Lecture Notes on Hyperbolic Conservation Laws

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Abstract

These notes provide an introduction to the theory of hyperbolic systems of conservation laws in one space dimension. The various chapters cover the following topics: **1.** Meaning of the conservation equations and definition of weak solutions. **2.** Shocks, Rankine-Hugoniot equations and admissibility conditions. **3.** Genuinely nonlinear and linearly degenerate characteristic fields. Solution to the Riemann problem. Wave interaction estimates. **4.** Weak solutions to the Cauchy problem, with initial data having small total variation. Proof of global existence via front-tracking approximations. **5.** Continuous dependence of solutions w.r.t. the initial data, in the $\mathbf{L}^1$ distance. **6.** Uniqueness of entropy-admissible weak solutions. **7.** Approximate solutions constructed by the Glimm scheme. **8.** Vanishing viscosity approximations. **9.** Counter-examples to global existence, uniqueness, and continuous dependence of solutions, when some key hypotheses are removed. The survey is concluded with an Appendix, reviewing some basic analytical tools used in the previous chapters.

1 Introduction

A scalar conservation law in one space dimension is a first order partial differential equation of the form

$$u_t + f(u)_x = 0\,. \tag{1.1}$$

Here u is called the *conserved quantity* while f is the *flux*. Equations of this type often describe transport phenomena. Integrating (1.1) over a given interval $[a,b]$ one obtains

$$\begin{aligned}\frac{d}{dt}\int_a^b u(t,x)\,dx \;&=\; \int_a^b u_t(t,x)\,dx \;=\; -\int_a^b f(u(t,x))_x\,dx \\ &=\; f(u(t,a)) - f(u(t,b)) \;=\; [\text{inflow at } a] - [\text{outflow at } b]\,.\end{aligned}$$

In other words, the quantity u is neither created nor destroyed: the total amount of u contained inside any given interval $[a,b]$ can change only due to the flow of u across boundary points (fig. 1).

Using the chain rule, (1.1) can be written in the quasilinear form

$$u_t + a(u)u_x = 0, \tag{1.2}$$

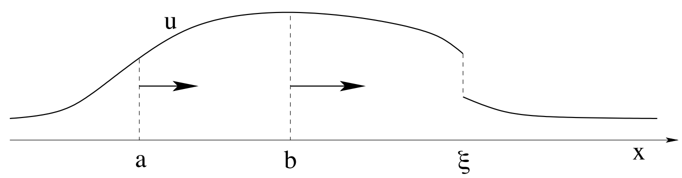

Figure 1: Flow across two points.

where $a = f'$ is the derivative of f. For smooth solutions, the two equations (1.1) and (1.2) are entirely equivalent. However, if u has a jump at a point ξ, the left hand side of (1.2) will contain the product of a discontinuous function $a(u)$ with the distributional derivative u_x, which in this case contains a Dirac mass at the point ξ. In general, such a product is not well defined. Hence (1.2) is meaningful only within a class of continuous functions. On the other hand, working with the equation in divergence form (1.1) allows us to consider discontinuous solutions as well, interpreted in distributional sense. More precisely, a locally integrable function $u = u(t,x)$ is a *weak solution* of (1.1) provided that

$$\iint \{u\phi_t + f(u)\phi_x\} \, dxdt \;=\; 0 \tag{1.3}$$

for every differentiable function with compact support $\phi \in \mathcal{C}^1_c$.

Example 1 (Traffic flow). Let $u(t,x)$ be the density of cars on a highway, at the point x at time t. For example, u may be the number of cars per kilometer. In first approximation, we shall assume that u is continuous and that the speed s of the cars depends only on their density, say

$$s = s(u), \qquad\qquad \text{with} \qquad \frac{ds}{du} < 0.$$

Given any two points a, b on the highway, the number of cars between a and b therefore varies according to the law

$$\begin{aligned} \int_a^b u_t(t,x)\,dx \;&=\; \frac{d}{dt}\int_a^b u(t,x)\,dx \;=\; [\text{inflow at } a] - \;[\text{outflow at } b] \\ &=\; s(u(t,a))\cdot u(t,a) - s(u(t,b))\cdot u(t,b) \;=\; -\int_a^b [s(u)\,u]_x \,dx\,. \end{aligned} \tag{1.4}$$

Since (1.4) holds for all a, b, this leads to the conservation law

$$u_t + [s(u)\,u]_x = 0,$$

where u is the conserved quantity and $f(u) = s(u)u$ is the flux function.

1.1 Strictly hyperbolic systems

The main object of our study will be the $n \times n$ *system of conservation laws*

$$\begin{cases} \dfrac{\partial}{\partial t}u_1 + \dfrac{\partial}{\partial x}[f_1(u_1,\ldots,u_n)] & = 0, \\ \qquad \cdot \quad \cdot \quad \cdot & \\ \dfrac{\partial}{\partial t}u_n + \dfrac{\partial}{\partial x}[f_n(u_1,\ldots,u_n)] & = 0. \end{cases} \tag{1.5}$$

For simplicity, this will still be written in the form (1.1), but keeping in mind that now $u = (u_1, \ldots, u_n)$ is a vector in $\mathbb{R}^n$ and that $f = (f_1, \ldots, f_n)$ is a map from $\mathbb{R}^n$ into $\mathbb{R}^n$. Calling $A(u) \doteq Df(u)$ the $n \times n$ Jacobian matrix of the map f at the point u, the system (1.5) can be written in the quasilinear form

$$u_t + A(u)u_x = 0. \tag{1.6}$$

We say that the above system is *strictly hyperbolic* if every matrix $A(u)$ has n real, distinct eigenvalues: $\lambda_1(u) < \cdots < \lambda_n(u)$. In this case, one can find bases of left and right eigenvectors of $A(u)$, denoted by $l_1(u), \ldots, l_n(u)$ and $r_1(u), \ldots, r_n(u)$, with

$$l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Example 2 (Gas Dynamics). The Euler equations describing the evolution of a non viscous gas take the form

$$\begin{cases} \rho_t + (\rho v)_x & = 0 \qquad \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x & = 0 \qquad \text{(conservation of momentum)} \\ (\rho E)_t + (\rho E v + p v)_x & = 0 \qquad \text{(conservation of energy)} \end{cases}$$

Here ρ is the mass density, v is the velocity while $E = e + v^2/2$ is the energy density per unit mass. The system is closed by a *constitutive relation* of the form $p = p(\rho, e)$, giving the pressure as a function of the density and the internal energy. The particular form of p depends on the gas under consideration.

1.2 Linear systems

We consider here two elementary cases where the solution of the Cauchy problem can be written explicitly.

The linear homogeneous scalar Cauchy problem with constant coefficients has the form

$$u_t + \lambda u_x = 0, \qquad\qquad u(0,x) = \bar{u}(x), \tag{1.7}$$

with $\lambda \in \mathbb{R}$. If $\bar{u} \in \mathcal{C}^1$, one easily checks that the travelling wave

$$u(t,x) = \bar{u}(x - \lambda t) \tag{1.8}$$

provides a classical solution to (1.7). In the case where the initial condition $\bar{u}$ is not differentiable and we only have $\bar{u} \in \mathbf{L}^1_{loc}$, the function u defined by (1.8) can still be interpreted as a solution, in distributional sense.

Next, consider the homogeneous system with constant coefficients

$$u_t + Au_x = 0, \qquad\qquad u(0,x) = \bar{u}(x), \tag{1.9}$$

where A is a $n \times n$ hyperbolic matrix, with real eigenvalues $\lambda_1 < \cdots < \lambda_n$ and right and left eigenvectors $r_i,\ l_i$, chosen so that

$$l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

Call $u_i \doteq l_i \cdot u$ the coordinates of a vector $u \in I\!R^n$ w.r.t. the basis of right eigenvectors $\{r_1, \cdots, r_n\}$. Multiplying (1.9) on the left by $l_1, \ldots, l_n$ we obtain

$$(u_i)_t + \lambda_i (u_i)_x \ = \ (l_i u)_t + \lambda_i (l_i u)_x \ = \ l_i u_t + l_i A u_x \ = \ 0, \qquad\qquad u_i(0,x) \ = \ l_i \bar{u}(x) \ \doteq \ \bar{u}_i(x).$$

Therefore, (1.9) decouples into n scalar Cauchy problems, which can be solved separately in the same way as (1.7). The function

$$u(t,x) \ = \ \sum_{i=1}^{n} \bar{u}_i(x - \lambda_i t)\, r_i \tag{1.10}$$

now provides the explicit solution to (1.9), because

$$u_t(t,x) \ = \ \sum_{i=1}^{n} -\lambda_i (l_i \cdot \bar{u}_x(x - \lambda_i t)) r_i \ = \ -\, A u_x(t,x).$$

Observe that in the scalar case (1.7) the initial profile is shifted with constant speed λ. For the system (1.9), the initial profile is decomposed as a sum of n waves, each travelling with one of the characteristic speeds $\lambda_1, \ldots, \lambda_n$.

As a special case, consider the Riemann initial data

$$\bar{u}(x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0. \end{cases}$$

The corresponding solution (1.10) can then be obtained as follows.

Write the vector $u^+ - u^-$ as a linear combination of eigenvectors of A, i.e.

$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j.$$

Define the intermediate states

$$\omega_i \doteq u^- + \sum_{j \leq i} c_j r_j, \qquad\qquad i = 0, \ldots, n,$$

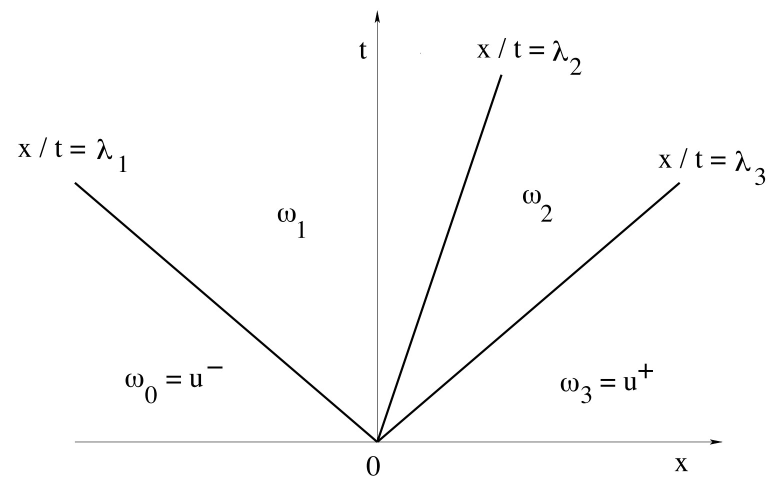

Figure 2: Solution of the Riemann problem for a linear system.

so that each difference $\omega_i - \omega_{i-1}$ is an i-eigenvector of A. The solution then takes the form (fig. 2):

$$u(t,x) = \begin{cases} \omega_0 = u^- & \text{for} \quad x/t \ < \lambda_1, \\ & \quad \dots \\ \omega_i & \text{for} \quad \lambda_i < x/t < \lambda_{i+1}, \\ & \quad \dots \\ \omega_n = u^+ & \text{for} \quad x/t \ > \lambda_n. \end{cases} \tag{1.11}$$

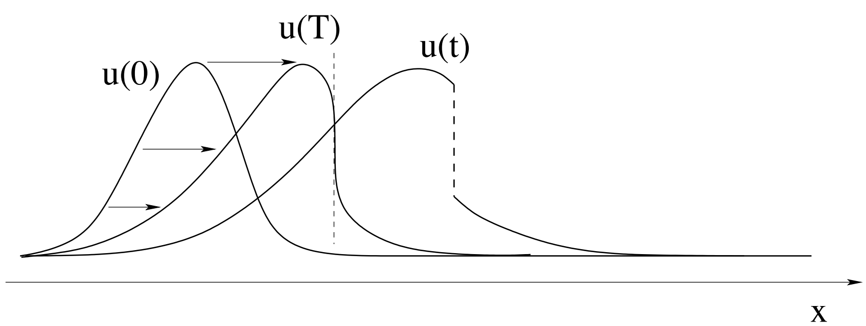

Figure 3: If the wave propagation speed depends on u, the profile of the solution changes in time, eventually leading to shock formation.

In the general nonlinear case (1.6) where the matrix A depends on the state u, new features will appear in the solutions.

(i) Since the eigenvalues λ_i now depend on u, the shape of the various components in the solution will vary in time (fig. 3). Rarefaction waves will decay, and compression waves will become steeper, possibly leading to shock formation in finite time.

(ii) Since the eigenvectors r_i also depend on u, nontrivial interactions between different waves will occur (fig. 4). The strength of the interacting waves may change, and new waves of different families can be created, as a result of the interaction.

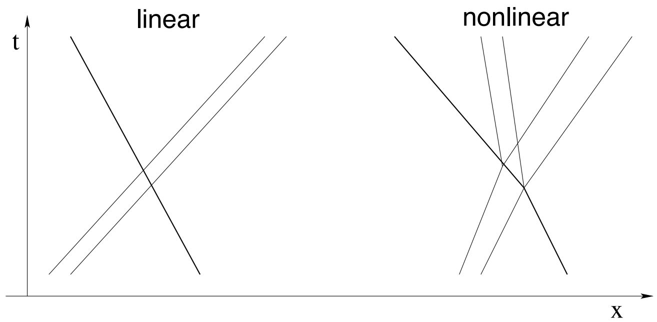

Figure 4: Left: for the linear hyperbolic system (1.9), the solution is a simple superposition of traveling waves. Right: For the non-linear system (1.5), waves of different families have nontrivial interactions.

The strong nonlinearity of the equations and the lack of regularity of solutions, also due to the absence of second order terms that could provide a smoothing effect, account for most of the difficulties encountered in a rigorous mathematical analysis of the system (1.1). It is well known that the main techniques of abstract functional analysis do not apply in this context. Solutions cannot be represented as fixed points of continuous transformations, or in variational form, as critical points of suitable functionals. Dealing with vector valued functions, comparison principles based on upper or lower solutions cannot be used. Moreover, the theory of accretive operators and contractive nonlinear semigroups works well in the scalar case [20], but does not apply to systems. For the above reasons, the theory of hyperbolic conservation laws has largely developed by *ad hoc* methods, along two main lines.

1. The BV setting, considered by J. Glimm [27]. Solutions are here constructed within a space of functions with bounded variation, controlling the BV norm by a wave interaction functional.

2. The $\mathbf{L}^\infty$ setting, considered by L. Tartar and R. DiPerna [24], based on weak convergence and a compensated compactness argument.

Both approaches yield results on the global existence of weak solutions. However, the method of compensated compactness appears to be suitable only for 2×2 systems. Moreover, it is only in the BV setting that the well-posedness of the Cauchy problem could recently be proved, as well as the stability and convergence of vanishing viscosity approximations. Throughout the following we thus restrict ourselves to the study of BV solutions, referring to [24] or [43, 48] for the alternative approach based on compensated compactness.

1.3 Loss of regularity

A basic feature of nonlinear systems of the form (1.1) is that, even for smooth initial data, the solution of the Cauchy problem may develop discontinuities in finite time. To achieve a global existence result, it is thus essential to work within a class of discontinuous functions, interpreting the equations (1.1) in their distributional sense (1.3).

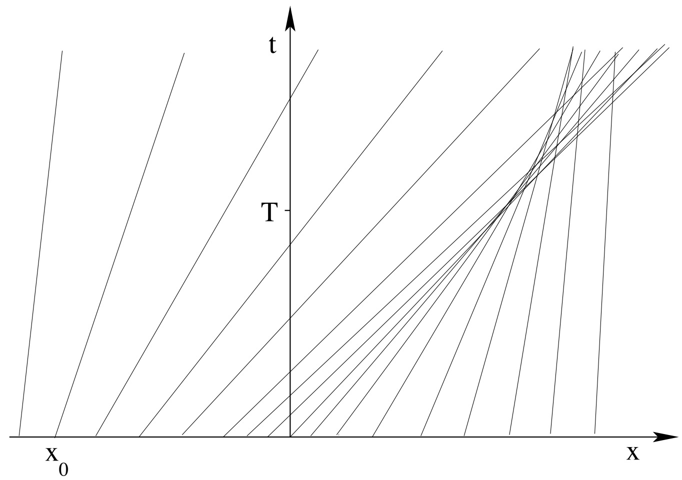

Figure 5: At time T when characteristics start to intersect, a shock is produced.

Example 3. Consider the scalar conservation law (inviscid Burgers' equation)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{1.12}$$

with initial condition

$$u(0,x) = \bar{u}(x) = \frac{1}{1+x^2}.$$

For $t > 0$ small the solution can be found by the method of characteristics. Indeed, if u is smooth, (1.12) is equivalent to

$$u_t + uu_x = 0. \tag{1.13}$$

By (1.13) the directional derivative of the function $u = u(t,x)$ along the vector $(1,u)$ vanishes. Therefore, u must be constant along the characteristic lines in the t-x plane:

$$t \mapsto (t,\ x + t\bar{u}(x)) = \left(t,\ x + \frac{t}{1+x^2}\right).$$

For $t < T \doteq 8/\sqrt{27}$, these lines do not intersect (fig. 5). The solution to our Cauchy problem is thus given implicitly by

$$u\left(t,\ x + \frac{t}{1+x^2}\right) = \frac{1}{1+x^2}. \tag{1.14}$$

On the other hand, when $t > 8/\sqrt{27}$, the characteristic lines start to intersect. As a result, the map

$$x \mapsto x + \frac{t}{1+x^2}$$

is not one-to-one and (1.14) no longer defines a single valued solution of our Cauchy problem.

An alternative point of view is the following (fig. 3). As time increases, points on the graph of $u(t,\cdot)$ move horizontally with speed u, equal to their distance from the x-axis. This determines a change in the profile of the solution. As t approaches the critical time $T \doteq 8/\sqrt{27}$, one has

$$\lim_{t \to T-} \left\{ \inf_{x \in I\!\!R} u_x(t,x) \right\} = -\infty,$$

and no classical solution exists beyond time T. The solution can be prolonged for all times $t \geq 0$ only within a class discontinuous functions.

2 Weak Solutions

2.1 Basic Definitions

A basic feature of nonlinear hyperbolic systems is the possible loss of regularity: solutions which are initially smooth may become discontinuous within finite time. In order to construct solutions globally in time, we are thus forced to work in a space of discontinuous functions, and interpret the conservation equations in a distributional sense.

Definition 2.1. Let $f : I\!\!R^n \mapsto I\!\!R^n$ be a smooth vector field. A measurable function $u = u(t,x)$, defined on an open set $\Omega \subseteq I\!\!R \times I\!\!R$ and with values in $I\!\!R^n$, is a *distributional solution* of the system of conservation laws

$$u_t + f(u)_x = 0 \tag{2.1}$$

if, for every $\mathcal{C}^1$ function $\phi : \Omega \mapsto I\!\!R^n$ with compact support, one has

$$\iint_\Omega \{u\, \phi_t + f(u)\, \phi_x\}\, dxdt = 0. \tag{2.2}$$

Observe that no continuity assumption is made on u. We only require u and $f(u)$ to be locally integrable in Ω. Notice also that weak solutions are defined up to $\mathbf{L}^1$ equivalence. A solution is not affected by changing its values on a set of measure zero in the t-x plane.

An easy consequence of the above definition is the closure of the set of solutions w.r.t. convergence in $\mathbf{L}^1_{\text{loc}}$.

Lemma 1. *Let $(u_m)_{m \geq 1}$ be a uniformly bounded sequence of distributional solutions of (2.1). If $u_m \to u$ in $\mathbf{L}^1_{\text{loc}}$, then the limit function u is itself a distributional solution.*

Indeed, the assumption of uniform boundedness implies $f(u_m) \to f(u)$ in $\mathbf{L}^1_{\text{loc}}$. For every $\phi \in \mathcal{C}^1_c$ we now have

$$\iint_\Omega \{u\, \phi_t + f(u)\, \phi_x\}\, dxdt = \lim_{m \to \infty} \iint_\Omega \{u_m\, \phi_t + f(u_m)\, \phi_x\}\, dxdt = 0.$$

□

In the following, we shall be mainly interested in solutions defined on a strip $[0,T] \times \mathbb{R}$, with an assigned initial condition

$$u(0,x) = \bar{u}(x). \tag{2.3}$$

Here $\bar{u} \in \mathbf{L}^1_{\rm loc}(\mathbb{R})$. To treat the initial value problem, it is convenient to require some additional regularity w.r.t. time.

Definition 2.2. A function $u : [0,T] \times \mathbb{R} \mapsto \mathbb{R}^n$ is a *weak solution* of the Cauchy problem (2.1), (2.3) if u is continuous as a function from $[0,T]$ into $\mathbf{L}^1_{\rm loc}$, the initial condition (2.3) holds and the restriction of u to the open strip $]0,T[\times\mathbb{R}$ is a distributional solution of (2.1).

2.2 Rankine-Hugoniot conditions

Let u be a weak solution of (2.1). If u is continuously differentiable restricted to an open domain Ω', then at every point $(t,x) \in \Omega'$, the function u must satisfy the quasilinear system

$$u_t + A(u)u_x = 0\,, \tag{2.4}$$

with $A(u) \doteq Df(u)$. Indeed, from (2.2) an integration by parts yields

$$\iint [u_t + A(u)u_x]\,\phi\,dxdt = 0.$$

Since this holds for every $\phi \in \mathcal{C}^1_c(\Omega')$, the identity (2.4) follows.

Next, we look at a discontinuous solution and derive some conditions which must be satisfied at points of jump. Consider first the simple case of a piecewise constant function, say

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t, \\ u^- & \text{if} \quad x < \lambda t, \end{cases} \tag{2.5}$$

for some $u^-, u^+ \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

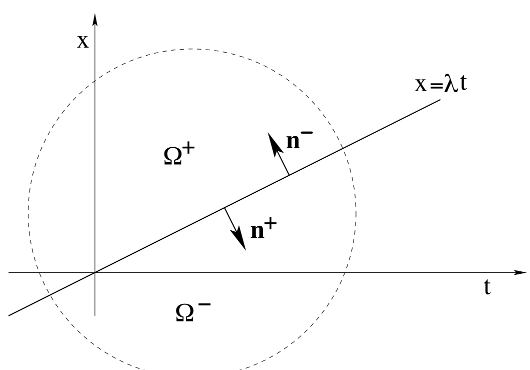

Figure 6: Deriving the Rankine-Hugoniot equations.

Lemma 2. *If the function U in (2.5) is a weak solution of the system of conservation laws (2.1), then*

$$\lambda\,(u^+ - u^-) = f(u^+) - f(u^-). \tag{2.6}$$

Proof. Let $\phi = \phi(t,x)$ be any continuously differentiable function with compact support. Let Ω be a ball containing the support of ϕ and consider the two domains

$$\Omega^+ \doteq \Omega \cap \{x > \lambda t\}, \qquad\qquad \Omega^- \doteq \Omega \cap \{x < \lambda t\},$$

as in fig. 6. Introducing the vector field $\mathbf{v} \doteq (U\phi,\ f(U)\phi)$, the identity (2.2) can be rewritten as

$$\iint_{\Omega^+ \cup \Omega^-} \operatorname{div} \mathbf{v}\ dxdt \ = \ 0. \tag{2.7}$$

We now apply the divergence theorem separately on the two domains Ω^+, Ω^-. Call $\mathbf{n}^+, \mathbf{n}^-$ the outer unit normals to Ω^+, Ω^-, respectively. Observe that $\phi = 0$ on the boundary $\partial\Omega$. Denoting by ds the differential of the arc-length, along the line $x = \lambda t$ we have

$$\mathbf{n}^+ \, ds = (\lambda,\ \ -1)\, dt \qquad\qquad \mathbf{n}^- \, ds = (-\lambda,\ 1)\, dt\,,$$

$$\begin{aligned} 0 \ &= \ \iint_{\Omega^+ \cup \Omega^-} \operatorname{div} \mathbf{v}\, dxdt \ = \ \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds \\ &= \ \int [\lambda u^+ - f(u^+)]\, \phi(t, \lambda t)\, dt + \int [-\lambda u^- + f(u^-)]\, \phi(t, \lambda t)\, dt\,. \end{aligned}$$

Therefore, the identity

$$\int [\lambda(u^+ - u^-) - f(u^+) + f(u^-)]\, \phi(t, \lambda t)\, dt \ = \ 0$$

must hold for every function $\phi \in \mathcal{C}^1_c$. This implies (2.6). □

The vector equations (2.6) are the famous *Rankine-Hugoniot conditions*. They form a set of n scalar equations relating the right and left states $u^+, u^- \in \mathbb{R}^n$ and the speed λ of the discontinuity. An alternative way of writing these conditions is as follows. Denote by $A(u) = Df(u)$ the $n \times n$ Jacobian matrix of f at u. For any $u, v \in \mathbb{R}^n$, define the averaged matrix

$$A(u, v) \doteq \int_0^1 A(\theta u + (1-\theta) v)\ d\theta \tag{2.8}$$

and call $\lambda_i(u, v)$, $i = 1, \ldots, n$, its eigenvalues. We can then write (2.6) in the equivalent form

$$\lambda\,(u^+ - u^-) = f(u^+) - f(u^-) = \int_0^1 Df(\theta u^+ + (1-\theta) u^-) \cdot (u^+ - u^-)\ d\theta = A(u^+, u^-) \cdot (u^+ - u^-). \tag{2.9}$$

In other words, the Rankine-Hugoniot conditions hold if and only if the jump $u^+ - u^-$ is an eigenvector of the averaged matrix $A(u^+, u^-)$ and the speed λ coincides with the corresponding eigenvalue.

Remark 1. In the scalar case, one arbitrarily assign the left and right states $u^-, u^+ \in \mathbb{R}$ and determine the shock speed as

$$\lambda \ = \ \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,. \tag{2.10}$$

Geometrically, this means that the shock speed is the slope of the secant line through the points $(u^-,\ f(u^-))$ and $(u^+,\ f(u^+))$ on the graph of the flux function f.

We now consider a more general solution $u = u(t,x)$ of (2.1) and show that the Rankine-Hugoniot equations are still satisfied at every point (τ, ξ) where u has an approximate jump, in the following sense.

Definition 2.3. We say that a function $u = u(t,x)$ has an *approximate jump discontinuity* at the point (τ, ξ) if there exists vectors $u^+ \neq u^-$ and a speed λ such that, defining U as in (2.5), there holds

$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r} \int_{-r}^{r} |u(\tau + t,\, \xi + x) - U(t,x)|\, dxdt \;=\; 0. \tag{2.11}$$

Moreover, we say that u is *approximately continuous* at the point (τ, ξ) if the above relations hold with $u^+ = u^-$ (and λ arbitrary).

Observe that the above definitions depend only on the $\mathbf{L}^1$ equivalence class of u. Indeed, the limit in (2.11) is unaffected if the values of u are changed on a set $\mathcal{N} \subset I\!R^2$ of Lebesgue measure zero.

Example 4. Let $g^-, g^+ : I\!R^2 \mapsto I\!R^n$ be any two continuous functions and let $x = \gamma(t)$ be a smooth curve, with derivative $\dot\gamma(t) \doteq \frac{d}{dt}\gamma(t)$. Define the function

$$u(t,x) \doteq \begin{cases} g^-(t,x) & \text{if} \quad x < \gamma(t), \\ g^+(t,x) & \text{if} \quad x > \gamma(t). \end{cases}$$

At a point (τ, ξ), with $\xi = \gamma(\tau)$, call $u^- \doteq g^-(\tau, \xi)$, $u^+ \doteq g^+(\tau, \xi)$. If $u^+ = u^-$, then u is continuous at (τ, ξ), hence also approximately continuous. On the other hand, if $u^+ \neq u^-$, then u has an approximate jump at (τ, ξ). Indeed, the limit (2.11) holds with $\lambda = \dot\gamma(\tau)$ and U as in (2.5).

We now prove the Rankine-Hugoniot conditions in the more general case of a point of approximate jump.

Theorem 1. *Let u be a bounded distributional solution of (2.1) having an approximate jump at a point (τ, ξ). In other words, assume that (2.11) holds, for some states u^-, u^+ and a speed λ, with U as in (2.5). Then the Rankine-Hugoniot equations (2.6) hold.*

Proof. For any given $\theta > 0$, one easily checks that the rescaled function

$$u^\theta(t,x) \doteq u(\tau + \theta t,\ \xi + \theta x)$$

is also a solution to the system of conservation laws. We claim that, as $\theta \to 0$, the convergence $u^\theta \to U$ holds in $\mathbf{L}^1_{\text{loc}}(I\!R^2;\ I\!R^n)$. Indeed, for any $R > 0$ one has

$$\lim_{\theta\to 0} \int_{-R}^{R} \int_{-R}^{R} |u^\theta(t,x) - U(t,x)|\, dxdt = \lim_{\theta\to 0} \frac{1}{\theta^2} \int_{-\theta R}^{\theta R} \int_{-\theta R}^{\theta R} |u(\tau + t,\ \xi + x) - U(t,x)|\, dxdt = 0$$

because of (2.11). Lemma 1 now implies that U itself is a distributional solution of (2.1), hence by Lemma 2 the Rankine-Hugoniot equations (2.6) hold. □

2.3 Admissibility conditions

To motivate the following discussion, we first observe that the concept of weak solution is usually not stringent enough to achieve uniqueness for a Cauchy problem. In some cases, infinitely many weak solutions can be found.

Example 5. For Burgers' equation

$$u_t + (u^2/2)_x = 0\,, \tag{2.12}$$

consider the Cauchy problem with initial data

$$u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

For every $0 < \alpha < 1$, a weak solution is

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2, \\ \alpha & \text{if} \quad \alpha t/2 \leq x < (1+\alpha)t/2, \\ 1 & \text{if} \quad x \geq (1+\alpha)t/2\,. \end{cases} \tag{2.13}$$

Indeed, the piecewise constant function u_α trivially satisfies the equation outside the jumps. Moreover, the Rankine-Hugoniot conditions hold along the two lines of discontinuity $\{x = \alpha t/2\}$ and $\{x = (1+\alpha)t/2\}$.

From the previous example it is clear that, in order to achieve the uniqueness of solutions and their continuous dependence on the initial data, the notion of weak solution must be supplemented with further "admissibility conditions", possibly motivated by physical considerations. Some of these conditions will be presently discussed.

Admissibility Condition 1 (Vanishing viscosity). A weak solution u of (2.1) is *admissible in the vanishing viscosity sense* if there exists a sequence of smooth solutions u^ε to

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \tag{2.14}$$

which converge to u in $\mathbf{L}^1_{\text{loc}}$ as $\varepsilon \to 0+$.

Unfortunately, it is very difficult to provide uniform estimates on solutions to the parabolic system (2.14) and characterize the corresponding limits as $\varepsilon \to 0+$. From the above condition, however, one can deduce other conditions which can be more easily verified in practice. A standard approach relies on the concept of entropy.

