Math 437, Homework 1

1. Consider $f(x) = x^3 - 2$.

   (a) Show that $f(x)$ has a root α in the interval $[1, 2]$.

   (b) Repeat, using fixed point iteration with $g_1(x) = x - f(x)/3$ and $g_2(x) = 2/x^2$. Take $x_0 = 1.5$ for the starting value.

   (c) Repeat, using Newton's method. Take $x_0 = 1.5$ for the starting value.

   For each method, present the results in a form of a table:

   column 1: n (step)

   column 2: x_n (approximation)

   column 3: $f(x_n)$ (residual)

   column 4: $|\alpha - x_n|$ (error)

2. Let α be a fixed point of $g(x)$. Consider the fixed point iteration $x_{n+1} = g(x_n)$ and suppose that $\max |g'(x)| = k < 1$. Prove the following error estimate:

$$|\alpha - x_{n+1}| \leq \frac{k}{1-k} |x_{n+1} - x_n|.$$

3. Show that the equation

$$x = 3 + 0.5 \cos x$$

   has a unique solution α. Show that the iteration $x_{n+1} = 3 + 0.5 \cos x_n$ will converge to α. Find a bound for the error.

4. Let α be the solution of $f(x) = 0$, and $\{x_n\}$ be the sequence of approximate solutions, generated by the Newton's method. Show that

$$\alpha - x_{n+1} = -\frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} (\alpha - x_n)^2,$$

   where ξ_n is between x_n and α.

5. Consider the following system of nonlinear equations:

$$f(x, y) = 2x^2 - 2xy + 2y^2 - x - y = 0$$

$$g(x, y) = 4x - y - 2 = 0.$$

   Find an approximation to the solution of this system by taking 1 step of Newton's method, starting from the initial guess $x_0 = 2$, $y_0 = 0$.