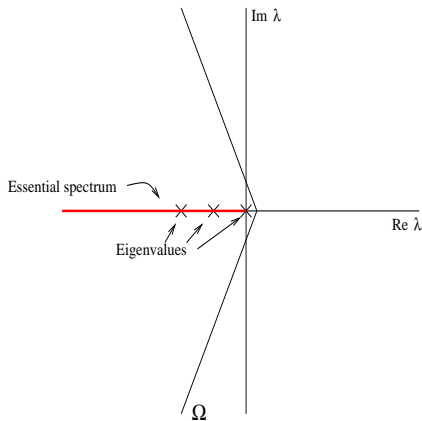


Figure: Essential spectrum for Cahn-Hilliard systems.

# The Pointwise Semigroup Framework

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Let  $G(x, t; y)$  denote a Green's function (distribution) for  $v_t = Lv$ :

$$G_t = LG$$

$$G(x, t; y) = \delta_y(x)I.$$

We can express our semigroup operator  $e^{Lt}$  as

$$e^{Lt}f = \int_{-\infty}^{+\infty} G(x, t; y)f(y)dy.$$

Our expression for  $v$  becomes

$$\begin{aligned} v(x, t) &= \int_{-\infty}^{+\infty} G(x, t; y)v_0(y)dy + \delta(t)\bar{u}'(x) \\ &+ \int_0^t \int_{-\infty}^{+\infty} G(x, t-s, y) \left[ \dot{\delta}(s)v(y, s) + Q \right]_y dy. \end{aligned}$$

## The Choice of Splitting

---

The main step in the analysis consists of deriving the splitting

$$G(x, t; y) = \bar{u}'(x)e(t; y) + \tilde{G}(x, t; y),$$

where  $\bar{u}'(x)e(t; y)$  is a leading order term associated with  $\lambda = 0$  that does not decay as  $t \rightarrow \infty$  and  $\tilde{G}(x, t; y)$  decays roughly like a heat kernel.

Intuitively,  $\bar{u}'(x)e(t; y)$  captures behavior associated with  $\lambda = 0$ , while  $\tilde{G}$  captures behavior associated with essential spectrum.

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## The Eigenvalue $\lambda = 0$

---

First, by definition

$$-\Gamma \bar{u}_{xx} + F'(\bar{u}) = 0 \Rightarrow -\Gamma \bar{u}_{xxx} + F''(\bar{u})\bar{u}_x = 0$$

so that  $\phi = \bar{u}_x$  solves

$$-\Gamma \phi_{xx} + F''(\bar{u})\phi = 0.$$

We see that

$$\left( M(\bar{u})(-\Gamma \phi_{xx} + F''(\bar{u})\phi)_x \right)_x = 0,$$

so that  $\lambda = 0$  is an eigenvalue with associated eigenfunction  $\bar{u}_x$ .

# The Evans Function

---

Our eigenvalue problem

$$\left( M(\bar{u})(-\Gamma\phi_{xx} + F'(\bar{u})\phi)_x \right)_x = \lambda\phi$$

has  $2m$  solutions that decay as  $x \rightarrow -\infty$ ,  $\{\phi_j^-\}_{j=1}^{2m}$  and  $2m$  solutions that decay as  $x \rightarrow +\infty$ ,  $\{\phi_j^+\}_{j=1}^{2m}$ . We will set

$$\Phi_j^\pm := \begin{pmatrix} \phi_j^\pm \\ \phi_j^{\pm'} \\ \phi_j^{\pm''} \\ \phi_j^{\pm'''} \end{pmatrix}, \quad \Phi^\pm := (\Phi_1^\pm, \dots, \Phi_{2m}^\pm).$$

The Evans function is

$$D(\lambda) = \det(\Phi^+(0; \lambda), \Phi^-(0; \lambda)).$$

## Characteristics of the Evans Function

---

Recall that any eigenfunction  $\phi$  must be in  $H^2$ , and so it must decay at both  $\pm\infty$ . In this way, there must exist constants  $\{\alpha_j\}_{j=1}^{2m}$  and  $\{\beta_j\}_{j=1}^{2m}$  so that

$$\sum_{j=1}^{2m} \alpha_j \phi_j^-(x; \lambda) = \phi(x; \lambda) = \sum_{j=1}^{2m} \beta_j \phi_j^+(x; \lambda).$$

By linear dependence  $D(\lambda) = 0$ . We know  $\lambda = 0$  is an eigenvalue, and it follows that  $D(0) = 0$ .

We would like to characterize this eigenvalue further by computing  $D'(0)$ ,  $D''(0)$ , etc. However,  $D$  is not differentiable at  $\lambda = 0$ .