Definition 2.4. A continuously differentiable function $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ is called an *entropy* for the system of conservation laws (2.1), with *entropy flux* $q : \mathbb{R}^n \mapsto \mathbb{R}$, if for all $u \in \mathbb{R}^n$ there holds

$$D\eta(u) \cdot Df(u) = Dq(u). \tag{2.15}$$

For $n \times n$ systems, (2.15) can be regarded as a first order system of n equations for the two scalar variables η, q. When $n \geq 3$, this system is overdetermined. In general, one should thus expect to find solutions only in the case $n \leq 2$. However, there are important physical examples of larger systems which admit a nontrivial entropy function.

An immediate consequence of (2.15) is that, if $u = u(t,x)$ is a $\mathcal{C}^1$ solution of (2.1), then

$$\eta(u)_t + q(u)_x = 0. \tag{2.16}$$

Indeed,

$$\eta(u)_t + q(u)_x \;=\; D\eta(u)u_t + Dq(u)u_x \;=\; D\eta(u)(-Df(u)u_x) + Dq(u)u_x \;=\; 0.$$

In other words, for a smooth solution u, not only the quantities $u_1, \ldots, u_n$ are conserved but the additional conservation law (2.16) holds as well. However one should be aware that, when u is discontinuous, the quantity $\eta(u)$ may not be conserved.

Example 6 Consider Burgers' equation (2.12). Here the flux is $f(u) = u^2/2$. Taking $\eta(u) = u^3$ and $q(u) = (3/4)u^4$, one checks that the equation (2.15) is satisfied. Hence η is an entropy and q is the corresponding entropy flux. We observe that the function

$$u(0,x) \;=\; \begin{cases} 1 & \text{if} \quad x < t/2, \\ 0 & \text{if} \quad x \geq t/2 \end{cases}$$

is a (discontinuous) weak solution of (2.12). However, it does not satisfy (2.16) in distribution sense. Indeed, calling $u^- = 1$, $u^+ = 0$ the left and right states, and $\lambda = 1/2$ the speed of the shock, one has

$$q(u^+) - q(u^-) \;\neq\; \lambda \Big[\eta(u^+) - \eta(u^-)\Big].$$

We now study how a convex entropy behaves in the presence of a small diffusion term. Assume $\eta, q \in \mathcal{C}^2$, with η convex. Multiplying both sides of (2.14) on the left by $D\eta(u^\varepsilon)$ one finds

$$[\eta(u^\varepsilon)]_t + [q(u^\varepsilon)]_x \;=\; \varepsilon D\eta(u^\varepsilon)u^\varepsilon_{xx} \;=\; \varepsilon\Big\{[\eta(u^\varepsilon)]_{xx} - D^2\eta(u^\varepsilon)\cdot(u^\varepsilon_x \otimes u^\varepsilon_x)\Big\}. \tag{2.17}$$

Observe that the last term in (2.17) satisfies

$$D^2\eta(u^\varepsilon)(u^\varepsilon_x \otimes u^\varepsilon_x) \;=\; \sum_{i,j=1}^{n} \frac{\partial^2\eta(u^\varepsilon)}{\partial u_i \partial u_j} \cdot \frac{\partial u^\varepsilon_i}{\partial x}\frac{\partial u^\varepsilon_j}{\partial x} \geq 0,$$

because η is convex, hence its second derivative at any point u^ε is a positive semidefinite quadratic form. Multiplying (2.17) by a nonnegative smooth function φ with compact support and integrating by parts, we thus have

$$\iint \{\eta(u^\varepsilon)\varphi_t + q(u^\varepsilon)\varphi_x\}\, dxdt \;\geq\; -\varepsilon \iint \eta(u^\varepsilon)\varphi_{xx}\, dxdt.$$

If $u^\varepsilon \to u$ in $\mathbf{L}^1$ as $\varepsilon \to 0$, the previous inequality yields

$$\iint \{\eta(u)\varphi_t + q(u)\varphi_x\}\, dxdt \;\geq\; 0 \tag{2.18}$$

whenever $\varphi \in \mathcal{C}^1_c$, $\varphi \geq 0$. The above can be restated by saying that $\eta(u)_t + q(u)_x \leq 0$ in distribution sense. The previous analysis leads to:

Admissibility Condition 2 (Entropy inequality). A weak solution u of (1.1) is *entropy-admissible* if

$$\eta(u)_t + q(u)_x \ \leq \ 0 \tag{2.19}$$

in the sense of distributions, for every pair (η, q), where η is a convex entropy for (2.1) and q is the corresponding entropy flux.

For the piecewise constant function U in (2.5), an application of the divergence theorem shows that $\eta(U)_t + q(U)_x \leq 0$ in distribution if and only if

$$\lambda\,[\eta(u^+) - \eta(u^-)] \ \geq \ q(u^+) - q(u^-). \tag{2.20}$$

More generally, let $u = u(t,x)$ be a bounded function which satisfies the conservation law (2.1). Assume that u has an approximate jump at (τ, ξ), so that (2.11) holds with U as in (2.5). Then, by the rescaling argument used in the proof of Theorem 1, one can show that the inequality (2.20) must again hold.

Of course, the above admissibility condition can be useful only if some nontrivial convex entropy for the system (2.1) is known. In the scalar case where $u \in I\!R$, convex entropies are easy to construct. In particular, for each $k \in I\!R$, consider the functions

$$\eta(u) = |u-k|, \qquad\qquad q(u) \ = \ \mathrm{sgn}(u-k)\cdot(f(u) - f(k)).$$

It is easily checked that η, q are locally Lipschitz continuous and satisfy (2.15) at every $u \neq k$. Although $\eta, q \notin \mathcal{C}^1$, we can still regard η as a convex entropy for (2.1), with entropy flux q. Following Volpert [53], we say that a bounded measurable function u is an *entropic solution* of (2.1) if

$$\iint \Big\{ |u-k|\varphi_t + \mathrm{sgn}(u-k)(f(u) - f(k))\varphi_x \Big\}\, dxdt \ \geq \ 0 \tag{2.21}$$

for every constant $k \in I\!R$ and every $\mathcal{C}^1$ function $\varphi \geq 0$ with compact support.

Based on (2.21), one can show that a shock connecting the left and right states u^-, u^+ is admissible if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \ \geq \ \frac{f(u^+) - f(u^*)}{u^+ - u^*} \tag{2.22}$$

for every $u^* = \alpha u^+ + (1-\alpha)u^-$, with $0 < \alpha < 1$. The above inequality can be interpreted as a stability condition. Indeed, let $u^* \in [u^-, u^+]$ be an intermediate state and consider a slightly perturbed solution (fig. 7), where the shock (u^-, u^+) is decomposed as two separate jumps, (u^-, u^*) and (u^*, u^+), say located at $\gamma^-(t) < \gamma^+(t)$ respectively. By the the Rankine-Hugoniot conditions (2.10), the two sides of (2.22) yield precisely the speeds of these jumps. If the inequality holds, then $\dot\gamma^- \geq \dot\gamma^+$, so that the backward shock travels at least as fast as the forward one. Therefore, the two shocks will not split apart as time increases, and the perturbed solution will remain close to the original solution possessing a single shock.

Observing that the Rankine-Hugoniot speed of a shock in (1.19) is given by the slope of the secant line to the graph of f through the points u^-, u^+, the condition (1.33) holds if and only if for every $\alpha \in [0,1]$ one has

$$\begin{cases} f(\alpha u^+ + (1-\alpha)u^-) \geq \alpha f(u^+) + (1-\alpha)f(u^-) & \text{if} \quad u^- < u^+, \\ f(\alpha u^+ + (1-\alpha)u^-) \leq \alpha f(u^+) + (1-\alpha)f(u^-) & \text{if} \quad u^- > u^+. \end{cases} \tag{2.23}$$

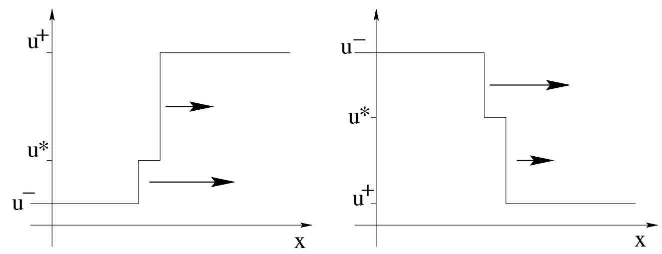

Figure 7: The solution is stable if the speed of the shock behind is greater or equal than the speed of the one ahead.

In other words, when $u^- < u^+$ the graph of f should remain above the secant line (fig. 8, left). When $u^- > u^+$, the graph of f should remain below the secant line (fig. 8, right).

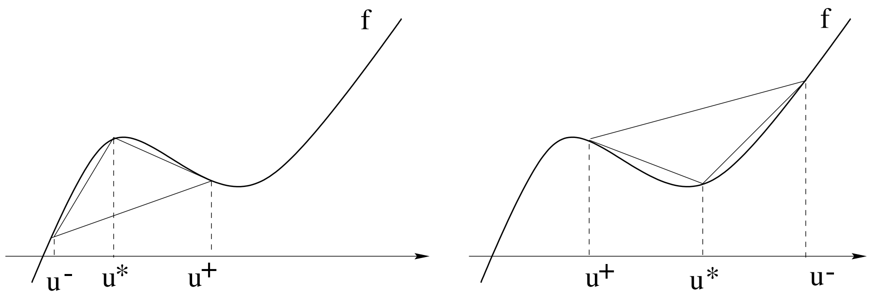

Figure 8: Left: an admissible upward shock. Right: an admissible downward shock.

Remark 2. One can show that the inequality (2.22) holds for every intermediate state $u^* = \alpha u^+ + (1-\alpha)u^-$ if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}. \tag{2.24}$$

In other words, the speed of the shock joining u^- with u^+ must be less or equal to the speed of every smaller shock, joining u^- with an intermediate state $u^* \in [u^-, u^+]$. While the condition (2.22) is meaningful only in the scalar case, the equivalent condition (2.24) admits a natural extension to the vector-valued case. This leads to the *Liu admissibility condition*, which will be discussed in Chapter 3.

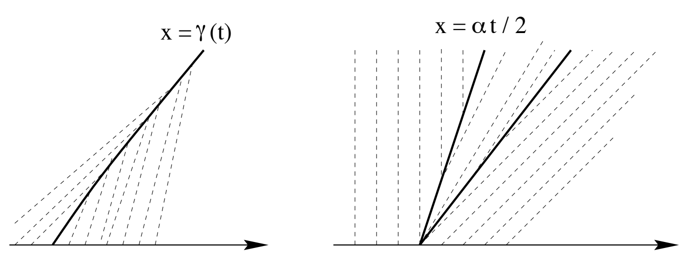

Figure 9: Left: a shock satisfying the Lax condition. Characteristics run toward the shock, from both sides. Right: the two shocks in the weak solution (2.13) violate this condition.

An alternative admissibility condition, due to P. Lax, is particularly useful because it can be applied to any system and has a simple geometrical interpretation. We recall that a function

u has an approximate jump at a point (τ, ξ) if (2.11) holds, for some U defined as in (2.5). In this case, by Theorem 1 the right and left states u^-, u^+ and the speed λ of the jump satisfy the Rankine-Hugoniot equations. In particular, λ must be an eigenvalue of the averaged matrix $A(u^-, u^+)$ defined at (2.8), i.e. $\lambda = \lambda_i(u^-, u^+)$ for some $i \in \{1, \ldots, n\}$.

Admissibility Condition 3 (Lax condition). A solution $u = u(t, x)$ of (2.1) satisfies the *Lax admissibility condition* if, at each point (τ, ξ) of approximate jump, the left and right states u^-, u^+ and the speed $\lambda = \lambda_i(u^-, u^+)$ of the jump satisfy

$$\lambda_i(u^-) \ \geq \ \lambda \ \geq \ \lambda_i(u^+). \tag{2.25}$$

To appreciate the geometric meaning of this condition, consider a piecewise smooth solution, having a discontinuity along the line $x = \gamma(t)$, where the solution jumps from a left state u^- to a right state u^+. According to (2.9), this discontinuity must travel with a speed $\lambda \doteq \dot{\gamma} = \lambda_i(u^-, u^+)$ equal to an eigenvalue of the averaged matrix $A(u^-, u^+)$. If we now look at the *i-characteristics*, i.e. at the solutions of the O.D.E.

$$\dot{x} = \lambda_i(u(t, x)),$$

we see that the Lax condition requires that these lines run into the shock, from both sides.

3 The Riemann Problem

In this chapter we construct the solution to the *Riemann problem*, consisting of the system of conservation laws

$$u_t + f(u)_x = 0 \tag{3.1}$$

with the simple, piecewise constant initial data

$$u(0, x) = \bar{u}(x) \doteq \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0. \end{cases} \tag{3.2}$$

This will provide the basic building block toward the solution of the Cauchy problem with more general initial data. Throughout our analysis, we shall adopt the following standard assumption, introduced by P. Lax [33].

(H) For each $i = 1, \ldots, n$, the i-th field is either *genuinely nonlinear*, so that $D\lambda_i(u) \cdot r_i(u) > 0$ for all u, or *linearly degenerate*, with $D\lambda_i(u) \cdot r_i(u) = 0$ for all u.

Notice that, in the genuinely nonlinear case, the i-th eigenvalue λ_i is strictly increasing along each integral curve of the corresponding field of eigenvectors r_i. In the linearly degenerate case, on the other hand, the eigenvalue λ_i is constant along each such curve. With the above assumption we are ruling out the possibility that, along some integral curve of an eigenvector r_i, the corresponding eigenvalue λ_i may partly increase and partly decrease, having several local maxima and minima.

Example 7 (the "p-system", modelling isentropic gas dynamics). Denote by ρ the density of a gas, by $v = \rho^{-1}$ its specific volume and by u its velocity. A simple model for isentropic gas dynamics (in Lagrangian coordinates) is then provided by the system

$$\begin{cases} v_t - u_x &= 0\,, \\ u_t + p(v)_x &= 0\,. \end{cases} \tag{3.3}$$

Here $p = p(v)$ is a function which determines the pressure in terms of of the specific volume. An appropriate choice is $p(v) = kv^{-\gamma}$, with $1 \leq \gamma \leq 3$. In the region where $v > 0$, the Jacobian matrix of the system is

$$A \doteq Df = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are found to be

$$\lambda_1 \;=\; -\sqrt{-p'(v)}\,, \qquad\qquad \lambda_2 \;=\; \sqrt{-p'(v)}\,, \tag{3.4}$$

$$r_1 \;=\; \begin{pmatrix} 1 \\ \\ \sqrt{-p'(v)} \end{pmatrix}, \qquad\qquad r_2 \;=\; \begin{pmatrix} -1 \\ \\ \sqrt{-p'(v)} \end{pmatrix}. \tag{3.5}$$

It is now clear that the system is strictly hyperbolic provided that $p'(v) < 0$ for all $v > 0$. Moreover, observing that

$$D\lambda_1 \cdot r_1 = \frac{p''(v)}{2\sqrt{-p'(v)}} = D\lambda_2 \cdot r_2\,,$$

we conclude that both characteristic fields are genuinely nonlinear if $p''(v) > 0$ for all $v > 0$.

As we shall see in the sequel, if the assumption (H) holds, then the solution of the Riemann problem has a simple structure consisting of the superposition of n elementary waves: shocks, rarefactions or contact discontinuities. This considerably simplifies all further analysis. On the other hand, for strictly hyperbolic systems that do not satisfy the condition (H), basic existence and stability results can still be obtained but at the price of heavier technicalities.

3.1 Shock and rarefaction waves

Fix a state $u_0 \in \mathbb{R}^n$ and an index $i \in \{1, \ldots, n\}$. As before, let $r_i(u)$ be an i-eigenvector of the Jacobian matrix $A(u) = Df(u)$. The integral curve of the vector field r_i through the point u_0 is called the *i-rarefaction curve* through u_0. It is obtained by solving the Cauchy problem in state space:

$$\frac{du}{d\sigma} \;=\; r_i(u), \qquad\qquad u(0) \;=\; u_0. \tag{3.6}$$

We shall denote this curve as

$$\sigma \mapsto R_i(\sigma)(u_0). \tag{3.7}$$

Clearly, the parametrization depends on the choice of the eigenvectors r_i. In particular, if we impose the normalization $|r_i(u)| \equiv 1$, then the rarefaction curve (3.7) will be parameterized by arc-length.

Next, for a fixed $u_0 \in I\!R^n$ and $i \in \{1, \dots, n\}$, we consider the curve of states u which can be connected to the right of u_0 by an i-shock, satisfying the Rankine-Hugoniot equations

$$\lambda(u - u_0) = f(u) - f(u_0)\,. \tag{3.8}$$

As in (2.9), we can write these equations in the form

$$A(u, u_0)(u - u_0) = \lambda(u - u_0)\,, \tag{3.9}$$

showing that $u - u_0$ must be a right i-eigenvector of the averaged matrix

$$A(u, u_0) \doteq \int_0^1 A(su + (1-s)u_0)\,ds\,.$$

By a theorem of linear algebra, this holds if and only if $u - u_0$ is orthogonal to every left j-eigenvector of $A(u, u_0)$, with $j \neq i$. The Rankine-Hugoniot conditions can thus be written in the equivalent form

$$l_j(u, u_0) \cdot (u - u_0) = 0 \qquad\qquad \text{for all} \quad j \neq i, \tag{3.10}$$

together with $\lambda = \lambda_i(u, u_0)$. Notice that (3.10) is a system of $n-1$ scalar equations in n variables (the n components of the vector u). Linearizing at the point $u = u_0$ one obtains the linear system

$$l_j(u_0) \cdot (w - u_0) = 0 \qquad\qquad j \neq i,$$

whose solutions are all the points $w = u_0 + c r_i(u_0)$, $c \in I\!R$. Since the vectors $l_j(u_0)$ are linearly independent, we can apply the implicit function theorem and conclude that the set of solutions of (3.10) is a smooth curve, tangent to the vector r_i at the point u_0. This will be called the *i-shock curve* through the point u_0 and denoted as

$$\sigma \mapsto S_i(\sigma)(u_0). \tag{3.11}$$

Using a suitable parametrization (say, by arclength), one can show that the two curves R_i, S_i have a second order contact at the point u_0 (fig. 10). More precisely, the following estimates hold.

$$\begin{cases} R_i(\sigma)(u_0) &= u_0 + \sigma r_i(u_0) + \mathcal{O}(1)\cdot\sigma^2, \\ S_i(\sigma)(u_0) &= u_0 + \sigma r_i(u_0) + \mathcal{O}(1)\cdot\sigma^2, \end{cases} \tag{3.12}$$

$$\Big| R_i(\sigma)(u_0) - S_i(\sigma)(u_0) \Big| = \mathcal{O}(1)\cdot\sigma^3, \tag{3.13}$$

$$\lambda_i\Big(S_i(\sigma)(u_0),\ u_0\Big) \;=\; \lambda_i(u_0) + \frac{\sigma}{2}(D\lambda_i(u_0))\cdot r_i(u_0) + \mathcal{O}(1)\cdot\sigma^2. \tag{3.14}$$

Here and throughout the following, the Landau symbol $\mathcal{O}(1)$ denotes a quantity whose absolute value satisfies a uniform bound, depending only on the system (3.1).

3.2 Three special cases

Before constructing the solution of the Riemann problem (3.1)-(3.2) with arbitrary data u^-, u^+, we first study three special cases:

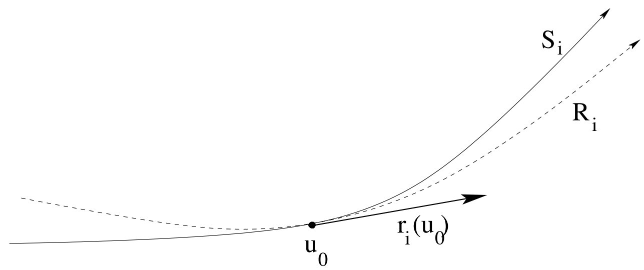

Figure 10: Shock and rarefaction curves.

1. Centered Rarefaction Waves. Let the i-th field be genuinely nonlinear, and assume that u^+ lies on the positive i-rarefaction curve through u^-, i.e. $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$. For each $s \in [0, \sigma]$, define the characteristic speed

$$\lambda_i(s) = \lambda_i(R_i(s)(u^-)).$$

Observe that, by genuine nonlinearity, the map $s \mapsto \lambda_i(s)$ is strictly increasing. Hence, for every $\lambda \in [\lambda_i(u^-),\ \lambda_i(u^+)]$, there is a unique value $s \in [0, \sigma]$ such that $\lambda = \lambda_i(s)$. For $t \geq 0$, we claim that the function

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x/t \ < \ \lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x/t \ = \ \lambda_i(s) \in [\lambda_i(u^-),\ \lambda_i(u^+)], \\ u^+ & \text{if} \quad x/t \ > \ \lambda_i(u^+), \end{cases} \tag{3.15}$$

is a piecewise smooth solution of the Riemann problem, continuous for $t > 0$. Indeed, from the definition it follows

$$\lim_{t \to 0+} \|u(t, \cdot) - \bar{u}\|_{\mathbf{L}^1} = 0.$$

Moreover, the equation (3.1) is trivially satisfied in the sectors where $x < t\lambda_i(u^-)$ or $x > t\lambda_i(u^+)$, because here $u_t = u_x = 0$. Next, assume $x = t\lambda_i(s)$ for some $s \in]0, \sigma[$. Since u is constant along each ray through the origin $\{x/t = c\}$, we have

$$u_t(t,x) + \frac{x}{t}\, u_x(t,x) = 0. \tag{3.16}$$

We now observe that the definition (3.15) implies $x/t = \lambda_i(u(t,x))$. By construction, the vector u_x has the same direction as $r_i(u)$, hence it is an eigenvector of the Jacobian matrix $A(u) \doteq Df(u)$ with eigenvalue $\lambda_i(u)$. On the sector of the t-x plane where $\lambda_i(u^-) < x/t < \lambda_i(u^+)$ we thus have

$$u_t + A(u)u_x = u_t + \lambda_i(u)u_x = 0\,,$$

proving our claim. Notice that the assumption $\sigma > 0$ is essential for the validity of this construction. In the opposite case $\sigma < 0$, the definition (3.15) would yield a triple-valued function in the region where $x/t \in [\lambda_i(u^+)\,,\ \lambda_i(u^-)]$.

2. Shocks. Assume again that the i-th family is genuinely nonlinear and that the state u^+ is connected to the right of u^- by an i-shock, i.e. $u^+ = S_i(\sigma)(u^-)$. Then, calling $\lambda \doteq \lambda_i(u^+, u^-)$ the Rankine-Hugoniot speed of the shock, the function

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases} \tag{3.17}$$

provides a piecewise constant solution to the Riemann problem. Observe that, if $\sigma < 0$, than this solution is entropy admissible in the sense of Lax. Indeed, since the speed is monotonically increasing along the shock curve, recalling (3.14) we have

$$\lambda_i(u^+) < \lambda_i(u^-, u^+) < \lambda_i(u^-). \tag{3.18}$$

Hence the conditions (2.25) hold. In the case $\sigma > 0$, however, one has $\lambda_i(u^-) < \lambda_i(u^+)$ and the admissibility conditions (2.25) are violated.

If η is a strictly convex entropy with flux q, one can also prove that the entropy admissibility conditions (2.19) hold for all $\sigma \leq 0$ small, but fail when $\sigma > 0$.

3. Contact discontinuities. Assume that the i-th field is linearly degenerate and that the state u^+ lies on the i-th rarefaction curve through u^-, i.e. $u^+ = R_i(\sigma)(u^-)$ for some σ. By assumption, the i-th characteristic speed λ_i is constant along this curve. Choosing $\lambda = \lambda(u^-)$, the piecewise constant function (3.17) then provides a solution to our Riemann problem. Indeed, the Rankine-Hugoniot conditions hold at the point of jump:

$$\begin{aligned} f(u^+) - f(u^-) &= \int_0^\sigma Df(R_i(s)(u^-))\, r_i(R_i(s)(u^-))\, ds \\ &= \int_0^\sigma \lambda(u^-)\, r_i(R_i(s)(u^-))\, ds = \lambda_i(u^-) \cdot [R_i(\sigma)(u^-) - u^-]. \end{aligned} \tag{3.19}$$

In this case, the Lax entropy condition holds regardless of the sign of σ. Indeed,

$$\lambda_i(u^+) = \lambda_i(u^-, u^+) = \lambda_i(u^-). \tag{3.20}$$

Moreover, if η is a strictly convex entropy with flux q, one can show that the entropy admissibility conditions (2.19) hold, for all values of σ.

Observe that, according to (3.19), for linearly degenerate fields the shock and rarefaction curves actually coincide, i.e. $S_i(\sigma)(u_0) = R_i(\sigma)(u_0)$ for all σ.

The above results can be summarized as follows. For a fixed left state u^- and $i \in \{1, \ldots, n\}$ define the mixed curve

$$\Psi_i(\sigma)(u^-) = \begin{cases} R_i(\sigma)(u^-) & \text{if} \quad \sigma \geq 0, \\ S_i(\sigma)(u^-) & \text{if} \quad \sigma < 0. \end{cases} \tag{3.21}$$

In the special case where $u^+ = \Psi_i(\sigma)(u^-)$ for some σ, the Riemann problem can then be solved by an elementary wave: a rarefaction, a shock or a contact discontinuity.

Remark 3. The shock admissibility condition, derived in (2.24) for the scalar case, has a natural generalization to the vector valued case. Namely:

Admissibility Condition 4 (Liu condition). Given two states $u^- \in \mathbb{R}^n$ and $u^+ = S_i(\sigma)(u^-)$, the weak solution (3.17) satisfies the T. P. Liu condition if

$$\lambda_i\Big(S_i(\sigma)(u^-),\ u^-\Big) \leq \lambda_i\Big(S_i(s)(u^-),\ u^-\Big) \qquad \text{for all} \quad s \in [0, \sigma]\,. \tag{3.22}$$

In other words, every i-shock joining u^- with some intermediate state $S_i(s)(u^-)$ must travel with speed greater or equal than the speed of our shock (u^-, u^+). A very remarkable fact is that, for general strictly hyperbolic systems (without any reference to the assumptions (H) on genuine nonlinearity or linear degeneracy), the Liu conditions completely characterize vanishing viscosity limits. Namely, a weak solution to (3.1) with small total variation can be obtained as limit of vanishing viscosity approximations if and only if all of its jumps satisfy the Liu admissibility condition.

3.3 General solution of the Riemann problem

Relying on the previous analysis, the solution of the general Riemann problem (3.1)-(3.2) can be obtained by finding intermediate states $\omega_0 = u^-,\ \omega_1, \ldots,\ \omega_n = u^+$ such that each pair of adiacent states ω_{i-1}, ω_i can be connected by an elementary wave, i. e.

$$\omega_i = \Psi_i(\sigma_i)(\omega_{i-1}) \qquad\qquad i = 1, \ldots, n. \tag{3.23}$$

This can be done whenever u^+ is sufficiently close to u^-. Indeed, for $|u^+ - u^-|$ small, the implicit function theorem provides the existence of unique wave strengths $\sigma_1, \ldots \sigma_n$ such that

$$u^+ = \Psi_n(\sigma_n) \circ \cdots \circ \Psi_1(\sigma_1)(u^-). \tag{3.24}$$

In turn, these determine the intermediate states ω_i in (3.23). The complete solution is now obtained by piecing together the solutions of the n Riemann problems

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \begin{cases} \omega_{i-1} & \text{if} \quad x < 0, \\ \omega_i & \text{if} \quad x > 0, \end{cases} \tag{3.25}$$

on different sectors of the t-x plane. By construction, each of these problems has an entropy-admissible solution consisting of a simple wave of the i-th characteristic family. More precisely:

CASE 1: The i-th characteristic field is genuinely nonlinear and $\sigma_i > 0$. Then the solution of (3.25) consists of a centered rarefaction wave. The i-th characteristic speeds range over the interval $[\lambda_i^-,\ \lambda_i^+]$, defined as

$$\lambda_i^- \doteq \lambda_i(\omega_{i-1}), \qquad\qquad \lambda_i^+ \doteq \lambda_i(\omega_i).$$

CASE 2: Either the i-th characteristic field is genuinely nonlinear and $\sigma_i \leq 0$, or else the i-th characteristic field is linearly degenerate (with σ_i arbitrary). Then the solution of (3.25) consists of an admissible shock or a contact discontinuity, travelling with Rankine-Hugoniot speed

$$\lambda_i^- \doteq \lambda_i^+ \doteq \lambda_i(\omega_{i-1}, \omega_i).$$

The solution to the original problem (3.1)-(3.2) can now be constructed (fig. 11) by piecing together the solutions of the n Riemann problems (3.25), $i = 1, \ldots, n$. Indeed, for $\sigma_1, \ldots, \sigma_n$ sufficiently small, the speeds λ_i^-, λ_i^+ introduced above remain close to the corresponding eigenvalues $\lambda_i(u^-)$ of the matrix $A(u^-)$. By strict hyperbolicity and continuity, we can thus assume that the intervals $[\lambda_i^-, \lambda_i^+]$ are disjoint, i.e.

$$\lambda_1^- \leq \lambda_1^+ \ < \ \lambda_2^- \leq \lambda_2^+ \ < \cdots \ < \ \lambda_n^- \leq \lambda_n^+.$$

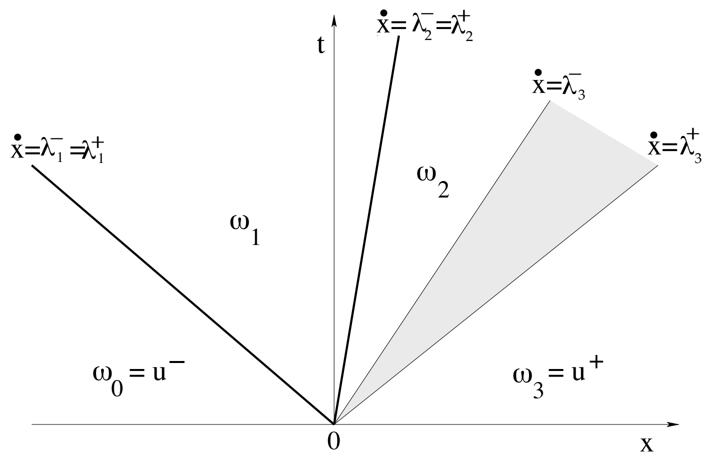

Figure 11: Typical solution of a Riemann problem, with the assumption (H).

Therefore, a piecewise smooth solution $u : [0, \infty) \times I\!R \mapsto I\!R^n$ is well defined by the assignment

$$u(t,x) \;=\; \begin{cases} u^- \;=\; \omega_0 & \text{if} \quad x/t \;\in\;]-\infty,\ \lambda_1^-[\,, \\[2ex] R_i(s)(\omega_{i-1}) & \text{if} \quad x/t \;=\; \lambda_i(R_i(s)(\omega_{i-1})) \in [\lambda_i^-,\ \lambda_i^+[\,, \\[2ex] \omega_i & \text{if} \quad x/t \;\in\; [\lambda_i^+,\ \lambda_{i+1}^-[\,, \\[2ex] u^+ = \omega_n & \text{if} \quad x/t \;\in\; [\lambda_n^+,\ \infty[\,. \end{cases} \tag{3.26}$$

Observe that this solution is self-similar, having the form $u(t,x) = \psi(x/t)$, with $\psi : I\!R \mapsto I\!R^n$ possibly discontinuous.

