1. PROBLEMS

This contains a set of possible solutions to all problems of HW-2. Be vigilant since typos are possible (and inevitable).

(1) Problem 1 (20 pts) For a matrix $A \in R^{n \times n}$ we define a norm by

$$\|A\| = \max_{x \in R^n, x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where $\|x\|$ is a norm in $R^n$. Show that the following is true:

(a) $\|A\| \geq 0$ and if $\|A\| = 0$ then $A = 0$ (here 0 is the zero matrix in $R^{n \times n}$);
(b) $\|\alpha A\| = |\alpha| \|A\|$, $\alpha$ any real number;
(c) $\|A + B\| \leq \|A\| + \|B\|;
(d) $\|AB\| \leq \|A\|\|B\|$.

(2) Problem 2 (10 pts) Show that a symmetric $n$ by $n$ matrix with positive element that is strictly row-wise diagonally dominant is also positive definite.

(3) Problem 3 (10 pts) Let $\rho(A)$ be the spectral radius of the matrix $A$. Show that for any integer $k > 0$, $\rho(A^k) = (\rho(A))^k$.

(4) Problem 4 (10 pts) Prove that $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$ for any matrix $A \in R^{n \times n}$.

(5) Problem 5 (10 pts) Prove that for any nonsingular matrices $A, B \in R^{n \times n}$,

$$\|B^{-1} - A^{-1}\| \leq cond(A) \frac{\|B - A\|}{\|B^{-1}\|}$$

where $\|\cdot\|$ is a matrix norm and $cond(A)$ is the condition number of $A$ w.r.t that norm.

(6) Problem 6 (10 pts) Show that any strictly diagonally dominant symmetric matrix with positive diagonal elements has only positive eigenvalues.

(7) Problem 7 (10 pts) Show that if $A$ is symmetric and positive definite matrix and $B$ is symmetric then the eigenvalues of $AB$ are real. If in addition $B$ is positive definite then the eigenvalues of $AB$ are positive.

(8) Problem 8 (20 pts) Show that

(a) $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$, where $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$;
(b) $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$, where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.
2. Solutions

(1) Problem 1:

Proof. (a) If \( A = 0 \), then clearly \( Ax = 0, \forall x \in \mathbb{R}^n \), so

\[
\|A\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} = 0.
\]

Otherwise \( \exists a_{ij} \neq 0, a_{ij} \in A \). Let \( e_j = [0, \cdots, 1, \cdots, 0]^T \), then

\[
\|A\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ae_j\|}{\|e_j\|} > 0.
\]

In summary, \( \|A\| \geq 0 \), and \( \|A\| = 0 \) iff \( A = 0 \).

(b) \( \|\alpha A\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|\alpha Ax\|}{\|x\|} = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} = |\alpha| \|A\| \)

(c) \( \|A + B\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|(A + B)x\|}{\|x\|} = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \leq \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} + \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\| \)

(d) According to the definition of the matrix norm, \( \|A\| = \max_{x \in \mathbb{R}^n} \|Ax\|/\|x\| \)

we have \( \|Ax\| \leq \|A\| \|x\|, \forall x \in \mathbb{R}^n \). So, \( \forall x \in \mathbb{R}^n \) we have the following inequalities:

\[
\|ABx\| \leq \|A\| \|Bx\| \leq \|A\| \|B\| \|x\|
\]

Therefore, \( \|AB\| = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|ABx\|}{\|x\|} \leq \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|A\| \|B\| \|x\|}{\|x\|} = \|A\| \|B\|. \)

\( \square \)

(2) Problem 2:

Proof. \( A \in \mathbb{R}^{n \times n} \) is symmetric, so \( \exists \) a real diagonal matrix \( \Lambda \) and an orthonormal matrix \( Q \) such that \( A = Q \Lambda Q^T \), where the diagonal values of \( \Lambda \) are \( A \)'s eigenvalues \( \lambda_i, \ i = 1, \cdots, n \) and the columns of \( Q \) are the corresponding eigenvectors.

According to Gerschgorin’s theorem, \( A \)'s eigenvalues are located in the union of disks

\[
d_i = \{z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|\}, \quad i = 1, \cdots, n.
\]
Because $A$ is positive and strictly row-wise diagonal dominant, so,
\[ d_i = \{ z \in C : 0 < a_{i,i} - \sum_{j \neq i} a_{i,j} \leq z \leq \sum_{j} a_{i,j}, \quad i = 1, \ldots, n. \] 

Therefore all $A$’s eigenvalues are positive, then $\forall x \in \mathbb{R}^n, \ x \neq 0$ we have,
\[ x^T A x = x^T Q \Lambda Q^T x = y^T \Lambda y > 0, \text{ where } y = Q^T x \neq 0. \]
i.e. $A$ is positive definite. \[ \square \]

(3) Problem 3: First, recall that the set of complex numbers $\sigma(A) = \{ \lambda : \det(A - \lambda I) = 0 \}$ is called spectrum of $A$.

