

AN OPTIMAL L^1 -MINIMIZATION ALGORITHM FOR STATIONARY HAMILTON-JACOBI EQUATIONS*

JEAN-LUC GUERMOND^{†‡} AND BOJAN POPOV[†]

Abstract. We describe an algorithm for solving steady Hamilton-Jacobi equations in dimension one using a L^1 -minimization technique on piecewise linear approximations. The algorithm is proved to have optimal complexity and to give approximations that converge to the viscosity solution of the problem. Numerical results are presented to illustrate the performance of the method.

Key words. Finite elements, best L^1 -approximation, viscosity solution, transport, ill-posed problem, HJ equation, Eikonal equation

AMS subject classifications. 65N35, 65N22, 65F05, 35J05

1. Introduction. This paper is concerned with the approximation of stationary Hamilton-Jacobi equations in one space dimension using piecewise linear finite elements and a minimization technique in L^1 . This work is part of a research program aiming at exploring the potential of nonlinear approximation techniques based on L^1 -minimization*.

L^1 -based approximation techniques have been introduced by Lavery [10], [11] and further explored in Guermond [4]. Numerical tests reported in these references suggest that L^1 -based minimization techniques can compute the viscosity solution of some first-order PDEs. This fact has been proved in one space dimension for linear first-order PDEs equipped with ill-posed boundary conditions in Lavery [11], Guermond and Popov [6], Guermond, Marpeau, and Popov [5]. This idea has been applied to stationary Hamilton-Jacobi equations in one and two space dimensions in Guermond and Popov [8, 7]. It is shown in [8, 7] that when equipped with an appropriate entropy, the L^1 -minimization algorithm constructs a sequence of approximate solutions that converges to the unique viscosity solution of the equation. The novelty of this approach is that the entropy is not viscosity-based and thus the graph of the approximate solution is not smeared in the vicinity of points where the gradient is discontinuous. The price to be paid for this highly-regarded property is that of solving a minimization problem in L^1 using a non-smooth and non-convex functional. In the wake of [5], we show in the present paper that it is possible to approximately solve the minimization problem associated with one-dimensional stationary Hamilton-Jacobi equations in $\mathcal{O}(N)$ operations, where N is the number of degrees of freedom involved, and the sequence of approximate minimizers (we henceforth call them almost minimizers) converges to the viscosity solution.

The paper is organized as follows. We describe the continuous and the discrete settings in §2. The algorithm that we propose for approximately solving the minimization problem is described in §3. We prove in §4 that under appropriate simplifying assumptions the algorithm yields an almost minimizer in $\mathcal{O}(N)$ operations. The main result of this paper is Theorem 4.6. We finish in §5 by numerically illustrating the capabilities of the proposed algorithm. Conclusions and comments are reported §6.

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2. The problem setting. We describe in this section the problem that we want to solve (i.e., one-dimensional stationary Hamilton-Jacobi equations) and we introduce the discrete setting that is used to construct an approximate solution.

2.1. The continuous problem. We consider the following one-dimensional stationary Hamilton-Jacobi equation

$$H(x, u, u') = 0, \text{ in } (0, 1), \text{ with } u(0) = 0, u(1) = 0, \quad (2.1)$$

where $[0, 1]$ is a bounded interval. We assume that the Hamiltonian is such that (2.1) has a unique viscosity solution u ; in other words, there is a unique u solving (2.1) with the following properties:

$$u \in \mathcal{C}^{0,1}[0, 1], \quad (2.2)$$

$$u \text{ is } q\text{-semiconcave for some } q > 1, \quad (2.3)$$

where we understand q -semiconcavity in the following sense:

DEFINITION 2.1. *A function u in $\mathcal{C}^{0,1}[0, 1]$ is said to be q -semiconcave, $q > 1$, if there is a concave function $v_c \in \mathcal{C}^{0,1}[0, 1]$ and a function $w \in W^{2,q}(0, 1)$ so that $u = v_c + w$.*

Remark 2.1. Recall that a function v in $\mathcal{C}^{0,1}[0, 1]$ is usually called uniformly semiconcave in textbooks if and only if it can be decomposed into $v(x) = v_c(x) + c_v x^2$ where c_v is a nonnegative constant and v_c is concave and in $\mathcal{C}^{0,1}[0, 1]$. Definition 2.1 is a slight generalization of semiconcavity.

A typical example of the above setting is the eikonal equation or any stationary Hamilton-Jacobi equations derived from scalar conservation laws with convex flux, see Barles [1], Evans [3], Kruřkov [9], or Lions and Souganidis [12]. The goal of this paper is to construct a sequence of approximate solutions to (2.1) by using continuous finite elements and by minimizing the residual in $L^1(\Omega)$.

To be able to collectively refer to (2.2)-(2.3) in the rest of the paper, we define

$$X = \{v \in \mathcal{C}^{0,1}[0, 1]; v \text{ is } q\text{-semiconcave}\}. \quad (2.4)$$

Remark 2.2. Note that (2.2)-(2.3) implies that $u' \in \text{BV}(a, b)$. To see this, observe that Definition 2.1 implies $u = v_c + w$ where $v_c \in \mathcal{C}^{0,1}[0, 1]$ is concave and $w \in W^{2,p}(0, 1) \subset W^{2,1}(0, 1)$. Hence,

$$|u'|_{\text{BV}(a,b)} \leq 2\|v'_c\|_{L^\infty(0,1)} + \|w'\|_{\text{BV}(a,b)}.$$

2.2. The discrete problem. Let $\{\mathcal{T}_h\}_{h>0}$ be an indexed family of finite element meshes. We assume that for any given $h > 0$, \mathcal{T}_h is a partition of the interval $[0, 1]$. Namely, for all index $h > 0$, there is an integer $n > 0$ such that $\mathcal{T}_h = \cup_{i=0}^n [x_i, x_{i+1}]$ with $x_0 = 0$, $x_{n+1} = 1$, and $h_i = x_{i+1} - x_i$. The quantity $h = \max_{0 \leq i \leq n} h_i$ is called the meshsize of the partition \mathcal{T}_h . Henceforth we refer to any cell $[x_i, x_{i+1}] \in \mathcal{T}_h$ by its index $i \in \{0, \dots, n\}$. We denote by $\mathcal{V}_h = \{x_i; 0 \leq i \leq n\}$ and $\mathcal{V}_h^i = \mathcal{V}_h \cap (0, 1) = \{x_i\}_{i=1}^n$ the collection of the mesh vertices and interior mesh vertices, respectively. To avoid extra technicalities, we assume that the mesh is quasi-uniform; that is, there is $c_m > 0$, uniform in h so that

$$c_m h \leq h_i \leq h, \quad \forall i \in \{0, \dots, n\}. \quad (2.5)$$

Let $k \geq 1$ be an integer and denote by \mathbb{P}_k the set of real-valued polynomials in $[0, 1]$ of total degree at most k . We introduce

$$X_h = \{v_h \in C^0[0, 1]; v_h|_K \in \mathbb{P}_k, \forall K \in \mathcal{T}_h; v_h(0) = 0, v_h(1) = 0\} \quad (2.6)$$

$$X_{(h)} = X + X_h \quad (2.7)$$

Let $(t)_+ := \frac{1}{2}(t + |t|)$ denote the positive part of t for all $t \in \mathbb{R}$. For every function v in $X_{(h)}$ we denote by $\{-\partial_n v\}_+ : \mathcal{V}_h^i \rightarrow \mathbb{R}^+$ the map such that for all $\{x\} = K_1 \cap K_2 \in \mathcal{V}_h^i$,

$$\{-\partial_n v\}_+(x) = \left(-\frac{1}{2}(v'_{|_{K_1}}(x) \cdot n_1 + v'_{|_{K_2}}(x) \cdot n_2) \right)_+,$$

where n_1 and n_2 are the unit outward normals to the mesh cells K_1 and K_2 at x , respectively. Note that, if $K_1 = [x_{i-1}, x_i]$ and $K_2 = [x_i, x_{i+1}]$, then $\{-\partial_n v\}(x_i)$ is the jump of v'_h at x_i , i.e., $\{-\partial_n v\}(x_i) = v'_h(x_i + 0) - v'_h(x_i - 0)$. That gives $\{-\partial_n v\}_+(x_i) = (v'_h(x_i + 0) - v'_h(x_i - 0))_+$.

Recalling that there is some $q > 1$ so that u is q -semi-concave, we now define a fixed real number p such that

$$1 < p \leq q. \quad (2.8)$$

We also define the following functional $J : X_{(h)} \ni v \mapsto J_h(v) \in \mathbb{R}^+$ by:

$$J(v) = \int_a^b |H(x, v, v')| dx + h \sum_{K \in \mathcal{T}_h} \int_K (v''(x))_+^p dx + h^{2-p} \sum_{x_i \in \mathcal{V}_h^i} (\{-\partial_n v\}_+(x_i))^p. \quad (2.9)$$

For every function v in $X_{(h)}$ we refer to $\int_0^1 |H(x, v, v')| dx$ as the residual. The two extra terms in the right-hand side above are referred to as the volume entropy term and the interface entropy term. The presence of these two terms is motivated by the fact that the viscosity solution u is q -semiconcave. Actually, an immediate consequence of q -semiconcavity is that there are $c > 0$ and $c' > 0$ such that for all $\delta > 0$ and all $\omega \subset (0, 1)$ so that $\omega \pm \delta \subset (0, 1)$, the following hold

$$u(x + \delta) - 2u(x) + u(x - \delta) \leq c \delta^{2-\frac{1}{p}}, \quad \forall x \in \omega \quad (2.10)$$

$$\|(u(\cdot + \delta) - 2u(\cdot) + u(\cdot - \delta))_+\|_{L^p(\omega)} \leq c' \delta^2. \quad (2.11)$$

Henceforth c, c', c'' denote generic constants which may vary at each occurrence but do not depend on δ nor on the mesh parameter h .

The volume entropy term and the interface entropy term in (2.9) are discrete versions of the L^p -norm of the positive part of the second derivative of v . According to (2.11), the discrete entropy of u is of order h . If we denote by $\mathcal{I}_h u$ the piecewise linear interpolant of u in X_h , one can show that $J(\mathcal{I}_h(u)) \leq c h$ and both the residual and the discrete entropy of $\mathcal{I}_h(u)$ are of order h ; i.e., they are balanced, (see [8, Lemma 4.2] for details).

Remark 2.3. Whenever $v \in C^{0,1}[0, 1]$ is a concave function, for example in the case of the eikonal equation, the volume and interface entropies in (2.9) are zero. These two terms do not add extra viscosity; they are meant to prevent the occurrence of large positive second derivatives.

We now focus our attention on the following minimization problem: Seek u_h in X_h such that

$$J(u_h) = \inf_{v_h \in X_h} J(v_h). \quad (2.12)$$

It is shown in [8] that $J(u_h) \leq ch$, where c does not depend on h . Since in practice u_h might not be computed exactly or might be approximated to some extent through some iterative process (see the details in the next section), we now define the notion of *almost minimizer*. A family of functions $\{v_h \in X_h\}_{h>0}$ is said to be a sequence of almost minimizers if there is $c > 0$, uniform in h , such that for all $h > 0$,

$$J(v_h) \leq ch. \quad (2.13)$$

For instance, the family $\{\mathcal{I}_h(u)\}_{h>0}$ is a sequence of almost minimizers, thus showing that the class of almost minimizers is not empty. It is shown in [8, Thm 6.2] that under the above assumptions on the mesh and nonrestrictive assumptions on the Hamiltonian, every sequences of almost minimizers for (2.13) converge strongly in $W^{1,1}(0,1) \cap C^0[0,1]$ to the unique viscosity solution to (2.1). This result has been extended to two-dimensional stationary Hamilton-Jacobi equations in [7].

The goal of the rest of the paper is to construct a fast algorithm to compute a sequence of almost minimizers for (2.12) using piecewise linear approximation, i.e., $k = 1$.

3. The algorithm. In this section we present an algorithm that constructs a sequence of almost minimizers for a discrete version of (2.13) where integrals have been replaced by quadratures. We henceforth assume that X_h is composed of piecewise linear functions, i.e., $k = 1$.