3.4 Error and interaction estimates

We conclude this section by proving two types of estimates, which will play a key role in the analysis of front tracking approximations.

Fix a left state u^- and consider a right state $u^+ \doteq R_k(\sigma)(u^-)$ on the k-rarefaction curve. In general, shock and rarefaction curves do not coincide; hence the function

$$u(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda_k(u^-)\,, \\ u^+ & \text{if} \quad x > \lambda_k(u^-)\,, \end{cases} \tag{3.27}$$

is not an exact solution of the system (3.1). However, we now show that the error in the Rankine-Hugoniot equations and the possible increase in a convex entropy are indeed very small. In the following, the Landau symbol $\mathcal{O}(1)$ denotes a quantity which remains uniformly bounded as u^-, σ range on bounded sets.

Lemma 3 (error estimates). *For $\sigma > 0$ small, one has the estimate*

$$\lambda_k(u^-)\Big[R_k(\sigma)(u^-) - u^-\Big] - \Big[f(R_k(\sigma)(u^-)) - f(u^-)\Big] = \mathcal{O}(1) \cdot \sigma^2\,. \tag{3.28}$$

Moreover, if η is a convex entropy with entropy flux q, then

$$\lambda_k(u^-)\Big[\eta(R_k(\sigma)(u^-)) - \eta(u^-)\Big] - \Big[q(R_k(\sigma)(u^-)) - q(u^-)\Big] = \mathcal{O}(1)\cdot\sigma^2\,. \tag{3.29}$$

Proof. Call $E(\sigma)$ the left hand side of (3.28). Clearly $E(0) = 0$. Differentiating w.r.t. σ at the point $\sigma = 0$ and recalling that $dR_k/d\sigma = r_k$, we find

$$\frac{dE}{d\sigma}\bigg|_{\sigma=0} = \lambda_k(u^-)r_k(u^-) - Df(u^-)r_k(u^-) = 0\,.$$

Since E varies smoothly with u^- and σ, the estimate (3.28) easily follows by Taylor's formula.

The second estimate is proved similarly. Call $E'(\sigma)$ the left hand side of (3.29) and observe that $E'(0) = 0$. Differentiating w.r.t. σ we now obtain

$$\frac{dE'}{d\sigma}\bigg|_{\sigma=0} = \lambda_k(u^-)D\eta(u^-)r_k(u^-) - Dq(u^-)r_k(u^-) = 0\,,$$

because $D\eta\lambda_k\, r_k = D\eta Df\, r_k = Dq\, r_k$. Taylor's formula now yields (3.29). □

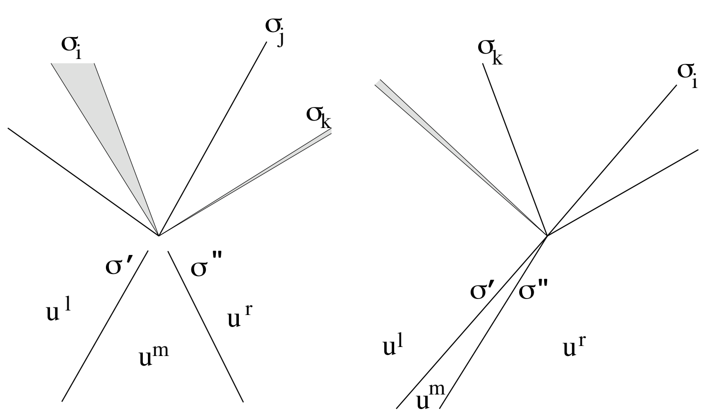

Figure 12: Wave interactions. Strengths of the incoming and outgoing waves.

Next, consider a left state u^l, a middle state u^m and a right state u^r (fig. 12, left). Assume that the pair (u^l, u^m) is connected by a j-wave of strength σ', while the pair (u^l, u^m) is connected by an i-wave of strength σ'', with $i < j$. We are interested in the strength of the waves $(\sigma_1, \ldots, \sigma_n)$ in the solution of the Riemann problem where $u^- = u^l$ and $u^+ = u^r$. Roughly speaking, these are the waves determined by the interaction of the σ' and σ''. The next lemma shows that $\sigma_i \approx \sigma''$, $\sigma_j \approx \sigma'$ while $\sigma_k \approx 0$ for $k \neq i, j$.

A different type of interaction is considered in fig. 12, right. Here the pair (u^l, u^m) is connected by an i-wave of strength σ', while the pair (u^l, u^m) is connected by a second i-wave, say of strength σ''. In this case, the strengths $(\sigma_1, \ldots, \sigma_n)$ of the outgoing waves satisfy $\sigma_i \approx \sigma' + \sigma''$ while $\sigma_k \approx 0$ for $k \neq i$. As usual, $\mathcal{O}(1)$ will denote a quantity which remains uniformly bounded as u^-, σ', σ'' range on bounded sets.

Lemma 4 (interaction estimates). *Consider the Riemann problem (3.1)-(3.2).*

(i) Recalling (3.21), assume that the right state is given by

$$u^+ = \Psi_i(\sigma'') \circ \Psi_j(\sigma')(u^-). \tag{3.30}$$

Let the solution consist of waves of size $(\sigma_1, \ldots, \sigma_n)$, as in (3.24). Then

$$|\sigma_i - \sigma''| + |\sigma_j - \sigma'| \sum_{k \neq i,j} |\sigma_k| = \mathcal{O}(1) \cdot |\sigma'\sigma''|\,. \tag{3.31}$$

(ii) Next, assume that the right state is given by

$$u^+ = \Psi_i(\sigma'') \circ \Psi_i(\sigma')(u^-), \tag{3.32}$$

Then the waves $(\sigma_1, \ldots, \sigma_n)$ in the solution of the Riemann problem can be estimated by

$$|\sigma_i - \sigma' - \sigma''| + \sum_{k \neq i} |\sigma_k| = \mathcal{O}(1) \cdot |\sigma'\sigma''|(|\sigma'| + |\sigma''|)\,. \tag{3.33}$$

Proof. (i) For $u^-, u^+ \in I\!R^n$, $k = 1, \ldots, n$, call $E_k(u^-, u^+) \doteq \sigma_k$ the size of the k-th wave in the solution of the corresponding Riemann problem. By our earlier analysis, the functions E_k are $\mathcal{C}^2$ with Lipschitz continuous second derivatives.

For a given left state u^-, we now define the composed functions Φ_k, $k = 1, \ldots, n$ by setting

$$\begin{aligned}
\Phi_i(\sigma', \sigma'') &\doteq \sigma_i - \sigma'' = E_i\Big(u^-,\ \Psi_j(\sigma') \circ \Psi_i(\sigma'')(u^-)\Big) - \sigma'', \\
\Phi_j(\sigma', \sigma'') &\doteq \sigma_j - \sigma' = E_j\Big(u^-,\ \Psi_j(\sigma') \circ \Psi_i(\sigma'')(u^-)\Big) - \sigma'_j, \\
\Phi_k(\sigma', \sigma'') &\doteq \sigma_k = E_k\Big(u^-,\ \Psi_j(\sigma') \circ \Psi_i(\sigma'')(u^-)\Big) \qquad \text{if } k \neq i, j.
\end{aligned}$$

The Φ_k are $\mathcal{C}^2$ functions of σ', σ'' with Lipschitz continuous second derivatives, depending continuously also on the left state u^-. Observing that

$$\Phi_k(\sigma', 0) = \Phi_k(0, \sigma'') = 0 \qquad \text{for all } \sigma', \sigma'', \quad k = 1, \ldots, n,$$

by Taylor's formula (see Lemma A.2 in the Appendix) we conclude that

$$\Phi_k(\sigma', \sigma'') = \mathcal{O}(1) \cdot |\sigma'\sigma''|$$

for all $k = 1, \ldots, n$. This establishes (3.31).

(ii) To prove (3.33), we consider the functions

$$\begin{aligned}
\Phi_i(\sigma', \sigma'') &\doteq \sigma_i - \sigma' - \sigma'' = E_i\Big(u^-,\ \Psi_i(\sigma'') \circ \Psi_i(\sigma')(u^-)\Big) - \sigma' - \sigma'', \\
\Phi_k(\sigma', \sigma'') &\doteq \sigma_k = E_k\Big(u^-,\ \Psi_i(\sigma'') \circ \Psi_i(\sigma')(u^-)\Big) \qquad \text{if } k \neq i\,.
\end{aligned}$$

In the case where $\sigma', \sigma'' \geq 0$, recalling (3.21) we have

$$u^+ = R_i(\sigma'') \circ R_i(\sigma')(u^-) = R_i(\sigma'' + \sigma')(u^-) .$$

Hence the Riemann problem is solved exactly by an i-rarefaction of strength $\sigma' + \sigma''$. Therefore

$$\Phi_k(\sigma', \sigma'') = 0 \qquad \text{whenever} \quad \sigma',\, \sigma'' \geq 0\,, \tag{3.34}$$

for all $k = 1, \ldots, n$. As before, the Φ_k are $\mathcal{C}^2$ functions of σ', σ'' with Lipschitz continuous second derivatives, depending continuously also on the left state u^-. We observe that

$$\Phi_k(\sigma', 0) \;=\; \Phi_k(0, \sigma'') = 0\,. \tag{3.35}$$

Moreover, by (3.34) the continuity of the second derivatives implies

$$\frac{\partial^2 \Phi_k}{\partial \sigma' \partial \sigma''}(0, 0) = 0\,. \tag{3.36}$$

By (3.35)-(3.36), using the estimate (10.16) in the Appendix we now conclude

$$\Phi_k(\sigma, \sigma') = \mathcal{O}(1) \cdot |\sigma' \sigma''| (|\sigma'| + |\sigma''|)$$

for all $k = 1, \ldots, n$. This establishes (ii). □

3.5 Two examples

Example 8 (chromatography). The 2×2 system of conservation laws

$$[u_1]_t + \left[\frac{u_1}{1 + u_1 + u_2}\right]_x = 0, \qquad [u_2]_t + \left[\frac{u_2}{1 + u_1 + u_2}\right]_x = 0, \qquad u_1, u_2 > 0, \tag{3.37}$$

arises in the study of two-component chromatography. Writing (3.37) in the quasilinear form (1.6), the eigenvalues and eigenvectors of the corresponding 2×2 matrix $A(u)$ are found to be

$$\lambda_1(u) = \frac{1}{(1 + u_1 + u_2)^2}, \qquad \lambda_2(u) = \frac{1}{1 + u_1 + u_2}\,,$$

$$r_1(u) = \frac{-1}{\sqrt{u_1^2 + u_2^2}} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad r_2(u) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The first characteristic field is genuinely nonlinear, the second is linearly degenerate. In this example, the two shock and rarefaction curves S_i, R_i always coincide, for $i = 1, 2$. Their computation is easy, because they are straight lines (fig. 13):

$$R_1(\sigma)(u) = u + \sigma r_1(u), \qquad R_2(\sigma)(u) = u + \sigma r_2(u). \tag{3.38}$$

Observe that the integral curves of the vector field r_1 are precisely the rays through the origin, while the integral curves of r_2 are the lines with slope -1. Now let two states $u^- = (u_1^-, u_2^-)$, $u^+ = (u_1^+, u_2^+)$ be given. To solve the Riemann problem (3.1)-(3.2), we first compute an intermediate state u^* such that $u^* = R_1(\sigma_1)(u^-)$, $\quad u^+ = R_2(\sigma_2)(u^*)$ for some σ_1, σ_2. By (3.38), the components of u^* satisfy

$$u_1^* + u_2^* = u_1^+ + u_2^+, \qquad u_1^* u_2^- = u_1^- u_2^*.$$

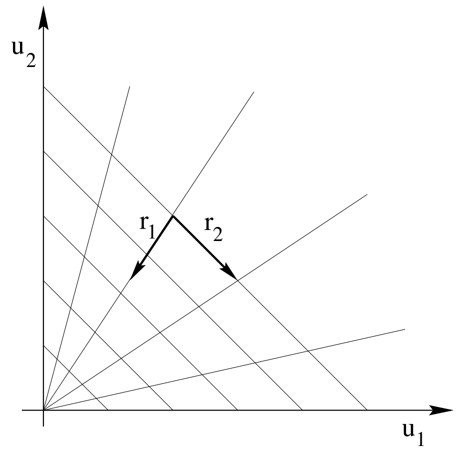

Figure 13: Integral curves of the eigenvectors r_1, r_2, for the system (3.37).

The solution of the Riemann problem thus takes two different forms, depending on the sign of $\sigma_1 = \sqrt{(u_1^-)^2 + (u_2^-)^2} - \sqrt{(u_1^*)^2 + (u_2^*)^2}$.

CASE 1: $\sigma_1 > 0$. Then the solution consists of a centered rarefaction wave of the first family and of a contact discontinuity of the second family:

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x/t < \lambda_1(u^-), \\ su^* + (1-s)u^- & \text{if} \quad x/t = \lambda_1(su^* + (1-s)u^-), \qquad s \in [0,1], \\ u^* & \text{if} \quad \lambda_1(u^*) < x/t < \lambda_2(u^+), \\ u^+ & \text{if} \quad x/t \geq \lambda_2(u^+). \end{cases} \tag{3.39}$$

CASE 2: $\sigma_1 \leq 0$. Then the solution contains a compressive shock of the first family (which vanishes if $\sigma_1 = 0$) and a contact discontinuity of the second family:

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x/t < \lambda_1(u^-, u^*), \\ u^* & \text{if} \quad \lambda_1(u^-, u^*) \leq x/t < \lambda_2(u^+), \\ u^+ & \text{if} \quad x/t \geq \lambda_2(u^+). \end{cases} \tag{3.40}$$

Observe that $\lambda_2(u^*) = \lambda_2(u^+) = (1 + u_1^+ + u_2^+)^{-1}$, because the second characteristic field is linearly degenerate. In this special case, since the integral curves of r_1 are straight lines, the shock speed in (3.40) can be computed as

$$\begin{aligned} \lambda_1(u^-, u^*) &= \int_0^1 \lambda_1(su^* + (1-s)u^-)\, ds \\ &= \int_0^1 \left[1 + s(u_1^* + u_2^*) + (1-s)(u_1^- + u_2^-)\right]^{-2} ds \\ &= \frac{1}{(1 + u_1^* + u_2^*)(1 + u_1^- + u_2^-)}\,. \end{aligned}$$

Example 9 (the p-system). Consider again the equations for isentropic gas dynamics (in

Lagrangian coordinates)

$$\begin{cases} v_t - u_x &= 0\,, \\ u_t + p(v)_x &= 0\,. \end{cases} \tag{3.41}$$

We now study the Riemann problem, for general initial data

$$U(0,x) = \begin{cases} U^- = (v^-, u^-) & \text{if} \quad x < 0, \\ U^+ = (v^+, u^+) & \text{if} \quad x > 0, \end{cases} \tag{3.42}$$

assuming, that $v^-, v^+ > 0$.

By (3.5), the 1-rarefaction curve through U^- is obtained by solving the Cauchy problem

$$\frac{du}{dv} = \sqrt{-p'(v)}, \qquad\qquad u(v^-) = u^-.$$

This yields the curve

$$R_1 = \left\{(v,u); \quad u - u^- = \int_{v^-}^{v} \sqrt{-p'(y)}\; dy\right\}. \tag{3.43}$$

Similarly, the 2-rarefaction curve through the point U^- is

$$R_2 = \left\{(v,u); \quad u - u^- = -\int_{v^-}^{v} \sqrt{-p'(y)}\; dy\right\}. \tag{3.44}$$

Next, the shock curves S_1, S_2 through U^- are derived from the Rankine-Hugoniot conditions

$$\lambda(v - v^-) = -(u - u^-), \qquad\qquad \lambda(u - u^-) = p(v) - p(v^-). \tag{3.45}$$

Using the first equation in (3.45) to eliminate λ, these shock curves are computed as

$$S_1 \;=\; \left\{(v,u); \qquad -(u-u^-)^2 = (v-v^-)(p(v)-p(v^-)), \quad \lambda \doteq -\frac{u-u^-}{v-v^-} < 0\right\}, \tag{3.46}$$

$$S_2 \;=\; \left\{(v,u); \qquad -(u-u^-)^2 = (v-v^-)(p(v)-p(v^-)), \quad \lambda \doteq -\frac{u-u^-}{v-v^-} > 0\right\}. \tag{3.47}$$

Recalling (3.4)-(3.5) and the assumptions $p'(v) < 0$, $p''(v) > 0$, we now compute the directional derivatives

$$(D\lambda_1)r_1 \;=\; (D\lambda_2)r_2 \;=\; \frac{p''(v)}{2\sqrt{-p'(v)}} \;>\; 0. \tag{3.48}$$

By (3.48) it is clear that the Riemann problem (3.41)-(3.42) admits a solution in the form of a centered rarefaction wave in the two cases $U^+ \in R_1$, $v^+ > v^-$, or else $U^+ \in R_2$, $v^+ < v^-$. On the other hand, a shock connecting U^- with U^+ will be admissible provided that either $U^+ \in S_1$ and $v^+ < v^-$, or else $U^+ \in S_2$ and $v^+ > v^-$.

Taking the above admissibility conditions into account, we thus obtain four lines originating from the point $U^- = (v^-, u^-)$, i.e. the two rarefaction curves

$$\sigma \mapsto R_1(\sigma),\ \ R_2(\sigma) \qquad\qquad \sigma \geq 0,$$

and the two shock curves

$$\sigma \mapsto S_1(\sigma),\ S_2(\sigma) \qquad\qquad \sigma \leq 0.$$

In turn, these curves divide a neighborhood of U^- into four regions (fig 14):

Ω_1, bordering on R_1, S_2, $\qquad \Omega_2$, bordering on R_1, R_2,
Ω_3, bordering on S_1, S_2, $\qquad \Omega_4$, bordering on S_1, R_2.

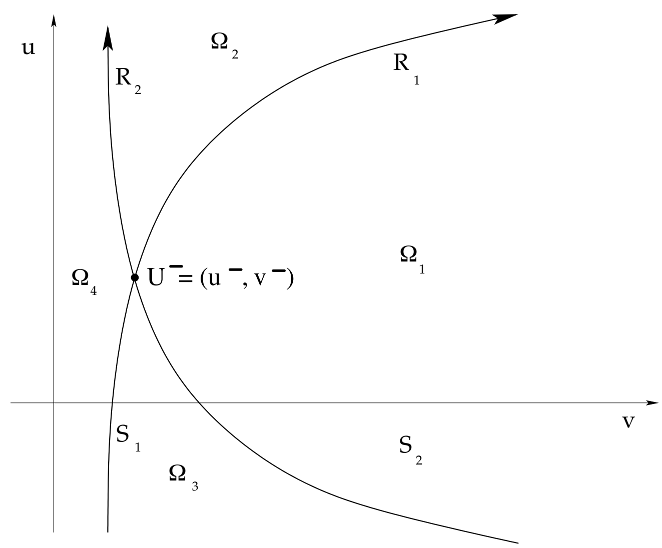

Figure 14: Shocks and rarefaction curves through the point $U^- = (v^-, u^-)$.

For U^+ sufficiently close to U^-, the structure of the general solution to the Riemann problem is now determined by the location of the state U^+, with respect to the curves R_i, S_i (fig 15).

CASE 1: $U^+ \in \Omega_1$. The solution consists of a 1-rarefaction wave and a 2-shock.

CASE 2: $U^+ \in \Omega_2$. The solution consists of two centered rarefaction waves.

CASE 3: $U^+ \in \Omega_3$. The solution consists of two shocks.

CASE 4: $U^+ \in \Omega_4$. The solution consists of a 1-shock and a 2-rarefaction wave.

Remark 4. Consider a 2×2 strictly hyperbolic system of conservation laws. Assume that the i-th characteristic field is genuinely nonlinear. The relative position of the i-shock and the i-rarefaction curve through a point u_0 can be determined as follows (fig. 10). Let $\sigma \mapsto R_i(\sigma)$ be the i-rarefaction curve, parameterized so that $\lambda_i(R_i(\sigma)) = \lambda_i(u_0) + \sigma$. Assume that, for some constant α, the point

$$S_i(\sigma) = R_i(\sigma) + (\alpha\sigma^3 + o(\sigma^3))r_j(u_0) \tag{3.49}$$

lies on the i-shock curve through u_0, for all σ. Here the Landau symbol $o(\sigma^3)$ denotes a higher order infinitesimal, as $\sigma \to 0$. The wedge product of two vectors in $I\!R^2$ is defined as

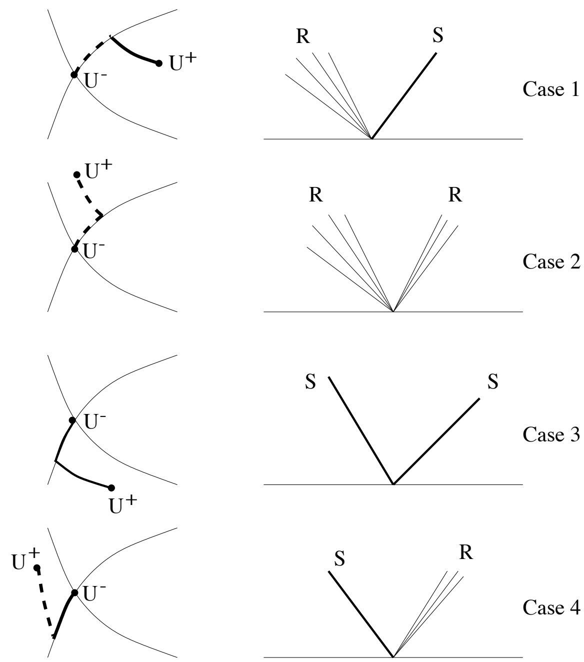

Figure 15: Solution to the Riemann problem for the p-system. The four different cases.

$\begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} c \\ d \end{pmatrix} \doteq ad - bc$. We then have

$$\begin{aligned}\Psi(\sigma) &\doteq \Big[R_i(\sigma) + (\alpha\sigma^3 + o(\sigma^3))r_j(u_0) - u_0\Big] \wedge \Big[f\Big(R_i(\sigma) + (\alpha\sigma^3 + o(\sigma^3))r_j(u_0)\Big) - f(u_0)\Big] \\ &\doteq A(\sigma) \wedge B(\sigma) \;\equiv\; 0\,.\end{aligned}$$

Indeed, the Rankine-Hugoniot equations imply that the vectors $A(\sigma)$ and $B(\sigma)$ are parallel. According to Leibnitz' rule, the fourth derivative is computed by

$$\begin{aligned}\frac{d^4}{d\sigma^4}\Psi &= \left(\frac{d^4}{d\sigma^4}A\right) \wedge B + 4\left(\frac{d^3}{d\sigma^3}A\right) \wedge \left(\frac{d}{d\sigma}B\right) + 6\left(\frac{d^2}{d\sigma^2}A\right) \wedge \left(\frac{d^2}{d\sigma^2}B\right) \\ &\quad +4\left(\frac{d}{d\sigma}A\right) \wedge \left(\frac{d^3}{d\sigma^3}B\right) + A \wedge \left(\frac{d^4}{d\sigma^4}B\right)\end{aligned}$$

By the choice of the parametrization, $\frac{d}{d\sigma}\lambda_i(R_i(\sigma)) \equiv 1$. Hence

$$\begin{aligned}\frac{d}{d\sigma}f(R_i(\sigma)) &= \lambda_i(R_i(\sigma))\,\frac{d}{d\sigma}R_i(\sigma)\,, \\ \frac{d^2}{d\sigma^2}f(R_i(\sigma)) &= \frac{d}{d\sigma}R_i(\sigma) + \lambda_i(R_i(\sigma))\,\frac{d^2}{d\sigma^2}R_i(\sigma)\,, \\ \frac{d^3}{d\sigma^3}f(R_i(\sigma)) &= 2\frac{d^2}{d\sigma^2}R_i(\sigma) + \lambda_i(R_i(\sigma))\,\frac{d^3}{d\sigma^3}R_i(\sigma)\,.\end{aligned}$$

For convenience, we write $r_i \bullet r_j \doteq (Dr_j)r_i$ to denote the directional derivative of r_j in the direction of r_i. At $\sigma = 0$ we have

$$A = B = 0\,, \qquad \frac{d}{d\sigma}R_i = r_i(u_0)\,, \qquad \frac{d^2}{d\sigma^2}R_i = (r_i \bullet r_i)(u_0)\,.$$

Using the above identities and the fact that the wedge product is anti-symmetric, we conclude

$$\begin{aligned}\frac{d^4}{d\sigma^4}\Psi\bigg|_{\sigma=0} &= 4\left(\frac{d^3}{d\sigma^3}R_i+6\alpha r_j\right)\wedge\left(\lambda_i\frac{d}{d\sigma}R_i\right)+6\left(\frac{d^2}{d\sigma^2}R_i\right)\wedge\left(\frac{d}{d\sigma}R_i+\lambda_i\frac{d^2}{d\sigma^2}R_i\right)\\ &\qquad +4\left(\frac{d}{d\sigma}R_i\right)\wedge\left(2\frac{d^2}{d\sigma^2}R_i+\lambda_i\frac{d^3}{d\sigma^3}R_i+6\alpha\lambda_j r_j\right)\\ &= 24\alpha(\lambda_i-\lambda_j)(r_j\wedge r_i)-2(r_i\bullet r_i)\wedge r_i \;=\; 0\,.\end{aligned}$$

The i-shock curve through u_0 is thus traced by points $S_i(\sigma)$ at (3.49), with

$$\alpha \;=\; \frac{(r_i\bullet r_i)\wedge r_i}{12(\lambda_i-\lambda_j)(r_j\wedge r_i)}\,. \tag{3.50}$$

The sign of α in (3.50) gives the position of the i-shock curve, relative to the i-rarefaction curve, near the point u_0. In particular, if $(r_i\bullet r_i)\wedge r_i\neq 0$, it is clear that these two curves do not coincide.

4 Front tracking approximations

4.1 Global existence of entropy weak solutions

In this chapter we describe in detail the construction of front tracking approximations and give the main application of this technique, proving the global existence of weak solutions. Consider the Cauchy problem

$$u_t+f(u)_x=0, \tag{4.1}$$

$$u(0,x)=\bar u(x). \tag{4.2}$$

where the flux function $f:\mathbb{R}^n\mapsto\mathbb{R}^n$ is smooth, defined on a neighborhood of the origin. We always assume that the system is strictly hyperbolic, and that the assumption (H) holds, so that each characteristic field is either genuinely nonlinear or linearly degenerate. The following basic theorem provides the global existence of an entropy weak solution, for all initial data with suitably small total variation.

Theorem 2. *Assume that the system (4.1) is strictly hyperbolic, and that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exists a constant $\delta_0>0$ such that, for every initial condition $\bar u\in\mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ with*

$$\textit{Tot.Var.}\{\bar u\}\leq\delta_0\,, \tag{4.3}$$

the Cauchy problem (4.1)-(4.2) has a weak solution $u=u(t,x)$ defined for all $t\geq 0$. This solution also satisfies the admissibility conditions

$$\eta(u)_t+q(u)_x\leq 0\,, \tag{4.4}$$

for every convex entropy η with entropy flux q.

The above theorem was first proved in fundamental paper of J. Glimm [27], where approximate solutions are constructed by piecing together Riemann solutions on the nodes of a fixed grid. Here we shall describe an alternative construction, based on wave-front tracking. This method was developed in [21], [23], and in [7] respectively for scalar, 2×2, and general $n \times n$ systems. Further versions of this construction can also be found in [3, 30, 47].

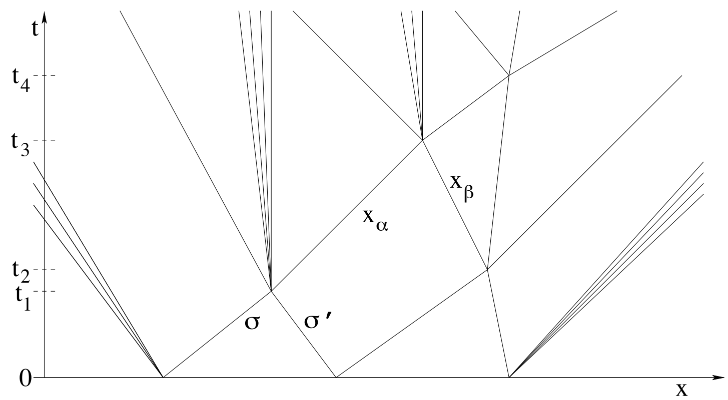

Figure 16: An approximate solution obtained by front tracking.

Roughly speaking, a front tracking ε-approximate solution (fig. 16) is a piecewise constant function $u = u(t,x)$ whose jumps are located along finitely many segments $x = x_\alpha(t)$ in the t-x plane. At any given time $t > 0$, all these jumps should approximately satisfy the Rankine-Hugoniot conditions, namely

$$\sum_\alpha \Big| \dot{x}_\alpha [u(t,x_\alpha+) - u(t,x_\alpha-)] - [f(u(t,x_\alpha+)) - f(u(t,x_\alpha-))] \Big| \;=\; \mathcal{O}(1)\cdot\varepsilon.$$

Moreover, if η is a convex entropy with flux q, at each time $t > 0$ in view of (2.18) we also require the approximate admissibility conditions

$$\sum_\alpha \Big\{ [q(u(t,x_\alpha+)) - q(u(t,x_\alpha-))] - \dot{x}_\alpha [\eta(u(t,x_\alpha+)) - \eta(u(t,x_\alpha-))] \Big\} \;\leq\; \mathcal{O}(1)\cdot\varepsilon.$$

The proof of Theorem 2 will be given in two steps. In the first part of the proof we describe a *naive front tracking algorithm* and derive uniform bounds on the total variation of the approximate solutions. Relying on Helly's compactness theorem, we also prove that a suitable subsequence of front tracking approximations converges to an entropy weak solution. This first step thus contains all the "heart of the matter". It would provide a complete proof, except for one gap: we are not considering here the possibility that infinitely many wave-fronts appear within finite time (fig. 22, left). If this happens, the naive front tracking algorithm breaks down, and solutions cannot be further prolonged in time.

The only purpose of the second part of the proof is to fix this technical problem. We show that a suitable *modified front tracking algorithm* will always produce approximate solutions having a finite number of wave-fronts for all times $t \geq 0$. By extending the previous estimates to this modified construction, one can complete the proof of Theorem 2.

