Math 437, Homework 2

1. (a) A monomial matrix is a square matrix in which each row and column contains exactly one nonzero entry. Prove that a monomial matrix is nonsingular.

   (b) A square matrix A is said to be skew-symmetric if $A^T = -A$. Prove that if A is skew-symmetric, then $x^T Ax = 0$ for all x.

2. (a) Given

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   Find $\tilde{E}_1$ such that $P_2E_1 = \tilde{E}_1P_2$. (This is used in showing that if A is nonsingular, there exists a permutation matrix P such that PA has an LU factorization.)

   (b) In general, let P^i_j denote the permutation matrix which interchanges row i and row j. Let E_k be an elementary row operation matrix used in Gaussian elimination. Show that if $i > j > k$, then $P^i_j E_k = \tilde{E}_k P^i_j$, where $\tilde{E}_k$ has the same form as E_k. What is the difference between E_k and $\tilde{E}_k$?

   (c) Let $(p_1, p_2, \ldots, p_n)$ be a permutation of $(1, 2, \ldots, n)$ and define the matrix P by $p_{ij} = \delta_{p_i j}$. Let A be an arbitrary $n \times n$ matrix. Describe PA, AP, P^{-1}, PAP^{-1}.

3. Given

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

   (a) Find a permutation matrix P such that PA has an LU factorization. Give the L, U factors of PA.

   (b) Solve $PAx = Pb$ by forward elimination ($Ly = Pb$) and back substitution ($Ux = y$).

4. Given

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix},$$

   find the LU factorization of A. Solve $Ax = b$ by forward elimination ($Ly = b$) and back substitution ($Ux = y$).

5. Consider the $N \times N$ tridiagonal matrix

$$A = \frac{1}{h^2} \begin{pmatrix} 2\epsilon + h^2 & -\epsilon & 0 & \ldots & 0 & 0 & 0 \\ -\epsilon & 2\epsilon + h^2 & -\epsilon & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -\epsilon & 2\epsilon + h^2 & -\epsilon \\ 0 & 0 & 0 & \ldots & 0 & -\epsilon & 2\epsilon + h^2 \end{pmatrix}$$

   and the vector $f = (f_1, f_2, \ldots, f_N)^T$, where $f_i = 2ih + 1$, $h = 2^{-n}$, $N = 2^n - 1$, $n = 1, 2, \ldots, 8$. Take $\epsilon = 10^{-3}$ and solve the system $Au = f$ using LU factorization for a tridiagonal matrix as discussed in class. In your code do not create the matrix A. For each value of $n = 1, 2, \ldots, 8$, plot the points (u_i, x_i), $i = 1, \ldots, N$, where u is the solution of the system and $x_i = ih$.