3.1. The approximate functional. We approximate the functional J by applying the midpoint rule to the integrals over the intervals $K \in \mathcal{T}_h$. For this purpose we set $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$ and for all $v \in X(h)$, we set $v_l = v(x_l)$ for all $l \in \{0, \frac{1}{2}, 1, \dots, n + \frac{1}{2}, n + 1\}$ and $\sigma_i(v) = (v_{i+1} - v_i)h_i^{-1}$ for all $i \in \{0, 1, \dots, n\}$. We then define

$$J_h(v) := \sum_{i=0}^n R_i(v) + h^{2-2p} \sum_{i=0}^n E_i(v), \quad (3.1)$$

where we use the notation

$$R_i(v) = h_i |H(x_{i+\frac{1}{2}}, v_{i+\frac{1}{2}}, v'_{i+\frac{1}{2}})|, \quad (3.2)$$

$$E_i(v) = \omega_i(v) (\sigma_i(v) - \sigma_{i-1}(v))_+^p, \quad (3.3)$$

and define $E_0(v) = h_0^p (\sigma_0(v))^p$ and $E_n(v) = h_n^p (\sigma_n(v))^p$ corresponding to the constant extensions $v_{-1} := v_0$ and $v_{n+2} := v_{n+1}$. The weight function ω_i is defined as follows:

$$\omega_i(v) = h_{i-1}^p S(\sigma_{i-1}(v), \sigma_i(v)) + h_i^p S(\sigma_i(v), \sigma_{i-1}(v)), \quad (3.4)$$

$$S(a, b) = \frac{1}{2}(\text{sgn}(|a| - |b|) + 1), \quad \text{where } \text{sgn} \text{ is the sign function.} \quad (3.5)$$

Note that $S(a, b)$ returns 1 if $|a| > |b|$, $\frac{1}{2}$ if $|a| = |b|$, and 0 otherwise. Therefore, the weight function ω_i returns h_{i-1}^p if the absolute value of $\sigma_{i-1}(v)$ is larger than the absolute value of $\sigma_i(v)$. When the mesh is uniform, i.e., $h_i = h$ for all $i \in \{0, \dots, n\}$,

$\omega_i(v) = h^p$ and the interior entropy terms in $J_h(v)$ coincide with the entropy terms of $J(v)$. If the mesh is not uniform, but quasi-uniform, the above entropy terms are equivalent. Note that the discrete functional contains entropy terms at the boundary corresponding to the constant extensions.

The discrete problem that we now consider is the following: Seek $u_h \in X_h$ so that

$$J_h(u_h) = \min_{v_h \in X_h} J_h(v_h). \quad (3.6)$$

3.2. Description of the algorithm. In this section we describe an algorithm for computing an almost minimizer of (3.6). The algorithm is composed of two parts described in Algorithm 1 and Algorithm 2, respectively. Algorithm 1 constructs an initial guess and Algorithm 2 improves on this guess using a local minimization strategy. Due to the nonlinearity of the Hamiltonian, problem (3.6) is not convex; as a result, the initialization must be done carefully in order to avoid being trapped in local minimums.

Algorithm 1 and Algorithm 2 involve set-valued maps $t_{i,l}$, $i \in \{0, \dots, n\}$, $l \in \{i, i+1\}$ defined as follows. Let us set

$$r_i(z, s) = h_i H(x_{i+\frac{1}{2}}, \frac{1}{2}(z+s), h_i^{-1}(s-z)), \quad \forall z, s \in \mathbb{R}. \quad (3.7)$$

Then we define the multi-valued (i.e., set-valued) nonlinear functions $t_{i,l}$, so that $r_i(z, t_{i,i}(z)) = 0$ and $r_i(t_{i,i+1}(s), s) = 0$. Note that due to the possible nonlinear character of the Hamiltonian, $t_{i,i}(z)$ and $t_{i,i+1}(z)$ are sets and these sets may be empty.

3.2.1. Description of Algorithm 1. Algorithm 1 proceeds from the boundary to the interior and on the way selects a guess by minimizing the functional J_h . The traversing of the domain can be done in many ways: from left to right, from right to left, or from both left and right. The traversing strategy used in Algorithm 1 is done simultaneously from the left and the right sides of the domain.

More specifically, Algorithm 1 proceeds as follows: First we define some arbitrarily large value u^{init} . Then we define a list of cells that must be dealt with (at most two), say `cell_list`; this list is initialized with cell 0 and cell n . Associated with `cell_list` there is the list of the nodes of the cells in `cell_list` that have been updated at the previous iteration, say `node_list`; this list is initialized with node 0 and node $n+1$, i.e., v_0 and v_{n+1} are set to zero, respectively (or the appropriate nonzero Dirichlet condition is enforced). Two ghost values v_{-1} and v_{n+2} , corresponding to constant extensions, are defined by $v_{-1} = v_0$ and $v_{n+2} = v_0$. Let \mathfrak{c} be the index of a cell in `cell_list` and $i \in \text{node_list}$ be the node of cell \mathfrak{c} that has already been updated. Note that $i = \mathfrak{c}$ or $i = \mathfrak{c} + 1$. If the second node of \mathfrak{c} , say j has been updated in the past, the algorithm stops and \mathfrak{c} is called the breakdown cell. The field v thus obtained will serve as the initial guess for Algorithm 2. If j has not been updated we then define \mathfrak{c}' to be the cell that touches \mathfrak{c} at node j . The second node of \mathfrak{c}' is denoted by k . We then define \mathfrak{c}'' to be the cell that touches \mathfrak{c}' at node k and we denote by l the second node of \mathfrak{c}'' (see Figure 3.1). The old residual is computed by setting $\tilde{v} = v$ and $\tilde{v}_j = \tilde{v}_k = \tilde{v}_l = u^{\text{init}}$; in other words \tilde{v} is equal to v up to node i and the node values of \tilde{v} beyond node i are set to u^{init} . Since we are going to modify only the node value v_j , the only parts of the residual that varies is

$$\begin{aligned} J_j(v) &:= R_{\mathfrak{c}}(v) + R_{\mathfrak{c}'}(v) + h^{2-2p}(E_i(v) + E_j(v) + E_k(v)) \\ &= R_{j-1}(v) + R_j(v) + h^{2-2p}(E_{j-1}(v) + E_j(v) + E_{j+1}(v)). \end{aligned}$$

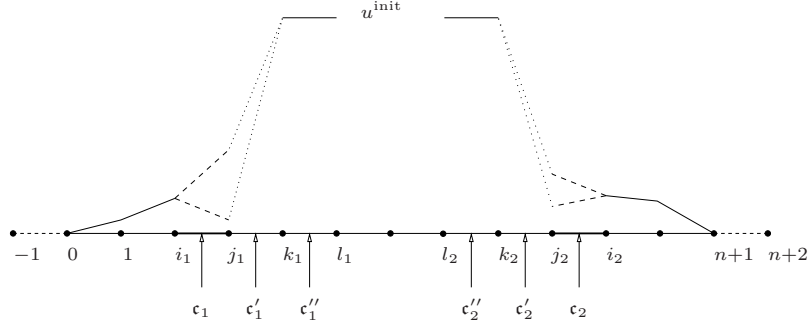


FIGURE 3.1. Notation and schematic representation of Algorithm 1.

Algorithm 1 Initialization for (3.6).

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- 1: Define u^{init} large enough; Set $v_0 = v_{n+1} = 0$ and define $v_{-1} = v_{n+2} = 0$
 - 2: Initialize array `updated(1:n) = false`
 - 3: Put cell 0 in `cell_list`; Put node 0 in `node_list`
 - 4: Put cell n in `cell_list`; Put node $n + 1$ in `node_list`
 - 5: **while** (`cell_list` not empty) **do**
 - 6: Take $\mathbf{c} \in \text{cell_list}$; Take $i \in \text{node_list}$; Let $j \neq i$ be the other node of \mathbf{c}
 - 7: Let $\mathbf{c}' \neq \mathbf{c}$ be s.t. $\mathbf{c}' \cap \mathbf{c} = \{x_j\}$; Let $k \neq j$ be the other node of \mathbf{c}'
 - 8: Let $\mathbf{c}'' \neq \mathbf{c}'$ be s.t. $\mathbf{c}'' \cap \mathbf{c}' = \{x_k\}$; Let $l \neq k$ be the other node of \mathbf{c}''
 - 9: **if** (`updated(j)=true`) **then**
 - 10: Store v and cell index $c_{\text{break}} \leftarrow \mathbf{c}$; Stop
 - 11: **end if**
 - 12: $\tilde{v} \leftarrow v$; $\tilde{v}_j \leftarrow u^{\text{init}}$; $\tilde{v}_k \leftarrow u^{\text{init}}$; $\tilde{v}_l \leftarrow u^{\text{init}}$
 - 13: $J_{\text{old}} \leftarrow |R_{\mathbf{c}}(\tilde{v})| + |R_{\mathbf{c}'}(\tilde{v})| + h^{2-2p}(E_i(\tilde{v}) + E_j(\tilde{v}) + E_k(\tilde{v}))$
 - 14: Remove cell \mathbf{c} from `cell_list`; Remove node i from `node_list`
 - 15: Compute the set $t_{\mathbf{c},i}(v_i)$
 - 16: **if** (set $t_{\mathbf{c},i}(v_i)$ not empty) **then**
 - 17: $\tilde{v} \leftarrow v$; $\tilde{v}_j \leftarrow t_{\mathbf{c},i}(v_i)$; $\tilde{v}_k \leftarrow u^{\text{init}}$; $\tilde{v}_l \leftarrow u^{\text{init}}$
 - 18: Pick $\bar{v} \in \tilde{v}$ with smallest residual $|R_{\mathbf{c}'}(\bar{v})| + h^{2-2p}(E_i(\bar{v}) + E_j(\bar{v}) + E_k(\bar{v}))$
 - 19: $J_{\text{new}} \leftarrow |R_{\mathbf{c}'}(\bar{v})| + h^{2-2p}(E_i(\bar{v}) + E_j(\bar{v}) + E_k(\bar{v}))$
 - 20: **if** ($J_{\text{new}} < J_{\text{old}}$) **then**
 - 21: $v_j \leftarrow \bar{v}_j$; `updated(j) = true`
 - 22: Put cell \mathbf{c}' in `cell_list`; Put node j in `node_list`
 - 23: **end if**
 - 24: **end if**
 - 25: **end while**
 - 26: Breakdown; Problem is ill-posed; Stop
-

We then set $J_{\text{old}} = J_j(\tilde{v})$.

Now we compute the set $t_{\mathbf{c},i}(v_i)$ and remove cell \mathbf{c} and node i from `cell_list` and `node_list`, respectively. If the set $t_{\mathbf{c},i}(v_i)$ is not empty, we consider all the values in $t_{\mathbf{c},i}(v_i)$ to be candidates for updating v_j . For all $z \in t_{\mathbf{c},i}(v_i)$ we define $\tilde{v}(z)$ by setting $\tilde{v}(z) = v$ and we correct the values at nodes j , k , and l by setting $\tilde{v}_j(z) = z$, $\tilde{v}_k(z) = \tilde{v}_l(z) = u^{\text{init}}$. We then define $\bar{z} := \operatorname{argmin}_{z \in t_{\mathbf{c},i}(v_i)} J_j(\tilde{v}(z))$, $\bar{v} := \tilde{v}(\bar{z})$ and set $J_{\text{new}} = J_j(\bar{v})$. If $J_{\text{old}} > J_{\text{new}}$, we set $v_j = \bar{v}_j$ and record the fact that node

j has been updated, and cell c' and node j are put in `cell_list` and `node_list`, respectively. Then the algorithm proceeds until every node has been updated. A schematics representation of the initialization algorithm is shown in Figure 3.1.

3.2.2. Description of Algorithm 2. At the end of Algorithm 1, we have a field v that satisfies $R_c(v) = 0$ for all $c \in \{0, \dots, n\} \setminus \{c_{\text{break}}\}$. The goal of Algorithm 2 is to move around the breakdown cell c_{break} by performing local L^1 -minimization until the functional J_h cannot be further minimized.

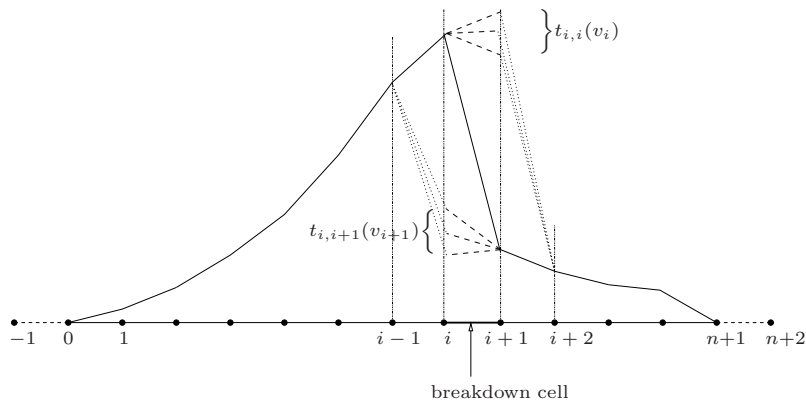


FIGURE 3.2. Notation and schematic representation of Algorithm 2.