# Analyticity of the Evans Function

---

The solutions  $\{\phi_j^-\}_{j=1}^{2m}$  and  $\{\phi_j^+\}_{j=1}^{2m}$  have the form

$$\phi_j^-(x; \lambda) = e^{\mu_{2m+j}^-(\lambda)x} (r_{2m+j}^- + \mathbf{O}(e^{-\eta|x|}))$$

$$\phi_j^+(x; \lambda) = e^{\mu_j^+(\lambda)x} (r_j^+ + \mathbf{O}(e^{-\eta|x|})),$$

where for  $j = 1, \dots, m$

$$\mu_j^\pm(\lambda) = -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|)$$

$$\mu_{m+j}^\pm(\lambda) = -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2})$$

$$\mu_{2m+j}^\pm(\lambda) = \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2})$$

$$\mu_{3m+j}^\pm(\lambda) = \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|).$$

## The Stability Condition

---

We can view the Evans function as an analytic function of  $\rho = \sqrt{\lambda}$ . We denote this function  $D_a(\rho)$ . It is straightforward to verify that

$$D_a(0) = D'_a(0) = \dots = D_a^{(m)}(0) = 0.$$

Our stability condition is

$$\frac{d^{m+1}D_a}{d\rho^{m+1}}(0) \neq 0.$$

For  $m = 1$  (the case of a single equation) it's easy to verify that this holds under standard physical assumptions. For  $m \geq 2$ , we have developed a framework for verifying this condition on a case-by-case basis.

## Spectral Stability

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It's relatively easy to verify that

$$\sigma_{ess} = (-\infty, 0].$$

If  $\bar{u}$  minimizes the Cahn-Hilliard energy, it's easy to show that

$$\sigma_{pt}(L) \setminus \{0\} \subset (-\infty, -\kappa]$$

for some  $\kappa > 0$ . If these conditions both hold, along with our stability condition

$$\frac{d^{m+1} D_a}{d\rho^{m+1}}(0) \neq 0.$$

we say  $\bar{u}(x)$  is spectrally stable.

## Stability Theorem

---

Suppose  $\bar{u}(x)$  is a spectrally stable transition front. Then for Hölder continuous initial conditions  $u_0(x) \in C^\gamma(\mathbb{R})$ ,  $0 < \gamma < 1$ , with

$$\|u(0, x) - \bar{u}(x)\|_{L^1} + \|u(0, x) - \bar{u}(x)\|_{L^\infty} \leq \epsilon,$$

for  $\epsilon > 0$  sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R} \times (0, \infty)) \cap C^{\gamma, \frac{\gamma}{4}}(\mathbb{R} \times [0, \infty))$$

and a shift  $\delta \in C^{1+\gamma}[0, \infty)$  so that

$$\begin{aligned} \|u(x + \delta(t), t) - \bar{u}(x)\|_{L^p} &\leq C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \\ |\delta(t) - \delta_\infty| &\leq C\epsilon(1+t)^{-1/4}. \end{aligned}$$

## Further Work

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- ▶ Extension to the case  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $n \geq 2$ ,  $m \geq 2$
- ▶ Periodic and pulse-type stationary solutions for  $m \geq 2$
- ▶ Genuinely multidimensional stationary solutions for  $n \geq 2$
- ▶ Transient dynamics
- ▶ Navier-Stokes-Cahn-Hilliard systems
- ▶ The functionalized Cahn-Hilliard equation

$$u_t = \Delta \left\{ (\epsilon^2 \Delta - F''(u) - \epsilon \eta_1)(\epsilon^2 \Delta u - F'(u)) + \epsilon(\eta_2 - \eta_1)F'(u) \right\}$$

## Notes on the Proof

---

Aside from our spectral argument, we can think of the proof in three steps:

1. Obtaining estimates on  $G$
2. Choosing the local shift  $\delta(t)$
3. Iterating a system of integral equations for  $v$ ,  $v_x$ , and  $\dot{\delta}$

Here, we briefly discuss each.

## Constructing $G$

---

We compute the Laplace transform of  $G$  ( $t \rightarrow \lambda$ ), writing  $\mathcal{L}\{G\} = G_\lambda(x; y)$ , so that

$$LG_\lambda - \lambda G_\lambda = -\delta_y(x).$$

Inverting, we find

$$G(x, t; y) = \frac{1}{2\pi i} \int_{\Omega} e^{\lambda t} G_\lambda(x; y) d\lambda,$$

where  $\Omega$  is the same contour previously described.

