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ON HOMOGENIZATION OF NONLINEAR HYPERBOLIC EQUATIONS

Y. Efendiev

Department of Mathematics
Texas A&M University, College Station, TX 77843-3368

B. Popov

Department of Mathematics
Texas A&M University, College Station, TX 77843-3368

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Abstract. In this paper we study homogenization of nonlinear hyperbolic equations. The weak limit of the solutions is investigated by approximating the flux functions with piecewise linear functions. We study mostly Riemann problems for layered velocity fields as well as for the heterogeneous divergence free velocity fields.

1. **Introduction.** The homogenization of transport phenomena plays an important role in many applications such as flow in porous media, turbulent diffusion [9, 20]. In many practical applications, the velocity field that transports the substance varies over a wide range of length scales, and, numerical flow models cannot in general resolve all of the scales of variations. Therefore, upscaling (numerical homogenization) approaches are needed for representing the effects of small scale variations on larger scale.

The homogenization of hyperbolic equations with periodic velocity field has been studied previously [21, 12] by considering the limit of the solutions as the period size approaches to zero. In [6] the homogenization of nonlinear hyperbolic equation using the two-scale convergence concept is studied, where strong two-scale convergence of the solutions was investigated. The homogenized equation is obtained at the expense of the velocity "averaging" along the trajectories. For example, for the layered velocity field this approach yields the same homogenized equation as the underlying fine-scale equation.

In this work our objective is to derive the homogenized equation for "the averages" of the solutions. A motivation for our paper stems from porous media applications, where the upscaled models for the solution on the coarse grid are needed. In a recent work [7], a coarse scale model, generalized convection-diffusion method is proposed. The starting point of this approach is the description of the homogenized equations. Once this description is postulated the calculations of the coarse scale quantities can be carried out. Our present paper helps to understand

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the form of the macroscopic equations which is important in designing numerical coarse models.

In this paper we consider two cases, layered flow and the flow driven by a divergence-free velocity field. The main idea in deriving a homogenized equation is to approximate the nonlinear flux function with piecewise linear functions. Similar ideas have used in investigating nonlinear hyperbolic equations with stochastic flux function [11]. In the case of the Riemann problem if the flux function is piecewise linear then the solution along each layer can be represented as a linear combination of linear waves [3]. We have extended the analysis to Cauchy problems for the layered media. For the homogenization in layered media we first obtain the homogenized solution. Then we propose a homogenized equation that has similar form as the one corresponding to the linear problem with unknown parameters (see [21]). These unknown parameters are found by substituting the homogenized solution. Properties of the effective parameters involved in homogenized equation are discussed in the paper. In the case of the Riemann problem the effective parameters depend only on one point statistics of the velocity field and the derivative of the flux function, and the effects of nonlinearities and heterogeneities can be separated. For the Cauchy problem we discretize the initial condition with piecewise constant function. In this case the effective parameters have more complex form and it is not possible in general to separate the heterogeneities from nonlinearities.

Using the flux discretization idea we also study the homogenization of nonlinear transport in a divergence free field. We consider the Riemann problem that is often used in applications of water flooding of the reservoir. By separating the nonlinearities from heterogeneities and using the homogenization results for linear transport we derive the average equation.

The paper is organized as follows. In the next section we present homogenization results for layered media. Section 3 is devoted to the homogenization in a divergence-free field.

2. **Homogenization in Layered Media.**

For simplicity we assume that the stratification is along the horizontal direction, and consider the following governing equation for a convected substance,

$$S_t^\epsilon + v^\epsilon(y) f(S^\epsilon)_x = 0, \tag{2.1}$$

where S^ϵ is the mass concentration of the transported substance and ϵ designates the small scale of the problem. Throughout the paper we will consider the two-dimensional case, though the results can be extended to higher dimensions. We assume that the domain for (2.1) is an infinite slab $\Omega = [-\infty, \infty] \times [0, L_y]$, and $S^\epsilon(x, y, 0) = H(x)$, where H does not depend on ϵ. In the section 2.2.2. we will also discuss the case $S^\epsilon(x, y, 0) = H(x, y)$. For the Riemann problem it is assumed that $H(x) = \{S_L, \text{if } x \le 0;\ S_R,\ \text{otherwise}\}$.

In the analysis of the layered flow we consider the velocity field to be discrete. In particular, we assume that the velocity field has n distinct values v_i with the volume fraction m_i, i.e., $v(z) = \{v_i\}$ and $m_i = meas\{v(z) = v_i\}$. We assume that the layering has periodic structure with period size ϵ. The homogenized field in this case $\overline{S} = \lim_{\epsilon \to 0} S^\epsilon$ is the average of S^ϵ across the layers in y direction, i.e.,

$$\overline{S}(x, t) = \int_0^{L_y} S^\epsilon(x, y, t) dy, \tag{2.2}$$

and $S^\epsilon \to \overline{S}$ as $\epsilon \to 0$ weakly in $L^\gamma(\mathbf{R})$ ($\gamma \geq 1$, see e.g. [13]) for any t. Note that the periodicity of layers is not necessary for our analysis. We will derive an equation for $\overline{S}$ that is the average solution of interest. The homogenization equation is derived based on solutions $\overline{S}$ (see [21]).