More precisely, Algorithm 2 proceeds as follows: We start with the field v that has been computed in Algorithm 1. Let i be the index of the breakdown cell. Define

$$J_{i,\text{loc}}(w) = R_{i-1}(w) + R_i(w) + R_{i+1}(w) + h^{2-2p}(E_{i-1}(w) + E_i(w) + E_{i+1}(w) + E_{i+2}(w)),$$

and set $J_{\text{old}} = J_{i,\text{loc}}(v)$. Now we compute the two sets $t_{i,i}(v_i)$ and $t_{i,i+1}(v_{i+1})$ and we denote $\Lambda_i = \{0\} \times t_{i,i}(v_i) \cup t_{i,i+1}(v_{i+1}) \times \{0\}$. If the set Λ_i is empty the algorithm stops and v is proposed as an almost minimizer. If the set Λ_i is not empty, we define for all $z := (z_1, z_2) \in \Lambda_i$ the function $w(z) \in X_h$ whose points values are

$$\begin{aligned} (v_0, \dots, v_{i-1}, v_i, z_2, v_{i+2}, \dots, v_{n+1}) & \quad \text{if } z_1 = 0, \\ (v_0, \dots, v_{i-1}, z_1, v_{i+1}, v_{i+2}, \dots, v_{n+1}) & \quad \text{if } z_2 = 0. \end{aligned}$$

We then define $\bar{z} = \operatorname{argmin}_{z \in \Lambda_i} J_{i,\text{loc}}(w(z))$ and we set $J_{\text{new}} = J_{i,\text{loc}}(w(\bar{z}))$. If $J_{\text{old}} > J_{\text{new}}$, then we update v as follows: If $\bar{z}_1 = 0$ we set $v_{i+1} = \bar{z}_2$ and cell $i+1$ becomes the new breakdown cell, otherwise we set $v_i = \bar{z}_1$ and cell $i-1$ becomes the new breakdown cell, in both cases we record the fact that something has been done. If $J_{\text{old}} \geq J_{\text{new}}$ we record the fact that nothing has been done.

Then the algorithm proceeds until nothing can be done to decrease the residual using the above strategy. A schematics representation of Algorithm 2 is shown in Figure 3.2.

Remark 3.1. Algorithm 1 has some resemblance with the fast marching method [13]. The fast marching algorithm consists of computing the extremal solutions starting from both ends of the interval and choosing the minimal one at each point in the domain. Algorithm 1 deviates from this technique by stopping when the two extremal branches meet. Algorithm 2 has some common features with fast sweeping [14]. The two main differences between our approach and the fast marching and fast

Algorithm 2 L^1 -switch for (3.6).

Start with initial guess from Algorithm 1: v and the breakdown cell i

loop

$J_{\text{old}} \leftarrow J_{i,\text{loc}}(v)$

Compute the set $\Lambda_i = \{0\} \times t_{i,i}(v_i) \cup t_{i,i+1}(v_{i+1}) \times \{0\}$; **nothing_done** \leftarrow **true**

if (set Λ_i not empty) **then**

Compute $\bar{z} = \operatorname{argmin}_{z \in \Lambda_i} J_{i,\text{loc}}(w(z))$

$J_{\text{new}} \leftarrow J_{i,\text{loc}}(w(\bar{z}))$

if ($J_{\text{new}} < J_{\text{old}}$) **then**

$v \leftarrow w(\bar{z})$; **nothing_done** \leftarrow **false**

if $\bar{z}_1 = 0$ **then**

Cell $i + 1$ is the new breakdown cell

else

Cell $i - 1$ is the new breakdown cell

end if

end if

end if

if (**nothing_done**=**true**) **then**

Done; Stop

end if

end loop

sweeping techniques are that we use central approximation of the equation (i.e., we do not use any special discretization of the Hamiltonian) and the selection principle is incorporated in the entropy part of the functional J_h .

4. Convergence analysis. The purpose of this section is to show that under appropriate simplifying assumptions, the combination Algorithm 1+Algorithm 2 yields a sequence of almost minimizers and the algorithmic complexity is $\mathcal{O}(\dim(X_h))$.

4.1. Additional structural hypotheses. In order to facilitate the analysis of the algorithms we henceforth assume that the Hamiltonian can be further decomposed as follows:

$$H(x, u(x), u'(x)) = u(x) + F(u'(x)) - g(x). \quad (4.1)$$

The function F is defined on \mathbb{R} , belongs to $\mathcal{C}^{0,1}[0, 1]$, and is assumed to satisfy the following properties:

$$F \text{ is convex}, \quad (4.2)$$

$$F(k) \geq c_1|k| + c_0^2 k^2, \quad \gamma := c_1 + c_0 > 0, \quad \min(c_1, c_0) \geq 0, \quad (4.3)$$

$$F(0) = 0, \quad (4.4)$$

where $\gamma := c_1 + c_0 > 0$, $\min(c_1, c_0) \geq 0$. We assume g to be in $\mathcal{C}^{0,1}[0, 1]$ and satisfies

$$\mu_0 := \frac{1}{4}e^{-\gamma^{-1}}g(0) - \frac{1}{2}\gamma^{-1}\sqrt{g(0)} - \int_0^1 (-g'(x))_+ dx > 0, \quad (4.5)$$

$$\mu_1 := \frac{1}{4}e^{-\gamma^{-1}}g(1) - \frac{1}{2}\gamma^{-1}\sqrt{g(1)} - \int_0^1 (g'(x))_+ dx > 0. \quad (4.6)$$

Note that (4.5) also implies $\lambda_0 := \frac{1}{4}e^{-\gamma^{-1}}g(0) - \frac{1}{2}\gamma^{-1}\sqrt{g(0)} > 0$ (resp. $\lambda_1 := \frac{1}{4}e^{-\gamma^{-1}}g(1) - \frac{1}{2}\gamma^{-1}\sqrt{g(1)} > 0$). Note also that (4.5) requires g to be a positive function, i.e., $g(x) \geq g(0) - \int_0^x (-g'(t))_+ dt > 0$ for all $x \in [0, 1]$ (a similar remark holds for (4.6)).

We now make a technical hypothesis concerning the mesh; namely, we assume that either one of the following statements is true:

- (i) The mesh is uniform.
- (ii) There are $c_g > 0$ and $\theta \geq 1$, s.t. $F(k) \leq c_g(|k|^\theta + 1)$ for all $k \in \mathbb{R}$ and $p > \theta + 1$. (4.7)

In other words, if the mesh is uniform no additional assumption on F is needed, but if the mesh is non-uniform, we require a growth condition on F .

4.2. Analysis of the extremal positive solutions. The purpose of this section is to define the so-called forward and backward extremal positive solutions, henceforth denoted by f and b , respectively, and to state some of their properties.

The extremal positive solutions will be used to generate the initial guess for the iterative technique. The extremal solutions f and b are defined to be piecewise linear on the mesh \mathcal{T}_h with $f_0 = 0$ and $b_{n+1} = 0$. To avoid technicalities we extend f on $[x_{-1}, 0]$ and b on $[1, x_{n+2}]$, where $x_{-1} := x_0 - h_0$ and $x_{n+2} := x_{n+1} + h_n$, by setting $f_{-1} = 0$ and $b_{n+2} = 0$. The node values f_i , $i \geq 1$, are defined inductively as follows: Consider the function

$$\phi_i(k) = f_i + \frac{1}{2}kh_i + F(k) - g_{i+\frac{1}{2}}, \quad (4.8)$$

and let k_i^+ be a positive root of $\phi_i(k) = 0$, then set $f_{i+1} = f_i + k_i^+ h_i$. Similarly the node values of b are defined backwards by looking for a negative root, κ_i^- , of

$$\psi_i(\kappa) = b_{i+1} - \frac{1}{2}\kappa h_i + F(\kappa) - g_{i+\frac{1}{2}}, \quad (4.9)$$

and by setting $b_i = b_{i+1} - \kappa_i^- h_i$. The following lemma justifies the above construction.

LEMMA 4.1. *Under the hypotheses (4.1) to (4.6), the following statements hold:*

- (i) $\phi_i(k) = 0$ has a unique positive and a unique negative root for all $i \in \{1, \dots, n\}$. The same holds for the equation $\psi_i(\kappa) = 0$. Therefore, f and b are uniquely defined.
- (ii) $\mu_0 \leq g_{i+\frac{1}{2}} - f_i$, (resp. $\mu_1 \leq g_{i+\frac{1}{2}} - b_{i+1}$) for all $i \in \{0, \dots, n\}$.
- (iii) $\max(\|f\|_{L^\infty}, \|b\|_{L^\infty}) \leq \|g\|_{L^\infty}$.
- (iv) $\max(\|f'\|_{L^\infty}, \|b'\|_{L^\infty}) \leq \gamma^{-1}(\|g\|_{L^\infty} + \|g\|_{L^\infty}^{\frac{1}{2}})$.

Proof. We prove the statements only for f . The proofs for b are analogous. We work by induction. Assume that $f_i - g_{i+\frac{1}{2}} < 0$; this true for $i = 0$ owing to (4.4) and since g is a positive function.

Observe that $\phi_i(0) = f_i - g_{i+\frac{1}{2}} < 0$ and ϕ_i is strictly increasing. Actually, owing to (4.3), ϕ_i goes to $+\infty$ when k goes to $-\infty$ and when k to $+\infty$. As a result, $\phi_i(k) = 0$ has a negative root and a positive root, say $k_i^- < 0$ and $k_i^+ > 0$, respectively. The convexity assumption (4.2) implies that the negative root is unique and the positive root is unique. To alleviate notation and since the context is clear we use k_i instead of k_i^+ in the rest of the proof. It is then legitimate to define $f_{i+1} = f_i + k_i h_i$.

We now need to prove that f_{i+1} satisfies the induction hypotheses, i.e., we want to prove that $f_{i+1} - g_{i+\frac{3}{2}} < 0$. Define

$$\varphi_i(k) = f_i + \frac{1}{2}kh_i + c_1 k + c_0^2 k^2 - g_{i+\frac{1}{2}}.$$

Hypothesis (4.3) implies that $\phi_i(k) \geq \varphi_i(k)$ for all $k \in \mathbb{R}$. Let l_i be the unique positive root of $\varphi_i(k) = 0$. By construction

$$k_i \leq l_i \leq l_{\max} := \min \left(\frac{g_{i+\frac{1}{2}} - f_i}{c_1 + \frac{1}{2}h_i}, \frac{(g_{i+\frac{1}{2}} - f_i)^{\frac{1}{2}}}{c_0} \right).$$

Moreover,

$$\begin{aligned} g_{i+\frac{3}{2}} - f_{i+1} &= g_{i+\frac{3}{2}} - g_{i+\frac{1}{2}} + g_{i+\frac{1}{2}} - f_i - k_i h_i \\ &\geq g_{i+\frac{3}{2}} - g_{i+\frac{1}{2}} + g_{i+\frac{1}{2}} - f_i - l_{\max} h_i. \end{aligned}$$

Setting $z_i = g_{i+\frac{1}{2}} - f_i$, $\Delta g_{i+\frac{3}{2}} = g_{i+\frac{3}{2}} - g_{i+\frac{1}{2}}$ and using the inequality $\min(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}) \leq \frac{\alpha+\gamma}{\beta+\delta}$ we infer

$$l_{\max} \leq \frac{z_i + z_i^{\frac{1}{2}}}{c_1 + c_0 + \frac{1}{2}h_i} \leq \gamma^{-1}(z_i + z_i^{\frac{1}{2}}),$$

which in turn implies

$$z_{i+1} \geq z_i + \Delta g_{i+\frac{3}{2}} - \gamma^{-1} h_i (z_i + z_i^{\frac{1}{2}}).$$

Let us define the sequence $(w_i)_{i \geq 0}$ by

$$w_0 = g_{\frac{1}{2}}, \quad \text{and} \quad w_{i+1} = w_i - \gamma^{-1} h_i (w_i + w_i^{\frac{1}{2}}),$$

where we have made the induction hypothesis that $w_l \geq 0$, for all $l \in \{0, \dots, i\}$, which is clearly true for $i = 0$ when h is small enough. Since the sequence $(w_i)_{i \geq 0}$ is decreasing: $w_{i+1} \leq w_i \leq \dots \leq w_0$, we infer

$$\begin{aligned} w_{i+1} &= w_0 \prod_{l=0}^i (1 - \gamma^{-1} h_l) - \gamma^{-1} \sum_{l=1}^{i+1} h_{l-1} w_{l-1}^{\frac{1}{2}} \prod_{m=l}^i (1 - \gamma^{-1} h_m) \\ &\geq \frac{1}{2} w_0 e^{-\gamma^{-1} x_{i+1}} - \gamma^{-1} x_{i+1} w_0^{\frac{1}{2}} \geq \frac{1}{2} e^{-\gamma^{-1}} w_0 - \gamma^{-1} w_0^{\frac{1}{2}} = \frac{1}{2} e^{-\gamma^{-1}} g_{\frac{1}{2}} - \gamma^{-1} g_{\frac{1}{2}}^{\frac{1}{2}} \end{aligned}$$

where we assumed that h is small enough (say $h \leq \frac{1}{2}\gamma$) and used $1 - t > \frac{1}{2}e^{-t}$ for all $t \in [0, \frac{1}{2}]$. Owing to the assumed regularity on g (i.e., $g \in W^{1,1}(0,1)$), $\frac{1}{2}e^{-\gamma^{-1}} g_{\frac{1}{2}} - \gamma^{-1} g_{\frac{1}{2}}^{\frac{1}{2}} \geq \lambda_0 := \frac{1}{2}(\frac{1}{2}e^{-\gamma^{-1}} g(0) - \gamma^{-1} \sqrt{g(0)}) > 0$ when h is small enough, i.e., see Hypothesis (4.5). This implies that $w_{i+1} \geq \lambda_0 > 0$, thus proving the induction hypothesis on w_{i+1} .