4.2 A naive front tracking algorithm

Let the initial condition $\bar{u}$ be given and fix $\varepsilon > 0$. We now describe an algorithm which produces a piecewise constant approximate solution to the Cauchy problem (4.1)-(4.2). The construction (fig. 16) starts at time $t = 0$ by taking a piecewise constant approximation $u(0, \cdot)$ of $\bar{u}$, such that

$$\text{Tot.Var.}\{u(0,\cdot)\} \leq \text{Tot.Var.}\{\bar{u}\}, \qquad \int |u(0,x) - \bar{u}(x)|\,dx \leq \varepsilon. \tag{4.5}$$

Let $x_1 < \cdots < x_N$ be the points where $u(0, \cdot)$ is discontinuous. For each $\alpha = 1, \ldots, N$, the Riemann problem generated by the jump $(u(0, x_\alpha-),\ u(0, x_\alpha+))$ is approximately solved on a forward neighborhood of $(0, x_\alpha)$ in the t-x plane by a piecewise constant function, according to the following procedure.

Accurate Riemann Solver. Consider the general Riemann problem at a point $(\bar{t}, \bar{x})$,

$$v_t + f(v)_x = 0, \qquad v(\bar{t}, x) = \begin{cases} u^- & \text{if } x < \bar{x}, \\ u^+ & \text{if } x > \bar{x}, \end{cases} \tag{4.6}$$

Recalling (3.21), let $\omega_0, \ldots, \omega_n$ be the intermediate states and $\sigma_1, \ldots, \sigma_n$ be the strengths of the waves in the solution, so that

$$\omega_0 = u^-, \quad \omega_n = u^+, \qquad \omega_i = \Psi_i(\sigma_i)(\omega_{i-1}) \qquad i = 1, \ldots, n. \tag{4.7}$$

If all jumps (ω_{i-1}, ω_i) were shocks or contact discontinuities, then this solution would be already piecewise constant. In general, the exact solution of (4.6) is not piecewise constant, because of the presence of centered rarefaction waves. These will be approximated by piecewise constant rarefaction fans, inserting additional states $\omega_{i,j}$ as follows.

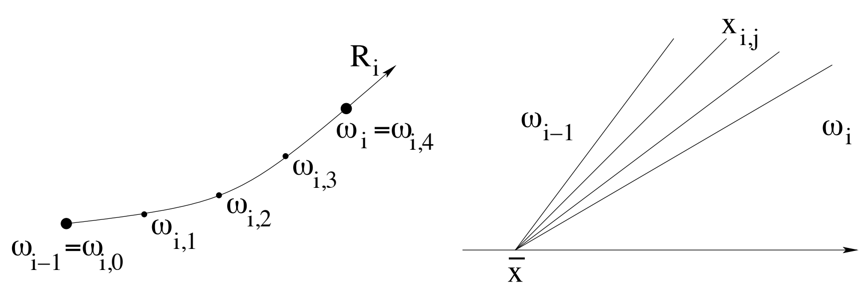

Figure 17: Replacing a centered rarefaction wave by a rarefaction fan.

If the i-th characteristic field is genuinely nonlinear and $\sigma_i > 0$, we divide the centered i-rarefaction into a number p_i of smaller i-waves, each with strength σ_i/p_i. Since we want $\sigma/p_i < \varepsilon$, we choose the integer

$$p_i \doteq 1 + [\![\sigma_i/\varepsilon]\!], \tag{4.8}$$

where $[\![s]\!]$ denotes the largest integer $\leq s$. For $j = 1, \ldots, p_i$, we now define the intermediate states and wave-fronts (fig. 17)

$$\omega_{i,j} = \Psi_i(j\sigma_i/p_i)(\omega_{i-1}), \qquad x_{i,j}(t) = \bar{x} + (t - \bar{t})\lambda_i(\omega_{i,j-1}). \tag{4.9}$$

For notational convenience, if the i-th characteristic field is genuinely nonlinear and $\sigma_i \leq 0$, or if the i-th characteristic field is linearly degenerate (with σ_i arbitrary), we define $p_i \doteq 1$ and set

$$\omega_{i,1} = \omega_i, \qquad x_{i,1}(t) = \bar{x} + (t - \bar{t})\lambda_i(\omega_{i-1}, \omega_i). \tag{4.10}$$

Here $\lambda_i(\omega_{i-1}, \omega_i)$ is the Rankine-Hugoniot speed of the jump connecting ω_{i-1} with ω_i, so that

$$\lambda_i(\omega_{i-1}, \omega_i) \cdot (\omega_i - \omega_{i-1}) = f(\omega_i) - f(\omega_{i-1}). \tag{4.11}$$

As soon as the intermediate states $\omega_{i,j}$ and the locations $x_{i,j}(t)$ of the jumps have been determined by (4.9) or (4.10), we can define a piecewise constant approximate solution to the Riemann problem (4.6) by setting (fig. 18)

$$v(t,x) = \begin{cases} u^- & \text{if } \ x < x_{1,1}(t), \\ u^+ & \text{if } \ x > x_{n,p_n}(t), \\ \omega_i \ \ (= \omega_{i,p_i}) & \text{if } \ x_{i,p_i}(t) < x < x_{i+1,1}(t), \\ \omega_{i,j} & \text{if } \ x_{i,j}(t) < x < x_{i,j+1}(t) \qquad (j = 1, \ldots, p_i - 1). \end{cases} \tag{4.12}$$

We observe that the difference between this function v and the exact solution is only due to the fact that every centered i-rarefaction wave is here divided into equal parts and replaced by a rarefaction fan containing p_i wave-fronts. Because of (4.8), the strength of each one of these fronts is $< \varepsilon$.

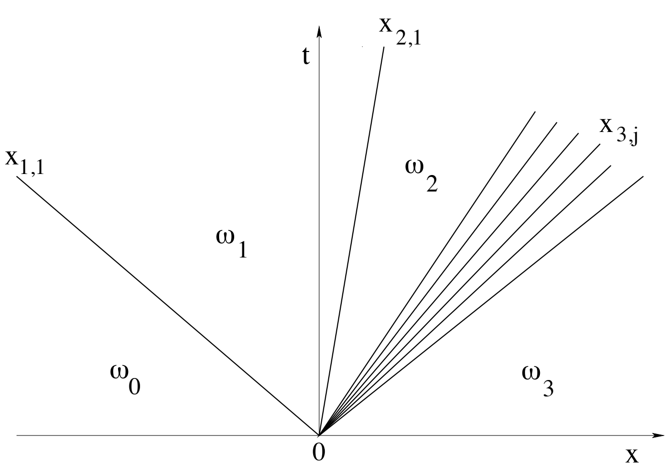

Figure 18: A piecewise constant approximate solution to the Riemann problem.

We now resume the construction of a front tracking solution to the original Cauchy problem (4.1)-(4.2). Having solved all the Riemann problems at time $t = 0$, the approximate solution u can be prolonged until a first time t_1 is reached, when two wave-fronts interact (fig. 16). Since $u(t_1, \cdot)$ is still a piecewise constant function, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions. The solution

u is then continued up to a time t_2 where a second interaction takes place, etc... We remark that, by an arbitrary small change in the speed of one of the wave fronts, it is not restrictive to assume that at most two incoming fronts collide, at each given time $t > 0$. This will considerably simplify all subsequent analysis, since we don't need to consider the case where three or more incoming fronts meet together.

4.3 Bounds on the total variation

In this section we derive bounds on the total variation of a front tracking approximation $u(t, \cdot)$, uniformly valid for all $t \geq 0$. These estimates will be obtained from Lemma 4, using an interaction functional.

We begin by introducing some notation. At a fixed time t, let x_α, $\alpha = 1, \ldots, N$, be the locations of the fronts in $u(t, \cdot)$. Moreover, let $|\sigma_\alpha|$ be the strength of the wave-front at x_α, say of the family $k_\alpha \in \{1, \ldots, n\}$. Consider the two functionals

$$V(t) \doteq V(u(t)) \doteq \sum_\alpha |\sigma_\alpha| \,, \tag{4.13}$$

measuring the *total strength of waves* in $u(t, \cdot)$, and

$$Q(t) \doteq Q(u(t)) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta| \,, \tag{4.14}$$

measuring the *wave interaction potential*. In (4.14), the summation ranges over the set $\mathcal{A}$ of all couples of approaching wave-fronts:

Definition 4.1. Two fronts, located at points $x_\alpha < x_\beta$ and belonging to the characteristic families $k_\alpha, k_\beta \in \{1, \ldots, n\}$ respectively, are *approaching* if $k_\alpha > k_\beta$ or else if $k_\alpha = k_\beta$ and at least one of the wave-fronts is a shock of a genuinely nonlinear family.

Roughly speaking, two fronts are approaching if the one behind has the larger speed (and hence it can collide with the other, at a future time).

Now consider the approximate solution $u = u(t, x)$ constructed by the front tracking algorithm. It is clear that the quantities $V(u(t))$, $Q(u(t))$ remain constant except at times where an interaction occurs. At a time τ where two fronts of strength $|\sigma'|, |\sigma''|$ collide, the interaction estimates (3.31) or (3.33) yield

$$\Delta V(\tau) \;\doteq\; V(\tau+) - V(\tau-) \;=\; \mathcal{O}(1) \cdot |\sigma'\sigma''|, \tag{4.15}$$

$$\Delta Q(\tau) \;\doteq\; Q(\tau+) - Q(\tau-) \;=\; -|\sigma'\sigma''| + \mathcal{O}(1) \cdot |\sigma'\sigma''| \cdot V(\tau-). \tag{4.16}$$

Indeed (fig. 19), after time τ the two colliding fronts σ', σ'' are no longer approaching. Hence the product $|\sigma'\sigma''|$ is no longer counted within the summation (4.14). On the other hand, the new waves emerging from the interaction (having strength $\mathcal{O}(1) \cdot |\sigma'\sigma''|$) can approach all the other fronts not involved in the interaction (which have total strength $\leq V(\tau-)$).

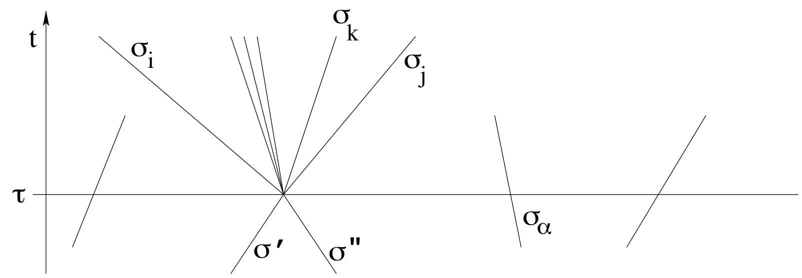

Figure 19: Estimating the change in the total variation at a time where two fronts interact.

If V remains sufficiently small, so that $\mathcal{O}(1) \cdot V(\tau-) \leq 1/2$, from (4.16) it follows

$$Q(\tau+) - Q(\tau-) \leq -\frac{|\sigma'\sigma''|}{2}. \tag{4.17}$$

By (4.15) and (4.17) we can thus choose a constant C_0 large enough so that the quantity

$$\Upsilon(t) \doteq V(t) + C_0 Q(t)$$

decreases at every interaction time, provided that V remains sufficiently small.

We now observe that the total strength of waves is an equivalent way of measuring the total variation. Indeed, for some constant κ one has

$$\text{Tot.Var.}\{u(t)\} \leq V(u(t)) \leq \kappa \cdot \text{Tot.Var.}\{u(t)\}. \tag{4.18}$$

Moreover, the definitions (4.13)-(4.14) trivially imply $Q \leq V^2$. If the total variation of the initial data $u(0, \cdot)$ is sufficiently small, the previous estimates show that the quantity $V + C_0 Q$ is nonincreasing in time. Therefore

$$\text{Tot.Var.}\{u(t)\} \leq V(u(t)) \leq V(u(0)) + C_0 Q(u(0)). \tag{4.19}$$

This provides a uniform bound on the total variation of $u(t, \cdot)$ valid for all times $t \geq 0$.

An important consequence of the bound (4.19) is that, at every time τ where two fronts interact, the corresponding Riemann problem can always be solved. Indeed, the left and right states differ by the quantity

$$|u^+ - u^-| \leq \text{Tot.Var.}\{u(\tau)\},$$

which remains small.

Another consequence of the bound on the total variation is the continuity of $t \mapsto u(t, \cdot)$ as a function with values in $\mathbf{L}^1_{loc}$. More precisely, there exists a Lipschitz constant L' such that

$$\int_{-\infty}^{\infty} |u(t,x) - u(t',x)|\, dx \leq L'|t - t'| \qquad \text{for all} \quad t, t' \geq 0. \tag{4.20}$$

Indeed, if no interaction occurs inside the interval $[t, t']$, the left hand side of (4.20) can be estimated simply as

$$\begin{aligned} \|u(t) - u(t')\|_{\mathbf{L}^1} &\leq |t - t'| \textstyle\sum_\alpha |\sigma_\alpha|\, |\dot{x}_\alpha| \\ &\leq |t - t'| \cdot [\text{total strength of all wave fronts}] \cdot [\text{maximum speed}] \qquad (4.21) \\ &\leq L' \cdot |t - t'|, \end{aligned}$$

for some uniform constant L'. The case where one or more interactions take place within $[t, t']$ is handled in the same way, observing that the map $t \mapsto u(t, \cdot)$ is continuous across interaction times.

4.4 Convergence to a limit solution

Fix any sequence $\varepsilon_\nu \to 0+$. For every $\nu \geq 1$ we apply the front tracking algorithm and construct an ε_ν-approximate solution u_ν of the Cauchy problem (4.1)-(4.2). By the previous analysis, the total variation of $u_\nu(t, \cdot)$ remains bounded, uniformly for all $t \geq 0$ and $\nu \geq 1$. Moreover, by (4.20) the maps $t \mapsto u_\nu(t, \cdot)$ are uniformly Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance. We can thus apply Helly's compactness theorem (see Theorem A.1 in the Appendix) and extract a subsequence which converges to some limit function u in $\mathbf{L}^1_{loc}$, also satisfying (4.20).

By the second relation in (4.5), as $\varepsilon_\nu \to 0$ we have $u_\nu(0) \to \bar{u}$ in $\mathbf{L}^1_{\rm loc}$. Hence the initial condition (4.2) is clearly attained. To prove that u is a weak solution of the Cauchy problem, it remains to show that, for every $\phi \in \mathcal{C}^1_c$ with compact support contained in the open half plane where $t > 0$, one has

$$\int_0^\infty \int_{-\infty}^\infty \phi_t(t,x)u(t,x) + \phi_x(t,x)f(u(t,x))\;dxdt = 0. \tag{4.22}$$

Since the u_ν are uniformly bounded and f is uniformly continuous on bounded sets, it suffices to prove that

$$\lim_{\nu\to 0} \int_0^\infty \int_{-\infty}^\infty \Big\{\phi_t(t,x)u_\nu(t,x) + \phi_x(t,x)f(u_\nu(t,x))\Big\}\;dxdt \;=\; 0. \tag{4.23}$$

Choose $T > 0$ such that $\phi(t,x) = 0$ whenever $t \notin [0,T]$. For a fixed ν, at any time t call $x_1(t) < \cdots < x_N(t)$ the points where $u_\nu(t, \cdot)$ has a jump, and set

$$\Delta u_\nu(t,x_\alpha) \doteq u_\nu(t,x_\alpha+) - u_\nu(t,x_\alpha-), \qquad \Delta f(u_\nu(t,x_\alpha)) \doteq f(u_\nu(t,x_\alpha+)) - f(u_\nu(t,x_\alpha-)).$$

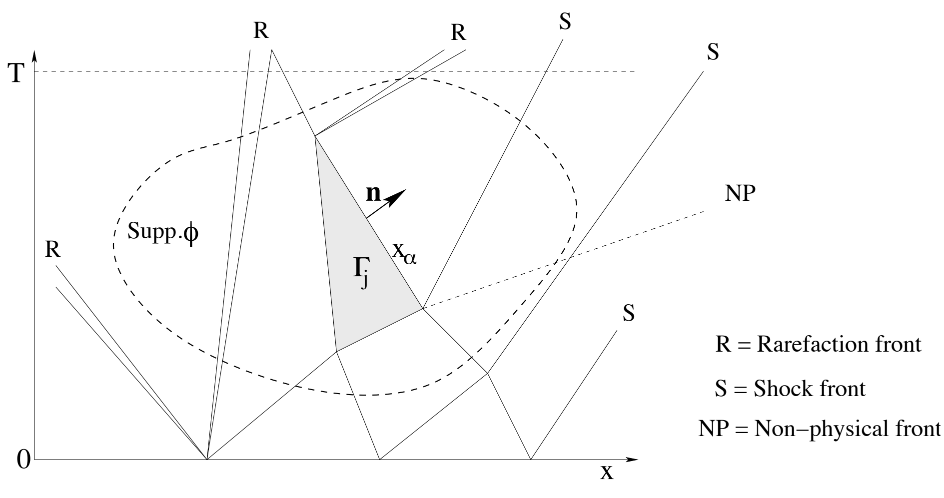

Figure 20: Estimating the error in an approximate solution obtained by front tracking.

Observe that the polygonal lines $x = x_\alpha(t)$ subdivide the strip $[0, T] \times I\!R$ into finitely many regions Γ_j where u_ν is constant (fig. 20). Introducing the vector

$$\Phi \doteq (\phi \cdot u_\nu \ , \ \ \phi \cdot f(u_\nu)),$$

by the divergence theorem the double integral in (4.23) can be written as

$$\sum_j \iint_{\Gamma_j} \text{div } \Phi(t, x) \, dtdx = \sum_j \int_{\partial\Gamma_j} \Phi \cdot \mathbf{n} \, d\sigma. \tag{4.24}$$

Here $\partial\Gamma_j$ is the oriented boundary of Γ_j, while $\mathbf{n}$ denotes an outer normal. Observe that $\mathbf{n} d\sigma = \pm(\dot{x}_\alpha, -1)dt$ along each polygonal line $x = x_\alpha(t)$, while $\phi(t, x) = 0$ along the lines $t = 0, t = T$. By (4.24) the expression within square brackets in (4.23) is computed by

$$\int_0^T \sum_\alpha \Big[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha))\Big] \phi(t, x_\alpha(t)) \ dt \, . \tag{4.25}$$

Here, for each $t \in [0, T]$, the sum ranges over all fronts of $u_\nu(t, \cdot)$. To estimate the above integral, let $|\sigma_\alpha|$ be the strength of the wave-front at x_α. If this wave is a shock or or contact discontinuity, by construction the Rankine-Hugoniot equations are satisfied exactly, i.e.

$$\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha)) = 0. \tag{4.26}$$

On the other hand, if the wave at x_α is a rarefaction front, its strength will satisfy $\sigma_\alpha \in]0, \varepsilon_\nu[$. Therefore, the error estimate (3.28) yields

$$\Big|\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha))\Big| \ = \ \mathcal{O}(1) \cdot |\sigma_\alpha|^2 \ \le \ C \cdot \varepsilon_\nu |\sigma_\alpha| \tag{4.27}$$

for some constant C. Summing over all wave-fronts and recalling that the total strength of waves in $u_\nu(t, \cdot)$ satisfies a uniform bound independent of t, ν, we obtain

$$\begin{aligned} \limsup_{\nu \to \infty} \ & \left|\sum_\alpha \Big[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t, x_\alpha) - \Delta f(u_\nu(t, x_\alpha))\Big] \phi(t, x_\alpha(t))\right| \\ & \le \ \Big(\max_{t,x} |\phi(t, x)|\Big) \cdot \limsup_{\nu \to \infty} \Big\{ \sum_{\alpha \in \mathcal{R}} C \, \varepsilon_\nu |\sigma_\alpha| \Big\} \\ & = \ 0. \end{aligned} \tag{4.28}$$

The limit (4.23) is now a consequence of (4.28). This shows that u is a weak solution to the Cauchy problem.

To conclude the proof, assume that a convex entropy η is given, with entropy flux q. To prove the admissibility condition (4.4) we need to show that

$$\liminf_{\nu \to \infty} \ \int_0^\infty \int_{-\infty}^\infty \Big\{\eta(u_\nu)\varphi_t + q(u_\nu)\varphi_x\Big\} \, dxdt \ \ge \ 0 \tag{4.29}$$

for every nonnegative $\varphi \in \mathcal{C}^1_c$ with compact support contained in the half plane where $t > 0$. Choose $T > 0$ so that φ vanishes outside the strip $[0, T] \times I\!R$. Using again the divergence theorem, for every ν the double integral in (4.29) can be computed as

$$\int_0^T \sum_\alpha \Big[\dot{x}_\alpha(t) \cdot \Delta\eta(u_\nu(t, x_\alpha)) - \Delta q(u_\nu(t, x_\alpha))\Big] \varphi(t, x_\alpha) \ dt \tag{4.30}$$

where, for each $t \in [0,T]$, the sum ranges over all jumps of $u_\nu(t,\cdot)$. We use here the notation

$$\Delta\eta \doteq \eta(u_\nu(t,x_\alpha+)) - \eta(u_\nu(t,x_\alpha-)), \qquad \Delta q \doteq q(u_\nu(t,x_\alpha+)) - q(u_\nu(t,x_\alpha-)).$$

To estimate the integral (4.30), let $|\sigma_\alpha|$ be the strength of the wave-front at x_α. If this wave is a shock or or contact discontinuity, by construction it satisfies the entropy condition

$$\dot{x}_\alpha(t)\cdot\Delta\eta(u_\nu(t,x_\alpha)) - \Delta q(u_\nu(t,x_\alpha)) \ \geq\ 0. \tag{4.31}$$

On the other hand, if the wave at x_α is a rarefaction front, its strength will satisfy $\sigma_\alpha \in]0,\varepsilon_\nu[$. Therefore, the estimate (3.29) yields

$$\dot{x}_\alpha(t)\cdot\Delta\eta(u_\nu(t,x_\alpha)) - \Delta q(u_\nu(t,x_\alpha)) \ =\ \mathcal{O}(1)\cdot\sigma_\alpha^2 \ \geq\ -C\cdot\varepsilon_\nu\,|\sigma_\alpha| \tag{4.32}$$

for some constant C. Summing over all wave-fronts and recalling that the total strength of waves in $u_\nu(t,\cdot)$ satisfies a uniform bound independent of t,ν, we obtain

$$\begin{aligned}\liminf_{\nu\to\infty}\ &\sum_\alpha \Big[\dot{x}_\alpha(t)\cdot\Delta\eta(u_\nu(t,x_\alpha)) - \Delta q(u_\nu(t,x_\alpha))\Big]\varphi(t,x_\alpha)\ dt\\ &\geq\ \Big(\max_{t,x}\ \varphi(t,x)\Big)\cdot\liminf_{\nu\to\infty}\Big\{-\sum_{\alpha\in\mathcal{R}} C\,\varepsilon_\nu|\sigma_\alpha|\Big\}\ \geq\ 0\,.\end{aligned} \tag{4.33}$$

The relation (4.29) now follows from (4.33).

4.5 A modified front tracking algorithm

For general $n\times n$ systems, the analysis given in the previous sections does not provide a complete proof of Theorem 2, because the naive front tracking algorithm can generate an infinite number of wave-fronts, within finite time. If this happens, the whole construction breaks down.

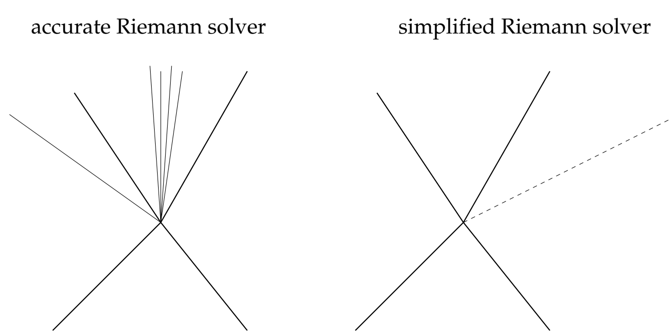

Figure 21: Two different ways of approximately solving the Riemann problem.

We thus need to modify the algorithm, to ensure that the total number of fronts remains uniformly bounded. For this purpose, we shall use two different procedures for solving a Riemann problem within the class of piecewise constant functions: the *Accurate Riemann*

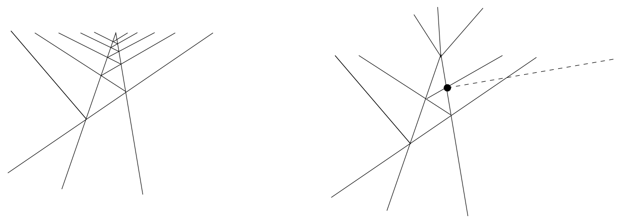

Figure 22: The use of the simplified Riemann solver prevents the number of wave fronts becoming infinite.

Solver (fig. 21, left), which introduces several new wave-fronts, and a *Simplified Riemann Solver* (fig. 21, right), which involves a minimum number of outgoing fronts. In this second case, all new waves are lumped together in a single *non-physical* front, travelling with a fixed speed $\hat{\lambda}$ strictly larger than all characteristic speeds. The main feature of this algorithm is illustrated in fig. 22. If all Riemann problems are solved accurately, the number of wave-fronts can become infinite (fig. 22, left). On the other hand, if at a certain point we use the Simplified Riemann Solver, the total number of fronts remains bounded for all times (fig. 22, right).

The Accurate Riemann Solver was described at (4.12). Next, we introduce a simplified way of solving a Riemann problem, with ≤ 3 outgoing fronts. Throughout the following, $\hat{\lambda}$ will denote a fixed constant, strictly larger than all characteristic speeds $\lambda_j(u)$.

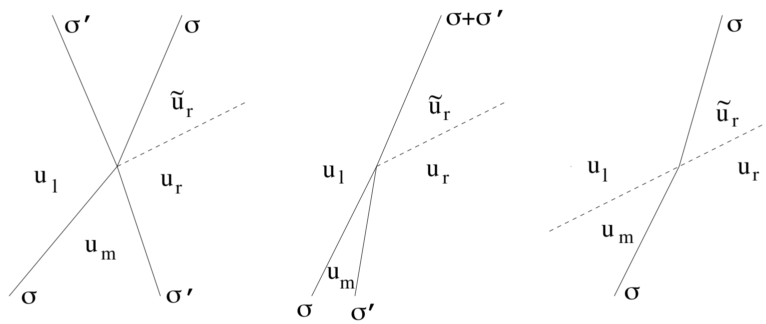

Figure 23: Three cases of the Simplified Riemann Solver.

Simplified Riemann Solver. Consider again the Riemann problem (4.6) at a point $(\bar{t}, \bar{x})$, say generated by the interaction of two incoming fronts, of strength σ, σ'. We distinguish two cases.

CASE 1: Let $j, j' \in \{1, \ldots, n\}$ be the families of the two incoming wave-fronts, with $j \geq j'$.

Assume that the left, middle and right states u_l, u_m, u_r before the interaction are related by

$$u_m = \Psi_j(\sigma)(u_l), \qquad u_r = \Psi_{j'}(\sigma')(u_m). \tag{4.34}$$

Define the auxiliary right state

$$\tilde{u}_r = \begin{cases} \Psi_j(\sigma) \circ \Psi_{j'}(\sigma')(u_l) & \text{if } j > j', \\ \\ \Psi_j(\sigma + \sigma')(u_r) & \text{if } j = j'. \end{cases} \tag{4.35}$$

Let $\tilde{v} = \tilde{v}(t, x)$ be the piecewise constant solution of the Riemann problem with data $u_l, \tilde{u}_r$, constructed as in (4.12). Because of (4.25), the piecewise constant function $\tilde{v}$ contains exactly two wave-fronts of sizes σ', σ if $j > j'$, or a single wave-front of size $\sigma + \sigma'$ if $j = j'$. It is important to observe that $\tilde{u}_r \neq u_r$, in general. We let the jump $(\tilde{u}_r,\ u_r)$ travel with the fixed speed $\hat{\lambda}$, strictly bigger than all characteristic speeds. In a forward neighborhood of the point $(\bar{t}, \bar{x})$, we thus define an approximate solution v as follows (fig. 23 left, center)

$$v(t, x) = \begin{cases} \tilde{v}(t, x) & \text{if} \quad x - \bar{x} < (t - \bar{t})\hat{\lambda}, \\ \\ u_r & \text{if} \quad x - \bar{x} > (t - \bar{t})\hat{\lambda}. \end{cases} \tag{4.36}$$

Notice that this simplified Riemann solver introduces a new *non-physical* wave-front, joining the states $\tilde{u}_r, u_r$ and travelling with constant speed $\hat{\lambda}$. In turn, this front may interact with other wave-fronts. One more case of interaction thus needs to be considered.

CASE 2: A non-physical front hits from the left a wave front of the i-the family (fig. 23 right), for some $i \in \{1, \ldots, n\}$.

Let u_l, u_m, u_r be the left, middle and right state before the interaction. If

$$u_r = \Psi_i(\sigma)(u_m), \tag{4.37}$$

define the auxiliary right state

$$\tilde{u}_r = \Psi_i(\sigma)(u_l). \tag{4.38}$$

Call $\tilde{v}$ the solution to the Riemann problem with data $u_l, \tilde{u}_r$, constructed as in (4.12). Because of (4.38), $\tilde{v}$ will contain a single i-wave with size σ. Since $\tilde{u}_r \neq u_r$ in general, we let the jump $(\tilde{u}_r,\ u_r)$ travel with the fixed speed $\hat{\lambda}$. In a forward neighborhood of the point $(\bar{t}, \bar{x})$, the approximate solution u is then defined again according to (4.36).

By construction, all non-physical fronts travel with the same speed $\hat{\lambda}$, hence they never interact with each other. The above cases therefore cover all possible interactions between two wave-fronts.

We observe that, in a non-physical front, the left and right states u^-, u^+ are essentially arbitrary. Therefore, the Rankine-Hugoniot conditions will not be satisfied, not even approximately. Calling

$$\sigma \doteq |u^+ - u^-| \tag{4.39}$$

the strength of the non-physical front, we only have the trivial estimate

$$\hat{\lambda}(u^+ - u^-) - \Big[f(u^+) - f(u^-) \Big] \;=\; \mathcal{O}(1) \cdot \sigma. \tag{4.40}$$

To ensure that our modified front tracking approximations u_ν converge to a solution of the original equations (4.1), the total strength of all non-physical fronts in each u_ν must approach

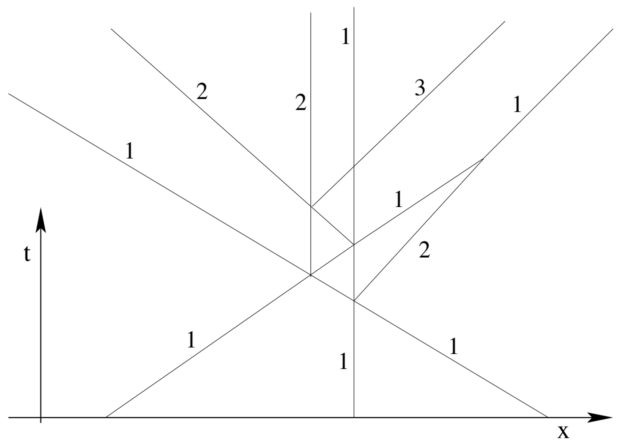

Figure 24: Generation order of various wave-fronts.

zero, as $\nu \to \infty$. This can be accomplished by carefully choosing which Riemann solver is used at any given interaction.

For this purpose, to each wave-front we attach a *generation order*, counting how many interactions were needed to produce it.

