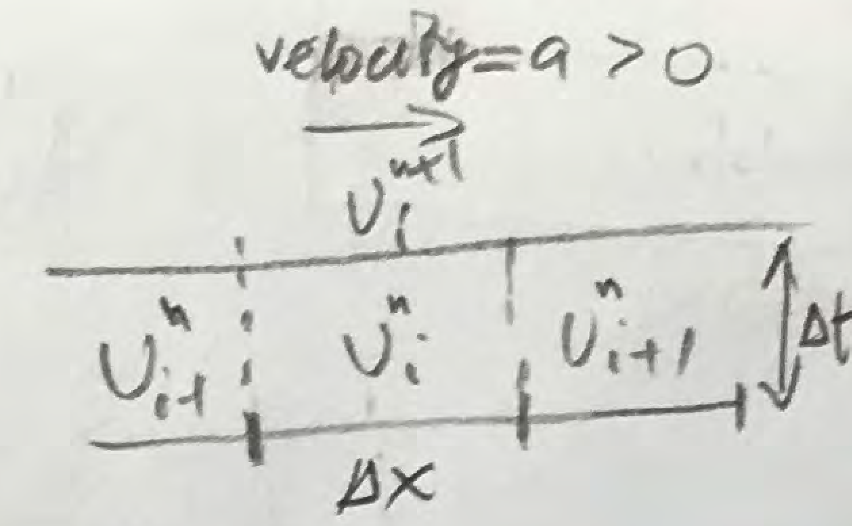
Lecture #8 – $\boxed{\text{motivation}}$ Math 638 – 2016



(*) Look at Godunov 1-d $f(u) = au$, $a > 0$, $a = \text{const.}$

Then $U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta X}\left(f(U_i) - f(U_{i-1})\right)$

$$\underline{U_i^{n+1} = U_i^n - \lambda \cdot a U_i^n + \lambda a U_{i-1}^n} \to \boxed{\text{form the PDE discretization}}$$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{aU_i^n - aU_{i-1}^n}{\Delta X} = 0$$

$$\underbrace{\frac{U_i^{n+1} - U_i^n}{\Delta t}}_{\partial_t u} + \underbrace{\frac{aU_{i+1}^n - aU_{i-1}^n}{2\Delta X}}_{\partial_x(au)} + \underbrace{\frac{-aU_{i-1}^n + 2aU_i^n - aU_{i+1}^n}{2\Delta X}}_{\text{perturbation terms}} = 0$$

"central" = "Galerkin"
derivative discretization

Viscosity

(*) $$\boxed{\frac{U_i^{n+1} - U_i^n}{\Delta t}} + \boxed{\frac{aU_{i+1}^n - aU_{i-1}^n}{2\Delta X}} = \boxed{\frac{a \cdot \Delta X}{2}} \cdot \boxed{\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta X^2}}$$

$$\partial_t u + \partial_x f(u) = \beta \frac{\Delta X}{2} \Delta u \quad \Bigg\| \quad \sum_{j=i-1}^{i+1} d_{ij} U_j^n$$

$\underline{\beta = \text{max lolal speed}}$

PDE: $\underline{\partial_t u + \partial_x f(u) = \partial_x\left(\underline{\underline{\varepsilon(u)}}\, u_x\right)}$

(*) Forward Euler time step (+) $\underline{\text{mass lumping}}$ FEM-term (+) $\boxed{\text{discrete artificial diffusion}}$

(some) FEM $\Leftrightarrow$ FDM on uniform meshes.

(!) If we repeat (*)—(*) with $a < 0 \Rightarrow$ Viscosity $= |a| \frac{\Delta X}{2} \cdot \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta X^2}$

Math 638 – 2016 Lecture #8

Invariant domain preserving schemes

1) $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$, $\vec{u} = \vec{u}(x,t)$ $x \in R^d$, $t > 0$

2) $\mathbf{f}(u)$ takes values in $(\mathbb{R}^m)^d$, i.e. $\mathbf{f}(u) = (\vec{f}_1(u), \vec{f}_2(u), \dots, \vec{f}_d(u))$

and $\vec{f}_j(u) = \begin{pmatrix} f_{1j}(u) \\ \vdots \\ f_{mj}(u) \end{pmatrix}$ and each $f_{ij}(u)$ = scalar function.