We now evaluate $\alpha_{i+1} := w_{i+1} - z_{i+1}$ as follows

$$\begin{aligned} \alpha_{i+1} &= w_{i+1} - z_{i+1} \leq w_i - z_i - \Delta g_{i+\frac{3}{2}} - \gamma^{-1} h_i (w_i - z_i + (w_i - z_i)(\sqrt{w_i} + \sqrt{z_i})^{-1}) \\ &\leq \alpha_i - \Delta g_{i+\frac{3}{2}} - \gamma^{-1} h_i \alpha_i (1 + (\sqrt{w_i} + \sqrt{z_i})^{-1}) \\ &\leq \alpha_i (1 - \gamma^{-1} h_i (1 + (\sqrt{w_i} + \sqrt{z_i})^{-1})) - \Delta g_{i+\frac{3}{2}} \end{aligned}$$

Observe that $1 - \gamma^{-1} h_i (1 + (\sqrt{w_i} + \sqrt{z_i})^{-1}) \geq 1 - \gamma^{-1} h_i (1 + \lambda_0^{-1/2}) \geq 0$ for h small enough. Then,

$$(\alpha_{i+1})_+ \leq (\alpha_i)_+ + (-\Delta g_{i+\frac{3}{2}})_+ \leq \sum_{l=0}^i (g_{l+\frac{1}{2}} - g_{l+\frac{3}{2}})_+ \leq \int_0^1 (-g'(x))_+ dx,$$

since $\alpha_0 = w_0 - z_0 = 0$. Then

$$\begin{aligned} z_{i+1} &= w_{i+1} + z_{i+1} - w_{i+1} \geq \lambda_0 - \alpha_{i+1} \geq \lambda_0 - (\alpha_{i+1})_+ \\ &\geq \lambda_0 - \int_0^1 (-g'(x))_+ dx := \mu_0 > 0, \end{aligned}$$

which owing to (4.5) proves item (ii) since $z_{i+1} \geq \mu_0 > 0$ and also proves the induction hypothesis on f_{i+1} , i.e., $f_{i+1} - g_{i+\frac{3}{2}} < 0$. Hence item (i) is proved.

Item (iii) is a consequence of $f_i < g_{i+\frac{1}{2}}$ for all $i \in \{0, \dots, n\}$. Item (iv) is a consequence of the bound $l_{\max} \leq \gamma^{-1}(z_i + z_i^{\frac{1}{2}})$ together with $f_i \geq 0$ for all $i \in \{0, \dots, n\}$. This completes the proof for f . Repeat the argument backwards for b . \square

For each i we denote by k_i^+ (resp. k_i^-) the unique positive (resp. negative) root of ϕ_i . We use the same convention for κ_i^+ and κ_i^- . For further references, we also use the following notation:

$$\begin{aligned} f_{i+1}^- &= f_i + k_i^- h_i & b_i^+ &= b_{i+1} - \kappa_i^+ h_i \\ f_{i+1}^+ &= f_i + k_i^+ h_i, & b_i^- &= b_{i+1} - \kappa_i^- h_i, \end{aligned} \quad (4.10)$$

Observe that $f_i^+ := f_i$ and $b_i^- := b_i$. We refer to f as the forward extremal positive solution and to b as the backward extremal positive solution.

Let us set

$$k_{\max} := \gamma^{-1} \left(\|g\|_{L^\infty} + \|g\|_{L^\infty}^{\frac{1}{2}} \right) \quad (4.11)$$

$$k_{\min} := \frac{1}{2} k_{\max} \min \left(\mu_0 F(k_{\max})^{-1}, \mu_1 F(-k_{\max})^{-1} \right). \quad (4.12)$$

Next we want to have a control on the entropy of f and b .

LEMMA 4.2. *Under the hypotheses (4.1) to (4.6), there is c , uniform in h , such that*

$$\max \left(|k_i^+ - k_{i-1}^+|, |\kappa_i^- - \kappa_{i-1}^-| \right) \leq ch, \quad (4.13)$$

$$k_{\min} \leq \min(k_i^+, -k_i^-, \kappa_i^+, -\kappa_i^-) \leq \max(k_i^+, -k_i^-, \kappa_i^+, -\kappa_i^-) \leq k_{\max}, \quad (4.14)$$

for all $i \in \{1, \dots, n\}$.

Proof. We do the proof for k_i^+ only because the rest is analogous. To alleviate notation and since the context is clear we use k_i instead of k_i^+ in this proof.

We first establish a bound from below for the k 's. By definition $\phi_i(k_i) = 0$ for all $i \in \{0, \dots, n\}$, which also gives $\phi_{i+1}(k_{i+1}) - \phi_i(k_i) = 0$. In other words,

$$\begin{aligned} 0 &= \phi_{i+1}(k_{i+1}) - \phi_i(k_i) \\ &= f_{i+1} - f_i + \frac{1}{2}(k_{i+1}h_{i+1} - k_i h_i) + F(k_{i+1}) - F(k_i) - g_{i+\frac{3}{2}} + g_{i+\frac{1}{2}} \\ &= \frac{1}{2}k_{i+1}h_{i+1} + \frac{1}{2}k_i h_i + \int_{k_i}^{k_{i+1}} F'(k) dk - \int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}} g'(x) dx \end{aligned}$$

Then,

$$\left(\operatorname{ess\,inf}_{[k_i, k_{i+1}]} |F'| \right) |k_{i+1} - k_i| \leq \frac{1}{2}k_{i+1}h_{i+1} + \frac{1}{2}k_i h_i + \int_{x_{i+\frac{1}{2}}}^{x_{i+\frac{3}{2}}} |g'(x)| dx.$$

Item (iv) in Lemma 4.1 implies that $|k_l| \leq k_{\max}$ for all $l \in \{0, \dots, n\}$. Then provided we can prove that the family $\{k_l\}_{0 \leq l \leq n}$ is bounded away from zero uniformly, the above inequality would yield an upper bound on $|k_{i+1} - k_i|$. By convexity of F we have

$$\phi_i(k) \leq f_i + \frac{1}{2}k_i h_i + k_{\max}^{-1} F(k_{\max})k - g_{i+\frac{1}{2}}, \quad \forall k \in [0, k_{\max}],$$

which, provided h is small enough, yields

$$k_i \geq \left(g_{i+\frac{1}{2}} - f_i - \frac{1}{2}k_i h_i \right) k_{\max} F(k_{\max})^{-1} \geq \frac{1}{2} \mu_0 k_{\max} F(k_{\max})^{-1} \geq k_{\min} > 0.$$

This proves the lower bound for k_i in (4.14); the upper bound was proved in Lemma 4.1. Then the convexity of F together with $F(0) = 0$ implies

$$\operatorname{ess\,inf}_{k \in [k_i, k_{i+1}]} F'(k) \geq F(k_{\min}) k_{\min}^{-1} \geq \frac{1}{2} \min(\gamma, \gamma^2) \min(1, k_{\min}) > 0,$$

which in turn yields

$$|k_{i+1} - k_i| \leq 2h (k_{\max} + \|g'\|_{L^\infty}) (\min(\gamma, \gamma^2))^{-1} \min(1, k_{\min})^{-1},$$

meaning that $h^{-1}|k_{i+1} - k_i|$ is bounded uniformly. The conclusion follows readily. Similar arguments hold for the backward extremal positive solution. \square

4.3. Initialization. Let $v \in X_h$ be the field obtained from Algorithm 1. The following lemma characterizes v .

LEMMA 4.3. *Assume that $u^{\text{init}} \geq 2\|g\|_{L^\infty}$ and hypotheses (4.1) to (4.7) hold. Then, the index of the breakdown cell m is $\lfloor \frac{1}{2}(n+1) \rfloor$ if Algorithm 1 starts from the left or $n - \lfloor \frac{1}{2}(n+1) \rfloor$ if the algorithm starts from the right, where $\lfloor \cdot \rfloor$ denotes the integer part. Moreover, the restriction of v on $[0, x_m]$ is equal to the forward extremal positive solution, the restriction of v on $[x_{m+1}, x_{n+1}]$ is equal to the backward extremal positive solution, and v connects linearly the forward and the backward solutions on $[x_m, x_{m+1}]$.*

Proof. We denote by v the field produced by Algorithm 1 at every step. This field is initialized so that $v_{-1} = v_0 = 0 = v_{n+1} = v_{n+2}$.

We work by induction. Let us assume that there is still work to be done from the left, i.e., there is an index $i \geq 0$ such that no value has yet been assigned to v_{i+1} (meaning that cell i is in `cell_list` and `update(i+1)` is false). We will prove by induction that if the restriction of v to $[x_{-1}, x_i]$ is equal to the forward extremal positive solution f , then Algorithm 1 updates v so that the restriction of v to $[x_i, x_{i+1}]$ is also equal to f , i.e., $v_{i+1} = f_{i+1}$. For $i = 0$, we only need to verify that $v_0 = f_0$, which is true by definition.

Let us now assume that the induction hypothesis holds for i , namely $v|_{[x_{-1}, x_i]} = f|_{[x_{-1}, x_i]}$. Let k_i^+ and k_i^- be the positive and the negative roots of $\phi_i(k) = 0$, respectively. The existence and uniqueness of k_i^+ and k_i^- have been established in Lemma 4.1. Algorithm 1 compares the functional J_h of the following three fields u^{old} , u^+ , and u^- whose vertex values are

$$\begin{aligned} &(0, f_0, \dots, f_i, u^{\text{init}}, u^{\text{init}}, \dots, u^{\text{init}}), \\ &(0, f_0, \dots, f_i, f_i + k_i^+ h_i, u^{\text{init}}, \dots, u^{\text{init}}), \\ &(0, f_0, \dots, f_i, f_i + k_i^- h_i, u^{\text{init}}, \dots, u^{\text{init}}), \end{aligned}$$

respectively and selects the field that minimizes J_h . Proving the induction hypothesis amounts to proving that $J_h(u^+) < \min(J_h(u^{\text{old}}), J_h(u^-))$ because $f_{i+1} = f_i + k_i^+ h_i := u_{i+1}^+$.

Step 1: We now want to prove that $J_h(u^+) \leq J_h(u^-)$. Since the two fields u^+ and u^- only differ at node x_{i+1} , it suffices to consider

$$J_i(u^\pm) := R_i(u^\pm) + R_{i+1}(u^\pm) + h^{2-2p}(E_i(u^\pm) + E_{i+1}(u^\pm) + E_{i+2}(u^\pm)). \quad (4.15)$$

Clearly $R_i(u^\pm) = 0$ by construction. Let us start by comparing $R_{i+1}(u^+)$ and $R_{i+1}(u^-)$. By definition, we have

$$\begin{aligned} \Delta R &:= R_{i+1}(u^-) - R_{i+1}(u^+) \\ &= h_{i+1} \left| \frac{1}{2}(u^{\text{init}} + u_{i+1}^-) + F(h_{i+1}^{-1}(u^{\text{init}} - u_{i+1}^-)) - g_{i+\frac{1}{2}} \right| \\ &\quad - h_{i+1} \left| \frac{1}{2}(u^{\text{init}} + u_{i+1}^+) + F(h_{i+1}^{-1}(u^{\text{init}} - u_{i+1}^+)) - g_{i+\frac{1}{2}} \right| \end{aligned}$$

Using the fact that $u^{\text{init}} \geq 2\|g\|_{L^\infty}$, we can remove the absolute values

$$\begin{aligned} \Delta R &= \frac{1}{2}h_{i+1}(u_{i+1}^- - u_{i+1}^+) \\ &\quad + h_{i+1} (F(h_{i+1}^{-1}(u^{\text{init}} - u_{i+1}^-)) - F(h_{i+1}^{-1}(u^{\text{init}} - u_{i+1}^+))) \\ &\geq (u_{i+1}^+ - u_{i+1}^-) \left(\underset{(h_{i+1}^{-1}\|g\|_{L^\infty, \infty})}{\text{ess inf}} |F'| - \frac{1}{2}h_{i+1} \right). \end{aligned}$$

Assuming that h is small enough, say $\min(\|g\|_{L^\infty}, \min(\gamma, \gamma^2)) \geq h$, we infer

$$\begin{aligned} \Delta R &\geq (u_{i+1}^+ - u_{i+1}^-)(F(1) - \frac{1}{2}h_{i+1}) \\ &\geq (u_{i+1}^+ - u_{i+1}^-)(\frac{1}{2} \min(\gamma, \gamma^2) - \frac{1}{2}h_{i+1}) \geq 0, \end{aligned}$$

giving $R_{i+1}(u^-) \geq R_{i+1}(u^+)$.