Definition 4.2. The *generation order* of a front is inductively defined as follows (fig. 24).

– All fronts generated by the Riemann problems at the initial time $t = 0$ are the "primal ancestors" and have generation order $k = 1$.

– Let two incoming fronts interact, say of the families $i, i' \in \{1, \dots, n\}$, with generation orders k, k'. The orders of the outgoing fronts are then defined as follows.

CASE 1: $i \neq i'$. Then

- the outgoing i-wave and i'-wave have the same orders k, k' as the incoming ones.

- the outgoing fronts of every other family $j \neq i, i'$ have order $\max\{k, k'\} + 1$.

CASE 2: $i = i'$. Then

- the outgoing front of the i-th family has order $\min\{k, k'\}$,

- the outgoing fronts of every family $j \neq i$ have order $\max\{k, k'\} + 1$.

In the above, we tacitly assumed that at most two incoming wave-fronts interact at any given time. This can always be achieved by an arbitrarily small change in the speed of the fronts. We shall also adopt the following important provision:

(P)*In every Riemann Solver, rarefaction waves of the same family as one of the incoming fronts are never partitioned*

In other words, if one of the incoming fronts belongs to the i-th family, we set $p_i = 1$ regardless of the strength of the i-th outgoing front. This guarantees that every wave-front can be uniquely continued forward in time, unless it gets completely cancelled by interacting with other fronts of the same family and opposite sign.

Given an integer $\nu \geq 1$, to construct the approximate solution $u_\nu(\cdot)$, at each point of interaction we choose:

• The Accurate Riemann Solver, if the two incoming fronts both have generation order $< \nu$.

• The Simplified Riemann Solver, if one of the incoming fronts has generation order $\geq \nu$, or if it is a non-physical front.

By implementing the above algorithm, one obtains a globally defined front tracking solution to the Cauchy problem. For future use, we record the following definition.

Definition 2. *Given $\varepsilon > 0$, we say that $u : [0, \infty[\mapsto \mathbf{L}^1(I\!\!R;\ I\!\!R^n)$ is an*ε-approximate front tracking solution *of (4.1) if the following holds:*

1. *As a function of two variables, $u = u(t, x)$ is piecewise constant, with discontinuities occurring along finitely many lines in the t-x plane. Only finitely many wave-front interactions occur, each involving exactly two incoming fronts. Jumps can be of three types: shocks (or contact discontinuities), rarefactions and non-physical waves, denoted as $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP}$.*

2. *Along each shock (or contact discontinuity) $x = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values $u^- \doteq u(t, x_\alpha -)$ and $u^+ \doteq u(t, x_\alpha +)$ are related by*

$$u^+ = S_{k_\alpha}(\sigma_\alpha)(u^-), \tag{4.41}$$

for some $k_\alpha \in \{1, \ldots, n\}$ and some wave size σ_α. If the k_α-th family is genuinely nonlinear, then the entropy admissibility condition $\sigma_\alpha < 0$ also holds. Moreover, the speed of the shock front satisfies

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+, u^-)| \leq \varepsilon. \tag{4.42}$$

3. *Along each rarefaction front $x = x_\alpha(t)$, $\alpha \in \mathcal{R}$, one has*

$$u^+ = R_{k_\alpha}(\sigma_\alpha)(u^-), \qquad \sigma_\alpha \in]0, \varepsilon] \tag{4.43}$$

for some genuinely nonlinear family k_α. Moreover,

$$|\dot{x}_\alpha(t) - \lambda_{k_\alpha}(u^+)| \leq \varepsilon. \tag{4.44}$$

4. *All non-physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ have the same speed:*

$$\dot{x}_\alpha(t) \equiv \hat{\lambda}, \tag{4.45}$$

where $\hat{\lambda}$ is a fixed constant strictly greater than all characteristic speeds. The total strength of all non-physical fronts in $u(t, \cdot)$ remains uniformly small, namely

$$\sum_{\alpha \in \mathcal{NP}} |u(t, x_\alpha+) - u(t, x_\alpha-)| \leq \varepsilon \qquad \text{for all } t \geq 0. \tag{4.46}$$

If, in addition, the initial value of u satisfies

$$\|u(0, \cdot) - \bar{u}\|_{\mathbf{L}^1} < \varepsilon, \tag{4.47}$$

we say that u is an ε-approximate solution to the Cauchy problem (4.1)-2.2).

Lemma 5. *Let the assumptions of Theorem 2 hold. If the initial data $\bar{u}$ has sufficiently small total variation, then for every $\varepsilon > 0$ the Cauchy problem (4.1)-(4.2) admits an ε-*approximate front tracking solution.

Using the above lemma, we construct a sequence of front tracking approximate solutions u_ν, with $\varepsilon_\nu \to 0$, having uniformly small total variation. As shown in Section 4.4, by taking a subsequence u_ν converging to some function $u = u(t, x)$ in $\mathbf{L}^1_{\text{loc}}$, one obtains an entropy-weak solution to the Cauchy problem. For further details we refer to [9].

5 A semigroup of solutions

We always assume that the system (4.1) is strictly hyperbolic, and satisfies the hypotheses (H), so that each characteristic field is either linearly degenerate or genuinely nonlinear. The analysis in the previous chapter has shown the existence of a global entropy weak solution of the Cauchy problem for every initial data with sufficiently small total variation. More precisely, recalling the definitions (4.13)-(4.14), consider a domain of the form

$$\mathcal{D} = cl\Big\{u \in \mathbf{L}^1(I\!R; I\!R^n); \quad u \text{ is piecewise constant}, \quad \Upsilon(u) \doteq V(u) + C_0 \cdot Q(u) < \delta_0\Big\}, \tag{5.1}$$

where cl denotes closure in $\mathbf{L}^1$. With a suitable choice of the constants C_0 and $\delta_0 > 0$, the proof of Theorem 2 for every $\bar{u} \in \mathcal{D}$, one can construct a sequence of ε-approximate front tracking solutions converging to a weak solution u taking values inside $\mathcal{D}$. Observe that, since the proof of convergence relied on a compactness argument, no information was obtained on the uniqueness of the limit. The main goal of the present chapter is to show that this limit is unique and depends continuously on the initial data. In the scalar case, this follows from the classical results of Volpert [53] and Kruzhkov [32]. For strictly hyperbolic systems we have:

Theorem 3. *For every $\bar{u} \in \mathcal{D}$, as $\varepsilon \to 0$ every sequence of ε-approximate solutions $u_\varepsilon : [0, \infty[\mapsto \mathcal{D}$ of the Cauchy problem (4.1)-(4.2), obtained by the front tracking method, converges to a unique limit solution $u : [0, \infty[\mapsto \mathcal{D}$. The map $(\bar{u}, t) \mapsto u(t, \cdot) \doteq S_t \bar{u}$ is a uniformly Lipschitz semigroup, i.e.:*

$$S_0 \bar{u} = \bar{u}, \qquad S_s(S_t \bar{u}) = S_{s+t} \bar{u}, \tag{5.2}$$

$$\|S_t\bar{u} - S_s\bar{v}\|_{\mathbf{L}^1} \ \leq \ L \cdot (\|\bar{u}-\bar{v}\|_{\mathbf{L}^1} + |t-s|) \qquad \text{for all} \ \ \bar{u},\bar{v} \in \mathcal{D}, \quad s,t \geq 0. \tag{5.3}$$

This result was first proved in [11] for 2×2 systems, then in [12] for general $n \times n$ systems, relying on a (lengthy and technical) homotopy method. Relying on some original ideas introduced by T.P.Liu and T.Yang in [41, 42], the paper [16] provided a much simpler proof of the continuous dependence result, which will be described here. An extension of the above result to initial-boundary value problems for hyperbolic conservation laws has recently appeared in [25]. All of the above results deal with solutions having small total variation. The existence of solutions, and the well posedness of the Cauchy problem for large BV data was studied respectively in [46] and in [34].

To prove the uniqueness of the limit of front tracking approximations, we need to estimate the distance between any two ε-approximate solutions u, v of (4.1). For this purpose we introduce a functional $\Phi = \Phi(u,v)$, uniformly equivalent to the $\mathbf{L}^1$ distance, which is "almost decreasing" along pairs of solutions. Recalling the construction of shock curves at (3.11), given two piecewise constant functions $u, v : I\!R \mapsto R^n$, we consider the scalar functions q_i defined implicitly by

$$v(x) = S_n(q_n(x)) \circ \ \cdots \ \circ S_1(q_1(x))(u(x)). \tag{5.4}$$

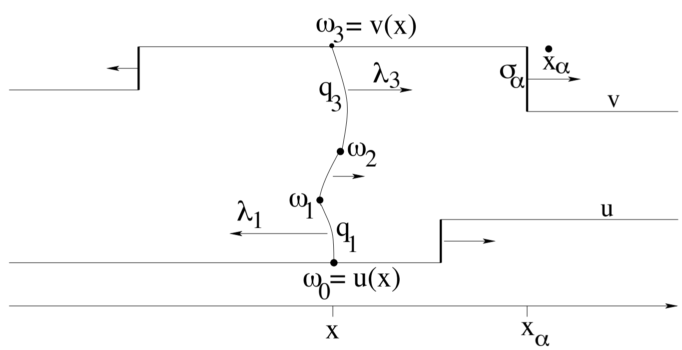

Figure 25: Decomposing a jump $(u(x),\ v(x))$ in terms of n (possibly non-admissible) shocks.

Remark 5. If we wanted to solve the Riemann problem with data $u^- = u(x)$ and $u^+ = v(x)$ only in terms of shock waves (possibly not entropy-admissible), then $q_1(x), \ldots, q_n(x)$ would be the sizes of these shocks (fig. 25). It is thus useful to think of $q_i(x)$ as the strength of the i-th component in the jump $(u(x),\ v(x))$.

On a compact neighborhood of the origin, we clearly have

$$\frac{1}{C_1} \cdot |v(x) - u(x)| \ \leq \ \sum_{i=1}^n |q_i(x)| \ \leq \ C_1 \cdot |v(x) - u(x)| \tag{5.5}$$

for some constant C_1. We now consider the functional

$$\Phi(u,v) \ \doteq \ \sum_{i=1}^n \int_{-\infty}^{\infty} |q_i(x)| W_i(x) \ dx, \tag{5.6}$$

where the weights W_i are defined by setting:

$$\begin{aligned} W_i(x) &\doteq 1+\kappa_1\cdot[\text{total strength of waves in } u \text{ and in } v \text{ which approach the } i\text{-wave } q_i(x)] \\ &\qquad +\kappa_2\cdot[\text{wave interaction potentials of } u \text{ and of } v] \\ &\doteq 1+\kappa_1 A_i(x)+\kappa_2[Q(u)+Q(v)]. \end{aligned} \tag{5.7}$$

The amount of waves approaching $q_i(x)$ is defined as follows. If the i-shock and i-rarefaction curves coincide, we simply take

$$A_i(x) \doteq \left[\sum_{x_\alpha<x,\ i<k_\alpha\le n} + \sum_{x_\alpha>x,\ 1\le k_\alpha<i}\right]|\sigma_\alpha|. \tag{5.8}$$

The summations here extend to waves both of u and of v. The definition (5.8) applies if the i-th field is linearly degenerate, or else if all i-rarefaction curves are straight lines. On the other hand, if the i-th field is genuinely nonlinear with shock and rarefactions curves not coinciding, our definition of A_i will contain an additional term, accounting for waves in u and in v of the same i-th family:

$$\begin{aligned} A_i(x) \doteq & \left[\sum_{\substack{\alpha\in\mathcal{J}(u)\cup\mathcal{J}(v)\\ x_\alpha<x,\ i<k_\alpha\le n}} + \sum_{\substack{\alpha\in\mathcal{J}(u)\cup\mathcal{J}(v)\\ x_\alpha>x,\ 1\le k_\alpha<i}}\right]|\sigma_\alpha| \\ & + \begin{cases} \left[\sum_{\substack{k_\alpha=i\\ \alpha\in\mathcal{J}(u),\ x_\alpha<x}} + \sum_{\substack{k_\alpha=i\\ \alpha\in\mathcal{J}(v),\ x_\alpha>x}}\right]|\sigma_\alpha| & \text{if } q_i(x)<0, \\ \left[\sum_{\substack{k_\alpha=i\\ \alpha\in\mathcal{J}(v),\ x_\alpha<x}} + \sum_{\substack{k_\alpha=i\\ \alpha\in\mathcal{J}(u),\ x_\alpha>x}}\right]|\sigma_\alpha| & \text{if } q_i(x)>0. \end{cases} \end{aligned} \tag{5.9}$$

Here and in the sequel, $\mathcal{J}(u)$ and $\mathcal{J}(v)$ denote the sets of all jumps in u and in v, while $\mathcal{J}\doteq\mathcal{J}(u)\cup\mathcal{J}(v)$. We recall that $k_\alpha\in\{1,\ldots,n+1\}$ is the family of the jump located at x_α with size σ_α. Notice that the strengths of non-physical waves do enter in the definition of Q. Indeed, a non-physical front located at x_α approaches all shock and rarefaction fronts located at points $x_\beta>x_\alpha$. On the other hand, non-physical fronts play no role in the definition of A_i.

The values of the large constants κ_1,κ_2 in (5.7) will be specified later. Observe that, as soon as these constants have been assigned, we can then impose a suitably small bound on the total variation of u,v so that

$$1\le W_i(x)\le 2 \qquad \text{for all } i,x. \tag{5.10}$$

From (5.5), (5.6) and (5.10) it thus follows

$$\frac{1}{C_1}\cdot\|v-u\|_{\mathbf{L}^1}\le\Phi(u,v)\le 2C_1\cdot\|v-u\|_{\mathbf{L}^1}. \tag{5.11}$$

Recalling the definition $\Upsilon(u)\doteq V(u)+C_0Q(u)$, the basic $\mathbf{L}^1$ stability estimate for front tracking approximations can now be stated as follows.

Theorem 4. *For suitable constants $C_2, \kappa_1, \kappa_2, \delta_0 > 0$ the following holds. Let u, v be ε-approximate front tracking solutions of (4.1), with small total variation, so that*

$$\Upsilon(u(t)) < \delta_0, \qquad \Upsilon(v(t)) < \delta_0 \qquad \textit{for all } \ t \geq 0. \tag{5.12}$$

Then the functional Φ in (5.6)–(5.9) satisfies

$$\Phi(u(t),\ v(t)) - \Phi(u(s),\ v(s)) \ \leq \ C_2\varepsilon(t-s) \qquad \textit{for all } \ 0 \leq s < t. \tag{5.13}$$

Relying on the above estimate, a proof of Theorem 3 can be easily worked out. Indeed, let $\bar u \in \mathcal{D}$ be given. Consider any sequence $\{u_\nu\}_{\nu\geq 1}$, such that each u_ν is a front tracking ε_ν-approximate solution of Cauchy problem (4.1)-(4.2), with

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0, \qquad \Upsilon(u_\nu(t)) < \delta_0 \qquad \text{for all } t \geq 0,\ \nu \geq 1.$$

For every $\mu, \nu \geq 1$ and $t \geq 0$, by (5.11) and (5.13) it now follows

$$\begin{aligned} \|u_\mu(t) - u_\nu(t)\|_{\mathbf{L}^1} \ &\leq \ C_1 \cdot \Phi(u_\mu(t),\ u_\nu(t)) \\ &\leq \ C_1 \cdot \Big[\Phi(u_\mu(0),\ u_\nu(0)) + C_2 t \cdot \max\{\varepsilon_\mu,\ \varepsilon_\nu\}\Big] \\ &\leq \ 2C_1^2 \|u_\mu(0) - u_\nu(0)\|_{\mathbf{L}^1} + C_1C_2 t \cdot \max\{\varepsilon_\mu,\ \varepsilon_\nu\}. \end{aligned} \tag{5.14}$$

Since the right hand side of (5.14) approaches zero as $\mu, \nu \to \infty$, the sequence is Cauchy and converges to a unique limit. The semigroup property (5.2) is an immediate consequence of uniqueness. Finally, let $\bar u, \bar v \in \mathcal{D}$ be given. For each $\nu \geq 1$, let u_ν, v_ν be front tracking ε_ν-approximate solutions of (2.1) with

$$\|u_\nu(0) - \bar u\|_{\mathbf{L}^1} < \varepsilon_\nu, \qquad \|v_\nu(0) - \bar v\|_{\mathbf{L}^1} < \varepsilon_\nu, \qquad \lim_{\nu\to\infty} \varepsilon_\nu = 0. \tag{5.15}$$

Using again (5.11) and (5.13) we deduce

$$\begin{aligned} \|u_\nu(t) - v_\nu(t)\|_{\mathbf{L}^1} \ &\leq \ C_1 \cdot \Phi(u_\nu(t),\ v_\nu(t)) \\ &\leq \ C_1 \cdot \Big[\Phi(u_\nu(0),\ v_\nu(0)) + C_2 t \varepsilon_\nu\Big] \\ &\leq \ 2C_1^2 \|u_\nu(0) - v_\nu(0)\|_{\mathbf{L}^1} + C_1C_2 t \varepsilon_\nu. \end{aligned} \tag{5.16}$$

Letting $\nu \to \infty$, by (5.15) it follows

$$\|u(t) - v(t)\|_{\mathbf{L}^1} \ \leq \ 2C_1^2 \cdot \|\bar u - \bar v\|_{\mathbf{L}^1}. \tag{5.17}$$

This establishes the uniform Lipschitz continuity of the semigroup with respect to the initial data. Recalling (4.21), the Lipschitz continuity with respect to time is clear. This completes the proof of Theorem 3.

In the remainder of this chapter we will sketch the main ideas in the proof of Theorem 4. The key point is to understand how the functional Φ evolves in time. In connection with (5.4), at each x define the intermediate states $\omega_0(x) = u(x)$, $\omega_1(x)$, $\ldots$, $\omega_n(x) = v(x)$ by setting

$$\omega_i(x) \doteq S_i(q_i(x)) \circ \ S_{i-1}(q_{i-1}(x)) \circ \ \cdots \ \circ S_1(q_1(x))(u(x)). \tag{5.18}$$

As remarked earlier, these are the intermediate states appearing in the solution of the Riemann problem with data $u^- = u(x)$ and $u^+ = v(x)$ obtained in terms of n (possibly not entropy-admissible) shock waves (fig. 25). Moreover, call

$$\lambda_i(x) \doteq \lambda_i(\omega_{i-1}(x),\ \omega_i(x)) \tag{5.19}$$

the speed of the i-shock connecting $\omega_{i-1}(x)$ with $\omega_i(x)$. A direct computation now yields

$$\begin{aligned} \frac{d}{dt}\Phi(u(t),\ v(t)) &= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{n}\Big\{|q_i(x_\alpha-)|W_i(x_\alpha-) - |q_i(x_\alpha+)|W_i(x_\alpha+)\Big\}\cdot\dot{x}_\alpha \\ &= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{n}\Big\{|q_i^{\alpha+}|W_i^{\alpha+}(\lambda_i^{\alpha+} - \dot{x}_\alpha) - |q_i^{\alpha-}|W_i^{\alpha-}(\lambda_i^{\alpha-} - \dot{x}_\alpha)\Big\}, \end{aligned} \tag{5.20}$$

with obvious meaning of notations. We regard the quantity $|q_i(x)|\lambda_i(x)$ as the flux of the i-th component of $|v-u|$ at x. For $x_{\alpha-1} < x < x_\alpha$, one clearly has

$$|q_i^{(\alpha-1)+}|\,\lambda_i^{(\alpha-1)+}\,W_i^{(\alpha-1)+} = |q_i(x)|\lambda_i(x)W_i(x) = |q_i^{\alpha-}|\,\lambda_i^{\alpha-}\,W_i^{\alpha-}.$$

Since for each $t>0$ the functions $u(t,\cdot), v(t,\cdot)$ are in $\mathbf{L}^1$ and piecewise constant, we must have $u(t,x) = v(t,x) = 0$, $q_i(t,x) = 0$ for all x outside a bounded interval. This allowed us to add and subtract the above terms in (5.20), without changing the overall sum.

In connection with (5.20), for each jump point $\alpha\in\mathcal{J}$ and every $i = 1,\ldots,n$, define

$$E_{\alpha,i} \doteq |q_i^{\alpha+}|W_i^{\alpha+}(\lambda_i^{\alpha+} - \dot{x}_\alpha) - |q_i^{\alpha-}|W_i^{\alpha-}(\lambda_i^{\alpha-} - \dot{x}_\alpha). \tag{5.21}$$

Our main goal will be to establish the bounds

$$\sum_{i=1}^{n} E_{\alpha,i} \le \mathcal{O}(1)\cdot|\sigma_\alpha| \qquad \alpha\in\mathcal{NP}, \tag{5.22}$$

$$\sum_{i=1}^{n} E_{\alpha,i} \le \mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha| \qquad \alpha\in\mathcal{R}\cup\mathcal{S}. \tag{5.23}$$

Here $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{NP}$ denote respectively the sets of rarefaction, shock, and non-physical fronts in u and v. As usual, by the Landau symbol $\mathcal{O}(1)$ we denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (2.1). In particular, this bound does not depend on ε or on the functions u, v. It is also independent of the choice of the constants κ_1, κ_2 in (5.7).

From (5.22)-(5.23), recalling (2.9) and the uniform bounds (5.12) on the total strength of waves, one obtains the key estimate

$$\frac{d}{dt}\Phi(u(t),\ v(t)) \le \mathcal{O}(1)\cdot\varepsilon. \tag{5.24}$$

If the constant κ_2 in (5.7) is chosen large enough, by the interaction estimates (3.31), (3.33) all weight functions $W_i(x)$ will decrease at each time τ where two fronts of u or two fronts of v interact. Integrating (5.24) over the interval $[s,t]$ we therefore obtain

$$\Phi(u(t),\ v(t)) \le \Phi(u(s),\ v(s)) + \mathcal{O}(1)\cdot\varepsilon(t-s), \tag{5.25}$$

proving the theorem.

All the remaining work is thus aimed at establishing (5.22)-(5.23).

If $\alpha \in \mathcal{NP}$, calling σ_α the strength of this jump as in (4.39), for $i = 1, \ldots, n$ one has the easy estimates

$$\begin{aligned} q_i^{\alpha+} - q_i^{\alpha-} &= \mathcal{O}(1) \cdot \sigma_\alpha, \\ \lambda_i^{\alpha+} - \lambda_i^{\alpha-} &= \mathcal{O}(1) \cdot \sigma_\alpha, \\ W_i^{\alpha+} - W_i^{\alpha-} &= 0 \qquad \text{if } \ q_i^{\alpha+} \cdot q_i^{\alpha-} > 0. \end{aligned}$$

Therefore, writing

$$\begin{aligned} E_{\alpha,i} = & \Big(|q_i^{\alpha+}| - |q_i^{\alpha-}|\Big) W_i^{\alpha+}(\lambda_i^{\alpha+} - \dot{x}_\alpha) \; + \; |q_i^{\alpha-}|(W_i^{\alpha+} - W_i^{\alpha-})(\lambda_i^{\alpha+} - \dot{x}_\alpha) \\ & + |q_i^{\alpha-}| W_i^{\alpha-}(\lambda_i^{\alpha+} - \lambda_i^{\alpha-}), \end{aligned} \tag{5.26}$$

the estimate (5.22) is clear.

The proof of (5.23) requires more work. Instead of writing down all computations, we will try to convey the main ideas with the help of a few pictures. For all details we refer to [9].

Given two piecewise constant functions u, v with compact support, for $i = 1, \ldots, n$ we can define the scalar components u_i, v_i by induction on the jump points of u, v. We start by setting $u_i(-\infty) = v_i(-\infty) = 0$. If $x_\alpha \in \mathcal{J}(u)$ is a jump point of u, then we let v_i be constant across x_α and set

$$u_i(x_\alpha+) \doteq u_i(x_\alpha-) - [q_i(x_\alpha+) - q_i(x_\alpha-)].$$

On the other hand, if $x_\alpha \in \mathcal{J}(v)$ is a jump point of v, then we let u_i be constant across x_α and set

$$v_i(x_\alpha+) \doteq v_i(x_\alpha-) + [q_i(x_\alpha+) - q_i(x_\alpha-)].$$

These definitions trivially imply

$$q_i(x) = v_i(x) - u_i(x) \qquad\qquad \text{for all } \ x \in I\!R, \quad i = 1, \ldots, n.$$

Observe that the functional Φ in (5.6)-(5.7) can also be written in the equivalent form

$$\begin{aligned} \Phi(u, v) = & [1 + \kappa_2 Q(u) + \kappa_2 Q(v)] \cdot \sum_{i=1}^n \int_{-\infty}^{\infty} |q_i(x)| \; dx \\ & + \kappa_1 \cdot \sum_{\sigma_\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v)} |\sigma_\alpha| \cdot \sum_i \int_{q_i(x) \text{ approaches } \sigma_\alpha} |q_i(x)| \; dx \,. \end{aligned} \tag{5.27}$$

According to the definition (5.9), the i-waves in u and v which approach $q_i(x)$ are those located within the thick portions of the graphs of u_i, v_i in fig. 26. Viceversa, for a given i-wave σ_α located at x_α, the regions where the jumps $q_i(x)$ approach σ_α are represented by the shaded areas in in fig. 27.

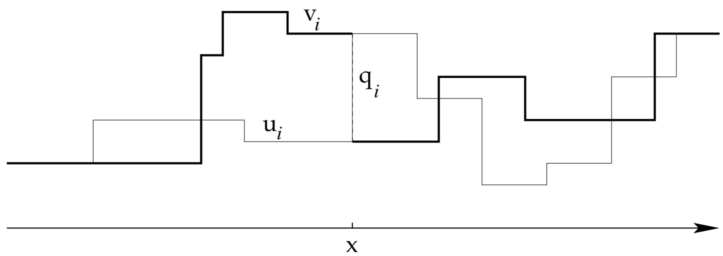

Figure 26: The i-waves in u and in v which are approaching the jump $q_i(x)$.

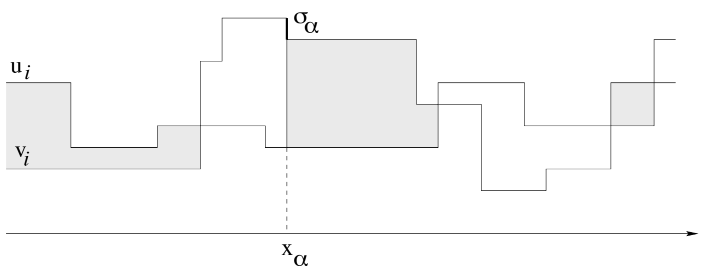

Figure 27: The jumps q_i in the shaded regions are the ones which approach the i-wave σ_α.

Now let v have a wave-front at x_α with strength σ_α, in the genuinely nonlinear k-th family. To fix the ideas, assume that $v_k(x_\alpha\pm) > u_k(x_\alpha)$. In connection with this front, for every $i < k$ the functional $\Phi(u, v)$ in (5.27) contains a term of the form (fig. 28)

$$A_{\alpha,i} \doteq \kappa_1 \cdot |\sigma_\alpha| \cdot [\text{area of the region between the graphs of } u_i \text{ and } v_i, \text{ to the right of } x_\alpha].$$

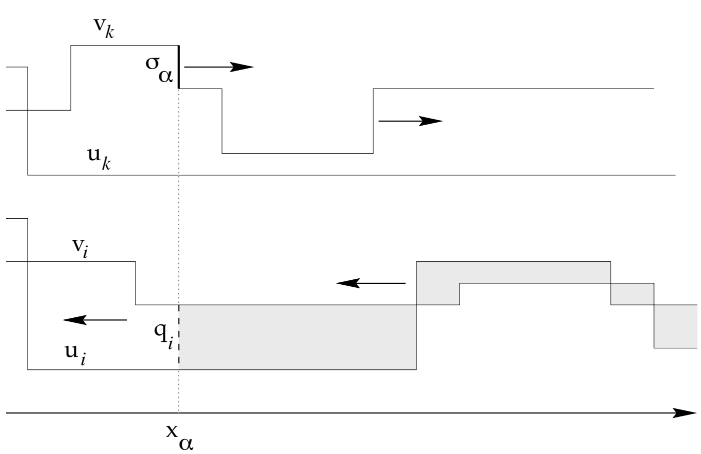

Figure 28: The jumps q_i in the shaded regions are the ones which approach the k-wave σ_α.

By strict hyperbolicity, the i-th and k-th characteristic speeds are strictly separated, say $\lambda_k - \lambda_i \geq c > 0$. If each component u_i, v_i, $i = 1, \ldots, n$, were an exact solution to a scalar conservation law:

$$(u_i)_t + F_i(u_i)_x = 0, \qquad (F_i' = \lambda_i)$$

uncoupled from all the other components, then we would have the estimate

$$\frac{dA_{\alpha,i}}{dt} \leq -\kappa_1|\sigma_\alpha||q_i^{\alpha+}|(\dot{x}_\alpha - \lambda_i^{\alpha+}) \leq -c\kappa_1|\sigma_\alpha||q_i^{\alpha+}|. \tag{5.28}$$

Here $\lambda_i^{\alpha+} \doteq \lambda_i(u_i(x_\alpha+),\ v_i(x_\alpha+))$ is a speed of an i-shock of strength $q_i^{\alpha+}$. In general, the estimate (5.28) must be supplemented with coupling and error terms, whose size is estimated as

$$\mathcal{O}(1)\cdot\left(\varepsilon + |q_k^{\alpha+}|(|q_k^{\alpha+}| + |\sigma_\alpha|) + \sum_{j\neq k}|q_j^{\alpha+}|\right)|\sigma_\alpha|. \tag{5.29}$$

A detailed computation thus yields

$$E_{\alpha,i} \leq \mathcal{O}(1)\cdot\left(\varepsilon + |q_k^{\alpha+}|(|q_k^{\alpha+}| + |\sigma_\alpha|) + \sum_{j\neq k}|q_j^{\alpha+}|\right)|\sigma_\alpha| - c\kappa_1|q_i^{\alpha+}||\sigma_\alpha| \qquad i \neq k. \tag{5.30}$$

Next, according to (5.9) the functional $\Phi(u,v)$ in (5.27) also contains a term of the form

$$A_{\alpha,k} \doteq \kappa_1 \cdot |\sigma_\alpha| \cdot [\text{area of the region between the graphs of } u_k \text{ and } v_k\text{, to the right of } x_\alpha].$$

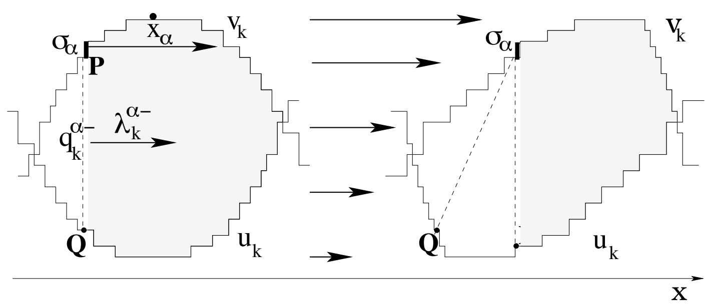

Figure 29: Due to genuine nonlinearity, the total amount of q_k approaching the k-wave σ_α (shaded region) decreases in time.

