My research focuses on analysis and probability on metric measure spaces. I am particularly interested in how the local (small scale) and global (large scale) geometries affect analysis on the space. My research seeks to discover and extend these connections. If the local geometry is sufficiently nice, we can explore difficult questions about abstract spaces. For carefully chosen spaces, this could form the basis of undergraduate research projects. A few interesting questions are:

> *Isoperimetry*: What is the minimum size of the boundary of a set with volume $V$? How does this change as we vary $V$?
>
> *Volume Doubling*: Let $B(x, r)$ be a ball of radius $r$ centered at $x$. Does there exist a constant $C$ such that:
>
> $$\text{Volume}(B(x, 2r)) \leq C\text{Volume}(B(x, r))?$$
>
> If not, can we find a constant, $C_r$, that depends only on the radius? Can we find a constant, $C_x$, that depends only on the center?
>
> *Poincaré Inequality*: Let f be an $L^2$ function with $L^2$ derivatives. Does there exist a constant $C$ (independent of $f$) such that:
>
> $$\int_{B(x,r)} |f(y) - \frac{1}{\text{Vol}(B(x, r))} \int_{B(x,r)} f(z)dz|^2 dy \leq Cr^2 \int_{B(x,r)} |\nabla f(y)|^2 dy?$$
>
> *Heat Kernel Bounds*: Let $p_t(x, y)$ be the heat kernel on the space. This is the fundamental solution to the heat equation $\partial_t u - \Delta u = 0$. The heat kernel can be used to describe the probability that a Brownian motion travels from $x$ to $y$ in time $t$. What is the behavior of $p_t(x, x)$ as $t \to 0$? What is the behavior of $p_t(x, x)$ as $t \to \infty$?

All of these questions are understood in $\mathbb{R}^n$. What happens in other metric spaces? Let's look at isoperimetry on a few examples.

> In $\mathbb{R}$, any connected set has a boundary consisting of two points. Here, the boundary size is bounded below by 2.
>
> In $\mathbb{R}^2$, for a fixed area, circles have the smallest perimeter. A circle with area $\pi r^2$ has perimeter length $2\pi r$. This means a set of volume $V$ will have boundary length greater than or equal to $2\sqrt{\pi}\sqrt{V}$.

A more complicated example is the two dimensional lattice; this is the subset of $\mathbb{R}^2$ where at least one coordinate is an integer. On a small scale, this space is similar to $\mathbb{R}$. In the interior of an edge, we have a line; at a vertex, we have a cross. Balls with radius less than one have boundaries consisting of either two or four points. When we look at the lattice on a large scale, we see a structure that is similar to a plane. Here, we find that balls of radius $r$ have a boundary consisting of approximately $4r$ points. This tells us sets with volume $V$ have a boundary size greater than or equal to

$$\begin{cases} 2 & \text{if } V < 1 \\ \text{Const}\sqrt{V} & \text{if } V \geq 1 \end{cases}$$

The third space demonstrates how the isoperimetric inequality captures a notion of dimension. While the space itself is locally one dimensional, its large scale geometry behaves like a two dimensional space. Another example of a space whose local geometry differs from its large scale geometry is a fractal "blow-up" which is an infinite fractal that is locally nice but globally has a self-similar structure (see Kigami [4].) Currently, I am working on a project with Lee Gibson to try to better understand how a notion of isoperimetry for a large scale geometry can be applied to certain types of fractals.

Another type of metric space that I study is called a Riemannian (Euclidean) complex [3]. This is a polytopal complex where each polytope has an associated Riemannian (Euclidean) metric. We make a few geometric assumptions on the complex:

- Any polytope of dimension less than $n$ is a face of an $n$-dimensional polytope.
- The complex is connected and remains so when we remove the $n-2$ dimensional faces.
- The Riemannian metrics are uniformly elliptic. Additionally, polytopes which share a lower dimensional face have metrics which coincide on the shared face.
- Interior angles and the distance between nonintersecting faces are bounded below by a constant.
- The number of $k$-dimensional faces that share a lower dimensional face is bounded above.