We now compare the entropies. By construction $u_{i+1}^+ = f_{i+1} < u^{\text{init}}$ and we thus have

$$(u_{i+3}^+ - u_{i+2}^+)h_{i+2}^{-1} - (u_{i+2}^+ - u_{i+1}^+)h_{i+1}^{-1} = -(u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1} < 0,$$

i.e., $E_{i+2}(u^+) = 0$. By the same argument we also have $E_{i+2}(u^-) = 0$. Finally, using the fact that $k_{i-1}^+ \geq 0$ and $k_i^- \leq 0$ we obtain $E_i(u^-) = 0$. Let us denote $\Delta E := E_{i+1}(u^-) - E_{i+1}(u^+) - E_i(u^+)$. If $i = 0$, it is easy to prove that $\Delta E := E_1(u^-) - E_1(u^+) - E_0(u^+) > 0$. Let us now assume that $i \geq 1$. If h is small enough, say $h \leq \|g\|_{L^\infty} k_{\max}^{-1}$, then $\omega_{i+1}(u^\pm) = h_{i+1}^p$

$$\begin{aligned} \Delta E &= h_{i+1}^p ((u^{\text{init}} - u_{i+1}^-)h_{i+1}^{-1} - (u_{i+1}^- - f_i)h_i^{-1})_+^p \\ &\quad - h_{i+1}^p ((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1} - (u_{i+1}^+ - f_i)h_i^{-1})_+^p \\ &\quad - \omega_i(u^-) ((u_{i+1}^+ - f_i)h_i^{-1} - (f_i - f_{i-1})h_{i-1}^{-1})_+^p. \end{aligned}$$

Using $u^{\text{init}} \geq 2\|g\|_{L^\infty}$, we conclude that $E_{i+1}(u^-)$ and $E_{i+1}(u^+)$ are positive. Owing to Lemma 4.2 and since $u_{i+1}^+ = f_{i+1}$, we deduce that $E_i(u^+)$ is bounded by ch^{2p} . This gives

$$\begin{aligned} \Delta E &\geq h_{i+1}^p ((u^{\text{init}} - u_{i+1}^-)h_{i+1}^{-1} - (u_{i+1}^- - f_i)h_i^{-1})^p \\ &\quad - h_{i+1}^p ((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1} - (u_{i+1}^+ - f_i)h_i^{-1})^p - ch^{2p}. \end{aligned}$$

Using the inequality $X^p - Y^p \geq (X - Y)Y^{p-1}$, which is valid for all $X \geq Y \geq 0$ and $p \geq 1$, and using the fact that $u_{i+1}^+ \geq u_{i+1}^-$, we infer

$$\begin{aligned} \Delta E &\geq h_{i+1}^p (h_{i+1}^{-1} + h_i^{-1})(u_{i+1}^+ - u_{i+1}^-)((u^{\text{init}} - f_{i+1})h_{i+1}^{-1} - (f_{i+1} - f_i)h_i^{-1})^{p-1} - ch^{2p} \\ &\geq c'(u_{i+1}^+ - u_{i+1}^-) - ch^{2p} \geq c''k_{\min}h - ch^{2p} \end{aligned}$$

where k_{\min} is defined in (4.12). Then $\Delta E > 0$ if h is small enough since $p > 1$. In conclusion $J_i(u^-) > J_i(u^+)$.

Step 2: We now need to show that $J_h(u^{\text{old}}) > J_h(u^+)$. Let us first compare the entropies. Clearly $E_{i+1}(u^{\text{old}}) = 0$, $E_{i+2}(u^{\text{old}}) = 0$, $E_{i+2}(u^+) = 0$. Let $\Delta E := E_i(u^{\text{old}}) - E_i(u^+) - E_{i+1}(u^+)$. If $i = 0$, an easy computation shows that $\Delta E > 0$. Let us assume now that $i \geq 1$. The smoothness of the forward extremal positive solution implies $E_i(u^+) \leq ch^{2p}$ and using the fact that u^{init} is large, say $\frac{1}{2}u^{\text{init}} \geq \|g\|_{L^\infty} \geq \|f\|_{L^\infty}$, we derive $\omega_i(u^{\text{old}}) = h_i$ and $\omega_{i+1}(u^+) = h_{i+1}$. The entropy variation is evaluated as follows:

$$\begin{aligned} \Delta E &= E_i(u^{\text{old}}) - E_i(u^+) - E_{i+1}(u^+) \\ &\geq h_i^p ((u^{\text{init}} - u_i^+)h_i^{-1} - k_{i-1}^+)^p - h_{i+1}^p ((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1} - k_i^+)^p - ch^{2p} \\ &\geq (u^{\text{init}} - u_i^+ - k_{i-1}^+h_i)^p - (u^{\text{init}} - u_{i+1}^+ - k_i^+(h_i + h_{i+1}))^p - ch^{2p}. \end{aligned}$$

Let $X = u^{\text{init}} - u_i^+ - k_{i-1}^+h_i$ and $Y = u^{\text{init}} - u_{i+1}^+ - k_i^+(h_i + h_{i+1})$, then

$$X - Y = -k_{i-1}^+h_i + k_i^+(h_i + h_{i+1}) = (k_i^+ - k_{i-1}^+)h_i + k_i^+h_{i+1} \geq -ch^2 + k_{\min}c_m h,$$

which implies $X - Y > ch$ provided h is small enough. As a result, we have

$$\Delta E \geq (X - Y)Y^{p-1} - c''h^{2p} \geq c'h - c''h^{2p} > ch > 0. \quad (4.16)$$

This means that the entropy of u^{old} is larger than that of u^+ . Now we compare the residuals. The residual variation is evaluated as follows:

$$\begin{aligned} \Delta R &:= R_i(u^{\text{old}}) + R_{i+1}(u^{\text{old}}) - R_{i+1}(u^+) \\ &= h_i \left| \frac{1}{2}(u^{\text{init}} + u_i^+) + F((u^{\text{init}} - u_i^+)h_i^{-1}) - g_{i+\frac{1}{2}} \right| + h_{i+1} \left| u^{\text{init}} - g_{i+\frac{3}{2}} \right| \\ &\quad - h_{i+1} \left| \frac{1}{2}(u^{\text{init}} + u_{i+1}^+) + F((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1}) - g_{i+\frac{3}{2}} \right| \\ &\geq h_i F((u^{\text{init}} - u_i^+)h_i^{-1}) - h_{i+1} F((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1}) \\ &\quad + \frac{1}{2}h_i(u^{\text{init}} + u_i^+ - 2g_{i+\frac{1}{2}}) + \frac{1}{2}h_{i+1}(u^{\text{init}} - u_{i+1}^+) \\ &\geq ch + h_i F((u^{\text{init}} - u_i^+)h_i^{-1}) - h_{i+1} F((u^{\text{init}} - u_{i+1}^+)h_{i+1}^{-1}), \end{aligned}$$

where to derive the last inequality we used that $u^{\text{init}} \geq 2\|g\|_{L^\infty}$ and that h is small. In the case of uniform meshes, we have

$$\Delta R \geq ch + h \left(F((u^{\text{init}} - u_i^+)h^{-1}) - F((u^{\text{init}} - u_i^+)h^{-1} - k_i^+) \right) \geq ch,$$

because $k_i^+ > 0$ and F is an increasing function for positive arguments. Therefore, we infer $J_i(u^{\text{old}}) \geq J_i(u^+)$ in the case of uniform meshes. For quasi-uniform meshes we use the growth hypothesis (4.7) to derive

$$|\Delta R| \leq ch^{1-\theta}. \quad (4.17)$$

Then, provided h is small enough and $p > 1 + \frac{1}{2}\theta$, we estimate

$$\Delta R + h^{2-2p}\Delta E \geq c(h^{3-2p} - h^{1-\theta}) > 0.$$

where we used the bound (4.17) for ΔR and the bound (4.16) for ΔE .

In conclusion $\min(J_h(u^{\text{old}}), J_h(u^-)) > J_h(u^+)$. That is to say the initialization algorithm selects u^+ to set the value of v at the node x_{i+1} , i.e., the algorithm sets $v_{i+1} = u_{i+1}^+$, which proves the induction hypothesis.

Since, as shown in the above argument, neither the forward update nor the backward one break down, the two branches collide in cell $[\frac{1}{2}(n+1)]$ or $n - [\frac{1}{2}(n+1)]$, depending on the initialization of `cell_list`. This concludes the proof. \square

4.4. Local L^1 -switch. We analyze Algorithm 2 in this section. We start with a preliminary technical result.

LEMMA 4.4. *Assume that the hypotheses (4.1) to (4.6) hold. There is c , uniform in h so that for all $i \in \{0, \dots, n\}$ such that $f_i \geq b_i$, the following hold*

$$\kappa_i^- - k_i^- \leq ch, \quad (4.18)$$

$$f_i^- - b_i + ch \leq f_{i+1}^- - b_{i+1}, \quad (4.19)$$

and a similar result holds if f and b are exchanged.

Proof. Obviously the set of such i 's is not empty because there are $c' > 0$ and $c'' > 0$ so that $f_n > c' > c''h > b_n$ if h is small enough. Let i be such that $f_i \geq b_i$. By definition we have

$$\begin{aligned} f_i + \frac{1}{2}(f_{i+1}^- - f_i) + F(k_i^-) - g_{i+\frac{1}{2}} &= 0, \\ b_i + \frac{1}{2}(b_{i+1} - b_i) + F(\kappa_i^-) - g_{i+\frac{1}{2}} &= 0. \end{aligned}$$

Subtracting these two equations gives

$$F(k_i^-) - F(\kappa_i^-) = b_i - f_i + \frac{1}{2}(\kappa_i^- - k_i^-)h_i.$$

If $\kappa_i^- \leq k_i^-$, there is nothing to prove. Otherwise, $k_i^- \leq \kappa_i^- < 0$ and the convexity assumptions (4.2)-(4.4) imply

$$(\kappa_i^- - k_i^-)|F'(-k_{\min})| \leq b_i - f_i + \frac{1}{2}(\kappa_i^- - k_i^-)h_i \leq \frac{1}{2}k_{\max}h.$$

This proves (4.18). To prove the second inequality, we proceed as follows:

$$\begin{aligned} f_{i+1}^- - b_{i+1} &= f_{i+1}^- - f_i + f_i - f_{i-1} + f_{i-1} - f_i^- + f_i^- - b_i + b_i - b_{i+1} \\ &= k_i^- h_i + k_{i-1}^+ h_{i-1} - k_{i-1}^- h_{i-1} - \kappa_i^- h_i + f_i^- - b_i \\ &= (k_i^- - \kappa_i^-)h_i + (k_{i-1}^+ - k_{i-1}^-)h_{i-1} + f_i^- - b_i \\ &\geq -c' h^2 + c'' h + f_i^- - b_i \geq ch + f_i^- - b_i, \end{aligned}$$

where we used (4.18), Lemma 4.1, and Lemma 4.2. \square

LEMMA 4.5. *Assume that the hypotheses (4.1) to (4.7) hold. Let i be the index of the breakdown cell. Assume that $v_l := f_l$ for all $0 \leq l \leq i$ and $v_l := b_l$ for all $i+1 \leq l \leq n+1$. Using the notation (4.10), if the mesh is fine enough and $f_i^- \geq b_i$ (resp. $b_i^+ \geq f_i$), Algorithm 2 updates the value of v_i to b_i (resp. v_{i+1} to f_{i+1}).*

Proof. We do the proof assuming $f_i^- \geq b_i$; the other case is similar.

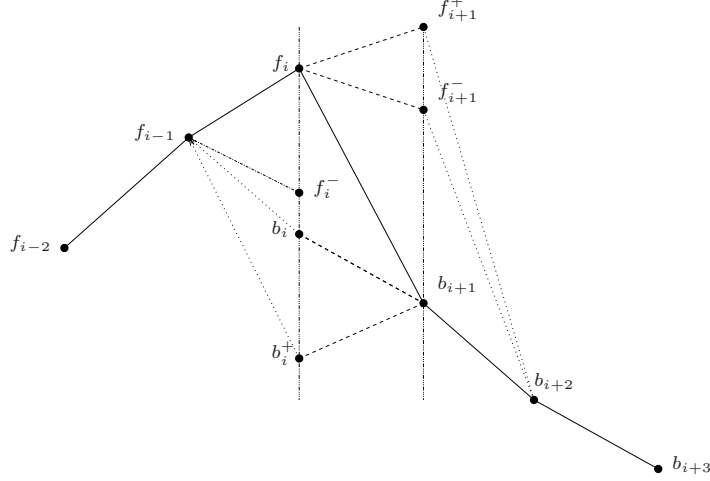


FIGURE 4.1. Notation for the proof of Lemma 4.5

(1) Let us recall that $\Lambda_i := \{0\} \times t_{i,i}(v_i) \cup t_{i,i+1}(v_{i+1}) \times \{0\}$ where $t_{i,i}(v_i) = \{f_{i+1}^+, f_{i+1}^-\}$ and $t_{i,i+1}(v_{i+1}) = \{b_i^+, b_i^-\}$. For all $z := (z_1, z_2) \in \Lambda_i$ we define the function $w(z) \in X_h$ whose points values are

$$\begin{aligned} (v_0, \dots, v_{i-1}, v_i, z_2, v_{i+2}, \dots, v_{n+1}) & \quad \text{if } z_1 = 0, \\ (v_0, \dots, v_{i-1}, z_1, v_{i+1}, v_{i+2}, \dots, v_{n+1}) & \quad \text{if } z_2 = 0. \end{aligned}$$

Let $\bar{z} \in \Lambda_i$ be such that $\bar{z} := \operatorname{argmin}_{z \in \Lambda_i} J_h(w(z))$. Then Algorithm 2 takes $w(\bar{z})$ as the new update if $J_h(w(\bar{z})) < J_h(v)$. Now we need to show that $\bar{z} = (b_i^-, 0)$ and $J_h(\bar{z})$ is less than $J_h(v)$, v being the current update.

