Stability of Transition Front Solutions in Cahn-Hilliard Systems

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SIAM Conference on Applications of Dynamical Systems,
May 22, 2011

The Cahn-Hilliard System

We consider the Cahn-Hilliard system for $x \in \mathbb{R}$ and $u \in \mathbb{R}^m$,

$$u_t = \Big(M(u)(-\Gamma u_{xx} + F'(u))_x \Big)_x.$$

This is a standard model of certain phase separation processes such as spinodal decomposition, where the components of u characterize m components of a mixture that contains $m+1$ components in all.

Here, $F \in \mathbb{R}$ is a measure of bulk free energy density, $M \in \mathbb{R}^{m \times m}$ is a measure of molecular mobility, and $\Gamma \in \mathbb{R}^{m \times m}$ characterizes interfacial energy.

We are interested in the asymptotic stability of transition fronts $\bar{u}(x)$.

Structural Observations

F will generally have $m+1$ distinct minimizers $\{\xi_j\}_{j=1}^{m+1}$ associated with energy-preferred phases of the mixture. A common example for $m=2$ is

$$F(u) = u_1^2 u_2^2 + u_1^2(1-u_1-u_2)^2 + u_2^2(1-u_1-u_2)^2,$$

with minimizers $(0,0)$, $(1,0)$, and $(0,1)$.

For u near a minimizer the matrix $F''(u)$ will be positive definite, and we expect asymptotic dynamics to be governed by second order diffusion.

Likewise, F will have at least one local maximum (here at $(\frac{1}{3}, \frac{1}{3})$), and for u near this maximizer $F''(u)$ will be negative definite.

We assume M and Γ are both positive definite (M uniformly).
$F \in C^4(\mathbb{R})$ and $M \in C^2(\mathbb{R})$.

Transition Fronts

We refer to a stationary solution $\bar{u}(x)$ that connects two minimizers of F as a transition wave. That is, $\bar{u}(x)$ will satisfy

$$\Big(M(\bar{u})(-\Gamma\bar{u}_{xx} + F'(\bar{u}))_x\Big)_x = 0$$
$$\lim_{x\to-\infty} \bar{u}(x) = \xi_j$$
$$\lim_{x\to+\infty} \bar{u}(x) = \xi_k,$$

for some $j \neq k$. It's easy to see that $\bar{u}(x)$ solves

$$-\Gamma\bar{u}_{xx} + F'(\bar{u}) = 0.$$

Example Case

For $m = 2$, $M(u) \equiv I$, $\Gamma = I$, and

$$F(u) = u_1^2 u_2^2 + u_1^2(1 - u_1 - u_2)^2 + u_2^2(1 - u_1 - u_2)^2.$$

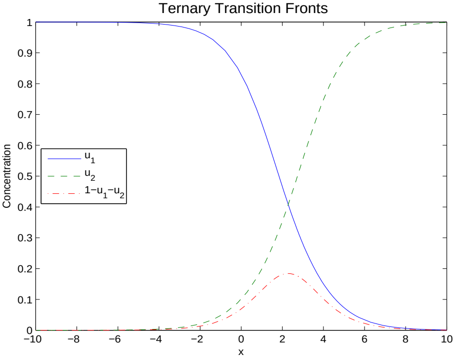

Figure: Ternary Transition Front.

Existence of Transition Fronts

The existence of transition fronts has been established under quite general conditions by:

- N. D. Alikakos, S. I. Betelu, and X. Chen (2006): for $m = 2$, using complex analysis
- N. D. Alikakos and G. Fusco (2008): $m \geq 2$, for $\Gamma = I$
- V. Stefanopoulos (2008): $m \geq 2$, for Γ positive definite and symmetric

In each of these references, the transition fronts arise as minimizers of the associated energy functional

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2} u_x \cdot \Gamma u_x dx.$$