3) Given a unit vector $\vec{n} \in S^{d-1}$ we consider

$$\mathbf{f}_{\vec{n}} = \mathbf{f} \cdot n = \sum_{j=1}^{d} \vec{f}_j \cdot n_j \quad ; \quad (\Sigma n_j^2 = 1)$$

and we assume that if $(u_L, u_R) \in \Omega \subset \mathbb{R}^m$ the problem

$$(1)_{\vec{n}} \begin{cases} \partial_t u + \partial_x (f_{\vec{n}}(u)) = 0 \\ u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \end{cases}$$

is solvable !

That is, the notion of $u(x,t)$ a unique solution of $(1)_{\vec{n}}$ is clear.

4) $u_t + \nabla_x \cdot \mathbf{f}(u) = 0$ is hyperbolic if $(1)_{\vec{n}}$ is hyperbolic, for any $u \in \Omega$.

i.e, $f_{\vec{n}}'$ has real eigenvalues

The solution $u(x,t) = \begin{cases} u_L & \text{if } \frac{x}{t} \le \lambda_{min} \\ \text{something} & \text{if } \lambda_{min} < \frac{x}{t} < \lambda_{max} \\ u_R & \text{if } \frac{x}{t} \ge \lambda_{max} \end{cases}$

Therefore, we assume finite speed of propagation

The (interface) Riemann solution is enclosed in the cone $\boxed{\lambda_{min} \le \frac{x}{t} \le \lambda_{max}}$

Let $\lambda_{max}(f_{\vec{n}}, u_L, u_R) := \max(|\lambda_{min}|, |\lambda_{max}|)$

Theorem: Let $t\,\lambda_{max}(f_{\vec{n}}, u_L, u_R) \le \frac{1}{2}$. Then

$$\bar{u}(t) = \int_{-1/2}^{1/2} u(x,t)\,dx = \frac{1}{2}(u_L + u_R) - t\left(\vec{f}_n(u_R) - \vec{f}_n(u_L)\right)$$

$\boxed{\text{This is the essential step in the Lax scheme (staggered average)}}$

Moreover, if $\partial_t \eta(u(x,t)) + \partial_x q_{\vec{n}}(u(x,t)) \le 0$ for an entropy pair $(\eta, q_{\vec{n}})$ then we have the following

$$(*) \quad \eta(\bar{u}(t)) \le \frac{1}{2}\left(\eta(u_L) + \eta(u_R)\right) - t\left(q_n(u_R) - q_n(u_L)\right).$$

$(*)$ is a discrete entropy inequality.

$\boxed{\text{Def}^n\text{:}}$ (Invariant domain) A convex set $A \subset \Omega$ is an invariant domain of the PDE if for any $\vec{n}$ and any $u_L, u_R \in A$ we have that $\bar{u}(t) \in A$ for all $0 \le t \le t_{max} = \dfrac{1}{2\lambda_{max}(f_n, u_L, u_R)}$.

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape regular sequence of affine matching meshes. Think of $\overset{P_1}{\boxed{\Delta\text{-s}}}$ or Q_1 in 2D.

The approximation space $\mathbb{P}$ is P_1 on triangles or Q_1 on quadrilaterals. Any $v \in \mathbb{P}(\mathcal{T}_h)$ can be expressed as a linear combination of the Lagrange shape functions.

