Math605

$\boxed{L\#8}$

Thursday
Sept 22, 2011

Convergence of the basic iterative methods

We already studied the convergence of Jacobi & Gauss-Seidel iterations for strictly diagonally dominant matrices A in $Ax=b$. This class is not exotic, but much larger is the class of SPD matrices, so we consider A an SPD.

In fact, we can study convergence of much more general class of iterative method based on the following more general splitting of A

(1) A is an SPD matrix, $(Ax, x) > 0$, $x \neq 0$ even when $x \in \mathbb{C}^n$!

(2) $A = D - C - C^T$, where

(a) D is an SPD matrix

(b) $\alpha D - C$ $\alpha > \frac{1}{2}$ is invertible

Examples of splittings

I. The simplest case is when

$D =$ diagonal of A

obviously $a_{ii} > 0 \Rightarrow D$ is an SPD

$C =$ strickly lower triangular part of A

obviously this is the classical SOR, G-S

II. Let A be written in the block form

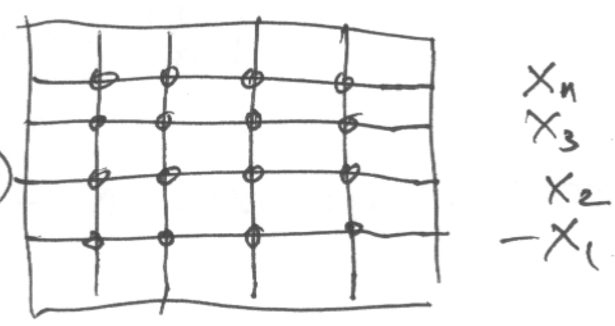
$$A = D - C - C^T$$

where D is block diagonal matrix

III Any other splitting that satisfies the above requirements

Example

$$X = \begin{vmatrix} X_1 \\ \vdots \\ X_n \end{vmatrix} \qquad X_j = (x_{j1}, x_{j2}, \dots, x_{jn})$$



$$D_1 = \begin{vmatrix} 4+h^2 & -1 & 0 & 0 & 0 \\ -1 & 4+h^2 & -1 & & \\ & & & & \\ 0 & 0 & & -1 & 4+h^2 \end{vmatrix}$$ n×n matrix

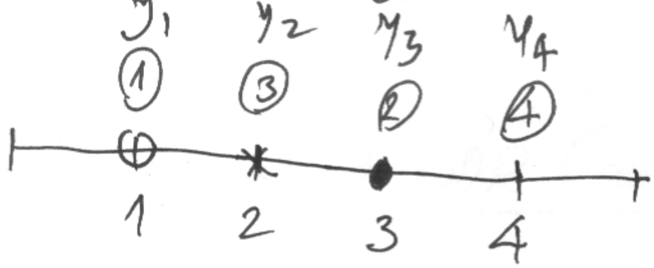
$$C_1 = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & 0 \\ & & & & \\ 0 & 0 & 0 & & -1 \end{vmatrix}$$ n×n matrix

$$A = \begin{vmatrix} D_1 & C_1 & 0 & 0 \\ C_1 & D_1 & & \\ & & \ddots & \\ 0 & 0 & \cdots & D_n \end{vmatrix}$$

$$Ax = A \begin{vmatrix} x_1 \\ x_2 \\ \\ x_n \end{vmatrix}$$ — n dimensional vectors

D_1^{-1} is essentially inverting the above tridiagonal matrix

Possible splittings



reordering of the unknowns in some applications

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = b$$

$$\begin{array}{cc} \left|\begin{array}{cc|cc} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ \hline -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{array}\right| \begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{vmatrix} & \begin{array}{l} \text{red} \\ \\ \text{black} \\ \\ \end{array} \end{array}$$

$$A = \begin{vmatrix} D_1 & C^T \\ C & D_1 \end{vmatrix}$$

red black ordering of the nodes/unknowns

$$D = \begin{vmatrix} D_1 & 0 \\ 0 & D_2 \end{vmatrix}$$

Convergence of SOR (and G-S) for A an SPD

Consider A an SPD matrix, i.e. $(Ax, x) > 0$, $x \neq 0$

$$A = D - C - C^T \quad \left(C - \text{lower triangular for example} \right.$$

$Q = \alpha D - C$

more general splitting is when C is such that D, SPD, & $D - C - C^T = A$

<u>Theorem:</u> If A is an SPD matrix, Q - SPD and $\alpha > \frac{1}{2}$ then the SOR iterations with $Q = \alpha D - C$ converges for any starting vector

$$Q\left(x^{(k+1)} - x^{(k)}\right) = b - Ax^{(k)}$$

equivalent $\quad x^{(k+1)} = \quad (I - Q^{-1}A)x^{(k)} + Q^{-1}b$

Remark: SOR $\quad \alpha = \frac{1}{\omega}$, D - diagonal C - lower Δ

G-S $\quad \alpha = 1$ $\qquad$ — " —

<u>Proof.</u> We write the error transfer matrix $G = I - Q^{-1}A$ and we need to prove that $\rho(G) < 1$.