We find that  $G_\lambda(x; y)$  can be expressed as

$$G_\lambda(x; y) = \bar{u}'(x)e_\lambda(y) + \tilde{G}_\lambda(x; y),$$

giving

$$G(x, t; y) = \bar{u}'(x)e(t; y) + \tilde{G}(x, t; y).$$

## Green's Function Estimates

---

Using our spectral information and the structure of our equation, we find for  $t \geq 1$

$$e(t; y) = \sum_{j=m+1}^{2m} c_j \int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2} dz + R(t; y)$$

$$|R(t; y)| \leq Ct^{-1/2} e^{-\frac{y^2}{Mt}},$$

and for  $t \geq 1$  and  $|x - y| \geq Kt$

$$|\tilde{G}(x, t; y)| \leq Ct^{-1/2} e^{-\frac{(x-y)^2}{Mt}}.$$

Here,  $\{c_j\}_{j=m+1}^{2m}$ ,  $C$ ,  $K$ , and  $M$  are fixed positive constants depending only on the spectrum of  $L$  and the structure of the equations.

## Choosing the Local Shift

---

We find (substituting  $G = \bar{u}'e + \tilde{G}$  and integrating by parts)

$$\begin{aligned} v(x, t) &= \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) v_0(y) dy + \delta(t) \bar{u}'(x) \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x, t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds \\ &+ \bar{u}'(x) \left\{ \int_{-\infty}^{+\infty} e(t; y) v_0(y) dy \right. \\ &\quad \left. - \int_0^t \int_{-\infty}^{+\infty} e_y(t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds \right\}. \end{aligned}$$

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## Choosing the Local Shift

---

We take

$$\begin{aligned}\delta(t) = & - \int_{-\infty}^{+\infty} e(t; y) v_0(y) dy \\ & + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds,\end{aligned}$$

which leaves

$$\begin{aligned}v(x, t) = & \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) v_0(y) dy \\ & - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x, t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds.\end{aligned}$$

We obtain similar integral equations for  $\dot{\delta}(t)$  and  $v_x(x, t)$ , making  $2m + 2$  equations.

# Integral Equations

---

Our full system of integral equations:

$$\begin{aligned}\dot{\delta}(t) &= - \int_{-\infty}^{+\infty} e_t(t; y) v_0(y) dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds, \\ v(x, t) &= \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) v_0(y) dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x, t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds \\ v_x(x, t) &= \int_{-\infty}^{+\infty} \tilde{G}_x(x, t; y) v_0(y) dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(x, t-s; y) \left[ \dot{\delta}(s) v(y, s) + Q \right] dy ds.\end{aligned}$$

## Linear Stability

---

Using the estimates we obtain on  $G$ , we can verify that

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} e_t(t; y) v_0(y) dy \right| &\leq C(1+t)^{-1} \\ \left\| \int_{-\infty}^{+\infty} \tilde{G}(x, t; y) v_0(y) dy \right\|_{L^p} &\leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \\ \left\| \int_{-\infty}^{+\infty} \tilde{G}_x(x, t; y) v_0(y) dy \right\|_{L^p} &\leq Ct^{-1/4}(1+t)^{-\frac{1}{4}}. \end{aligned}$$

## Bounding Function

---

We take these values as preliminary estimates and set

$$\zeta(t) := \sup_{\substack{0 \leq s \leq t \\ 1 \leq p \leq \infty}} \left\{ \|v(\cdot, s)\|_{L^p} (1+s)^{\frac{1}{2}(1-\frac{1}{p})} \right. \\ \left. + \|v_x(\cdot, s)\|_{L^p} s^{1/4} (1+s)^{1/4} + |\dot{\delta}(s)|(1+s) \right\}.$$

Clearly,

$$\|v(\cdot, s)\|_{L^p} \leq \zeta(t) (1+s)^{-\frac{1}{2}(1-\frac{1}{p})} \\ \|v_x(\cdot, s)\|_{L^p} \leq \zeta(t) s^{-1/4} (1+s)^{-1/4} \\ |\dot{\delta}(s)| \leq \zeta(t) (1+s)^{-1}.$$

## Nonlinear Stability

---

Upon substitution of these bounds into our integral equations, we find

$$\zeta(t) \leq C(\epsilon + \zeta(t)^2).$$

Here  $C$  is a new constant and we recall

$$\|v(\cdot, 0)\|_{L^1} + \|v(\cdot, 0)\|_{L^\infty} \leq \epsilon.$$

It's straightforward to verify that this implies

$$\zeta(t) < 2C\epsilon$$

for all  $t \geq 0$ . This inequality is equivalent with the statement of our theorem.