To simplify notation and whenever confusion is not possible, we henceforth use $w(f_{i+1}^+)$, $w(f_{i+1}^-)$, $w(b_i^+)$, $w(b_i^-)$ instead of the tensor product notation $w((0, f_{i+1}^+))$, $w((0, f_{i+1}^-))$, $w((b_i^+, 0))$, $w((b_i^-, 0))$. With this notation we need to show that the function $w(b_i^-)$ has the smallest functional among the four possible candidates $w(z)$ and the current update v , see Figure 4.1.

(2) We first prove that $J_h(w(f_{i+1}^+)) \geq J_h(w(f_{i+1}^-))$. Using the notation set in (4.15), we need to compare $J_{i+1}(w(f_{i+1}^+)) \geq J_{i+1}(w(f_{i+1}^-))$.

(2.a) We first compare the residuals. Clearly $R_i(w(f_{i+1}^\pm)) = 0$ and

$$\begin{aligned} \Delta R &:= R_{i+1}(w(f_{i+1}^+)) - R_{i+1}(w(f_{i+1}^-)) \\ &= h_{i+1} \left| \frac{1}{2}(f_{i+1}^+ + b_{i+2}) + F((b_{i+2} - f_{i+1}^+)h_{i+1}^{-1}) - g_{i+\frac{3}{2}} \right| \\ &\quad - h_{i+1} \left| \frac{1}{2}(f_{i+1}^- + b_{i+2}) + F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1}) - g_{i+\frac{3}{2}} \right|. \end{aligned}$$

Now we use the fact that $g_{i+\frac{3}{2}} = \frac{1}{2}(b_{i+1} + b_{i+2}) + F((b_{i+2} - b_{i+1})h_{i+1}^{-1})$. Then

$$\begin{aligned} \Delta R &:= h_{i+1} \left| \frac{1}{2}(f_{i+1}^+ - b_{i+1}) + F((b_{i+2} - f_{i+1}^+)h_{i+1}^{-1}) - F((b_{i+2} - b_{i+1})h_{i+1}^{-1}) \right| \\ &\quad - h_{i+1} \left| \frac{1}{2}(f_{i+1}^- - b_{i+1}) + F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1}) - F((b_{i+2} - b_{i+1})h_{i+1}^{-1}) \right|. \end{aligned}$$

The assumption $f_i^- \geq b_i$ together with (4.19) implies that $f_{i+1}^+ > f_{i+1}^- > b_{i+1} > b_{i+2}$. This in turn, together with the property $F(X) > F(Y) > 0$, for all $X < Y < 0$, which

is a consequence of (4.2)-(4.4), implies that we can remove the absolute values,

$$\Delta R = h_{i+1}(\frac{1}{2}(f_{i+1}^+ - f_{i+1}^-) + F((b_{i+2} - f_{i+1}^+)h_{i+1}^{-1}) - F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1})) \geq 0.$$

(2.b) Next we compare the entropies. Note that $E_{i+1}(w(f_{i+1}^+)) = 0$, $E_i(w(f_{i+1}^-)) = 0$. We also have $E_{i+1}(w(f_{i+1}^-)) = 0$, since

$$\begin{aligned} (b_{i+2} - f_{i+1}^-)h_{i+1}^{-1} - k_i^- &= \kappa_{i+1}^- + (b_{i+1} - f_{i+1}^-)h_{i+1}^{-1} - k_i^- \\ &= \kappa_{i+1}^- - \kappa_i^- + \kappa_i^- - k_i^- + (b_{i+1} - f_{i+1}^-)h_{i+1}^{-1} \\ &\leq c'h - c'' < 0, \end{aligned}$$

where where c' and c'' are two positive constants and we used (4.18) and (4.19). The above observations imply,

$$\begin{aligned} \Delta E &:= E_i(w(f_{i+1}^+)) + E_{i+2}(w(f_{i+1}^+)) - E_{i+2}(w(f_{i+1}^-)) \\ &\geq E_{i+2}(w(f_{i+1}^+)) - E_{i+2}(w(f_{i+1}^-)). \end{aligned}$$

Now we evaluate the weights $\omega_{i+2}(w(f_{i+1}^-))$ and $\omega_{i+2}(w(f_{i+1}^+))$. Note that owing to (4.19) and the assumption $f_i^- \geq b_i$, we have

$$\begin{aligned} (b_{i+2} - f_{i+1}^+)h_{i+1}^{-1} &< (b_{i+2} - f_{i+1}^-)h_{i+1}^{-1} = \kappa_{i+1} - \kappa_{i+2} + \kappa_{i+2} + (b_{i+1} - f_{i+1}^-)h_{i+1}^{-1} \\ &\leq \kappa_{i+2} + c'h - c'' < \kappa_{i+2} < 0, \end{aligned}$$

where c' and c'' are two positive constants. This implies $\omega_{i+2}(w(f_{i+1}^+)) = h_{i+1}^p = \omega_{i+2}(w(f_{i+1}^-))$. The above observations yield

$$\begin{aligned} \Delta E &\geq h_{i+1}^p(\kappa_{i+2} - (b_{i+2} - f_{i+1}^+)h_{i+1}^{-1})^p - h_{i+1}^p(\kappa_{i+2} - (b_{i+2} - f_{i+1}^-)h_{i+1}^{-1})^p - ch^{2p} \\ &\geq h_{i+1}^p((f_{i+1}^+ - f_{i+1}^-)h_{i+1}^{-1})^p - ch^{2p} \geq (2k_{\min})^p h_i^p - ch^{2p}, \\ &\geq ch^p, \end{aligned}$$

where we used the inequality $X^p - Y^p \geq (X - Y)^p$, for all $X, Y \geq 0$, and the mesh regularity assumption. Steps (2.a) and (2.b) prove $J_h(w(f_{i+1}^+)) \geq J_h(w(f_{i+1}^-))$.

(3) Similar arguments as in step (2) can be used to prove that $J_h(w(b_i^+)) \geq J_h(w(b_i^-))$. We omit the details.

(4) The remaining of the argument consists now of proving that $J_h(w(f_{i+1}^-)) \geq J_h(w(b_i^-))$.

(4.a) Let us start by comparing the entropies. Observe first that $E_i(w(f_{i+1}^-)) = 0$ and $E_{i-1}(w(b_i^-)) = 0$. We also have $E_{i+1}(w(f_{i+1}^-)) = 0$, as shown in (2.b). Note also that Lemma 4.2 implies

$$E_{i-1}(w(f_{i+1}^-)) + E_{i+1}(w(b_i^-)) + E_{i+2}(w(b_i^-)) \leq ch^{2p}.$$

The entropy variation is then

$$\begin{aligned} \Delta E &:= E_{i-1}(w(f_{i+1}^-)) + E_{i+2}(w(f_{i+1}^-)) - E_i(w(b_i^-)) - E_{i+1}(w(b_i^-)) - E_{i+2}(w(b_i^-)) \\ &\geq E_{i+2}(w(f_{i+1}^-)) - E_i(w(b_i^-)) - ch^{2p}. \end{aligned}$$

We have already computed $E_{i+2}(w(f_{i+1}^-))$ in (2.b). Let us now compute the weight $\omega_i(w(b_i^-))$. If $\kappa_i \leq (b_i - f_{i-1})h_{i-1}^{-1}$, then $E_i(w(b_i^-)) = 0$ and the conclusion is evident

since $E_{i+2}(w(f_{i+1}^-)) \geq ch^p$. Otherwise, $\omega_i(w(b_i^-)) = h_{i-1}^p$ and

$$\begin{aligned} \Delta E &\geq h_{i+1}^p(\kappa_{i+2}^- - \kappa_{i+1}^- - (b_{i+1} - f_{i+1}^-)h_{i+1}^{-1})^p - h_{i-1}^p(\kappa_i^- - (b_i - f_i^-)h_{i-1}^{-1} - k_i^-)^p - ch^{2p} \\ &\geq ((\kappa_{i+2}^- - \kappa_{i+1}^-)h_{i+1} - (b_{i+1} - f_{i+1}^-))^p - ((\kappa_i^- - k_i^-)h_{i-1} - (b_i - f_i^-))^p - ch^{2p} \quad \blacksquare \end{aligned}$$

Let $X = (\kappa_{i+2}^- - \kappa_{i+1}^-)h_{i+1} - (b_{i+1} - f_{i+1}^-)$ and $Y = (\kappa_i^- - k_i^-)h_{i-1} - (b_i - f_i^-)$, then using Lemma 4.2 and (4.18)-(4.19), we deduce $X - Y \geq c'h - c''h^2 > 0$. Using the inequality $X^p - Y^p \geq (X - Y)^p$, this implies

$$\Delta E \geq (X - Y)^p \geq c'h^p - c''h^{2p} > ch^p > 0. \quad (4.20)$$

(4.b) Let us now compare the residuals. We have

$$\begin{aligned} \Delta R &:= R_{i+1}(w(f_{i+1}^-)) - R_{i-1}(w(b_i^-)) \\ &= h_{i+1}|\frac{1}{2}(f_{i+1}^- + b_{i+2}) + F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1}) - g_{i+\frac{3}{2}}| \\ &\quad - h_{i-1}|\frac{1}{2}(b_i^- + f_{i-1}) + F((b_i^- - f_{i-1})h_{i-1}^{-1}) - g_{i-\frac{1}{2}}|. \end{aligned}$$

Using the definition of b_{i+1} and f_i^- , and recalling that $b_i := b_i^-$, the residual variation can also be recast into

$$\begin{aligned} \Delta R &= h_{i+1}|\frac{1}{2}(f_{i+1}^- - b_{i+1}) + F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1}) - F((b_{i+2} - b_{i+1})h_{i+1}^{-1})| \\ &\quad - h_{i-1}|\frac{1}{2}(b_i - f_i^-) + F((b_i - f_{i-1})h_{i-1}^{-1}) - F((f_i^- - f_{i-1})h_{i-1}^{-1})| \\ &= h_{i+1}|\frac{1}{2}(f_{i+1}^- - b_{i+1}) + F((b_{i+2} - f_{i+1}^-)h_{i+1}^{-1}) - F((b_{i+2} - b_{i+1})h_{i+1}^{-1})| \\ &\quad - h_{i-1}|\frac{1}{2}(b_i - f_i^-) + F((b_i - f_{i-1})h_{i-1}^{-1}) - F((f_i^- - f_{i-1})h_{i-1}^{-1})|. \end{aligned}$$

The absolute values have been removed since the arguments are nonnegative. It is clear for the first argument, but for the second one we proceeded as follows:

$$\begin{aligned} I &:= \frac{1}{2}(b_i - f_i^-) + F((b_i - f_{i-1})h_{i-1}^{-1}) - F((f_i^- - f_{i-1})h_{i-1}^{-1}) \\ &\geq \frac{1}{2}(b_i - f_i^-) + |F'(-k_{\min})|(f_i^- - b_i)h_{i-1}^{-1} \\ &\geq (-\frac{1}{2} + ch^{-1})(f_i^- - b_i) \geq 0. \end{aligned}$$

If the mesh is uniform, ΔR can be recast into

$$\begin{aligned} h^{-1}\Delta R &= \frac{1}{2}(f_{i+1}^- + b_{i+2} - b_i^- - f_{i-1}) \\ &\quad + F((b_{i+2} - f_{i+1}^-)h^{-1}) - F((b_i^- - f_{i-1})h^{-1}) - g_{i+\frac{3}{2}} + g_{i-\frac{1}{2}} \\ &\geq -ch + |F'(-k_{\min})|(b_i^- - f_{i-1} - b_{i+2} + f_{i+1}^-)h^{-1} \\ &= -ch + |F'(-k_{\min})|(b_i - b_{i+1} + b_{i+1} - b_{i+2} - f_{i-1} + f_i - f_i + f_{i+1}^-)h^{-1} \\ &= -ch + |F'(-k_{\min})|(-\kappa_i - \kappa_{i+1} + k_{i-1} + k_i^-) \\ &\geq -ch + |F'(-k_{\min})|(-c'h + 2k_{\min}) \geq c'' > 0, \end{aligned}$$

where we used $g \in \mathcal{C}^{0,1}[0, 1]$, (4.14), and (4.18).