If the components u_k, v_k were exact solutions of a genuinely nonlinear scalar conservation law, say

$$(u_k)_t + F_k(u_k)_x = 0, \qquad\qquad (F_k' = \lambda_k) \tag{5.31}$$

with $F_k'' \geq c' > 0$, then one would have the estimate

$$\frac{dA_{\alpha,k}}{dt} \leq -\kappa_1|\sigma_\alpha|\,|q_k^{\alpha+}|(\dot{x}_\alpha - \lambda_k^{\alpha+}) \leq -\kappa_1|\sigma_\alpha|\,|q_k^{\alpha+}|\cdot\frac{c'}{2}(|q_k^{\alpha+}| + |\sigma_\alpha|). \tag{5.32}$$

The decrease of this area is illustrated in fig. 29. In general, the estimate (5.32) must be supplemented with coupling and error terms, whose size is again estimated as (5.29). A detailed computation thus yields

$$E_{\alpha,k} \leq \mathcal{O}(1)\cdot\left(\varepsilon + |q_k^{\alpha+}|\Big(|q_k^{\alpha+}| + |\sigma_\alpha|\Big) + \sum_{j\neq k}|q_j^{\alpha+}|\right)|\sigma_\alpha| - \frac{c'\kappa_1}{2}|q_k^{\alpha+}||\sigma_\alpha|\Big(|q_k^{\alpha+}| + |\sigma_\alpha|\Big). \tag{5.33}$$

Choosing κ_1 sufficiently large, (5.30) and (5.33) together yield (5.23).

A different estimate is needed in the case where the jump in v_k crosses the graph of u_k, say $v_k(x_\alpha+) < u_k(x_\alpha) < v_k(x_\alpha-)$. To fix the ideas, assume

$$|q_k^{\alpha+}| \;=\; |v_k(x_\alpha+) - u_k(x_\alpha)| \le |v_k(x_\alpha-) - u_k(x_\alpha)| \;=\; |q_k^{\alpha-}|. \tag{5.34}$$

In this case, the estimates (5.30) remain valid. In connection with the k-th field, the functional Φ contains a term of the form (fig. 30):

$$A_{\alpha,k} \doteq [\text{area of the region between the graphs of } u_k \text{ and } v_k], \tag{5.35}$$

where the above area includes points both to the right and to the left of x_α.

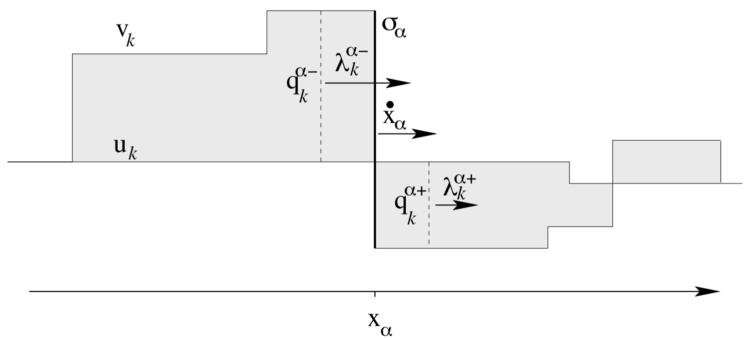

Figure 30: Due to genuine nonlinearity, also in this case the total amount of q_k approaching the k-wave σ_α (shaded area) decreases in time.

If the components u_k, v_k provided an exact solution to the genuinely nonlinear scalar conservation law (5.31), due to genuine nonlinearity we would have

$$\frac{dA_{\alpha,k}}{dt} \le -\frac{c'}{2}|q_k^{\alpha-}| \cdot |q_k^{\alpha+}| \le -\frac{c'}{4}|\sigma_\alpha||q_k^{\alpha+}|. \tag{5.36}$$

Indeed, by (5.34),

$$|\sigma_\alpha| = |q_k^{\alpha-}| + |q_k^{\alpha+}| \le 2|q_k^{\alpha-}|.$$

In general, the estimate (5.36) must be supplemented with coupling and error terms, whose size is again estimated as (5.29). A detailed computation thus yields

$$E_{\alpha,k} \le \mathcal{O}(1) \cdot \left(\varepsilon + |q_k^{\alpha+}|(|q_k^{\alpha+}| + |\sigma_\alpha|) + \sum_{j \neq k} |q_j^{\alpha+}| \right) |\sigma_\alpha| - \frac{c'}{4}|q_k^{\alpha+}||\sigma_\alpha|. \tag{5.37}$$

Assuming that the total strength of waves remains sufficiently small, we have

$$|q_k^{\alpha+}| + |\sigma_\alpha| \ll \frac{c'}{4},$$

hence (5.30) and (5.37) together yield (5.23). For details we again refer to [9].

6 Uniqueness of solutions.

According to the analysis in the previous chapters, the solution of the Cauchy problem (4.1)-(4.2) obtained as limit of front tracking approximations is unique and depends Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm. This basic result, however, leaves open the question whether other weak solutions may exist, possibly constructed by different approximation algorithms. We will show that this is not the case: indeed, every entropy admissible solution, satisfying some minimal regularity assumptions, necessarily coincides with the one obtained as limit of front tracking approximations.

As a first step, we estimate the distance between an approximate solution, obtained by the front tracking method, and the exact solution of the Cauchy problem (4.1)-(4.2), given by the semigroup trajectory $t \mapsto u(t, \cdot) = S_t \bar{u}$. Let $u^\varepsilon : [0, T] \mapsto \mathcal{D}$ be an ε-approximate front tracking solution, according to Definition 2. We claim that the corresponding error can then be estimated as

$$\|u^\varepsilon(T, \cdot) - S_T \bar{u}\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \varepsilon(1 + T). \tag{6.1}$$

To see this, we first estimate the limit

$$\lim_{h \to 0+} \frac{\|u^\varepsilon(\tau + h) - S_h u^\varepsilon(\tau)\|_{\mathbf{L}^1}}{h}$$

at any time $\tau \in [0, T]$ where no wave-front interaction takes place. Let $u(\tau, \cdot)$ have jumps at points $x_1 < \cdots < x_N$. Call $\mathcal{S}$ the set of indices $\alpha \in \{1, \ldots, N\}$ such that $u(\tau, x_\alpha -)$ and $u(\tau, x_\alpha +)$ are connected by a shock or by a contact discontinuity, so that (2.4)-(2.5) hold. Moreover, call $\mathcal{R}$ the set of indices α corresponding to a rarefaction wave of a genuinely nonlinear family, so that (2.6)-(2.7) hold. Finally, call $\mathcal{NP}$ the set of indices α corresponding to non-physical fronts.

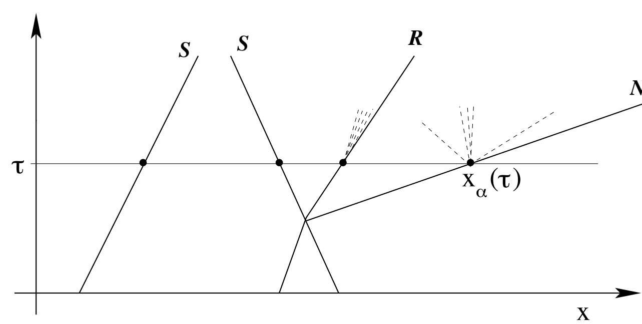

Figure 31: The exact solution (dotted lines) which, at time τ, coincides with the value of a piecewise constant front tracking approximation.

For each α, call ω_α the self-similar solution of the Riemann problem with data $u^\pm = u(\tau,\ x_\alpha \pm)$. We observe that, for $h > 0$ small enough, the semigroup trajectory $h \mapsto S_h u(\tau)$ is obtained by piecing together the solutions of these Riemann problems (fig. 31). Using the properties

(4.41)–(4.46) one now derives the bound

$$\begin{aligned}
&\lim_{h\to 0+} \frac{\|u(\tau+h) - \widetilde{S}_h u(\tau)\|_{\mathbf{L}^1}}{h} \\
&\qquad = \sum_{\alpha\in\mathcal{R}\cup\mathcal{S}\cup\mathcal{NP}} \left(\lim_{h\to 0+} \frac{1}{h} \int_{x_\alpha-\rho}^{x_\alpha+\rho} |u(\tau+h,x) - \omega_\alpha(h,\ x-x_\alpha)|\ dx \right) \\
&\qquad = \sum_{\alpha\in\mathcal{S}} \mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha| + \sum_{\alpha\in\mathcal{R}} \mathcal{O}(1)\cdot|\sigma_\alpha|\Big(|\sigma_\alpha|+\varepsilon\Big) + \sum_{\alpha\in\mathcal{NP}} \mathcal{O}(1)\cdot|\sigma_\alpha| \\
&\qquad = \mathcal{O}(1)\cdot\varepsilon\,.
\end{aligned} \tag{6.2}$$

Here ρ can be any suitably small positive number. From the assumption (4.47), the bound (6.2) and the error formula (10.6) in the Appendix, we finally obtain

$$\begin{aligned}
\|u^\varepsilon(T,\cdot) - S_T\bar u\|_{\mathbf{L}^1} &\leq \|S_T u^\varepsilon(0,\cdot) - S_T\bar u\|_{\mathbf{L}^1} + \|u^\varepsilon(T,\cdot) - S_T u^\varepsilon(0,\cdot)\|_{\mathbf{L}^1} \\
&\leq L\cdot\|u^\varepsilon(0,\cdot) - \bar u\|_{\mathbf{L}^1} + L\cdot\int_0^T \left\{ \liminf_{h\to 0+} \frac{\|u^\varepsilon(\tau+h) - S_h u^\varepsilon(\tau)\|_{\mathbf{L}^1}}{h} \right\} d\tau \\
&= \mathcal{O}(1)\cdot\varepsilon + \mathcal{O}(1)\cdot\varepsilon T.
\end{aligned}$$

We shall now state our main uniqueness theorem. For sake of clarity, a complete set of assumptions is listed below.

(A1) (Conservation Equations) The function $u = u(t,x)$ is a weak solution of the Cauchy problem (4.1)-(4.2), taking values within the domain $\mathcal{D}$ of a semigroup S. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The initial condition (2.2) holds, together with

$$\iint \big(u\varphi_t + f(u)\varphi_x\big)\ dxdt = 0 \tag{6.3}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0,T[\,\times I\!R$.

(A2) (Lax Entropy Conditions) Let u have an approximate jump discontinuity at some point $(\tau,\xi)\in]0,T[\times I\!R$. More precisely, let there exists states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases} \tag{6.4}$$

there holds

$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r} \int_{-r}^{r} \Big| u(\tau+t,\ \xi+x) - U(t,x) \Big|\ dxdt = 0. \tag{6.5}$$

Then, for some $i \in \{1,\ldots,n\}$, one has the inequalities

$$\lambda_i(u^-)\ \geq\ \lambda\ \geq\ \lambda_i(u^+). \tag{6.6}$$

(A3) (Tame Oscillation Condition) For some constants C, $\hat{\lambda}$ the following holds. For every point $x \in I\!R$ and every $t, h > 0$ one has

$$|u(t+h,x) - u(t,x)| \;\leq\; C \cdot \text{Tot.Var.}\Big\{u(t,\cdot)\,; \quad [x - \hat{\lambda}h,\; x + \hat{\lambda}h]\Big\}. \tag{6.7}$$

(A4) (Bounded Variation Condition) There exists $\delta > 0$ such that, for every space-like curve $\{t = \tau(x)\}$ with $|d\tau/dx| \leq \delta$ a.e., the function $x \mapsto u(\tau(x), x)$ has locally bounded variation.

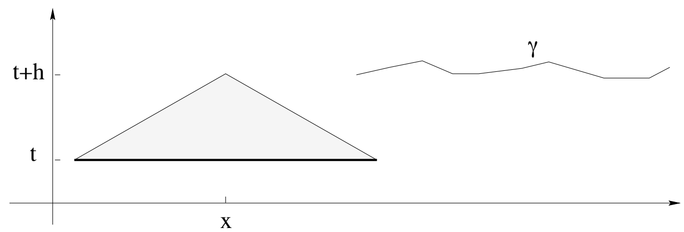

Figure 32: The tame oscillation and the bounded variation condition.

Remark 6. The condition (A3) restricts the oscillation of the solution. An equivalent, more intuitive formulation is the following. For some constant $\hat{\lambda}$ larger than all characteristic speeds, given any interval $[a,b]$ and $t \geq 0$, the oscillation of u on the triangle (fig. 32) $\Delta \doteq \{(s,y) : \;\; s \geq t,\;\; a + \hat{\lambda}(s-t) < y < b - \hat{\lambda}(s-t)\}$, defined as

$$\text{Osc}\{u;\; \Delta\} \doteq \sup_{(s,y),(s',y') \in \Delta} |u(s,y) - u(s',y')|,$$

is bounded by a constant multiple of the total variation of $u(t,\cdot)$ on $[a,b]$.

Assumption (A4) simply requires that, for some fixed $\delta > 0$, the function u has bounded variation along every space-like curve γ which is "almost horizontal" (fig. 32). Indeed, the condition is imposed only along curves of the form $\{t = \tau(x);\; x \in [a,b]\}$ with

$$|\tau(x) - \tau(x')| \;\leq\; \delta |x - x'| \qquad\qquad \text{for all} \;\; x, x' \in [a,b].$$

One can prove that all of the above assumptions are satisfied by weak solutions obtained as limits of Glimm or wave-front tracking approximations [9]. The following result shows that the entropy weak solution of the Cauchy problem (4.1)-(4.2) is unique within the class of functions that satisfy either the additional regularity condition (A3), or (A4).

Theorem 5. *Let the map $u : [0,T] \mapsto \mathcal{D}$ be continuous (w.r.t. the $\mathbf{L}^1$ distance), taking values in the domain of the semigroup S generated by the system (4.1). If* (A1), (A2) *and* (A3) *hold, then*

$$u(t,\cdot) = S_t \bar{u} \qquad\qquad \textit{for all} \;\; t \in [0,T]. \tag{6.8}$$

In particular, the weak solution that satisfies these conditions is unique. The same conclusion holds if the assumption (A3) *is replaced by* (A4).

The main steps of the proof are given below.

1. Since u takes values inside the domain $\mathcal{D}$ of the semigroup, the total variation of $u(t,\cdot)$ remains uniformly bounded. From the basic equation (4.1), it follows that u is Lipschitz continuous with values in $\mathbf{L}^1$, namely

$$\|u(t) - u(s)\|_{\mathbf{L}^1} \leq L \cdot |t - s| \tag{6.9}$$

for some Lipschitz constant L. More precisely, if M and λ^* are constants such that

$$\text{Tot.Var.}\{u\} \leq M \qquad \text{for all} \quad u \in \mathcal{D},$$

$$|f(\omega) - f(\omega')| \ \leq \ \lambda^*|\omega - \omega'| \qquad \text{whenever} \quad |\omega|, |\omega'| \leq M,$$

as Lipschitz constant in (6.9) one can take $L \doteq \lambda^* M$. As a consequence, $u = u(t,x)$ can be regarded as a BV function of the two variables t, x, in the sense that the distributional derivatives $D_t u, D_x u$ are Radon measures. By a well known structure theorem [26], there exists a set $\widetilde{\mathcal{N}} \subset]0,T[\, \times I\!\!R$ of 1-dimensional Hausdorff measure zero such that, at every point $(\tau,\xi) \notin \widetilde{\mathcal{N}}$, u either is approximately continuous or has an approximate jump discontinuity. Taking the projection of $\widetilde{\mathcal{N}}$ on the t-axis, we conclude that there exists a set $\mathcal{N} \subset [0,T]$ of measure zero, containing the endpoints 0 and T, such that, at every point $(\tau,\xi) \in [0,T] \times I\!\!R$ with $\tau \notin \mathcal{N}$, setting $u^- \doteq u(\tau, \xi-)$, $u^+ \doteq u(\tau,\xi+)$ the following property holds.

(P) Either $u^+ = u^-$, in which case (6.4)-(6.5) hold with λ arbitrary. Or else $u^+ \neq u^-$, in which case (6.4)-(6.5) hold for some particular $\lambda \in I\!\!R$. In this second case, for some $i \in \{1,\ldots,n\}$ the Rankine-Hugoniot equations and the Lax entropy condition hold:

$$\lambda_i(u^-,u^+) \cdot (u^+ - u^-) = f(u^+) - f(u^-), \qquad \lambda_i(u^+) \leq \lambda_i(u^-,u^+) \leq \lambda_i(u^-). \tag{6.10}$$

2. According to Theorem A.2 in the Appendix, at any fixed time $T > 0$ the distance between the solution $u(T)$ and the unique semigroup trajectory $S_T\bar{u}$ is estimated by

$$\|u^\varepsilon(T,\cdot) - S_T\bar{u}\|_{\mathbf{L}^1} \ \leq \ L\cdot\|u^\varepsilon(0,\cdot) - \bar{u}\|_{\mathbf{L}^1} + L\cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\|u^\varepsilon(\tau+h) - S_h u^\varepsilon(\tau)\|_{\mathbf{L}^1}}{h} \right\} d\tau\,. \tag{6.11}$$

We will establish (6.8) by showing that the integrand on the right hand side of (6.11) vanishes at each time $t \notin \mathcal{N}$. Because of the finite speed of propagation, it actually suffices to show that, for each $t \notin \mathcal{N}$, $\varepsilon > 0$ and every interval $[a,b]$, there holds

$$\limsup_{h\to 0+} \frac{1}{h} \int_a^b \Big| u(t+h,\ x) - (S_h u(t))(x) \Big|\, dx = \mathcal{O}(1)\cdot\varepsilon. \tag{6.12}$$

3. Let $u = u(t,x)$ be as in Theorem 5. In the following, for any given point (τ,ξ), we denote by $U^\sharp = U^\sharp_{(\tau,\xi)}$ the solution of the Riemann problem

$$w_t + f(w)_x = 0, \qquad w(\tau,x) = \begin{cases} u^+ \doteq u(\tau,\xi+) & \text{if} \quad x > \xi, \\ u^- \doteq u(\tau,\xi-) & \text{if} \quad x < \xi. \end{cases} \tag{6.13}$$

By the property (P), apart from the trivial case where $u^+ = u^-$, this solution consists of a single entropy admissible shock. The function $U^\sharp$ provides a good approximation to the solution u in a forward neighborhood of the point (τ, ξ). More precisely, using the Lipschitz continuity (6.9), one can prove that

$$\lim_{h\to 0+} \frac{1}{h} \int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda} \Big| u(\tau+h,\ x) - U^\sharp_{(\tau,\xi)}(\tau+h,\ x) \Big|\, dx \;=\; 0, \tag{6.14}$$

for every $\hat\lambda > 0$ and every (τ, ξ) with $\tau \notin \mathcal{N}$.

4. Next, for a given point (τ, ξ) we denote by $U^\flat = U^\flat_{(\tau,\xi)}$ the solution of the linear system with constant coefficients

$$w_t + \widetilde A w_x = 0, \qquad\qquad w(\tau, x) = u(\tau, x), \tag{6.15}$$

where $\widetilde A \doteq A(u(\tau, \xi))$. As in the previous step, we need to estimate the difference between u and $U^\flat$, in a forward neighborhood of the point (τ, ξ). Consider any open interval $]a, b[$ containing the point ξ and fix a speed $\hat\lambda$ strictly larger than the absolute values of all characteristic speeds. For $t \geq \tau$ define the open intervals

$$J(t) \doteq\ ]a + (t-\tau)\hat\lambda,\quad b - (t-\tau)\hat\lambda[\tag{6.16}$$

and the region

$$\Gamma(t) \doteq \{(s, x);\quad s \in [\tau,\ t],\quad x \in J(s)\}. \tag{6.17}$$

With the above notation, for every $\tau' \geq \tau$ one can prove the estimate

$$\int_{J(\tau')} \Big| u(\tau',\ x) - U^\flat(\tau',\ x) \Big|\, dx = \mathcal{O}(1)\cdot \sup_{(t,x)\in\Gamma(\tau')} |u(t,x) - u(\tau,\xi)| \cdot \int_\tau^{\tau'} \text{Tot.Var.}\{u(t,\cdot);\ J(t)\}\, dt. \tag{6.18}$$

5. Given $\tau \notin \mathcal{N}$, $\varepsilon > 0$ and $a < b$, using either one of the assumptions (A3) or (A4) we can cover a neighborhood of the interval $[a, b]$ with finitely many points ξ_i and open intervals $I_j \doteq]a_j,\ b_j[$ such that the following conditions hold (fig. 33).

(i) Each point x is contained in at most two of the open intervals I_j.

(ii) The total variation of $u(\tau, \cdot)$ on each I_j is $\leq \varepsilon$.

(iii) For some $\tau' > \tau$, calling $\zeta_j \doteq (a_j + b_j)/2$ and

$$\Gamma_j \doteq \Big\{(t, x);\quad t \in [\tau, \tau'],\quad a_j + (t-\tau)\hat\lambda \ <\ x\ <\ b_j - (t-\tau)\hat\lambda\Big\},$$

there holds

$$\sup_{(t,x)\in\Gamma_j} |u(t,x) - u(\tau, \zeta_j)| \leq \varepsilon. \tag{6.19}$$

6. We now construct a function $U = U(t, x)$ which coincides with $U^\sharp_{(\tau,\xi_i)}$ near each point (τ, ξ_i) and with $U^\flat_{(\tau,\zeta_j)}$ in a forward neighborhood of each point (τ, ζ_j).

$$U(t,x) \doteq \begin{cases} U^\sharp_{(\tau,\xi_i)}(t,x) & \text{if} \quad |x - \xi_i| \leq (t-\tau)\hat\lambda, \\[2ex] U^\flat_{(\tau,\zeta_j)}(t,x) & \text{if} \quad (t,x) \in \Gamma_j \setminus \bigcup_{k<j} \Gamma_k. \end{cases}$$

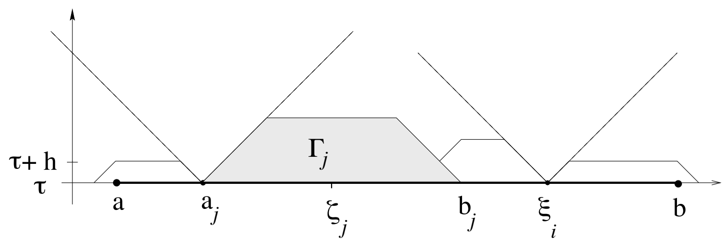

Figure 33: Covering the interval $[a, b]$ with subintervals where the solutions $u(\tau + h)$ and $S_h u(\tau)$ can both be compared with the approximations $U^\sharp$ or $U^\flat$.

By (6.14) and (6.18), this function U provides a good approximation of u, for times $t = \tau + h$ with $h > 0$ small. Indeed, recalling (6.19) and the property (i) of the covering, we have

$$
\begin{aligned}
\limsup_{h\to 0+} \quad & \frac{1}{h}\int_a^b \Big|u(\tau+h,\ x) - U(\tau+h,\ x)\Big|\, dx \\
& \leq \sum_i \limsup_{h\to 0+} \frac{1}{h}\int_{\xi_i - h\hat\lambda}^{\xi_i + h\hat\lambda} \Big|u(\tau+h,\ x) - U^\sharp_{(\tau,\xi_i)}(\tau+h,\ x)\Big|\, dx \\
& \qquad + \sum_j \limsup_{h\to 0+} \frac{1}{h}\int_{a_j + h\hat\lambda}^{b_j - h\hat\lambda} \Big|u(\tau+h,\ x) - U^\flat_{(\tau,\zeta_j)}(\tau+h,\ x)\Big|\, dx \\
& \leq 0 + \limsup_{h\to 0+} \left\{ \mathcal{O}(1)\cdot \sum_j \frac{\varepsilon}{h}\int_\tau^{\tau+h} \text{Tot.Var.}\{u(t);\quad I_j\}\; dt \right\} \\
& \leq \limsup_{h\to 0+} \left\{ \mathcal{O}(1)\cdot \frac{\varepsilon}{h}\int_\tau^{\tau+h} 2\cdot \text{Tot.Var.}\{u(t);\quad I\!\!R\}\; dt \right\} \\
& = \mathcal{O}(1)\cdot\varepsilon.
\end{aligned}
\tag{6.20}
$$

7. We now observe that the semigroup trajectory $v(t,\cdot) \doteq S_t\bar u$ is also an entropy weak solution of the Cauchy problem (4.1)-(4.2), and satisfies all the assumptions (A1)–(A3). In particular, the total variation of $v(t,\cdot)$ remains uniformly bounded, and its oscillation on each domain $\Gamma(t)$ of the form (6.17) is bounded by

$$
\sup_{(t,x)\in\Gamma(\tau')} |v(t,x) - v(\tau,\xi)| = \mathcal{O}(1)\cdot \text{Tot.Var.}\{v(\tau);\quad]a,b[\,\}. \tag{6.21}
$$

As a consequence, we can repeat the estimate (6.20) with v in the role of u and obtain

$$
\limsup_{h\to 0+} \frac{1}{h}\int_a^b \Big|v(\tau+h,\ x) - U(\tau+h,\ x)\Big|\, dx = \mathcal{O}(1)\cdot\varepsilon. \tag{6.22}
$$

Together, (6.20) and (6.22) imply (6.12). Since $\varepsilon > 0$ and the interval $[a, b]$ were arbitrary, this achieves a proof of Theorem 5.

7 The Glimm scheme

The fundamental paper of Glimm [27] contained the first rigorous proof of existence of global weak solutions to hyperbolic systems of conservation laws. For several years, the Glimm approximation scheme has provided the foundation for most of the theoretical results on the subject. We shall now describe this algorithm in a somewhat simplified setting, for systems where all characteristic speeds remain inside the interval $[0,1]$. This is not a restrictive assumption. Indeed, consider any hyperbolic system of the form

$$u_t + A(u)u_x = 0,$$

and assume that all eigenvalues of A remain inside the interval $[-M, M]$. Performing the linear change of independent variables

$$y = x + Mt, \qquad\qquad \tau = 2Mt,$$

we obtain a new system

$$u_\tau + A^*(u)u_y = 0, \qquad\qquad A^*(u) \doteq \frac{1}{2M}\,A(u) + \frac{1}{2}\,I$$

where all eigenvalues of the matrix A^* now lie inside the interval $[0,1]$.

To construct an approximate solution to the Cauchy problem

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar u(x), \tag{7.1}$$

we start with a grid in the t-x plane having step size $\Delta t = \Delta x$, with nodes at the points

$$P_{jk} = (t_j, x_k) \doteq (j\Delta t,\ k\Delta x) \qquad\qquad j,k \in \mathbb{Z}\,.$$

Moreover, we shall need a sequence of real numbers $\theta_1, \theta_2, \theta_3, \ldots$ *uniformly distributed* over the interval $[0,1]$. This means that, for every $\lambda \in [0,1]$, the percentage of points θ_i, $1 \le i \le N$ which fall inside $[0,\lambda]$ should approach λ as $N \to \infty$, i.e.:

$$\lim_{N\to\infty} \frac{\#\{j\ ;\ \ 1 \le j \le N,\ \ \theta_j \in [0,\lambda]\ \}}{N} = \lambda \qquad\qquad \text{for each } \lambda \in [0,1]. \tag{7.2}$$

By $\#I$ we denote here the cardinality of a set I.

At time $t = 0$, the Glimm algorithm starts by taking an approximation of the initial data $\bar u$, which is constant on each interval of the form $]x_{k-1},\ x_k[$, and has jumps only at the nodal points $x_k \doteq k\,\Delta x$. To fix the ideas, we shall take

$$u(0,x) = \bar u(x_k) \qquad\qquad \text{for all } \ x \in [x_k,\ x_{k+1}[\,. \tag{7.3}$$

For times $t > 0$ sufficiently small, the solution is then obtained by solving the Riemann problems corresponding to the jumps of the initial approximation $u(0,\cdot)$ at the nodes x_k. Since by assumption all waves speeds are contained in $[0,1]$, waves generated from different nodes remain separated at least until the time $t_1 = \Delta t$. The solution can thus be prolonged on the whole time interval $[0,\ \Delta t[$. For bigger times, waves emerging from different nodes may cross each other, and the solution would become extremely complicated. To avoid this unpleasant situation, a restarting procedure is adopted. Namely, at time $t_1 = \Delta t$ the function $u(t_1-,\cdot)$ is approximated by a new function $u(t_1+,\cdot)$ which is piecewise constant, having jumps exactly at the nodes $x_k \doteq k\,\Delta x$. Our approximate solution u can now be constructed on the further time interval $[\Delta t,\ 2\Delta t[$, again by piecing together the solutions of the various Riemann problems determined by the jumps at the nodal points x_k. At time $t_2 = 2\Delta t$, this solution is again approximated by a piecewise constant function, etc...

A key aspect of the construction is the restarting procedure. At each time $t_j \doteq j\,\Delta t$, we need to approximate $u(t_j-,\cdot)$ with a a piecewise constant function $u(t_j+,\cdot)$, having jumps precisely at the nodal points x_k. This is achieved by a random sampling technique. More precisely, we look at the number θ_j in our uniformly distributed sequence. On each interval $[x_{k-1},\ x_k[$, the old value of our solution at the intermediate point $x_k^* = \theta_1 x_k + (1-\theta_1)x_{k-1}$ becomes the new value over the whole interval:

$$u(t_j+,\ x) = u(t_j-,\ \theta_j x_k + (1-\theta_j)x_{k-1}) \qquad \text{for all } x \in [x_{k-1},\ x_k[\,. \tag{7.4}$$

An approximate solution constructed in this way is illustrated in fig. 34. The asterisks mark the points where the function is sampled, assuming $\theta_1 = 1/2$, $\theta_2 = 1/3$.

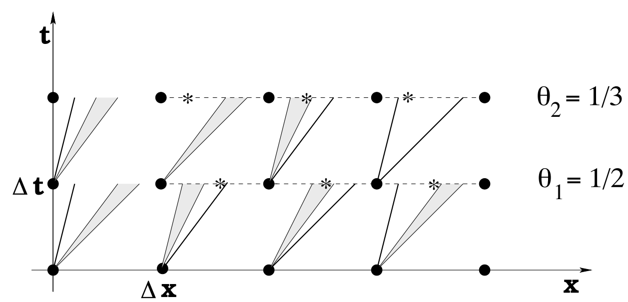

Figure 34: An approximate solution constructed by the Glimm scheme.

For a strictly hyperbolic system of conservation laws, satisfying the hypotheses (H) in Section 3, the fundamental results of J.Glimm [27] and T.P.Liu [38] have established that

1. If the initial data $\bar u$ has small total variation, then an approximate solution can be constructed by the above algorithm for all times $t \geq 0$. The total variation of $u(t,\cdot)$ remains small.

2. Letting the grid size $\Delta t = \Delta x$ tend to zero and using always the same sequence of numbers $\theta_j \in [0,1]$, one obtains a sequence of approximate solutions u_ν. By Helly's compactness theorem, one can extract a subsequence that converges to some limit function $u = u(t,x)$ in $\mathbf{L}^1_{\rm loc}$.