# The Essential Spectrum

---

The essential spectrum is determined by the asymptotic ( $x \rightarrow \pm\infty$ ) eigenvalue equations

$$-M(u_{\pm})\Gamma\phi_{xxxx} + M(u_{\pm})F''(u_{\pm})\phi_{xx} = \lambda\phi.$$

It corresponds with solutions

$$\phi(x) = e^{i\xi x} w,$$

so that

$$\left( -M(u_{\pm})\Gamma\xi^4 - M(u_{\pm})F''(u_{\pm})\xi^2 \right) w = \lambda(\xi)w.$$

By positivity and symmetry of the matrices  $\Gamma$ ,  $M(u_{\pm})$ , and  $F''(u_{\pm})$ ,

$$\sigma_{\text{ess}} = (-\infty, 0].$$

## The Point Spectrum

---

First, for  $\lambda \neq 0$ , if  $\phi(x; \lambda)$  solves

$$\left( M(\bar{u})(-\Gamma\phi_{xx} + F''(\bar{u})\phi)_x \right)_x = \lambda\phi,$$

we can show by integrating both sides that  $\int_{-\infty}^{+\infty} \phi(x; \lambda) dx = 0$ .  
We set

$$\varphi(x; \lambda) := \int_{-\infty}^x \phi(y; \lambda) dy,$$

so that the integrated eigenvalue problem is

$$M(\bar{u})(-\Gamma\varphi_{xxx} + F''(\bar{u})\varphi_x)_x = \lambda\varphi.$$

Multiply by  $M(\bar{u})^{-1}$  and take  $L^2$  inner product with  $\varphi$ :

$$-\langle \varphi_x, -\Gamma\varphi_{xxx} + F''(\bar{u})\varphi_x \rangle = \lambda \langle \varphi, M(\bar{u})^{-1}\varphi \rangle.$$

Here,  $\langle \cdot, \cdot \rangle$  denotes  $L^2$  inner product.

## The Eigenvalues $\lambda \neq 0$

---

We can express this last equation as

$$-\langle \varphi_x, H\varphi_x \rangle = \lambda \langle \varphi, M(\bar{u})^{-1}\varphi \rangle,$$

where  $H$  is the Schrödinger type operator

$$H := -\Gamma \partial_x^2 + F''(\bar{u}).$$

Here, we recall that  $\bar{u}$  minimizes the energy

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} \langle u_x, \Gamma u_x \rangle dx,$$

and so  $\frac{\delta^2 E}{\delta u^2} = H$  is a non-negative operator. We conclude

$$\sigma_{pt} \subset (-\infty, 0].$$

## Notes

---

Set  $B_{\pm} := F''(u_{\pm})$  and  $M_{\pm} := M(u_{\pm})$ . Then  $\sigma(\Gamma^{-1}B_{\pm}) = \{\nu_j^{\pm}\}_{j=1}^m$  and  $\sigma(M_{\pm}B_{\pm}) = \{\beta_j^{\pm}\}_{j=1}^m$ . Also,

$$\left( -(\mu_j^{\pm})^4 M_{\pm} \Gamma + (\mu_j^{\pm})^2 M_{\pm} B_{\pm} - \lambda I \right) r_j^{\pm} = 0.$$

We refer to the solutions for which  $\mu_j^{\pm}(0) \neq 0$  as *fast* and the solutions for which  $\mu_j^{\pm}(0) = 0$  as *slow*.

In the remainder of the talk  $r_j^{\pm}$  will denote the evaluation of  $r_j^{\pm}$  at  $\lambda = 0$ .

By a choice of indexing, we take

$$\phi_{2m}^{-}(x; 0) = \bar{u}'(x) = \phi_1^{+}(x; 0).$$

## Semigroup Notes

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The estimates on  $G$  will be determined by the spectrum of  $L$ . Since  $\bar{u}'$  is a stationary solution for this equation,

$$e^{L(t-s)}\bar{u}'(\cdot) = \bar{u}'(x),$$

and assuming  $\delta(0) = 0$  we have

$$v(x, t) = e^{Lt}v_0 + \delta(t)\bar{u}'(x) + \int_0^t e^{L(t-s)} \left[ \dot{\delta}(s)v_y(y, s) + Q_y \right] ds.$$

## Intuition about the System

---

Recall that the system is

$$u_t = \left( M(u)(-\Gamma u_{xx} + F'(u)) \right)_x,$$

which we can express this as

$$u_t = \left( M(u)(-\Gamma u_{xxx}) \right)_x + \left( M(u)F''(u) \right)_x u_x + M(u)F''(u)u_{xx}.$$

For  $u$  near a minimizer of  $F$  the matrix  $F''(u)$  will be positive definite, and we expect asymptotic dynamics to be governed by second order diffusion.

Likewise,  $F$  will have at least one local maximum (in our example at  $(\frac{1}{3}, \frac{1}{3})$ ), and for  $u$  near this maximizer  $F''(u)$  will be negative definite.

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