Now let us consider the case when the mesh is only quasi-uniform. In that case, we are not capable of proving that $\Delta R \geq 0$, but if we assume some maximum growth condition on F at infinity, we are going to be able to dominate $|\Delta R|$ by the entropy variation, i.e., $h^{2-2p}\Delta E$. Let us assume the growth condition $F(k) \leq c_g(|k|^\theta + 1)$. We majorize ΔR as follows:

$$|\Delta R| \leq h(2\|g\|_{L^\infty} + F(\|g\|_{L^\infty}(c_m h)^{-1})) \leq ch(1 + h^{-\theta}) \leq c'h^{1-\theta}.$$

Then, provided h is small enough and $p > 1 + \theta$,

$$\Delta R + h^{2-2p}\Delta E \geq c(h^{2-p} - h^{1-\theta}) > 0.$$

where we used (4.20).

(5) Now we prove that $J_h(w(b_i^-)) < J_h(v)$. We evaluate the entropy variation as follows:

$$\begin{aligned} \Delta E &:= E_{i-1}(v) + E_{i+1}(v) + E_{i+2}(v) - E_i(w(b_i^-)) + E_{i+1}(w(b_i^-)) + E_{i+2}(w(b_i^-)) \\ &\geq -ch^{2p} + E_{i+1}(v) - E_i(w(b_i^-)). \end{aligned}$$

If $E_i(w(b_i^-)) = 0$ then $\Delta E \geq ch^p$ since $E_{i+1}(v) \geq ch^p$ as can be shown using arguments similar to those in (2.b). Otherwise $\omega_i(w(b_i^-)) = h_{i-1}$ and we get

$$\begin{aligned} \Delta E &\geq -ch^{2p} + h_i^p (\kappa_{i+1}^- - \kappa_i^- - (b_i - f_i^-)h_i^{-1} + h_{i-1}(k_{i-1}^+ - k_{i-1}^-)h_i^{-1})^p \\ &\quad - h_{i-1}^p (\kappa_i^- - (b_i - f_i^-)h_{i-1}^{-1} - k_i^-)^p \\ &= -ch^{2p} + (f_i^- - b_i + h_{i-1}(k_{i-1}^+ - k_{i-1}^-) + h_i(\kappa_{i+1}^- - \kappa_i^-))^p \\ &\quad - (f_i^- - b_i + h_{i-1}(\kappa_i^- - k_i^-))^p. \end{aligned}$$

Let $X = f_i^- - b_i + h_{i-1}(k_{i-1}^+ - k_{i-1}^-) + h_i(\kappa_{i+1}^- - \kappa_i^-)$ and $Y = f_i^- - b_i + h_{i-1}(\kappa_i^- - k_i^-)$. It is easy to see that $X \geq 0$ and by our assumption $E_i(w(b_i^-)) > 0$ we have $Y > 0$. Now, we consider the difference

$$\begin{aligned} X - Y &= h_{i-1}(k_{i-1}^+ - k_{i-1}^-) + h_i(\kappa_{i+1}^- - \kappa_i^-) - h_{i-1}(\kappa_i^- - k_i^-) \\ &\geq 2h_{i-1}k_{\min} - c'h^2 - c''h^2 > ch \end{aligned}$$

for h small enough, where in the above inequality we used (4.18) and the properties of forward and backward characteristics. We use the inequality $X^p - Y^p \geq (X - Y)^p$ and conclude

$$\Delta E \geq -c'h^{2p} + c''h^p > ch^p > 0. \quad (4.21)$$

Let us now compare the residuals of v and $w(b_i^-)$. We have

$$\begin{aligned} \Delta R &:= R_i(v) - R_{i-1}(w(b_i^-)) \\ &= h_i|\frac{1}{2}(f_i + b_{i+1}) + F((b_{i+1} - f_i)h_i^{-1}) - g_{i+\frac{1}{2}}| \\ &\quad - h_{i-1}|\frac{1}{2}(f_{i-1} + b_i) + F((b_i - f_{i-1})h_{i-1}^{-1}) - g_{i-\frac{1}{2}}| \\ &= h_i|\frac{1}{2}(f_i - b_i) + F((b_{i+1} - f_i)h_i^{-1}) - F((\kappa_i^-)| \\ &\quad - h_{i-1}|\frac{1}{2}(b_i - f_{i-1}) + F((b_i - f_{i-1})h_{i-1}^{-1}) - F(k_{i-1}^-)|. \end{aligned}$$

Proceeding as in (4.b), we can remove the absolute values and we obtain

$$\begin{aligned} \Delta R &= h_i(\frac{1}{2}(f_i + b_{i+1}) + F((b_{i+1} - f_i)h_i^{-1}) - g_{i+\frac{1}{2}}) \\ &\quad - h_{i-1}(\frac{1}{2}(f_{i-1} + b_i) + F((b_i - f_{i-1})h_{i-1}^{-1}) - g_{i-\frac{1}{2}}). \end{aligned}$$

If the mesh is uniform the above equality simplifies as follows:

$$\begin{aligned} \Delta R &= \frac{1}{2}h(k_{i-1}^+ + \kappa_i^-) + h(g_{i-\frac{1}{2}} - g_{i+\frac{1}{2}}) + F((b_{i+1} - f_i)h_i^{-1}) - F((b_i - f_{i-1})h_{i-1}^{-1}) \\ &\geq -c'h + 2|F'(-k_{\min})|k_{\min} > 0 \end{aligned}$$

when h is small. This proves that $J_h(w(b_i^-)) < J_h(v)$ in the uniform case. Now let us consider the case when the mesh is only quasi-uniform. In that case, we use the growth assumption (4.7) and estimate $|\Delta R|$ as follows:

$$|\Delta R| \leq c'h + c''h^{1-\theta} \leq ch^{1-\theta}.$$

Therefore,

$$J_h(v) - J_h(w(b_i^-)) = \Delta R + h^{2-2p}\Delta E \geq c'h^{2-p} - c''h^{1-\theta} > 0$$

provided h is small enough and $p > 1 + \theta$. This concludes the proof. \square

4.5. The final argument. We prove in this section that the combination Algorithm 1+Algorithm 2 give a sequence of almost minimizers and is optimally complex.

THEOREM 4.6. *Assume (4.1) to (4.7) and assume that the initial guess for Algorithm 2 is the field computed by Algorithm 1. Then Algorithm 2 gives an almost minimizer, v_h , after at most n steps and stops after at most $2n$ steps. The sequence $(v_h)_{h>0}$ converges to the unique viscosity solution to (2.1) in $W^{1,1}(0,1) \cap C^0[0,1]$.*

Proof. Observe first that the field generated by Algorithm 1, say v , is such that $v_l = f_l$ for all $0 \leq l \leq i$ and $v_l = b_l$ for all $i+1 \leq l \leq n$ where the index of the breakdown cell is either $i := \lfloor \frac{1}{2}(n+1) \rfloor$ or $i := n - \lfloor \frac{1}{2}(n+1) \rfloor$ depending whether the algorithms starts on the left side or the right side of the domain. In other words, the first set of hypotheses of Lemma 4.5 is satisfied.

After initialization, we are in one of the following four possible situations:

$$\begin{array}{ll} \text{case 1: } & b_i \leq f_i^-, & \text{case 2: } & f_i^- < b_i \leq f_i, \\ \text{case 3: } & b_i^+ < f_i \leq b_i, & \text{case 4: } & f_i \leq b_i^+. \end{array}$$

Assume that we are in case 1 or 4. Let us consider case 1, the other case being symmetric. Then we are under the conditions of application of Lemma 4.5, and Algorithm 2 pushes the breakdown cell to the left and reduces the gap $f_i^- - b_i$. The new breakdown cell is $i-1$ and

$$\begin{aligned} b_{i-1}^+ &= f_{i-1} - f_{i-1} + f_i^- - f_i^- + b_i - b_i + b_{i-1}^+ \\ &= f_{i-1} + k_{i-1}^- h_{i-1} - \kappa_{i-1}^+ h_{i-1} + b_i - f_i^- \\ &\leq f_{i-1} - 2k_{\min} h_{i-1} + b_i - f_i^- < f_{i-1}, \end{aligned}$$

i.e., the new update cannot satisfy the condition of case 4. As a result jumping from case 1 to case 4 is impossible in one step. After each iteration, the new update v either satisfies the condition of case 1, in which case the breakdown cell moves further to the left, or v satisfies the conditions of case 2 or 3. In conclusion, Algorithm 2 produces an update that satisfies the conditions of case 2 or 3 in less than $n/2$ iteration (it can be shown that when the distance of the breakdown cell to the boundary is less than ch we are in case 4).

Assume now that we are in case 2 or 3. Then, in one step only, Algorithm 2 produces an update v so that $J_h(v) \leq ch$ with c uniform in h . To convince ourselves that this is true, assume that we are in case 2, the other case is symmetric. The index of the breakdown cell being i , Algorithm 2 compares five fields among which is the following w , with $w_l = f_l$, $0 \leq l \leq i-1$ and $w_l = b_l$, $i \leq l \leq n+1$. It clear that

$$J_h(w) \leq R_i(w) + h^{2-2p}(E_{i-1}(w) + E_i(w)) + ch.$$

A simple computation shows $R_i(w) \leq ch$. Also, using $b_i \geq f_i^-$, we infer

$$\begin{aligned} E_i(w) &= \omega_i(w)(\kappa_i^- - (b_i - f_{i-1})h_{i-1}^{-1})_+^p \\ &= \omega_i(w)(\kappa_i^- - (f_i^- - f_{i-1})h_{i-1}^{-1} - (b_i - f_i^-)h_{i-1}^{-1})_+^p \\ &\leq \omega_i(w)(\kappa_i^- - k_i^-)_+^p. \end{aligned}$$

Using arguments similar to those in the proof of (4.18), we can prove that $|\kappa_i^- - k_i^-| \leq c_1 h$, which means $E_i(w) \leq c_2 h^{2p}$. To compute $E_{i-1}(w)$ we use Lemma 4.2 as follows:

$$\begin{aligned} E_{i-1}(w) &= \omega_{i-1}(w)((b_i - f_{i-1})h_{i-1}^{-1} - k_{i-1})_+^p \\ &\leq \omega_{i-1}(w)((f_i - f_{i-1})h_{i-1}^{-1} + (b_i - f_i)h_{i-1}^{-1} - k_{i-1})_+^p \\ &\leq \omega_{i-1}(w)(k_i - k_{i-1})_+^p \leq ch^{2p}. \end{aligned}$$

In conclusion, recalling that the update v computed by Algorithm 2 has the smallest functional, we infer $J_h(v) \leq J_h(w) \leq ch$.

The above arguments prove that $J_h(v) \leq ch$ after at most $n/2$ steps in Algorithm 2.

The first time the algorithm reaches cases 2 or 3, we have $J_h(v) \leq ch$ and the point values of v are $(f_0, f_1, \dots, f_i, b_{i+1}, \dots, b_{n+1})$, where i is the breakdown cell. If the algorithm does not stop after that step, the new breakdown cell \mathfrak{c} is either $i - 1$ or $i + 1$, and two possibilities occur. Either we keep the above pattern

$$(f_0, f_1, \dots, f_{\mathfrak{c}}, b_{\mathfrak{c}+1}, \dots, b_{n+1}), \quad (4.22)$$

or the pattern changes into one of the following two possibilities

$$(f_0, f_1, \dots, f_{\mathfrak{c}-1}, f_{\mathfrak{c}}, f_{\mathfrak{c}+1}^-, b_{\mathfrak{c}+2}, \dots, b_{n+1}), \quad (4.23)$$

$$(f_0, f_1, \dots, f_{\mathfrak{c}-1}, b_{\mathfrak{c}}^+, b_{\mathfrak{c}+1}, b_{\mathfrak{c}+2}, \dots, b_{n+1}). \quad (4.24)$$

If Algorithm 2 keeps producing the pattern (4.22), it eventually stops for the following reasons. The algorithm can move the breakdown cell in one direction only, since if the breakdown cell were the same as it was two steps back keeping the pattern (4.22), the algorithm would revisit a configuration with a strictly higher value of J_h . Moreover, the algorithm stops after at most $\frac{2k_{\max}}{c_m k_{\min}}$ steps, since it would otherwise reach configurations corresponding to case 1 or 4, which is impossible since it would imply that the breakdown cell would have to move back.