Let $(\lambda, x)^{\in(\mathbb{C},\mathbb{C}^n)}$ be an eigenpair of G, i.e.

$$Gx = \lambda x \quad \Rightarrow \text{define } y = (I-G)x$$

(λ, x) complex in general

or (1) $y = x - Gx = (1-\lambda)x = x - (I - Q^{-1}A)x = Q^{-1}Ax$

$$\Rightarrow \quad \boxed{Qy = Ax} \equiv (\alpha D - C)y = Ax$$

and also

(2) $Q - A = \alpha D - C - (D - C - C^T) = \alpha D - D + C^T$

(a) $(\alpha D - C)y = Ax$

$(\alpha D - D + C^T)y = (Q-A)y = Ax - Ay = A(x-y) = A(\underbrace{x - Q^{-1}Ax}_{Gx})$

(b) $(\alpha D - D + C^T)y = AGx$

(a) take inner product with y to get

$((\alpha D - C)y, y) = (Ax, y)$ $\qquad (\cdot,\cdot)$ – L^2-inner prod

(b) take inner product with y to get

$((\alpha D - D + C^T)y, y) = (AGx, y)$

$$\alpha(Dy,y)-(Cy,y)=(Ax,y)$$

$$\alpha(Dy,y)-(Dy,y)+(Cy,y)=(y,AGx)$$

$$\boxed{Gx=\lambda x}$$

Now add these two to get

$$(2\alpha-1)(Dy,y)=(Ax,y)+(y,AGx)$$

$$(2\alpha-1)(Dy,y)=(Ax,(1-\lambda)x)+((1-\lambda)x,\lambda Ax)$$

$$(2\alpha-1)(Dy,y)=(1-\bar\lambda)(Ax,x)+(1-\lambda)\bar\lambda(x,Ax)$$

$$y=(1-\lambda)x$$

$$\underbrace{(2\alpha-1)}_{0}|1-\lambda|^2\underbrace{(Dx,x)}_{\text{real}>0}=(1-|\lambda|^2)\underbrace{(Ax,x)}_{\text{real}>0}$$

$$1-|\lambda|^2\geq 0 \qquad |\lambda|^2\leq 1$$

Is it possible $\lambda=1$?

$$Gx=x \Rightarrow (I-G)x=0$$

$$(I-I+Q^{-1}A)x=0 \quad \underbrace{Q^{-1}A}x=0$$

$Q^{-1}A$ singular impossible

Thus $\left.\begin{array}{l}|1-\lambda|^2>0\\ (Ax,x)>0\\ (Dx,x)>0\end{array}\right\}\Rightarrow \begin{array}{l}1-|\lambda|^2>0\\ |\lambda|<1\end{array}$

$$\rho(G)<1$$

Converges

There is a way to optimize the choice of the parameter in SOR. For the case of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & 2 \end{pmatrix} \qquad \omega_{opt} = 2\left(1 - \frac{\pi}{n+1}\right)$$

for this choice $\rho(G_{SOR}) = 1 - \frac{2\pi}{n+1}$

Please, run your programs of PA # 2 with this choice of ω to see the difference in the number of iterations.

The same example Jacobi has

$$\rho(G_{Jacobi}) \cong 1 - \frac{\pi^2}{(n+1)^2}$$ much closer to 1 and much slower to converge.

$\rho(G_J) = \|G_J\|_2$

I would like you to experiment with these two methods for this matrix to see the difference in the number of iterations when you run it for $n = 20, 40, 80$

Warm-up

Quizzing

A any matrix

$\|A\|_2 = \frac{\|Ax\|_2}{\|x\|_2} = ?$

A symmetric $\|A\|_2 = \rho(A) = \max_{\lambda \in \sigma(A)} |\lambda(A)|$ eigenvalue

$\|A\|_2^2 = \sup \frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax, Ax)}{(x,x)} = \frac{(A^TAx, x)}{(x,x)} = \rho(A^*A)$

$$\boxed{\|A\|_2 = \rho(A^*A)^{1/2}}$$

A strictly diagonally dominant Jacobi converges

A strictly columnwise diagonally dominant does Jacobi converge ?

A SPD Richardson $0 < \lambda = \lambda_1 \le \dots \le \lambda_n = \Lambda$

$\boxed{\tau = \frac{2}{\lambda_1 + \lambda_n}}$ $\rho(G_{Rich}) = |1 - \tau\lambda_1| = \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}$