15 pts 1) Use curl to determine whether the vector field $F(x, y, z) = x\,\mathbf{i} + e^y \sin z\,\mathbf{j} + e^y \cos z\,\mathbf{k}$ is conservative. If it is, find the potential.

- $\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & e^y \sin z & e^y \cos z \end{vmatrix}$

$= (e^y \cos z - e^y \cos z)\,i + (0-0)\,j + (0-0)\,k = 0$

- F defined on $\mathbb{R}^3$
- partial derivatives of components are continuous

$\Longrightarrow$ F conservative.

$f(x,y,z) = \frac{1}{2}x^2 + e^y \sin z + k$

$f = \nabla F$

10 pts 2) Is there a vector field F on $\mathbb{R}^3$ such that $\operatorname{curl} F = y\,\mathbf{i} + zy\,\mathbf{j} - x^3z\,\mathbf{k}$? Justify your answer.

$$\operatorname{div} F = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(zy) - \frac{\partial}{\partial z} x^3 z$$

$$= 0 + z + x^3 \neq 0$$

if $F = \operatorname{curl} G$

then $\operatorname{div} F = \operatorname{div} \operatorname{curl} G = 0.$

So: <u>no</u>

20 pts 3) Compute the integral $\iint_R x\, dA$ where R is bounded by the ellipse $4x^2 + 25y^2 = 100$. Use the transformation $x = 5u$ and $y = 2v$.

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix} = 10$

Set: $4x^2 + 25y^2 = 100$

$4\cdot(5u)^2 + 25(2v)^2 = 100$

$4\cdot 25u^2 + 100v^2 = 100$

$u^2 + v^2 = 1$

$$\iint_R x\, dA = \iint_{u^2+v^2 \leq 1} 5u \cdot 10\, du\, dv$$

$$= 50\int_0^{2\pi}\int_0^1 r\cos\theta \cdot r\, dr\, d\theta = 50\left(\int_0^{2\pi} \cos\theta\, d\theta\right)\left(\int_0^1 r^2 dr\right)$$

$$= 0$$

20 pts 5) Evaluate $\iint_S \sqrt{1+x^2+y^2}\, dS$, where S is the helicoid with vector equation $r(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, where $0 \le u \le 1$ and $0 \le v \le \pi$.

$$r_u = \cos v\, i + \sin v\, j$$

$$r_v = -u\sin v\, i + u\cos v\, j + k$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} =$$

$$= i\left(\sin v\right) - j\left(\cos v\right) + k\left(u\cos^2 v + u\sin^2 v\right)$$

$$= \sin v\, i - \cos v\, j + u k$$

$$|r_u \times r_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\iint_S \sqrt{1+x^2+y^2}\, dS = \int_0^1 \int_0^{\pi} \sqrt{1 + u^2\cos^2 v + u^2 \sin^2 v} \cdot \sqrt{1+u^2}\, du\, dv$$

$$= \int_0^1 \int_0^{\pi} \sqrt{1+u^2} \cdot \sqrt{1+u^2}\, du\, dv = \int_0^1 \int_0^{\pi} 1+u^2\, du\, dv$$

$$= \frac{4}{3}\pi.$$

15 pts 6) Use Green's theorem to evaluate the integral $\int_C x^2y\,dx - 3y^2\,dy$ along the positively oriented circle C: $x^2+y^2=1$.

statement not necessary

Green's Theorem:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

so

$$\int_C x^2y\,dx - 3y^2\,dy = \iint_{x^2+y^2\le 1} 0 - x^2\,dA$$

$$= -\iint_{x^2+y^2\le 1} x^2\,dA = -\int_0^{2\pi}\int_0^1 (r\cos\theta)^2\,r\,dr\,d\theta$$

$$= -\left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^1 r^3\,dr\right) =$$

$$= -\left(\int_0^{2\pi}\frac{1}{2} + \frac{\cos 2\theta}{2}\,d\theta\right)\left(\int_0^1 r^3\,dr\right)$$

$$= -\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi}\right)\cdot\frac{1}{4} = -\pi\left(\frac{1}{4}\right) = -\frac{\pi}{4}$$

20 pts 4) Find the surface area of the part of the paraboloid $z = -x^2 - y^2 + 5$ which lies above the xy-plane.

Parametrization:

$$r(x,y) = xi + yj + (-x^2 - y^2 + 5)k$$

where $x^2 + y^2 \leq 5$

Then $r_x = i + (-2x)k$

$$r_y = j + (-2y)k$$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2xi - (-2y)j + k$$

$$|r_x \times r_y| = \sqrt{1 + (2x)^2 + (2y)^2} = \cancel{\sqrt{5}} \sqrt{1 + 4(x^2 + y^2)}$$

or straigth from the formula

$$\iint_S 1 \, dS = \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{2\pi} \int_0^{\sqrt{5}} \sqrt{1 + 4r^2} \cdot r \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\frac{2}{3} \frac{1}{8} \left[1 + 4r^2 \right]_0^{3/2} \right)$$

$$= 2\pi \cdot \frac{2}{3} \left((1 + 4 \cdot 5)^{3/2} - 1^{3/2} \right) = 2\pi \cdot \frac{2}{3} \left((21)^{3/2} - 1 \right)$$

after substitution $u = 1 + 4r^2$