With the given geometric assumptions, for any radius, $r$, the complex will have a corresponding volume doubling constant, $C_r$ [3]. Additionally, a

Poincaré inequality holds on balls with a constant that depends on the radius, $r$, but not on the center (Riemannian case: [7], Euclidean case: [6]). When we have a local volume doubling constant and a local Poincaré inequality, we can describe the heat kernel for small times [8].

**Theorem 1.** *Let $X$ be a Riemannian [7] (Euclidean [6]) complex of dimension $n$ that satisfies the geometric assumptions. Then*

$$p_t(x,x) \approx \frac{1}{t^{n/2}} \text{ as } t \to 0.$$

Note that the heat kernel behaves, up to a constant that is independent of the point $x$, like the heat kernel on $\mathbb{R}^n$ asymptotically as $t \to 0$. The local behavior reflects the local geometry and structure of our space. This work is related to results in a paper of Brin and Kifer [2] which constructs Brownian motion on two dimensional Euclidean complexes.

An interesting example of these complexes are ones which come from biology. In a paper by Billera, Holmes, and Vogtmann [1] they describe a way of classifying distances between phylogenetic trees, which are trees that describe evolution of species. One can form an Euclidean complex, where each of the faces corresponds to a different tree, and one moves through the points in the face by changing the edge lengths in the tree. One can then consider probability distributions on this space to determine likely genetic ancestry. A natural direction would be to look at Brownian motion on this specific set of complexes; this is intimately connected with understanding heat kernels.

A collection of examples with a wide variety of global geometries can be found by considering metric spaces, $X$, which are acted upon by a finitely generated group of isometries, $G$. When we take the space and mod out by the group, we obtain a compact set $N = X/G$. When $N$ can be expressed as a finite Riemannian complex, then $X$ is a Riemannian complex as well. A simple example of this is $X = \mathbb{R}^2$, $N =$ the unit square, and $G = \mathbb{Z}^2$. A more complicated example can be constructed when $G$ is the free group and $N$ has an Euclidean metric; the space will be globally hyperbolic, but locally Euclidean.

In [5], Pittet and Saloff-Coste consider a manifold, $X$, which has a finitely generated group of isometries, $G$. In this situation, the heat kernel on the manifold ($p_t^X(x,x)$) and the return probability of the random walk on the group ($p_t^G(e,e)$) satisfy an asymptotic equivalence as $t \to \infty$. A natural question is whether this holds on a Riemannian complex with a finitely generated group of isometries.

**Theorem 2.** *[6] Let $X$ be a Euclidean complex of dimension $n$ that satisfies the geometric assumptions, and let $G$ be finitely generated group. If $X/G$ is a Euclidean complex consisting of a finite number of polytopes, we have constants $c, C$, and $T$ such that for all $t > T$,*

$$cp^G_{ct}(e,e) \le \sup_{x \in X} p^X_t(x,x) \le Cp^G_{Ct}(e,e).$$

By theorem 2, the heat kernel on the complex and the return probability on the group have similar large time behavior. Currently, I am extending this result from Euclidean complexes to Riemannian complexes as well as other metric spaces which satisfy a set of geometric assumptions locally.

# References

[1] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann. Geometry of the space of phylogenetic trees. *Adv. in Appl. Math.*, 27(4):733–767, 2001.

[2] Michael Brin and Yuri Kifer. Brownian motion, harmonic functions and hyperbolicity for Euclidean complexes. *Math. Z.*, 237(3):421–468, 2001.

[3] James Eells and Bent Fuglede. *Harmonic Maps Between Riemannian Polyhedra*. Cambridge University Press, Cambridge, UK, 2001.

[4] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[5] Christophe Pittet and Laurent Saloff-Coste. On the stability of the behavior of random walks on groups. *J. Geom. Anal.*, 10(4):713–737, 2000.

[6] Melanie Pivarski. Heat kernels on Euclidean complexes. *Ph.D. Dissertation, Cornell University*, 2006.

[7] Melanie Pivarski and Laurent Saloff-Coste. Small time heat kernel behavior on Riemannian complexes. *New York Journal of Mathematics*, 14:459–494, 2008.

[8] K. T. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl. (9)*, 75(3):273–297, 1996.