3. If the numbers θ_j are uniformly distributed over the interval $[0,1]$, i.e. if (7.2) holds, then the limit function u provides a weak solution to the Cauchy problem (7.1).

The importance of the sequence θ_j being uniformly distributed can be best appreciated in the following example.

Example 10. Consider a Cauchy problem of the form (7.1). Assume that the exact solution consists of exactly one single shock, travelling with speed $\lambda \in [0,1]$, say

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t, \\ u^- & \text{if} \quad x < \lambda t. \end{cases}$$

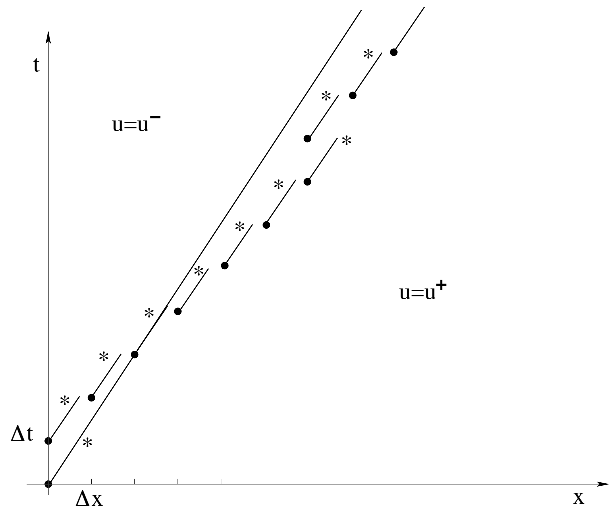

Figure 35: Applying the Glimm scheme to a solution consisting of a single shock.

Consider an approximation of this solution obtained by implementing the Glimm algorithm (fig. 35). By construction, at each time $t_j \doteq j\Delta t$, the position of the shock in this approximate solution must coincide with one of the nodes of the grid. Observe that, passing from t_{j-1} to t_j, the position $x(t)$ of the shock remains the same if the j-th sampling point lies on the left, while it moves forward by Δx if the j-th sampling point lies on the right. In other words,

$$x(t_j) = \begin{cases} x(t_{j-1}) & \text{if} \quad \theta_j \in]\lambda, 1], \\ x(t_{j-1}) + \Delta x & \text{if} \quad \theta_j \in [0, \lambda]. \end{cases} \tag{7.5}$$

Let us fix a time $T > 0$, and take $\Delta t \doteq T/N$. From (7.5) it now follows

$$\begin{aligned} x(T) &= \#\{j \; ; \;\; 1 \leq j \leq N, \;\; \theta_j \in [0,\lambda] \;\} \cdot \Delta t \\ &= \frac{\#\{j \; ; \;\; 1 \leq j \leq N, \;\; \theta_j \in [0,\lambda] \;\}}{N} \cdot T\,. \end{aligned}$$

It is now clear that the assumption (7.2) on the uniform distribution of the sequence $\{\theta_j\}_{j\geq 1}$ is precisely what is needed to guarantee that, as $N \to \infty$ (equivalently, as $\Delta t \to 0$), the location $x(T)$ of the shock in the approximate solution converges to the exact value λT.

Remark 7. At each restarting time t_j we need to approximate the BV function $u(t_j-, \cdot)$ with a new function which is piecewise constant on each interval $[x_{k-1}, \; x_k[$. Instead of the sampling procedure (7.4), an alternative method consists of taking average values:

$$u(t_j+, x) \doteq \frac{1}{\Delta x} \int_{x_{k-1}}^{x_k} u(t_j-, y)\, dy \qquad \text{for all} \;\; x \in [x_{k-1}, \, x_k[\,. \tag{7.6}$$

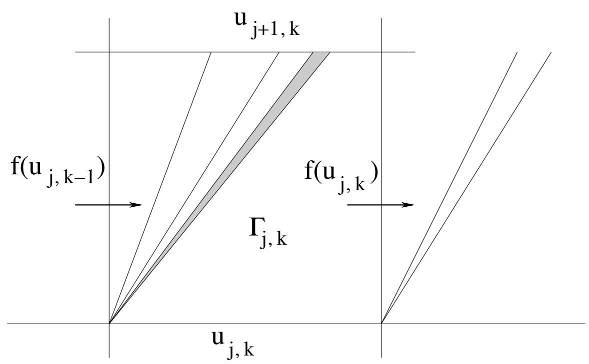

Figure 36: Approximations leading to the Godunov scheme.

Calling u_{jk} the constant value of $u(t_j+)$ on the interval $[x_{k-1},\, x_k[$, an application of the divergence theorem on the square Γ_{jk} (fig. 36) yields

$$u_{j+1,k} = u_{j,k} + [f(u_{j,k-1}) - f(u_{j,k})] \tag{7.7}$$

Indeed, all wave speeds are in $[0, 1]$, hence

$$u(t, x_{k-1}) = u_{j,k-1}, \qquad\qquad u(t, x_k) = u_{j,k} \qquad \text{for all} \quad t \in [t_j, t_{j+1}[\,.$$

The finite difference scheme (7.6) is the simplest version of the Godunov (upwind) scheme. It is very easy to implement numerically, since it does not require the solution of any Riemann problem. Unfortunately, it is very difficult to obtain a priori bounds on the total variation of solutions constructed by the Godunov method. Proving the convergence of these approximations remains an outstanding open problem.

The remaining part of this chapter will be concerned with error bounds, for solutions generated by the Glimm scheme.

Observe that, at each restarting time $t_j = j\,\Delta t$, the replacement of $u(t_j-)$ with the piecewise constant function $u(t_j+)$ produces an error measured by

$$\|u(t_j+) - u(t_j-)\|_{\mathbf{L}^1}$$

As the time step $\Delta t = T/N$ approaches zero, the total sum of all these errors does not converge to zero, in general. This can be easily seen in Example 10, where we have

$$\begin{aligned}\sum_{j=1}^{N} \|u(t_j+) - u(t_j-)\|_{\mathbf{L}^1} &\geq \sum_{j=1}^{N} |u^+ - u^-| \cdot \Delta t \cdot \min\{(1-\lambda),\ \lambda\} \\ &= |u^+ - u^-| \cdot T \cdot \min\{(1-\lambda),\ \lambda\}.\end{aligned}$$

This fact makes it difficult to obtain sharp error estimates for solutions generated by the Glimm scheme. Roughly speaking, the approximate solutions converge to the correct one not because the total errors become small, but because, by a sort of randomness in the sampling choice, small errors eventually cancel each other in the limit.

Clearly, the speed of convergence of the Glimm approximate solutions as $\Delta t, \Delta x \to 0$ strongly depends on how well the sequence $\{\theta_i\}$ approximates a uniform distribution on the interval $[0, 1]$. In this connection, let us introduce

Definition 3. Let a sequence of numbers $\theta_j \in [0, 1]$ be given. For fixed integers $0 \le m < n$, the *discrepancy* of the set $\{\theta_m, \ldots, \theta_{n-1}\}$ is defined as

$$D_{m,n} \doteq \sup_{\lambda \in [0,1]} \left| \lambda - \frac{\#\{j\ ;\ \ m \le j < n,\ \ \theta_j \in [0, \lambda]\ \}}{n - m} \right| . \tag{7.8}$$

We now describe a simple method for defining the numbers θ_j, so that the corresponding discrepancies $D_{m,n}$ approach zero as $n - m \to \infty$ at a nearly optimal rate.

Fix an integer $r \ge 2$. Every integer $k \ge 1$ can then be uniquely written in base r, i.e. as a sum of powers of r of the form

$$k = k_0 + k_1 r + \ldots + k_M r^M \qquad\qquad 0 \le k_i \le r - 1. \tag{7.9}$$

In connection with (7.9) we then define

$$\theta_k \doteq \frac{k_0}{r} + \frac{k_1}{r^2} + \ldots + \frac{k_M}{r^{M+1}} \ \in [0, 1]. \tag{7.10}$$

For example, writing integer numbers in the usual basis $r = 10$, the computation of θ_k is as follows. One simply has to invert the order of the digits of k and put a zero in front:

$$\theta_1 = 0.1\,, \quad \ldots \quad , \quad \theta_{759} = 0.957\,, \quad \ldots \quad , \quad \theta_{39022} = 0.22093\,, \quad \ldots$$

For the sequence (7.9)-(7.10), one can prove that the discrepancies satisfy

$$D_{m,n} \le C \cdot \frac{1 + \ln(n - m)}{n - m} \qquad\qquad \text{for all} \ \ n > m \ge 0 \tag{7.11}$$

for some constant $C = C(r)$. See [17] for details. For approximate solutions constructed in terms of the above sequences (θ_j), using the restarting procedures (7.3)-(7.4), sharp error estimates can be proved.

Theorem 6. *Given any initial data $\bar u \in \mathbf{L}^1$ with small total variation, call $u^{\text{exact}}(t, \cdot) = S_t \bar u$ the exact solution of the Cauchy problem (7.1). Moreover, let $u^{\text{Glimm}}(t, \cdot)$ be the approximate solution generated by the Glimm scheme, in connection with a grid of size $\Delta t = \Delta x$ and a fixed sequence $(\theta_j)_{j \ge 0}$ satisfying (7.11). For every fixed time $T \ge 0$, letting the grid size tend to zero, one has the error estimate*

$$\lim_{\Delta x \to 0} \frac{\|u^{\text{Glimm}}(T, \cdot) - u^{\text{exact}}(T, \cdot)\|_{\mathbf{L}^1}}{\sqrt{\Delta x} \cdot |\ln \Delta x|} = 0\,. \tag{7.12}$$

In other words, the $\mathbf{L}^1$ error tends to zero faster then $\sqrt{\Delta x} \cdot |\ln \Delta x|$, i.e. slightly slower than the square root of the grid size.

To prove Theorem 6, one first constructs a front tracking approximate solution $u = u(t,x)$ that coincides with u^{Glimm} at the initial time $t = 0$ and at the terminal time $t = T$. This construction is based on a fundamental lemma of T.P.Liu [38]. The $\mathbf{L}^1$ distance between $u(T,\cdot)$ and the exact solution $S_T\bar{u}$ can then be estimated using the error formula (10.6). For all details we refer to [17].

Extensions to more general hyperbolic systems, without the assumption of genuine nonlinearity, have recently appeared in [2, 29].

8 The Vanishing Viscosity Approach

In view of the previous uniqueness and stability results, it is natural to expect that the entropy weak solutions of the hyperbolic system

$$u_t + f(u)_x \;=\; 0 \tag{8.1}$$

should coincide with the unique limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\, u^\varepsilon_{xx} \tag{8.2}$$

letting the viscosity coefficient $\varepsilon \to 0$. For smooth solutions, this convergence is easy to show. However, one should keep in mind that a weak solution of the hyperbolic system (8.1) in general is only a function with bounded variation, possibly with a countable number of discontinuities. In this case, as the smooth functions u^ε approach the discontinuous solution u, near points of jump their gradients u^ε_x must tend to infinity (fig. 37), while their second derivatives u^ε_{xx} become even more singular. Therefore, establishing the convergence $u^\varepsilon \to u$ is a highly nontrivial matter. In earlier literature, results in this direction relied on three different approaches:

1 - Comparison principles for parabolic equations. For a scalar conservation law, the existence, uniqueness and global stability of vanishing viscosity solutions was first established by Oleinik [44] in one space dimension. The famous paper by Kruzhkov [32] covers the more general class of $\mathbf{L}^\infty$ solutions and is also valid in several space dimensions.

2 - Singular perturbations. This technique was developed by Goodman and Xin [28], and covers the case where the limit solution u is piecewise smooth, with a finite number of non-interacting, entropy admissible shocks. See also [52] and [45], for further results in this direction.

3 - Compensated compactness. With this approach, introduced by Tartar and DiPerna [24], one first considers a weakly convergent subsequence $u^\varepsilon \rightharpoonup u$. For a class of 2×2 systems, one can show that this weak limit u actually provides a distributional solution to the nonlinear system (8.1). The proof relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies.

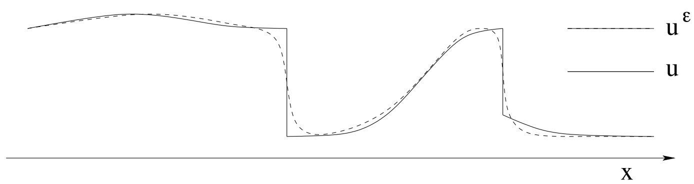

Figure 37: A discontinuous solution to the hyperbolic system and a viscous approximation.

Since the hyperbolic Cauchy problem is known to be well posed within a space of functions with small total variation, it seems natural to develop a theory of vanishing viscosity approximations within the same space BV. This was indeed accomplished in [5], in the more general framework of nonlinear hyperbolic systems not necessarily in conservation form. The only assumptions needed here are the strict hyperbolicity of the system and the small total variation of the initial data.

Theorem 7. *Consider the Cauchy problem for the hyperbolic system with viscosity*

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon\, u^\varepsilon_{xx} \qquad\qquad u^\varepsilon(0,x) = \bar u(x)\,. \tag{8.3}$$

Assume that the matrices $A(u)$ are strictly hyperbolic, smoothly depending on u in a neighborhood of the origin. Then there exist constants C, L, L' and $\delta > 0$ such that the following holds. If

$$\textit{Tot.Var.}\{\bar u\} < \delta\,, \qquad\qquad \|\bar u\|_{\mathbf{L}^\infty} < \delta, \tag{8.4}$$

then for each $\varepsilon > 0$ the Cauchy problem (8.3)$_\varepsilon$ has a unique solution u^ε, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^\varepsilon(t,\cdot) \doteq S^\varepsilon_t \bar u$.

In addition, one has:

BV bounds :
$$\textit{Tot.Var.}\{S^\varepsilon_t \bar u\} \leq C\ \textit{Tot.Var.}\{\bar u\}\,. \tag{8.5}$$

$\mathbf{L}^1$ **stability** :
$$\|S^\varepsilon_t \bar u - S^\varepsilon_t \bar v\|_{\mathbf{L}^1} \leq L\,\|\bar u - \bar v\|_{\mathbf{L}^1}\,, \tag{8.6}$$

$$\|S^\varepsilon_t \bar u - S^\varepsilon_s \bar u\|_{\mathbf{L}^1} \leq L'\left(|t-s| + |\sqrt{\varepsilon t} - \sqrt{\varepsilon s}\,|\right). \tag{8.7}$$

Convergence: *As $\varepsilon \to 0+$, the solutions u^ε converge to the trajectories of a semigroup S such that*

$$\|S_t \bar u - S_s \bar v\|_{\mathbf{L}^1} \leq L\,\|\bar u - \bar v\|_{\mathbf{L}^1} + L'\,|t-s|\,. \tag{8.8}$$

These vanishing viscosity limits can be regarded as the unique **vanishing viscosity solutions** *of the hyperbolic Cauchy problem*

$$u_t + A(u)u_x = 0, \qquad\qquad u(0,x) = \bar u(x)\,. \tag{8.9}$$

In the conservative case $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)\,, \tag{8.10}$$

satisfying the Liu admissibility conditions.

Assuming, in addition, that each characteristic field is genuinely nonlinear or linearly degenerate, the vanishing viscosity solutions coincide with the unique limits of Glimm and front tracking approximations.

In the genuinely nonlinear case, an estimate on the rate of convergence of these viscous approximations was provided in [19]:

Theorem 8. *For the strictly hyperbolic system of conservation laws (8.10), assume that every characteristic field is genuinely nonlinear. At any time $t > 0$, the difference between the corresponding solutions of (8.3) and (8.10) can be estimated as*

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot(1+t)\sqrt{\varepsilon}\,|\ln\varepsilon|\ \text{Tot.Var.}\{\bar{u}\}\,.$$

The following remarks relate Theorem 7 with previous literature. We recall again a standard assumption on the characteristic fields, introduced by P.Lax:

(H) For each $i = 1,\ldots,n$, the i-th field is either *genuinely nonlinear*, so that $D\lambda_i(u)\cdot r_i(u) > 0$ for all u, or *linearly degenerate*, with $D\lambda_i(u)\cdot r_i(u) = 0$ for all u.

(i) Concerning the global existence of weak solutions to the strictly hyperbolic system of conservation laws (8.10), Theorem 7 contains the famous result of J.Glimm [27]. It also slightly extends the existence theorem of T.P.Liu [39], which does not require the assumption (H) and is satisfied by "generic" hyperbolic systems. The main novelty of Theorem 7 lies in the case where the system (8.9) is non-conservative. However, one should be aware that for non-conservative systems the limit solution might change if we choose a viscosity matrix different from the identity, say

$$u_t + A(u)u_x = \varepsilon\left(B(u)u_x\right)_x.$$

The analysis of viscous solutions, with a more general viscosity matrix $B(u)$, is still a wide open problem.

(ii) Concerning the uniform stability of entropy weak solutions, the results previously available for $n\times n$ hyperbolic systems [11, 12, 16] always required the assumption (H). For 2×2 systems, this condition was greatly relaxed in [1]. On the other hand, Theorem 7 establishes this uniform stability for completely general $n\times n$ strictly hyperbolic systems, without any reference to the assumption (H).

(iii) For the viscous system (8.11), previous results in [40, 50, 51, 52] have established the stability of special types of solutions, such as travelling viscous shocks or viscous rarefactions, w.r.t. suitably small perturbations. Taking $\varepsilon = 1$, our present theorem yields at once the uniform Lipschitz stability of all viscous solutions with sufficiently small total variation, w.r.t. the $\mathbf{L}^1$ distance.

(iv) For piecewise smooth solutions of the hyperbolic system (8.1), convergence of vanishing viscosity approximations was studied in [28], using singular perturbation techniques. In this special case, better convergent rates are established. On the other hand, Theorem 8 applies to arbitrary solutions, with the only assumption of small total variation.

We give below a general outline of the proof of Theorem 7. For details, wee [5] or the lecture notes [10].

1 - Rescaling. As a preliminary, we observe that u^ε is a solution of (8.3) if and only if the rescaled function $u(t,x) \doteq u^\varepsilon(\varepsilon t, \varepsilon x)$ is a solution of the parabolic system with unit viscosity

$$u_t + A(u)u_x = u_{xx}\,, \tag{8.11}$$

with initial data $u(0,x) = \bar u(\varepsilon x)$. Clearly, the stretching of the space variable has no effect on the total variation. Notice however that the values of u^ε on a fixed time interval $[0,T]$ correspond to the values of u on the much longer time interval $[0,\ T/\varepsilon]$. To obtain the desired BV bounds for the viscous solutions u^ε, we can confine all our analysis to solutions of (8.11), but we need estimates uniformly valid for all times $t \geq 0$, depending only on the total variation of the initial data $\bar u$.

2 - Parabolic estimates. Consider a solution of (8.11), with initial data

$$u(0,x) = \bar u(x) \tag{8.12}$$

having small total variation, say

$$\text{Tot.Var.}\{\bar u\} \leq \delta_0\,.$$

Setting

$$u^* \doteq \lim_{x\to-\infty} \bar u(x)\,, \qquad\qquad A^* \doteq A(u^*)\,, \tag{8.13}$$

we can rewrite the parabolic equation (8.11) in the form

$$u_t + A^* u_x - u_{xx} \;=\; [A^* - A(u)]u_x\,. \tag{8.14}$$

Call $G^*(t)$ the Green function corresponding to the linear homogeneous equation

$$u_t + A^* u_x - u_{xx} \;=\; 0\,. \tag{8.15}$$

The matrix valued function G^* can be explicitly computed. If w solves (8.15), then the i-th component $w_i \doteq l_i^* \cdot w$ satisfies the scalar equation

$$w_{i,t} + \lambda_i^* w_{i,x} - w_{i,xx} = 0\,.$$

Therefore $w_i(t) = G_i^*(t) * w_i(0)$, where the convolution kernel is

$$G_i^*(t,x) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{(x-\lambda_i^* t)^2}{4t}\right\}.$$

We observe that this Green kernel $G^* = G^*(t,x)$ satisfies the bounds

$$\|G^*(t)\|_{\mathbf{L}^1} \leq \kappa\,, \qquad \|G^*_x(t)\|_{\mathbf{L}^1} \leq \frac{\kappa}{\sqrt{t}}\,, \qquad \|G^*_{xx}(t)\|_{\mathbf{L}^1} \leq \frac{\kappa}{t}\,, \tag{8.16}$$

for some constant κ and all $t > 0$.

The solution of the non-homogeneous Cauchy problem (8.14)-(8.12) now admits the representation

$$u(t) \;=\; G(t) * \bar{u} + \int_0^t G(t-s) * \Big([A^* - A(u(s))]u_x(s)\Big)\, ds\,. \tag{8.17}$$

Notice that, if the total variation of $u(s,\cdot)$ remains small, the right hand side of (8.17) can be regarded as a small perturbation of the homogeneous equation. We thus expect that similar regularity estimates will hold.

Indeed, one can then prove that the Cauchy problem (8.11)-(8.12) has a unique solution, defined on a sufficiently small time interval $[0,\hat{t}]$, with $\hat{t} \approx \delta_0^{-2}$. For $t \in [0,\hat{t}]$ the total variation remains small:

$$\text{Tot.Var.}\{u(t)\} = \|u_x(t)\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0\,,$$

while all higher derivatives decay quickly. In particular

$$\|u_x(t)\|_{\mathbf{L}^\infty}\,,\ \|u_{xx}(t)\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot\frac{\delta_0}{\sqrt{t}}\,, \qquad \|u_{xx}(t)\|_{\mathbf{L}^\infty}\,,\ \|u_{xxx}(t)\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot\frac{\delta_0}{t}\,.$$

As long as the total variation remains small, one can prolong the solution also for larger times $t > \hat{t}$. In this case, uniform bounds on higher derivatives remain valid (fig. 38).

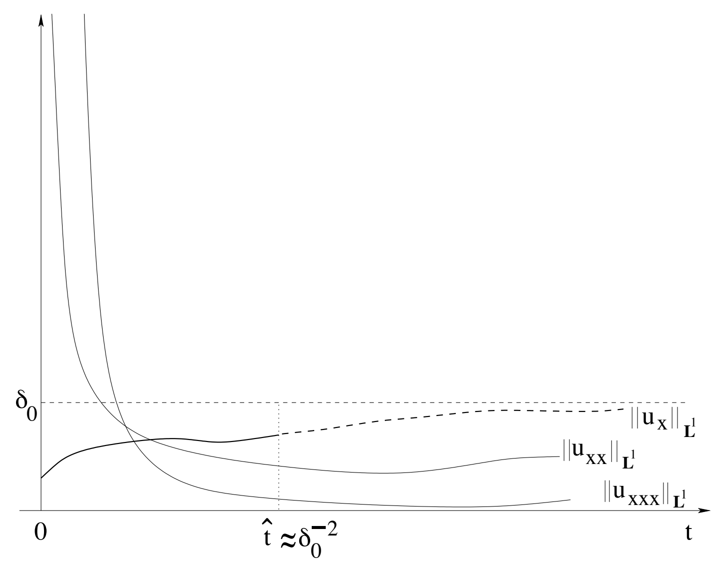

Figure 38: On an initial time interval, the higher order norms of the solution decay rapidly. For t large, these norms remain small as long as the total variation of the solution is small.

3 - Wave decomposition. To establish uniform bounds on the total variation, valid for arbitrarily large times, we decompose the gradient along a basis of unit vectors $\tilde{r}_i = \tilde{r}_i(u, u_x, u_{xx})$, say

$$u_x = \sum_i v_i \tilde{r}_i \tag{8.18}$$

We then derive an evolution equation for these gradient components, of the form

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \qquad\qquad i = 1, \ldots, n\,, \tag{8.19}$$

A crucial point in the entire analysis is the careful choice of the unit vectors $\tilde{r}_i$. A natural guess would be to take $\tilde{r}_i = r_i(u)$ as the i-th eigenvector of the hyperbolic matrix $A(u)$. Unfortunately, this choice does not work. Indeed, the corresponding equation (8.19) would then contain source terms ϕ_i which are NOT integrable:

$$\int_0^\infty \int_{-\infty}^\infty |\phi_i(t,x)|\, dx\, dt \;\; = \;\; +\infty.$$

The key idea, introduced in [5], is to decompose the gradient u_x as a sum of gradients of visocus travelling waves, namely

$$u_x(x) \;\; = \;\; \sum_i v_i \tilde{r}_i = \sum_i U_i'(x)\,. \tag{8.20}$$

We recall that, for the parabolic system (8.11), a travelling wave profile is a solution of the special form

$$u(t,x) = U(x - \sigma t).$$

Of course, the function U must then satisfy the second order O.D.E.

$$U'' = (A(U) - \sigma)U'.$$

At each point x, looking at the third order jet (u, u_x, u_{xx}) we seek travelling profiles $U_1, \ldots, U_n$ such that

$$U_i'' = (A(U_i) - \sigma_i)U_i' \tag{8.21}$$

for some speed σ_i close to the characteristic speed λ_i, and moreover

$$U_i(x) = u(x) \qquad\qquad i = 1, \ldots, n\,, \tag{8.22}$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad\qquad \sum_i U_i''(x) = u_{xx}(x)\,. \tag{8.23}$$

It turns out that this decomposition is unique provided that the travelling profiles are chosen within suitable center manifolds. We let $\tilde{r}_i$ be the unit vector parallel to U_i', so that $U_i' = v_i \tilde{r}_i$ for some scalar v_i. One can show that $\tilde{r}_i$ remains close to the eigenvector $r_i(u)$ of the Jacobian matrix $A(u) \doteq Df(u)$, but $\tilde{r}_i \neq r_i(u)$ in general. The first equation in (8.23) now yields the decomposition (7.11). If $u = u(t,x)$ is a solution of (8.11), we can think of v_i as the density of i-waves in u. The remarkable fact is that these components satisfy a system of evolution equations (8.19) where the source terms ϕ_i on the right hand side are INTEGRABLE over the whole domain $\{x \in \mathbb{R}\,,\ t > 0\}$. Indeed, we can think of the sources ϕ_i as new waves produced by interactions between viscous waves. Their total strength is controlled by means of viscous interaction functionals, somewhat similar to the one introduced by Glimm in [G] to study the hyperbolic case.

4 - Bounds on the source terms. A careful examination of the source terms ϕ_i in (8.19) shows that they arise mainly because of interactions among different viscous waves. Their total strength can thus be estimated in terms of suitable interaction potentials. This will allow us to prove the implication

$$\begin{aligned} &\|u_x(t)\|_{\mathbf{L}^1} \leq \delta_0 \quad \text{for all } \ t\in[\hat t,T], \qquad \sum_i \int_{\hat t}^T \int |\phi_i(t,x)|\,dxdt \leq \delta_0 \\ &\Longrightarrow \quad \int_{\hat t}^T \int |\phi_i(t,x)|\,dxdt = \mathcal{O}(1)\cdot\delta_0^2 \qquad i=1,\ldots,n\,. \end{aligned} \tag{8.24}$$

Since the left hand side of (8.19) is in conservation form, for every $t\in[\hat t,\,T]$ one has

$$\|u_x(t)\|_{\mathbf{L}^1} \;\leq\; \sum_i \|v_i(t)\|_{\mathbf{L}^1} \;\leq\; \sum_i \left(\|v_i(\hat t)\|_{\mathbf{L}^1} + \int_{\hat t}^t \int |\phi_i(s,x)|\,dxds\right). \tag{8.25}$$

By the implication (8.24), for δ_0 sufficiently small we obtain uniform bounds on the total strength of the source terms ϕ_i, and hence on the BV norm of $u(t,\cdot)$.

5 - Analysis of the linearized variational equation. Similar techniques can also be applied to a solution $z=z(t,x)$ of the variational equation

$$z_t + [DA(u)\cdot z]u_x + A(u)z_x \;=\; z_{xx}\,, \tag{8.26}$$

which describes the evolution of a first order perturbation to a solution u of (8.12). Assuming that the total variation of u remains small, one proves an estimate of the form

$$\|z(t,\cdot)\|_{\mathbf{L}^1} \leq L\,\|z(0,\cdot)\|_{\mathbf{L}^1} \qquad \text{for all } \ t\geq 0\,, \tag{8.27}$$

valid for every solution u of (8.11) having small total variation and every $\mathbf{L}^1$ solution of the corresponding system (8.26).

6 - Lipschitz continuity w.r.t. the initial data. Relying on (8.27), a standard homotopy argument yields the Lipschitz continuity of the flow of (8.11) w.r.t. the initial data, uniformly in time. Indeed, let any two solutions u,v of (8.11) be given (fig. 39). We can connect them by a smooth path of solutions u^θ, whose initial data satisfy

$$u^\theta(0,x) \doteq \theta u(0,x) + (1-\theta)v(0,x)\,, \qquad \theta\in[0,1]\,.$$

The distance $\|u(t,\cdot)-v(t,\cdot)\|_{\mathbf{L}^1}$ at any later time $t>0$ is clearly bounded by the length of the path $\theta\mapsto u^\theta(t)$. In turn, this can be computed by integrating the norm of a tangent vector. Calling $z^\theta \doteq du^\theta/d\theta$, each vector z^θ is a solution of the corresponding equation (8.26), with u replaced by u^θ. Using (8.28) we thus obtain

$$\begin{aligned} \|u(t,\cdot)-v(t,\cdot)\|_{\mathbf{L}^1} \;&\leq\; \int_0^1 \left\|\frac{d}{d\theta}u^\theta(t)\right\|_{\mathbf{L}^1} d\theta \;\leq\; \int_0^1 \|z^\theta(t)\|_{\mathbf{L}^1}\,d\theta \\ &\leq\; L\textstyle\int_0^1 \|z^\theta(0)\|_{\mathbf{L}^1}\,d\theta \;=\; L\,\|u(0,\cdot)-v(0,\cdot)\|_{\mathbf{L}^1}\,. \end{aligned} \tag{8.28}$$

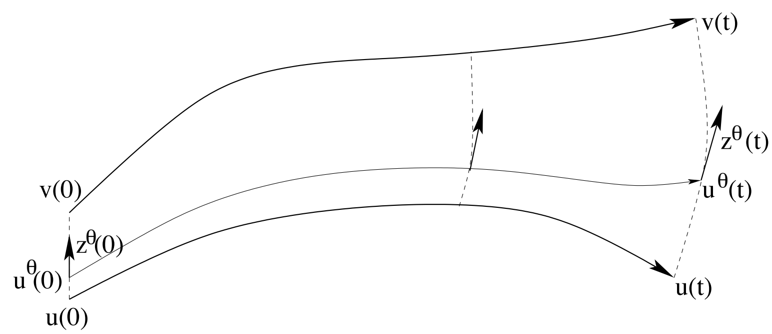

Figure 39: Estimating the distance between two viscous solution by a homotopy method.

By the simple rescaling of coordinates $t \mapsto \varepsilon t$, $x \mapsto \varepsilon x$, all of the above estimates remain valid for solutions u^ε of the system $(8.3)_\varepsilon$. In particular this yields the bounds (8.5) and (8.6).