Local Tracking

We define our perturbation as

$$v(x,t) := u(x+\delta(t),t) - \bar{u}(x),$$

where $\delta(t)$ tracks the shift between $u(x,t)$ and $\bar{u}(x)$. The perturbation equation is

$$v_t = \Big(M(\bar{u})(-\Gamma v_{xx} + F''(\bar{u})v)_x \Big)_x + \dot{\delta}(t)(\bar{u}_x + v_x) + Q_x,$$

where

$$|Q| \leq C\Big[e^{-\eta|x|}|v|^2 + |v||v_x| + |v||v_{xxx}| \Big].$$

The Evans Function

The associated eigenvalue problem

$$L\phi = \Big(M(\bar{u})(-\Gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x = \lambda\phi$$

has $2m$ solutions that decay as $x \to -\infty$, $\{\phi_j^-\}_{j=1}^{2m}$ and $2m$ solutions that decay as $x \to +\infty$, $\{\phi_j^+\}_{j=1}^{2m}$. We set

$$\Phi_j^\pm := \begin{pmatrix} \phi_j^\pm \\ \phi_j^{\pm\prime} \\ \phi_j^{\pm\prime\prime} \\ \phi_j^{\pm\prime\prime\prime} \end{pmatrix}, \quad \Phi^\pm := (\Phi_1^\pm, \dots, \Phi_{2m}^\pm).$$

The Evans function is

$$D(\lambda) = \det(\Phi^+(0;\lambda), \Phi^-(0;\lambda)).$$

Analyticity of the Evans Function

The solutions $\{\phi_j^-\}_{j=1}^{2m}$ and $\{\phi_j^+\}_{j=1}^{2m}$ have the form

$$\phi_j^-(x;\lambda) = e^{\mu_{2m+j}^-(\lambda)x}(r_{2m+j}^- + \mathbf{O}(e^{-\eta|x|}))$$
$$\phi_j^+(x;\lambda) = e^{\mu_j^+(\lambda)x}(r_j^+ + \mathbf{O}(e^{-\eta|x|})),$$

where for $j = 1, \ldots, m$

$$\mu_j^\pm(\lambda) = -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|); \quad \mu_{m+j}^\pm(\lambda) = -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2});$$

$$\mu_{2m+j}^\pm(\lambda) = \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2}); \quad \mu_{3m+j}^\pm(\lambda) = \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|).$$

The Evans function is analytic as a function of $\rho := \sqrt{\lambda}$.

Technical Note

The slow growth solutions $\{\psi_j^-\}_{j=m+1}^{2m}$ and $\{\psi_j^+\}_{j=1}^{m}$ of

$$L\phi = \Big(M(\bar{u})(-\Gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x = \lambda\phi$$

have the form

$$\psi_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x}(r_j^- + \mathbf{O}(e^{-\eta|x|}))$$
$$\psi_j^+(x;\lambda) = e^{\mu_{2m+j}^+(\lambda)x}(r_{2m+j}^+ + \mathbf{O}(e^{-\eta|x|})).$$

These coalesce with the slow decay solutions as $|\lambda| \to 0$.

Spectral Conditions

For the linear operator

$$L\phi := \Big(M(\bar{u})(-\Gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x,$$

we assume:

1. $\sigma_e(L) \subset (-\infty, 0]$

   This is easy to verify if $M(u_\pm)$ are symmetric matrices.

2. $\mathrm{Re}\sigma_{pt}(L)\backslash\{0\} < 0$.

   This follows from the construction of transition fronts as energy minimizers.

3. $$\frac{d^{m+1}D_a}{d\rho^{m+1}}(0) \neq 0.$$

   This can be verified analyically in certain cases, and it is straightforward to verify numerically.

Stability Theorem

Suppose $\bar{u}(x)$ is a transition front solution to (CH) for which our spectral conditions hold. Then for Holder continuous initial conditions $u_0(x) \in C^\gamma(\mathbb{R})$, $\gamma > 0$, with

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

for $\epsilon > 0$ sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4,1}(\mathbb{R} \times (0,\infty))$$
$$\lim_{\substack{(t,x)\to(0,x_0)\\ x\in\mathbb{R},t>0}} u(t,x) = u(0,x_0), \quad \text{for each } x_0 \in \mathbb{R},$$

and a shift $\delta \in C^1[0,\infty)$ so that

$$|u(x+\delta(t),t) - \bar{u}(x)| \le C\epsilon\Big[(1+t)^{-1/2} e^{-\frac{x^2}{Lt}} + (1+|x|+\sqrt{t})^{-3/2}\Big]$$
$$|\delta(t) - \delta_\infty| \le C\epsilon(1+t)^{-1/4}.$$