Proof. Then, $\forall \lambda \in \sigma(A)$, we show that $\lambda^k \in \sigma(A^k)$. Indeed, let $A u = \lambda u$ with $u \neq 0$.

\[ A^k u = \lambda A^{k-1} u = \cdots = \lambda^k u. \]

For any $\mu \in \sigma(A^k)$, we have $A^k v = \mu v, \ v \neq 0$. Since, $v$ belong to the range of $A$, so at least there exists one eigenvalue $\lambda = \mu^{1/k}$ s.t.
\[ A v = \mu^{1/k} v. \]

Thus, we always have that for any $|\mu|$ where $A^k u = \mu u, \ u \neq 0$, there exists a $|\lambda| = |\mu|^{1/k}$ where $A u = \lambda u, \ u \neq 0$; for any $|\lambda|$ where $A u = \lambda u, \ u \neq 0$, there exists a $|\mu| = |\lambda^k| = |\lambda|^k$ where $A^k u = \mu u, \ u \neq 0$. So,
\[ \rho(A^k) = \max_{\mu \in \sigma(A^k)} |\mu| \equiv \max_{\lambda \in \sigma(A)} |\lambda|^k = (\max_{\lambda \in \sigma(A)} |\lambda|)^k = (\rho(A))^k. \]

\[ \square \]

(4) Problem 4:

Proof. It follows from the obvious string of inequalities
\[
\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \max_{\|x\|_2 = 1} \left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j \right)^2 \right)^{1/2} \\
\leq \max_{\|x\|_2 = 1} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 x_j^2 \right)^{1/2} \quad \text{Schwartz inequality} \\
= \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \leq \left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |a_{ij}| \right)^2 \right)^{1/2} \\
\leq (n \max_{i} \sum_{j=1}^{n} |a_{ij}|^2)^{1/2} \leq (n \|A\|_\infty)^{1/2} = \sqrt{n} \|A\|_\infty \\
\square \\

(5) Problem 5:

Proof. Assume the norm $\| \cdot \|$ satisfies the submultiplicaitve property, i.e. $\|AB\| \leq \|A\| \|B\|$ (subordinate matrix norms have this property).
\[
\|B^{-1} - A^{-1}\| = \|A^{-1} - B^{-1}\| = \|B^{-1}(B - A)A^{-1}\| \\
\leq \|B^{-1}\| \|B - A\| \|A^{-1}\| \\
\]
$A$, $B$ are nonsingular matrices, divide $\|B^{-1}\|$ on both sides,

$$\frac{\|B^{-1} - A^{-1}\|}{\|B^{-1}\|} \leq \|B - A\| \|A^{-1}\| = \frac{\|B - A\| \|A^{-1}\| \|A\|}{\|A\|} = \text{cond}(A) \|B - A\| \|A\|$$

(6) Problem 6:

Proof. For any symmetric matrix $A$, all its eigenvalues are real.

According to Gershgorin’s theorem, $A$’s eigenvalues are located in the union of disks

$$d_i = \{ z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}| \}, \quad i = 1, \cdots, n.$$  

i.e.,

$$d_i = \{ z \in \mathbb{C} : a_{i,i} - \sum_{j \neq i} |a_{i,j}| \leq z \leq a_{i,i} + \sum_{j \neq i} |a_{i,j}| \}, \quad i = 1, \cdots, n.$$  

Because $A$’s diagonal elements are positive and $A$ is strictly diagonal dominant, so,

$$a_{i,i} - \sum_{j \neq i} |a_{i,j}| = |a_{i,i} - \sum_{j \neq i} |a_{i,j}| | > 0$$  

$$a_{i,i} + \sum_{j \neq i} |a_{i,j}| = |a_{i,i} + \sum_{j \neq i} |a_{i,j}| | > 0, \quad i = 1, \cdots, n.$$  

So, all this disks are located at the right side of the $y$-axis in the complex plane, so all eigenvalues of $A$ are positive.\qed

(7) Problem 7:

Proof. First possible solution:

(1) Consider an eigenvalue $\lambda$ and its eigenvector $\psi$ of the matrix $AB$, that is $AB\psi = \lambda \psi$. We do not know whether $\lambda$ and $\psi$ are real, so we assume that they are complex. Recall that the inner product of two complex vectors $\psi$ and $\phi$ is defined as $(\phi, \psi) = \sum_i \phi_i \bar{\psi}_i$, where $\bar{\psi}$ is the complex conjugate to $\psi$. Recall, that $(\phi, \psi) = (\psi, \phi)$. For the inner product of the complex vectors $AB\psi$ and $\psi$ we get

$$AB\psi = \lambda \psi \Rightarrow BAB\psi = \lambda B\psi, \quad (BAB\psi, \psi) = \lambda (B\psi, \psi).$$

Now using the symmetry of $A$ and $B$ we get

$$(BAB\psi, \psi) = (\psi, BAB\psi) = (\overline{BAB\psi}, \overline{\psi}),$$

which means that the complex number $(BAB\psi, \psi)$ is equal to its complex conjugate, i.e. the number is real. In the same way we prove that $(B\psi, \psi)$ is real as well, from where we conclude that $\lambda$ is real. Then we conclude that $\psi$ is real as well.

(2) If $A$ and $B$ are positive definite, then $(B\psi, \psi) > 0$ and $(BAB\psi, \psi) > 0$, therefore from $(BAB\psi, \psi) = \lambda (B\psi, \psi)$ it follows that $\lambda > 0$. 

Another possible solution (for those with more advanced knowledge in linear algebra):

1. $A$ is SPD, its square root exists $A^{1/2}$ which is SPD, then

$$AB \sim A^{-1/2}ABA^{1/2} = A^{1/2}BA^{1/2}$$

Because both $A^{1/2}$ and $B$ are symmetric, then

$$(A^{1/2}BA^{1/2})^T = (A^{1/2})^T B (A^{1/2})^T = A^{1/2}BA^{1/2},$$

so $A^{1/2}BA^{1/2}$ is symmetric, and the eigenvalues of $A^{1/2}BA^{1/2}$ are real. $AB \sim A^{1/2}BA^{1/2}$, they have the same spectrum, so all eigenvalues of $AB$ are real.

2. If $B$ is also SPD, we can show that $A^{1/2}BA^{1/2}$ is SPD. The symmetry is shown in (1). Now show positive definite. \( \forall x \in \mathbb{R}^n, x \neq 0 \) with $y = A^{1/2}x \neq 0$ we have

$$(A^{1/2}BA^{1/2}x, x) = (BA^{1/2}x, A^{1/2}x) = (By, y) > 0.$$ 

So $A^{1/2}BA^{1/2}$ is SPD, and all its eigenvalues are positive. $AB \sim A^{1/2}BA^{1/2}$, they have the same spectrum, so all eigenvalues of $AB$ are positive.

\[ \square \]

we show (b) as well

(8) Problem 8:

**Proof.** We prove (a). First we show that $\|A\|_\infty \leq \max_i \sum_j |a_{i,j}|$. Indeed,

$$\|A\|_\infty \leq \max_{\|x\|_\infty = 1} \|Ax\|_\infty$$

$$= \max_{\|x\|_\infty = 1} \max_i \sum_{j=1}^n |a_{ij}x_j|$$

$$\leq \max_{\|x\|_\infty = 1} \max_i \sum_{j=1}^n |a_{ij}| |x_j|$$

$$\leq \max_{\|x\|_\infty = 1} \max_i \sum_{j=1}^n |a_{ij}| \sum_{j=1}^n |a_{ij}|$$

$$= \max_i \sum_{j=1}^n |a_{ij}|$$

Next we show also that $\max_i \sum_j |a_{i,j}| \leq \|A\|_\infty$. Assume that for some integer $k \in [1, n]$ we have

$$\sum_{j=1}^n |a_{kj}| = \max_i \sum_{j=1}^n |a_{ij}|.$$

By choosing a vector $y$ s.t.,

$$y_j = \begin{cases} 1, & \text{for } a_{k,j} \geq 0; \\ -1, & \text{for } a_{k,j} < 0; \end{cases}$$
Then, $\|y\|_{\infty} = 1$ and
$$\|A\|_{\infty} = \sup_{x \in \mathbb{R}^n, \|x\|_{\infty} = 1} \|Ax\|_{\infty} \geq \|Ay\|_{\infty} = \sum_j |a_{k,j}| = \max_i \sum_j |a_{i,j}|.$$

Thus, from the inequalities $\max_i \sum_j |a_{i,j}| \leq \|A\|_{\infty} \leq \max_i \sum_j |a_{i,j}|$, it follows that
$$\|A\|_{\infty} = \max_{i} \sum_j |a_{i,j}|.$$ 

The inequality (b) is shown in a similar way. $\square$