7 - Existence of a vanishing viscosity limit. By a compactness argument, these BV bounds imply the existence of a strong limit $u^{\varepsilon_m} \to u$ in $\mathbf{L}^1_{\text{loc}}$, at least for some subsequence $\varepsilon_m \to 0$. In the conservative case where $A = Df$, it is now easy to show that this limit u provides a weak solution to the Cauchy problem (8.10).

8 - Characterization and uniqueness of the vanishing viscosity limit. At this stage, it only remains to prove that the limit is unique, i.e. it does not depend on the choice of the sequence $\varepsilon_m \to 0$. For a system in conservative form, and with the standard assumption (H) that each field is either genuinely nonlinear or linearly degenerate, we can apply the uniqueness theorem in [13] and conclude that the limit of vanishing viscosity approximations coincides with the limit of Glimm and of front tracking approximations.

9 - The non-conservative case. To handle the general non-conservative case, some additional work is required. We first consider Riemann initial data and show that in this special case the vanishing viscosity solution is unique and can be accurately described. Then we prove that any weak solution obtained as limit vanishing viscosity approximations is also a "viscosity solution" in the sense that it satisfies the local integral estimates (6.14)-(6.18), where $U^\sharp$ is now the unique non-conservative solution of a Riemann problem. By an argument introduced in [8], a Lipschitz semigroup is completely determined as soon as one specifies its local behavior for piecewise constant initial data. Characterizing its trajectories as "viscosity solutions" we thus obtain the uniqueness of the semigroup of vanishing viscosity limits.

9 Counterexamples

In the previous chapters, the global existence and uniqueness theory for weak solutions to systems of conservation laws was developed under certain key assumptions. In particular:

- The system is strictly hyperbolic.

- The initial data is small in BV.

In this final chapter, we consider three examples showing that:

1. If the system is not strictly hyperbolic, BV solutions may not depend continuously on the initial data, in the $\mathbf{L}^1$ norm.

2. If the initial data is in $\mathbf{L}^\infty$ but with possibly unbounded variation, then the solution may become measure-valued in finite time. In such case, the uniqueness and continuous dependence of solutions may also fail.

3. If the total variation of the initial data is large, then the $\mathbf{L}^\infty$ norm of the solution may blow up in finite time. In particular, no global solution exists in BV.

Example 11. Consider the following simple model of traffic flow. Denote by x the space coordinate along a highway. Call $k(x)$ the number of lanes open to traffic. Typically, one will have $k(x) = 2$ for most x, except along some short stretches of the highway where $k(x) = 1$ because one lane is closed due to repair work. Calling $u = u(t,x)$ the density of cars (= number of cars per kilometer), we consider the system

$$\begin{cases} u_t + f(k,u)_x &= 0, \\ k_t &= 0, \end{cases} \qquad\qquad f(k,u) \doteq 8ku - u^2. \tag{9.1}$$

Consider first the family of stationary solutions

$$u^\varepsilon(t,x) = \begin{cases} 3 & \text{if} \quad x \in [0,\varepsilon], \\ 1 & \text{if} \quad x \notin [0,\varepsilon], \end{cases} \qquad\qquad k_\varepsilon(t,x) = \begin{cases} 1 & \text{if} \quad x \in [0,\varepsilon], \\ 2 & \text{if} \quad x \notin [0,\varepsilon], \end{cases}$$

Clearly, as $\varepsilon \to 0$, the functions $u^\varepsilon, k^\varepsilon$ converge in $\mathbf{L}^1_{\text{loc}}$ to the constant solutions

$$u(t,x) \equiv 1, \qquad\qquad k(t,x) \equiv 2.$$

On the other hand, consider now the family of (time-dependent) solutions (fig. 40)

$$(k^\varepsilon, u^\varepsilon)(t,x) = \begin{cases} \omega_0 & \text{if} \quad x/t < -6/(3+2\sqrt{3}), \\ \omega_1 & \text{if} \quad -6/(3+2\sqrt{3}) < x/t < 0, \\ \omega_2 & \text{if} \quad 0 < x < \varepsilon, \\ \omega_3 & \text{if} \quad 0 < (x-\varepsilon)/t < 6/(2\sqrt{3}-3), \\ \omega_0 & \text{if} \quad 6/(2\sqrt{3}-3) < (x-\varepsilon)/t, \end{cases}$$

where (fig. 41)

$$\omega_0 = (2,\ 2), \qquad \omega_1 = (8+4\sqrt{3},\ 2), \qquad \omega_2 = (4,\ 1), \qquad \omega_3 = (8-4\sqrt{3},\ 2).$$

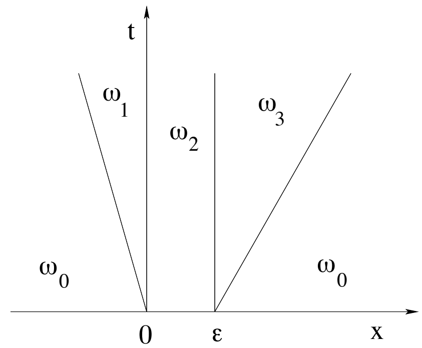

Figure 40: For a system which is not strictly hyperbolic, the solution may not depend continuously on the initial data.

Observe that, as $\varepsilon \to 0$, the initial data converges to the constant function

$$\bar{u}(x) \equiv 2, \qquad \bar{k}(x) \equiv 2.$$

However, for times $t > 0$, the solutions u^ε do not converge to the corresponding constant solution

$$u(t,x) \equiv 2, \qquad k(t,x) \equiv 2.$$

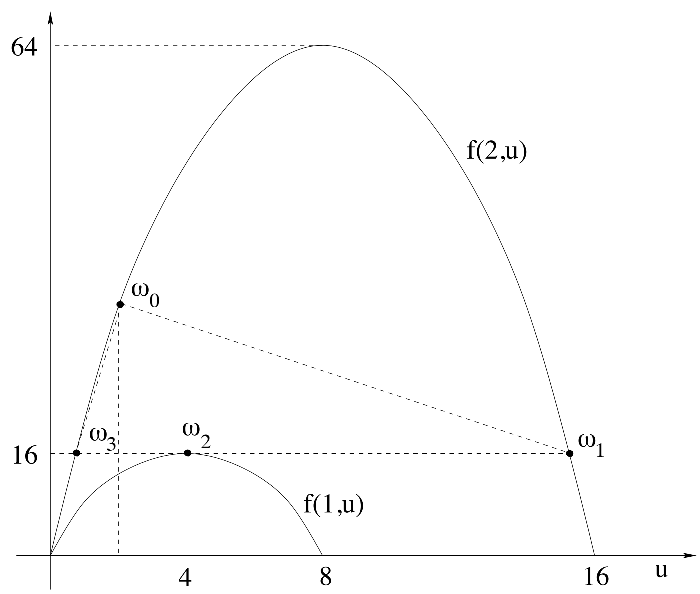

Figure 41: Different flux functions corresponding to $k = 1$ and $k = 2$.

Example 12. We now consider the Cauchy problem for a strictly hyperbolic 3×3 system of the form

$$\begin{cases} u_t + u_x &= 0, \\ v_t - v_x &= 0, \\ w_t + [\phi(u,v) \cdot w]_x &= 0, \end{cases} \qquad \begin{cases} u(0,x) &= \bar{u}(x), \\ v(0,x) &= \bar{v}(x), \\ w(0,x) &= \bar{w}(x). \end{cases} \tag{9.2}$$

for some smooth function $\phi : I\!R^2 \mapsto [-1/2,\ 1/2]$. Observe that the above system is strictly hyperbolic, with characteristic speeds

$$\lambda_1 = -1, \qquad \lambda_2 = \phi(u,v), \qquad \lambda_3 = 1.$$

All fields are linearly degenerate.

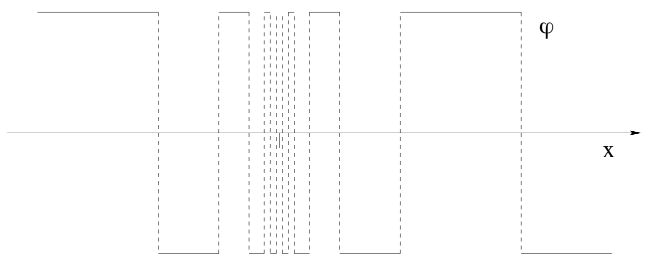

Figure 42: The initial data considered at (9.4).

If the initial data $\bar u, \bar v$ are in BV, then by a result in [6] it follows that the O.D.E. for characteristics

$$\dot x = \psi(t,x) \doteq \phi(u(t,x), v(t,x)) \tag{9.3}$$

has unique solutions. On the other hand, it is possible to choose $\bar u, \bar v \notin BV$ and $\phi \in \mathcal{C}^\infty$ such that the corresponding O.D.E. (9.3) has multiple (forward and backward) solutions through a given point (fig. 43). In this case, if we look at the conservation equation for w, we see that all the mass initially located within the interval $[a,b]$ will be concentrated at the single point at time τ. The $\mathbf{L}^\infty$ norm of the solution thus blows up, and the solution itself becomes measure-valued. After time τ, the solution may be prolonged as a measure-valued solution in many different ways. Indeed, the Dirac mass at P may be split among different characteristics emerging from P, or else continued as a single Dirac mass moving along any one particular characteristic originating from P.

An example where this happens is the following. Take (fig. 42)

$$\bar u(x) = \bar v(x) = \varphi(x) \doteq \begin{cases} 1 & \text{if} \quad 2^{-2n-1} < |x| < 2^{-2n} \\ -1 & \text{if} \quad 2^{-2n} < |x| < 2^{-2n+1} \end{cases} \tag{9.4}$$

$$\phi(u,v) \doteq \frac{1-uv}{6}\,.$$

In this case, it is easy to see that the O.D.E. (9.3) admits infinitely many solutions through the origin (fig. 43).

Example 13. Following [31], consider a 3×3 system of the form

$$U_t + F(U)_x = 0, \tag{9.5}$$

where $U = (u,v,w)$ and

$$F(U) = F(u,v,w) = \begin{pmatrix} uv + w \\ g(v) \\ u(1-v^2) - vw \end{pmatrix} \tag{9.6}$$

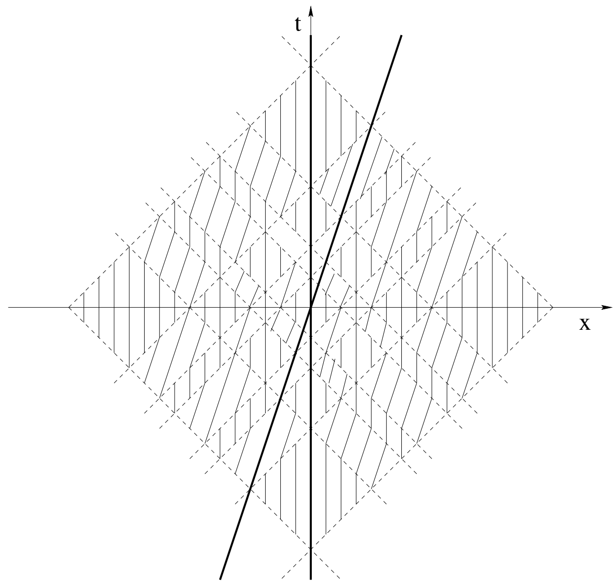

Figure 43: Since the total variation of the solution to (9.2)-(9.4) is unbounded, the O.D.E. for characteristics can have multiple forward and backward solutions through a given point.

We assume that the scalar function g is strictly convex, with

$$g(0) = 0, \qquad g(v) = g(-v), \qquad -1 < g'(v) < 1 \tag{9.7}$$

for all v. It is then clear that the system is strictly hyperbolic in the region where $|v| < 1$. Indeed

$$A(U) \doteq DF(u, v, w) = \begin{pmatrix} v & u & 1 \\ 0 & g'(v) & 0 \\ 1 - v^2 & 2uv - w & -v \end{pmatrix}$$

is a matrix with real distinct eigenvalues

$$\lambda_1 = -1, \qquad \lambda_2 = g'(v), \qquad \lambda_3 = 1.$$

The corresponding right eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ 0 \\ -1 - v \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad r_3 = \begin{pmatrix} 1 \\ 0 \\ 1 - v \end{pmatrix}.$$

The first and third characteristic fields are linearly degenerate, the second is genuinely nonlinear.

By a suitable choice of the function g, the analysis in [31] shows that one can construct a solution $U = U(t, x)$ of the following form (fig. 44). The initial data contains two approaching 2-shocks, and a 1-wave between them, of unit strength. Subsequent interactions produce alternatively a 3-wave and a 1-wave, with strengths given as in fig. 44, with $\beta > 1$. The $\mathbf{L}^\infty$ norm of the solution, and hence also its total variation, approach infinity as $t \to T-$, the time where the two 2-shocks hit each other.

This example shows that, in general, if the initial data are large enough, the solution can blow up in finite time. One should remark, however, that the system (9.6) does not come from any

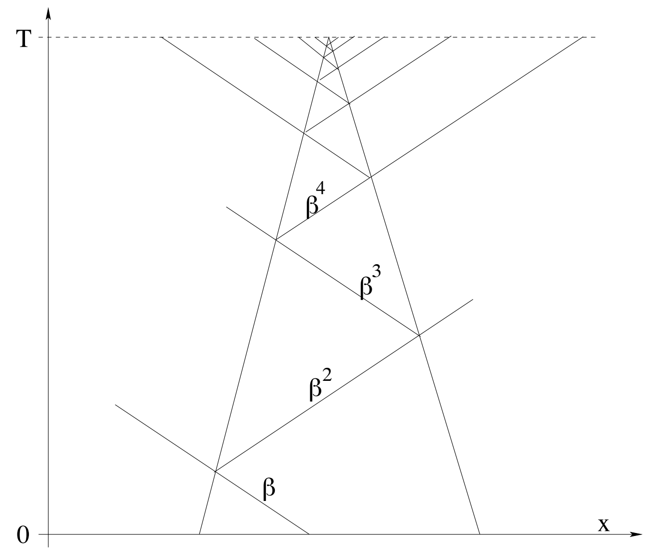

Figure 44: Structure of a solution where the total variation blows up in finite time.

realistic physical model. In particular, it does not admit any strictly convex entropy. It is a major open problem to establish whether blow up in the total variation norm can occur for solutions to the equations of gas dynamics.

10 Appendix

We collect here some results of basic mathematical analysis, which were used in previous sections.

A.1 - A compactness theorem

We state below a version of Helly's compactness theorem, which provides the basic tool in the proof of existence of weak solutions.

Theorem A.1. *Consider a sequence of functions* $u_\nu : [0,\infty[\,\times I\!R \mapsto I\!R^n$ *with the following properties.*

$$\text{Tot.Var.}\{u_\nu(t,\cdot)\} \leq C, \qquad |u_\nu(t,x)| \leq M \qquad \text{for all } t,x, \tag{10.1}$$

$$\int_{-\infty}^{\infty} |u_\nu(t,x) - u_\nu(s,x)|\,dx \leq L|t-s| \qquad \text{for all } t,s \geq 0, \tag{10.2}$$

for some constants C, M, L. *Then there exists a subsequence* u_μ *which converges to some function* u *in* $\mathbf{L}^1_{\text{loc}}([0,\infty) \times I\!R;\ I\!R^n)$. *This limit function satisfies*

$$\int_{-\infty}^{\infty} |u(t,x) - u(s,x)|\,dx \leq L|t-s| \qquad \text{for all } t,s \geq 0. \tag{10.3}$$

The point values of the limit function u can be uniquely determined by requiring that

$$u(t,x) = u(t,x+) \doteq \lim_{y\to x+} u(t,y) \qquad \textit{for all } t,x. \tag{10.4}$$

In this case, one has

$$\text{Tot.Var.}\{u(t,\cdot)\} \le C, \qquad |u(t,x)| \le M \qquad \text{for all } t,x. \tag{10.5}$$

A detailed proof can be found in [9].

A.2 - An elementary error estimate

Let $\mathcal{D}$ be a closed subset of a Banach space E and consider a Lipschitz continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$. More precisely, assume that

(i) $S_0 u = u, \qquad S_s S_t u = S_{s+t} u,$

(ii) $\|S_t u - S_s v\| \le L \cdot \|u - v\| + L' \cdot |t-s|.$

Given a Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}$, we wish to estimate the difference between w and the trajectory of the semigroup S starting at $w(0)$.

Theorem A.2. *Let $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ be a continuous flow satisfying the properties (i)-(ii). For every Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}$ one then has the estimate*

$$\|w(T) - S_T w(0)\| \le L \cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\|w(t+h) - S_h w(t)\|}{h} \right\} dt. \tag{10.6}$$

Remark 8. The integrand in (10.6) can be regarded as the instantaneous error rate for w at time t. Since the flow is uniformly Lipschitz continuous, during the time interval $[t,T]$ this error is amplified at most by a factor L (see fig. 45).

Proof. We first show that the integrand function

$$\phi(t) \doteq \liminf_{h\to 0+} \frac{\|w(t+h) - S_h w(t)\|}{h}$$

is bounded and measurable. Indeed, for every $h > 0$, the function

$$\phi_h(t) \doteq \frac{\|w(t+h) - S_h w(t)\|}{h} \tag{10.7}$$

is continuous. By the continuity of the maps $h \mapsto \phi_h(t)$, we have

$$\phi(t) = \lim_{\varepsilon\to 0+} \inf_{h\in \mathbf{Q}\cap]0,\varepsilon]} \phi_h(t),$$

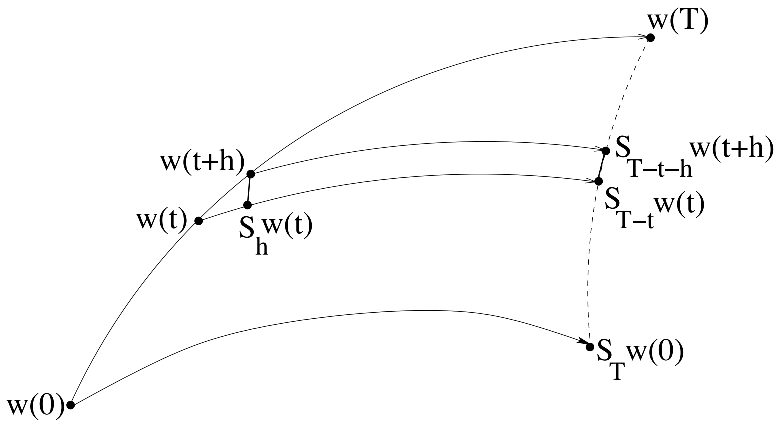

Figure 45: Comparing the approximate solution w with the trajectory of the semigroup having the same initial data.

where the infimum is taken only over rational values of h. Hence the function ϕ is Borel measurable. Its boundedness is a consequence of the uniform Lipschitz continuity of w and of all trajectories of the semigroup S.

Next, define

$$\Phi(t) \doteq \|S_{T-t}w(t) - S_T w(0)\|, \tag{10.8}$$

$$z(t) \doteq \Phi(t) - L\int_0^t \phi(s)\, ds. \tag{10.9}$$

We need to show that $z(T) \leq 0$. Since z is Lipschitz continuous and $z(0) = 0$, it suffices to show that the time derivative satisfies $\dot{z}(s) \leq 0$ for almost every $s \in [0, T]$. By the theorems of Rademacher and of Lebesgue, there exists a null set $\mathcal{N} \subset [0, T]$ such that, for each $s \notin \mathcal{N}$, both functions z, Φ are differentiable at time s while ϕ is quasicontinuous at s. For $s \notin \mathcal{N}$ we thus have

$$\dot{z}(s) \;=\; \lim_{h\to 0} \frac{z(s+h) - z(s)}{h} \;=\; \lim_{h\to 0} \frac{\Phi(s+h) - \Phi(s)}{h} - L \cdot \phi(s). \tag{10.10}$$

By the definition of Φ and the properties of S it follows

$$\begin{aligned}
\Phi(s+h) - \Phi(s) \;&=\; \|S_{T-s-h}w(s+h) - S_T w(0)\| - \|S_{T-s}w(s) - S_T w(0)\| \\
&\leq\; \|S_{T-s-h}w(s+h) - S_{T-s}w(s)\| \\
&\leq\; \|S_{T-s-h}w(s+h) - S_{T-s-h}S_h w(s)\| \\
&\leq\; L \cdot \|w(s+h) - S_h w(s)\|\,.
\end{aligned}$$

Therefore

$$\lim_{h\to 0+} \frac{\Phi(s+h) - \Phi(s)}{h} \;\leq\; L \cdot \liminf_{h\to 0+} \frac{\|w(s+h) - S_h w(s)\|}{h} = L \cdot \phi(s).$$

By (10.10) this implies

$$\dot{z}(s) \leq 0 \qquad\qquad \text{for all} \quad s \notin \mathcal{N}.$$

Hence $z(T) \leq 0$, proving the theorem. □

A.3 - Taylor estimates

This final section contains some useful bounds on the size of a function Ψ of several variables, valid in a neighborhood of the origin. The assumption that Ψ vanishes on certain sets, such as the coordinate axes or the diagonals, implies that some of the Taylor coefficients of Ψ at the origin are equal to zero. Writing a Taylor approximation and using this additional information, one obtains the desired estimates. As usual, the Landau notation $\mathcal{O}(1)$ here indicates a function whose absolute value remains uniformly bounded.

Lemma A.1. *Let $\Phi : \mathbb{R}^m \mapsto \mathbb{R}^n$ be differentiable with Lipschitz continuous derivative. If*

$$\Phi(0) = 0, \qquad \frac{\partial \Phi}{\partial x}(0) = 0, \tag{10.11}$$

then one has

$$\Phi(x) = \mathcal{O}(1) \cdot |x|^2. \tag{10.12}$$

Lemma A.2. *Let $\Phi : \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^n$ be twice differentiable, with Lipschitz continuous second derivatives. If*

$$\Phi(x,0) = \Phi(0,y) = 0 \qquad \textit{for all} \;\; x \in \mathbb{R}^p, \;\, y \in \mathbb{R}^q, \tag{10.13}$$

then

$$\Phi(x,y) = \mathcal{O}(1) \cdot |x|\,|y| \tag{10.14}$$

for all x, y in a neighborhood of the origin. If in addition there holds

$$\frac{\partial^2 \Phi}{\partial x \partial y}(0,0) = 0, \tag{10.15}$$

then one has the sharper estimate

$$\Phi(x,y) = \mathcal{O}(1) \cdot |x|\,|y|\,(|x| + |y|). \tag{10.16}$$

Proof. Observe that (10.13) implies

$$\begin{aligned} \Phi(x,y) &= \int_0^1 \left(\frac{\partial \Phi}{\partial y}(x, \theta y) \right) \cdot y \, d\theta \\ &= \int_0^1 \int_0^1 \left(\frac{\partial^2 \Phi}{\partial x \partial y}(\theta' x, \;\theta y) \right) \cdot (x \otimes y) \, d\theta' d\theta \\ &= \mathcal{O}(1) \cdot |x|\,|y|. \end{aligned} \tag{10.17}$$

This gives (10.14). If (10.15) also holds, then the Lipschitz continuity of the second derivative implies

$$\left| \frac{\partial^2 \Phi}{\partial x \partial y}(x,y) \right| = \mathcal{O}(1) \cdot (|x| + |y|).$$

Using this bound inside the double integral in (10.17), we obtain (10.16). □

Lemma A.3. *Let* $\Phi = \Phi(\tilde q, q^*, \sigma)$ *be a* $\mathcal{C}^2$ *mapping from* $I\!R^{n-1} \times I\!R \times I\!R$ *into* $I\!R$, *with Lipschitz continuous second derivatives. Assume that, for all* $\tilde q \in I\!R^{n-1}$, $s, q^*, \sigma \in I\!R$ *there holds*

$$\Phi(\tilde q, q^*, 0) = \Phi(0, s, -s) = \Phi(0, 0, \sigma) = 0. \tag{10.18}$$

Then one has the estimate

$$\Phi(\tilde q, q^*, \sigma) = \mathcal{O}(1) \cdot (|\tilde q|\,|\sigma| + |q^*|\,|\sigma|\,|q^* + \sigma|). \tag{10.19}$$

If, in addition, for all q^*, σ *there holds*

$$\Phi(0, q^*, \sigma) = 0, \tag{10.20}$$

then one has

$$\Phi(\tilde q, q^*, \sigma) = \mathcal{O}(1) \cdot |\tilde q|\,|\sigma|. \tag{10.21}$$

Proof. To derive (10.19) we write

$$\Phi(\tilde q, q^*, \sigma) = \Phi(0, q^*, \sigma) + [\Phi(\tilde q, q^*, \sigma) - \Phi(0, q^*, \sigma)]. \tag{10.22}$$

By (10.18), for all $\tilde q, q^*$ one has

$$\frac{\partial \Phi}{\partial \tilde q}(\tilde q, q^*, 0) = 0.$$

Hence, by the Lipschitz continuity of the first derivative,

$$\frac{\partial \Phi}{\partial \tilde q}(\tilde q, q^*, \sigma) = \mathcal{O}(1) \cdot |\sigma|.$$

Using this last estimate we obtain

$$\begin{aligned} \Phi(\tilde q, q^*, \sigma) - \Phi(0, q^*, \sigma) &= \int_0^1 \left(\frac{\partial \Phi}{\partial \tilde q}(\theta \tilde q, q^*, \sigma) \right) \cdot \tilde q \, d\theta \\ &= \mathcal{O}(1) \cdot |\tilde q|\,|\sigma|. \end{aligned} \tag{10.23}$$

To establish (10.19) it thus remains to prove

$$\Phi(0, q^*, \sigma) = \mathcal{O}(1) \cdot |q^*|\,|\sigma|\,|q^* + \sigma|. \tag{10.24}$$

Observe that (10.18) implies

$$\frac{\partial^2 \Phi}{\partial q^* \partial \sigma}(0, 0, 0) = 0.$$

In the case where q^* and σ have the same sign, one has $|q^* + \sigma| = |q^*| + |\sigma|$. Therefore (10.24) follows from Lemma A.2, being equivalent to the estimate (10.16).

To see what happens when q^* and σ have opposite signs, to fix the ideas assume $q^* < 0 < \sigma$, $|q^*| < \sigma$, the other cases being entirely similar. We consider two possibilities:

If $|q^*| < \sigma/2$, then

$$|q^* + \sigma| \geq \frac{|\sigma|}{2} \geq \frac{|q^*| + |\sigma|}{3}$$

and (10.24) again follows from (10.16) in Lemma A.2.

In the remaining case where $\sigma/2 \leq |q^*| \leq \sigma$, we write

$$\begin{aligned}
\Phi(0, q^*, \sigma) &= \Phi(0, -\sigma, \sigma) + \int_{-\sigma}^{q^*} \frac{\partial \Phi}{\partial q^*}(0, s, \sigma)\, ds \\
&= 0 + \int_{-\sigma}^{q^*} \int_0^{\sigma} \frac{\partial^2 \Phi}{\partial q^* \partial \sigma}(0, s, s')\, ds ds' \\
&= \int_{-\sigma}^{q^*} \int_0^{\sigma} \mathcal{O}(1) \cdot (|s| + |s'|)\, ds ds' \\
&= \mathcal{O}(1) \cdot |\sigma + q^*|\, |\sigma|\, (|\sigma| + |\sigma|).
\end{aligned}$$

This yields again (10.24), completing the proof of (10.19).

If the additional assumption (10.20) holds, then the estimate (10.21) is an immediate consequence of (10.22) and (10.23). □

Lemma A.4. *Let $\Phi = \Phi(\tilde{q}, q^*, \sigma)$ be a $\mathcal{C}^1$ mapping from $I\!R^{n-1} \times I\!R \times I\!R$ into $I\!R$, with Lipschitz continuous derivatives.*

(a) If Φ satisfies the assumptions

$$\frac{\partial \Phi}{\partial q^*}(0,0,0) = \frac{\partial \Phi}{\partial \sigma}(0,0,0) = \frac{1}{2}, \qquad \Phi(0, s, -s) = 0 \quad \textit{for all } s. \tag{10.25}$$

then one has the estimate

$$\Phi(\tilde{q}, q^*, \sigma) = \frac{q^* + \sigma}{2} + \mathcal{O}(1) \cdot \Big(|\tilde{q}| + |q^* + \sigma| (|q^*| + |\sigma|) \Big). \tag{10.26}$$

(b) If instead Φ satisfies the assumptions

$$\frac{\partial \Phi}{\partial q^*}(0,0,0) = \frac{1}{2}, \qquad \Phi(0, 0, \sigma) = 0 \quad \textit{for all } \sigma, \tag{10.27}$$

then one has the estimate

$$\Phi(\tilde{q}, q^*, \sigma) = \frac{q^*}{2} + \mathcal{O}(1) \cdot \Big(|\tilde{q}| + |q^*| (|q^*| + |\sigma|) \Big). \tag{10.28}$$

Proof. We write Φ in the form

$$\Phi(\tilde{q}, q^*, \sigma) = [\Phi(\tilde{q}, q^*, \sigma) - \Phi(0, q^*, \sigma)] + \Phi(0, q^*, \sigma).$$

The first term on the right hand side clearly satisfies

$$\Phi(\tilde{q}, q^*, \sigma) - \Phi(0, q^*, \sigma) = \mathcal{O}(1) \cdot |\tilde{q}|. \tag{10.29}$$

To estimate the second term, in case *(a)* we define $s \doteq (q^* - \sigma)/2$ and compute

$$\begin{aligned} \Phi(0, q^*, \sigma) &= \Phi(0, s, -s) + \int_0^{(q^*+\sigma)/2} \left[\frac{d}{d\theta}\Phi(0,\ s+\theta,\ \ -s+\theta)\right] d\theta \\ &= \int_0^{(q^*+\sigma)/2} \Big[1 + \mathcal{O}(1) \cdot (|q^*| + |\sigma|)\Big] \, d\theta \qquad (10.30) \\ &= \frac{q^* + \sigma}{2} + \mathcal{O}(1) \cdot |q^* + \sigma| \, (|q^*| + |\sigma|). \end{aligned}$$

Combining (10.29) with (10.30) one obtains (10.26).

In case *(b)*, we write

$$\begin{aligned} \Phi(0, q^*, \sigma) &= \int_0^{q^*} \left[\frac{d}{d\theta}\Phi(0,\ \theta,\ \sigma)\right] d\theta \\ &= \int_0^{q^*} \left[\frac{1}{2} + \mathcal{O}(1) \cdot (|q^*| + |\sigma|)\right] d\theta \qquad (10.31) \\ &= \frac{q^*}{2} + \mathcal{O}(1) \cdot |q^*| \, (|q^*| + |\sigma|). \end{aligned}$$

Combining (10.29) with (10.31) one obtains (10.28). □

Remark 9. In the above lemmas, assume that the functions Φ and their derivatives depend continuously on an additional parameter ω and satisfy the given assumptions for all values of ω. Then all of the above estimates are still valid, for quantities $\mathcal{O}(1)$ which remain uniformly bounded as ω ranges on compact sets.

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