Method of Proof

We proceed by integrating our perturbation equation to the form

$$v(x,t) = \int_{-\infty}^{+\infty} G(x,t;y)v_0(y) + \delta(t)\bar{u}'(x) - \int_0^t \int_{-\infty}^{+\infty} G_y(x,t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big] dyds,$$

where G denotes a Green's function for the linear equation $w_t = Lw$. The main step in the analysis consists of deriving the splitting

$$G(x,t;y) = \bar{u}'(x)e(t;y) + \tilde{G}(x,t;y),$$

where $\bar{u}'(x)e(t;y)$ is a leading order term associated with $\lambda = 0$ that does not decay as $t \to \infty$ and $\tilde{G}(x,t;y)$ decays roughly like a heat kernel.

Choosing the Local Shift

We find

$$
\begin{aligned}
v(x,t) = \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y) + \delta(t)\bar{u}'(x) \\
- \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds \\
+\bar{u}'(x)\Big\{ \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy \\
- - \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds\Big\}.
\end{aligned}
$$

Choosing the Local Shift

We take

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\left[\dot{\delta}(s)v(y,s)+Q\right]dyds,$$

which leaves

$$v(x,t) = \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y) - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\left[\dot{\delta}(s)v(y,s)+Q\right]dyds.$$

We obtain similar integral equations for $\dot{\delta}(t)$ and $v_x(x,t)$, making $2m+2$ equations.

Short Time

We can obtain estimates on v, v_x, δ, and $\dot{\delta}$, provided we can control v_{xxx}. For this, we write the Cahn-Hilliard equation as

$$u_t = (M(u)(-\Gamma u_{xxx} + F''(u)u_x))_x.$$

Set

$$\begin{aligned} \tilde{M}(x,t) &:= M(u(x,t)) \\ \tilde{A}(x,t) &:= F''(u(x,t)), \end{aligned}$$

and consider the linear equation

$$u_t = (\tilde{M}(x,t)(-\Gamma u_{xxx} + \tilde{A}(x,t)u_x))_x.$$

Let $G^u(x,t;\xi,\tau)$ denote a Green's function for a weak formulation of this linear equation (given u) and set

$$\mathcal{T}u(x,t) := \int_{-\infty}^{+\infty} G^u(x,t;\xi,\tau)u^\tau(\xi)d\xi.$$

Short Time

For $t \in [\tau, T]$, with $T - \tau$ sufficiently small, we show that $\mathcal{T}$ is a contraction on

$$\mathcal{S} := \{u \in C^{\alpha, \frac{\alpha}{4}}(\mathbb{R} \times [\tau, T]) : u(x, \tau) = u^{\tau}(x), \|u\|_{C^{\alpha, \frac{\alpha}{4}}} \leq K\}.$$

Increased regularity can be obtained by bootstrapping (i.e., u has been shown to solve a linear problem with Holder continuous coefficients). We now have

$$u(x, t) = \int_{-\infty}^{+\infty} G^{u}(x, t; \xi, t - t_0) u^{t - t_0}(\xi) d\xi,$$

and we easily justify

$$u_{xxx}(x, t) = \int_{-\infty}^{+\infty} G^{u}_{xxx}(x, t, \xi, t - t_0) u^{t - t_0}(\xi) d\xi.$$

Notes

Set $B_\pm := f'(u_\pm)$ and $M_\pm := M(u_\pm)$. Then $\sigma(\Gamma^{-1}B_\pm) = \{\nu_j^\pm\}_{j=1}^m$ and $\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m$. Also,

$$\Big(-(\mu_j^\pm)^4 M_\pm \Gamma + (\mu_j^\pm)^2 M_\pm B_\pm - \lambda I \Big) r_j^\pm = 0.$$

We refer to the solutions for which $\mu_j^\pm(0) \neq 0$ as *fast* and the solutions for which $\mu_j^\pm(0) = 0$ as *slow*.

In the remainder of the talk $r_j^\pm$ will denote the evaluation of $r_j^\pm$ at $\lambda = 0$.

By a choice of indexing, we take

$$\phi_{2m}^-(x;0) = \bar{u}'(x) = \phi_1^+(x;0).$$