MATH 638 PROJECTS

Question 1. Consider the initial-value problem for a scalar conservation law:

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+ \\ u = u^0 & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

where u is a bounded and measurable function. Prove that the following two definitions are equivalent if $u \in C([0,\infty); L^1_{loc}(\mathbb{R}))$ (assume that u is piecewise smooth if you do not know/understand the notation).

Definition 1. u is a weak solution, if $\forall \phi \in C^1_0(\mathbb{R} \times [0,\infty))$, u satisfies

$$\int_{\mathbb{R}} \int_0^\infty u(x,t)\phi_t(x,t) + f(u(x,t))\phi_x(x,t)\ dt\ dx + \int_{\mathbb{R}} u^0(x)\phi(x,0)\ dx = 0$$

Definition 2. u is a weak solution, if $\forall \phi \in C^1_0(\mathbb{R} \times [0,\infty))$ and $\forall T \in [0,\infty)$, u satisfies

$$\int_{\mathbb{R}} \int_0^T u(x,t)\phi_t(x,t) + f(u(x,t))\phi_x(x,t)\ dt\ dx = \int_{\mathbb{R}} u(x,T)\phi(x,T) - \int_{\mathbb{R}} u^0(x)\phi(x,0)\ dx$$

Question 2. Let f in (1) be smooth and strictly convex (or just take $f(u) = 0.5u^2$). Assume that the initial condition $u^0 : \mathbb{R} \to \mathbb{R}$ satisfies:

- u^0 has a compact support, i.e., $\text{supp}(u^0) \in [-C, C]$,
- $|u^0(x)| \le C,\ \forall x \in \mathbb{R}$,
- $\dfrac{u^0(x) - u^0(y)}{x - y} \le C,\ \forall x, y \in \mathbb{R}$ such that $\ x \ne y$,

where C is a fixed positive constant. Consider one-step of the LxF scheme

$$v_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^0(x)\ dx, \text{ where } x_j = j\Delta x,$$

$$v^1_{j+\frac{1}{2}} = \frac{1}{2} v^0_{j+1} + \frac{1}{2} v^0_j - \frac{\Delta t}{\Delta x}\big(f(v^0_{j+1}) - f(v^0_j)\big).$$

Prove the following stability results:

a. For all $j \in \mathbb{Z}$, we have: $\max\limits_j \dfrac{v^0_{j+1} - v^0_j}{\Delta x} \le C$ and $|v^0_j| \le C$.

b. For all $j \in \mathbb{Z}$, we have: $\max\limits_j \dfrac{v^1_{j+\frac{1}{2}} - v^1_{j-\frac{1}{2}}}{\Delta x} \le C$ and $|v^1_{j+\frac{1}{2}}| \le C$.

c. Let $\{g_j\}_{j\in\Lambda}$ be a finite sequence such that $|g_j| \le C$, $\dfrac{g_{j+1}-g_j}{\Delta x} \le C$ for all $j, j+1 \in \Lambda$, and the number of elements in the sequence $|\Lambda|$ is such that $|\Lambda| \le \dfrac{C}{\Delta x} + 1$. Then we have that

$$\sum_{j,j+1\in\Lambda} |g_{j+1} - g_j| \le \tilde{C},$$

where $\tilde{C}$ is a fixed constant that depends only on C, i.e., $\tilde{C}$ is independent of Δx.

Question 3. Implement NT and LxF numerical schemes for the following scalar conservation law:

$$\begin{cases} u_t + f(u)_x = 0, & x \in (-2,2),\ t > 0 \\ u(x,0) = u_0(x), & x \in (-2,2) \end{cases}$$

with periodic boundary conditions.

Test cases:

(1) Linear Transport: $f(u) = u$,

$$u_0(x) = \begin{cases} 1 & , \ x > 0 \\ -1 & , \ x \le 0 \end{cases}$$

(2) Burgers' Equation: $f(u) = u^2/2$,

$$u_0(x) = \begin{cases} 1+x & , \ x \in [-1,0) \\ 1-x & , \ x \in [0,1] \\ 0 & , \ \text{otherwise} \end{cases}$$

(3) Buckley-Leverett. $f(u) = \frac{u^2}{u^2+(1-u)^2}$,

$$u_0(x) = \begin{cases} 1 & , \ x < 0 \\ 0 & , \ x \ge 0 \end{cases}$$

Set the number of cells to 200, the CFL condition to 0.3, and test the schemes at times $t = 0.5, 2$ and 20. Use the minmod limiter in the NT scheme.