The remaining case is that when the algorithm bifurcates into pattern (4.23) or (4.24). Let us consider the bifurcation (4.23) with $\mathfrak{c} = i$, the other case being symmetric. The situation right after bifurcation is depicted in Figure 4.2. The bifurcation occurred in cell $i - 1$ and the new breakdown cell is i . Algorithm 2 must either stop or update v by choosing among the four possible candidates, $w_1, w_2, w_3, w_4 \in X_h$ whose vertex values are respectively the following:

$$\begin{aligned} &(f_0, f_1, \dots, f_{i-1}, f_i^-, f_{i+1}^{-+}, b_{i+2}, \dots, b_{n+1}), \\ &(f_0, f_1, \dots, f_{i-1}, f_i^-, f_{i+1}^{--}, b_{i+2}, \dots, b_{n+1}), \\ &(f_0, f_1, \dots, f_{i-1}, b_i, b_{i+1}, b_{i+2}, \dots, b_{n+1}), \\ &(f_0, f_1, \dots, f_{i-1}, b_i^+, b_{i+1}, b_{i+2}, \dots, b_{n+1}), \end{aligned}$$

where $f_{i+1}^{-+} = f_i^- + k_i^{-+} h_i$ and $f_{i+1}^{--} = f_i^- + k_i^{--} h_i$. The parameters k_i^{-+} and k_i^{--} are the positive and negative roots of $f_i^- + \frac{1}{2} k h_i + F(k) - g_{i+\frac{1}{2}} = 0$, respectively. Repeating

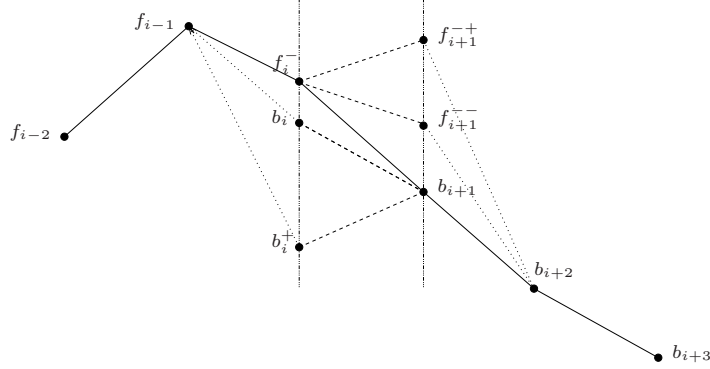


FIGURE 4.2. Notation for the proof of Theorem 4.6 right after bifurcation

arguments similar to those in step (2.a) of the proof of Lemma 4.5, we deduce that $J_h(w_1)$ and $J_h(w_4)$ are larger than $ch^{2-p} \gg c'h$. The only two candidates left are w_2 and w_3 . Assume that either $J_h(w_3) < J_h(w_2)$ or w_3 is chosen as the update in the event $J_h(w_3) = J_h(w_2)$. Let J_{old} be the value of the functional J_h at the previous step. Since the algorithm did not stop, we have $J_h(w_3) < J_{\text{old}}$ and the new breakdown cell is $i - 1$. Then the new pattern for v is exactly the same as it was right before the bifurcation. i.e., two steps back. At the next step the algorithm will bifurcate again and we will be back to the same situation with $J_{\text{old}} := J_h(w_3)$. This is a contradiction to the fact that the value of J_h is strictly decreasing. The conclusion of this argument is that either the algorithm stops or it continues with $v = w_2$.

Now we start an induction argument. Assume that current pattern is

$$(f_0, \dots, f_l, f_{l+1}^-, f_{l+2}^-, \dots, f_{i-1}^-, f_i^-, b_{i+1}, \dots, b_n),$$

where $i \geq l + 1$ is the index of the breakdown cell. The induction hypothesis is that if $i \geq l + 1$ then either the algorithm continues with the f^{--} family or stops. We verified the induction hypothesis for $i = l + 1$ above. Assume now that $i \geq l + 2$. We are now going to prove that either the algorithm stops or continues down using the f^{--} family. If the algorithm does not stop, then it updates v by choosing among the four possible candidates, $w_1, w_2, w_3, w_4 \in X_h$, whose vertex values are respectively the following:

$$\begin{aligned} & (f_0, \dots, f_l, f_{l+1}^-, f_{l+2}^-, \dots, f_{i-1}^-, f_i^-, f_{i+1}^{-+}, b_{i+2}, \dots, b_n), \\ & (f_0, \dots, f_l, f_{l+1}^-, f_{l+2}^-, \dots, f_{i-1}^-, f_i^-, f_{i+1}^-, b_{i+2}, \dots, b_n), \\ & (f_0, \dots, f_l, f_{l+1}^-, f_{l+2}^-, \dots, f_{i-1}^-, b_i, b_{i+1}, b_{i+2}, \dots, b_n), \\ & (f_0, \dots, f_l, f_{l+1}^-, f_{l+2}^-, \dots, f_{i-1}^-, b_i^+, b_{i+1}, b_{i+2}, \dots, b_n), \end{aligned}$$

where $f_{i+1}^{-+} = f_i^- + k_i^{-+}h_i$ and $f_{i+1}^- = f_i^- + k_i^{-}h_i$. The parameters k_i^{-+} and k_i^{-} are the positive and negative roots of $f_i^- + \frac{1}{2}kh_i + F(k) - g_{i+\frac{1}{2}} = 0$, respectively. The fact that the above equation has a unique positive and a unique negative root follows from the observation that $|f_i^- - b_i| \leq ch$, otherwise the algorithm would have stopped since the value of J_h would have been large. Repeating arguments similar to those in step (2.a) of the proof of Lemma 4.5, we deduce that $J_h(w_1)$ and $J_h(w_4)$ are larger than $ch^{2-p} \gg c'h$. The only two candidates left are w_2 and w_3 . Assume that w_3 is the new update. Let J_{old} be the value of the functional J_h at the previous

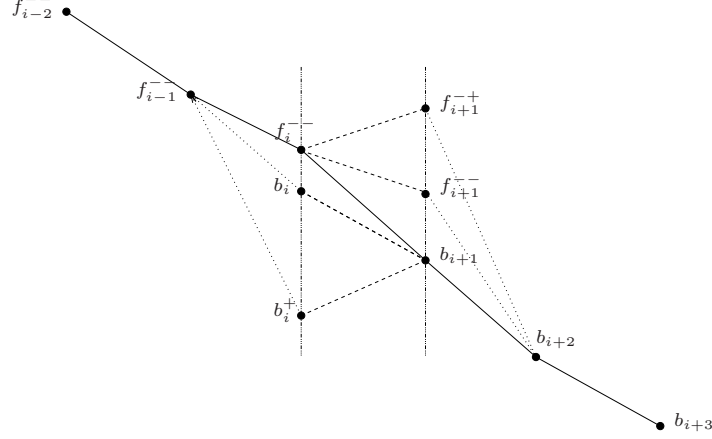


FIGURE 4.3. Notation for the proof of Theorem 4.6. The sliding mechanism.

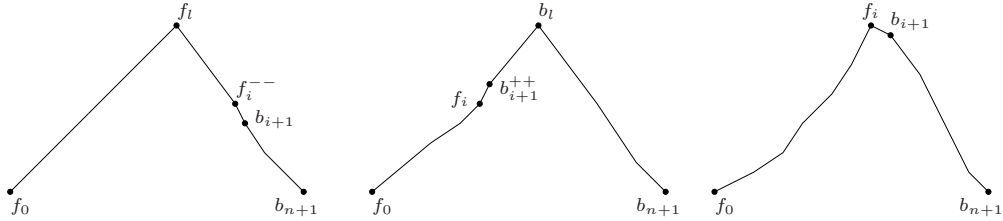


FIGURE 4.4. Notation for the proof of Theorem 4.6. The BV argument.

step. Since the algorithm did not stop, we have $J_h(w_3) < J_{\text{old}}$. This cannot be true since w_3 was one of the four possible choices for updating v two steps back when the breakdown cell was $i-2$. Therefore the new update is w_2 , i.e., we continue down using the f^- family. This concludes the induction. In the worst case the algorithm stops when the breakdown cell reaches the boundary. In conclusion, Algorithm 2 always stops in less than $2n$ steps and $J_h(v) \leq ch$.

We now need to prove that after Algorithm 2 stops, v is an almost minimizer. The point values of v follow one of the three patterns shown in Figure 4.4. It is now clear that the following bound holds (details are omitted):

$$\|v\|_{\text{BV}[0,1]} \leq 2\|g\|_{L^\infty}.$$

Now we derive an upper bound on $J(v)$ by estimating $\Delta J := J(v) - J_h(v)$

$$\begin{aligned} |\Delta J| &\leq h^{2-2p}(E_0(v) + E_{n+1}(v)) \\ &\quad + \left| \int_0^1 |v + H(v') - g| dx - \sum_{i=0}^n h_i |v_{i+\frac{1}{2}} + H(v'_{i+\frac{1}{2}}) - g_{i+\frac{1}{2}}| \right| \\ &\leq ch + \left| \sum_{i=0}^n \int_{x_i}^{x_{i+1}} (|v(x) + H(v'_{i+\frac{1}{2}}) - g(x)| - |v_{i+\frac{1}{2}} + H(v'_{i+\frac{1}{2}}) - g_{i+\frac{1}{2}}|) dx \right| \\ &\leq ch + \sum_{i=0}^n \int_{x_i}^{x_{i+1}} |v(x) - v_{i+\frac{1}{2}} + g(x) - g_{i+\frac{1}{2}}| dx \\ &\leq ch(1 + \|v\|_{\text{BV}[0,1]} + \|g\|_{\text{BV}[0,1]}) \leq c' h. \end{aligned}$$

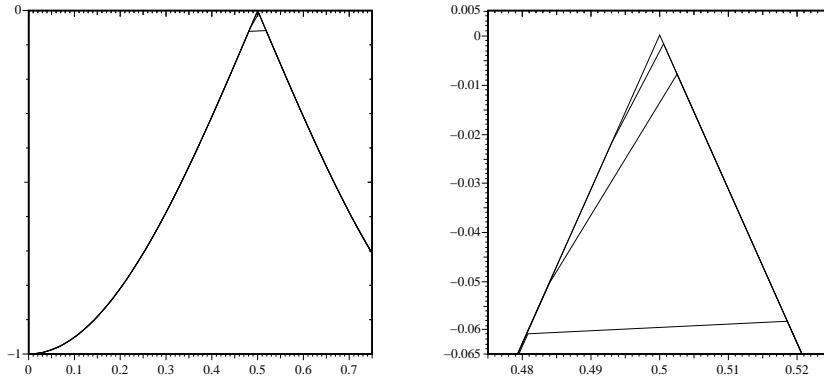


FIGURE 5.1. Left panel: Comparison of the exact solution of (5.1) with numerical solutions using quasi-uniform random meshes with 20, 40, and 100 points. Right panel: Zoom of solutions.

This implies $J(v) \leq ch$. Let $v_h := v$, then owing to [8, Thm 6.2], we conclude that the sequence $(v_h)_{h>0}$ converges in $W^{1,1}(0,1) \cap C^0[0,1]$ to the unique viscosity solution to (2.1). \square

Remark 4.1. One can show that if F is homogeneous of degree 1, the assumption (4.7) is not necessary in the proof of the above theorem. The assumption (4.7) is also not needed to prove Theorem 4.6 if in Algorithm 1 and Algorithm 2 the functionals J_{old} and J_{new} are computed using the entropy terms only, i.e., leaving out the residuals.

5. Numerical illustrations. In this section, we illustrate the performance of the method using random quasi-uniform meshes.

We consider the following equation:

$$u + H(u') = f, \quad \text{on } [0, \frac{3}{4}], \quad u(0) = -1, \quad u(\frac{3}{4}) = -2^{-\frac{1}{2}}, \quad (5.1)$$

with $H(p) = \pi^{-2}p^2$, $f(x) = -|\cos(\pi x)| + \sin^2(\pi x)$. The viscosity solution is $u(x) = -|\cos(\pi x)|$ (other details on this test case are given in [2]).

We report in Figure 5.1 the results of Algorithm 1+Algorithm 2 on random quasi-uniform meshes for this problem. We compare in this figure the viscosity solution and approximates solutions using meshes composed of 20, 40, and 100 points. It is clear that the method is very accurate. Our numerical tests have revealed that, when working with non-uniform meshes, it is important to use the entropy weights described in (3.4)-(3.5), otherwise, Algorithm 1 or Algorithm 2 may terminate before producing an approximate minimizer. We refer to [5, 8] for convergence tests and more numerical examples.

6. Conclusions. We have shown in this paper that it is indeed possible to approximately solve the minimization problem (2.12) in $\mathcal{O}(N)$ operations for one-dimensional stationary Hamilton-Jacobi equations. This confirms the idea that L^1 -based approximation techniques are not only optimal from the theoretical point of view (i.e., they naturally yield viscosity solutions without adding artificial viscosity), but they can also be made computationally practical and optimal complexity is achievable. Of course, this conclusion is still modest since we have dealt only with one-dimensional problems and piecewise linear approximations. In view of [7] though we expect the algorithm described in the present paper to be extendable in two space dimensions at least.

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