$$V_h(x) = \sum_{i=1}^{I} v(\hat{a}_e)\, \varphi_i(x)$$

$\varphi(\hat{a}_i) = 1$

$\varphi(\hat{a}_j) = 0 \quad \forall j \neq i$

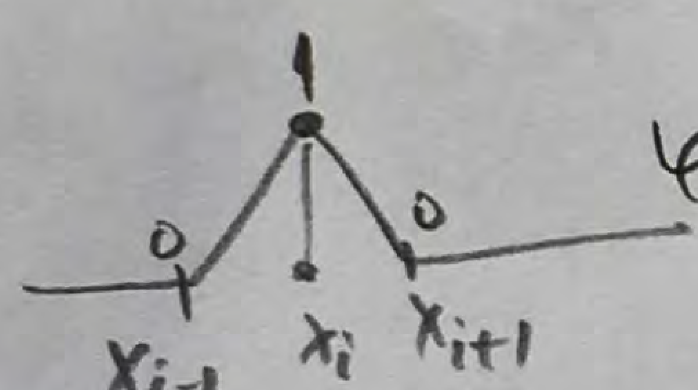
Key property:

$$\min V_h(x) = \min_e v(\hat{a}_e)$$

$$\max V_h(x) = \max_e v(\hat{a}_e)$$

(2D) $v_h/_K$ = linear $= a + bx + cy$, $K = \Delta$-gle (2D) or $v_h/_K = a + bx + cy + dxy$, K = rectangle (2D)

(1D)



$\varphi_i(x)$ $\longrightarrow$ p.w. linear $V_h(x)$ continuous

Recall the equation:

$$m_i = \int \varphi_i(x)\,dx$$

$\varphi_i \cdot \Big|$ $\partial_t u + \nabla \cdot f(u) = 0$

integrate

Discretization:

$$\underbrace{m_i \frac{U_i^{n+1} - U_i^n}{\tau}}_{\text{time derivative discretization}} + \int \Big(\underbrace{\nabla \cdot \Pi_h f(u_h^n)}_{\boxed{\Pi_h f = \sum_j f(U_j^n)\cdot \varphi_j}}\Big) \varphi_i\, dx - \underbrace{\sum_{j \in I(S_i)} d_{ij} U_j^n}_{\text{stabilization term}} = 0$$

(interpolant)

artificial viscosity!

The scheme:

$$m_i \frac{U_i^{n+1} - U_i^n}{\tau} = - \sum_{j \neq i} f(U_j^n) \cdot c_{ij} + \sum_{j \neq i} d_{ij} U_j \text{, where}$$

$$c_{ij} = \int \nabla \varphi_j \cdot \varphi_i$$

We require $\sum_j d_{ij} = 0$, note $d_{ij} = 0$ if $j \notin I(S_i)$;

$$m_i \frac{U_i^{n+1} - U_i^n}{\tau} = - \sum_{j \in I(S_i)} \Big(f(U_j^n) - \underbrace{f(U_i^n)}_{\text{adds zero}} \Big) \cdot c_{ij} + \sum_{j \in I(S_i)} d_{ij} \Big(U_j + \underbrace{U_i}_{\text{adds zero}} \Big)$$

Then:

$$U_i^{n+1} = U_i^n \Big(1 - \sum_{j \neq i} \frac{2\tau}{m_i} d_{ij} \Big) + \sum_{j \neq i} \frac{2\tau d_{ij}}{m_i} \cdot \overline{U}_{ij} \text{, where}$$

$$\overline{U}_{ij} = \frac{1}{2}(U_j + U_i) - \frac{\|c_{ij}\|}{2 d_{ij}} \Big(f(U_j^n) - f(U_i^n) \Big) \cdot \overset{\text{unit vector}}{\frac{c_{ij}}{\|c_{ij}\|}} = n_{ij}$$

As long as $\frac{\|c_{ij}\|}{2 d_{ij}}$ = artificial time $\tau \le \frac{1}{2 \cdot \lambda_{max}(f \cdot n_{ij}, U_i, U_j)}$

we have that

$\overline{U}_{ij}$ = the average of the exact solution of ~~the~~ Riemann problem with $U_L = U_i$, $U_R = U_j$ and flux $= f \cdot n_{ij}$
!

Then $\overline{U}_{ij} \in A$ if A is an invariant domain for U_i^n & U_j^n for this fixed $j \in I(S_i)$

Theorem If A is an invariant domain s.t. $U_j^n \in A$ for all $j \in I(S_i)$, then

$\overline{U}_{ij} \in A$ $\forall j \in I(S_i)$ and $U_i^{n+1} \in A$! □