2.1. **Riemann problem.** To study the numerical homogenization of nonlinear hyperbolic equation we will use a piecewise linear approximation of the flux functions $f(S)$. It is known that [3] if $f_k(w)$ is a piecewise linear approximation to $f(w)$ such that $f_k \to f$ in Lipschitz norm then the solution of

$$\frac{\partial w_k}{\partial t} + f_k(w_k)_x = 0$$

converges to the solution of

$$w_t + f(w)_x = 0$$

in L_1 norm, provided $w_k(t=0)$ converges to $w(t=0)$ in L_1 norm. One can also estimate the convergence rate (see [18])

$$\|w_k(t) - w(t)\|_{L_1(\mathbf{R})} \leq \|w_k(t=0) - w(t=0)\|_{L_1(\mathbf{R})} \\ +t\|f_k - f\|_{Lip} \min(|w_k(t=0)|_{BV}, |w(t=0)|_{BV}), \quad (2.3)$$

where the norm on the left hand side is taken with respect to the spatial variable. Here $|g|_{BV} = \limsup_{t>0} \frac{1}{t}\int_R |g(x+t) - g(x)|dx$. Taking into account that $\|f_k - f\|_{Lip} \leq \frac{C}{k}\|f''\|_{L^\infty(R)}$ if f is piecewise C^2 (see [18]) and assuming accurate initial approximation (2.3) becomes

$$\|w_k(t) - w(t)\|_{L_1(\mathbf{R})} \leq \frac{Ct}{k}|w(t=0)|_{BV}. \quad (2.4)$$

Our study of numerical homogenization will focus on piecewise linear flux functions,

$$\frac{\partial S_k^\epsilon}{\partial t} + v(\frac{y}{\epsilon})\frac{\partial f_k(S_k^\epsilon)}{\partial x} = 0.$$

We recall that the velocity is assumed to be a discrete value function, $v(z) = v_i$, $m_i = meas(v(z) = v_i)$, $i = 1, \dots, n$. Using (2.4) we have

$$\left\| \int_0^{L_y} S_k^\epsilon(x,y,t)dy - \int_0^{L_y} S^\epsilon(x,y,t)dy \right\|_{L_1(\mathbf{R})} \leq \frac{Ct}{k}|S^\epsilon(x,t=0)|_{BV}, \quad (2.5)$$

where C also depends on max of the velocity field. Denoting the homogenized field for S_k^ϵ by $\overline{S}_k = \int_0^{L_y} S_k^\epsilon(x,y,t)dy$ we have

$$\|\overline{S}_k(t) - \overline{S}(t)\|_{L_1(\mathbf{R})} \leq \frac{Ct}{k}|S^\epsilon(t=0)|_{BV}.$$

In the rest of this section our focus is on the derivation of the equation for $\overline{S}_k$. First we restrict ourselves to the Riemann problem, i.e., $S^\epsilon(t=0) = S_k^\epsilon(t=0) = \{S_L, \text{ if } x \leq 0; S_R, \text{ if } x > 0\}$.

Next we present the result from [3] where the solution of the Riemann problem for piecewise linear flux is computed. Consider

$$\frac{\partial w_k}{\partial t} + f_k(w_k)_x = 0,$$

$w_k(t=0) = \{w_L, \text{ if } x \leq 0; w_R, \text{if } x > 0\}$. Without loss of generality we assume that $w_L > w_R$. Assume that the boundary of the convex hull $\{(w,v)|w_R \leq w \leq w_L, v \leq f(w)\}$ is discretized at points $(w_R, f(w_R)), (w_{k-1}, f(w_{k-1})), \cdots, (w_1, f(w_1)), (w_L, f(w_L))$,

$w_R < w_{k-1} < \cdots < w_1 < w_L$. Then the admissible weak solution of the Riemann problem

$$\frac{\partial w_k}{\partial t} + f_k(w_k)_x = 0,$$

$w_k(t=0) = \{w_L, \text{ if } x \le 0; w_R, \text{ if } x > 0\}$ is given by

$$\begin{aligned}
&w_k(x,t) = w_L, \quad \text{for } -\infty < \frac{x}{t} \le \frac{f(w_1) - f(w_L)}{w_1 - w_L},\\
&w_k(x,t) = w_1, \quad \text{for } \frac{f(w_1) - f(w_L)}{w_1 - w_L} < \frac{x}{t} \le \frac{f(w_2) - f(w_1)}{w_2 - w_1},\\
&\cdots\\
&w_k(x,t) = w_{k-1}, \quad \text{for } \frac{f(w_{k-1}) - f(w_{k-2})}{w_{k-1} - w_{k-2}} < \frac{x}{t} \le \frac{f(w_R) - f(w_{k-1})}{w_R - w_{k-1}},\\
&w_k(x,t) = w_R, \quad \text{for } \frac{f(w_R) - f(w_{k-1})}{w_R - w_{k-1}} < \frac{x}{t} < \infty.
\end{aligned} \tag{2.6}$$

For our analysis we write the solution $w_k(x,t)$ in the following form,

$$w_k(x,t) = \sum_{i=0}^{k} h(x - u_i t)\Delta_i + w_R, \tag{2.7}$$

where

$$u_i = \frac{f(w_i) - f(w_{i-1})}{w_i - w_{i-1}}, \quad \Delta_i = w_i - w_{i-1},$$

$w_0 = w_L$, $w_k = w_R$, and $h(x) = \{1, \text{ if } x \le 0; \ 0 \text{ if } x > 0\}$. Here Δ_i are the strength of shock waves, and u_i can be regarded as $f'(w_i^*)$, where $w_{i-1} \le w_i^* \le w_i$. Note that f' from now on will refer to the derivative of the modified flux.

For further convenience assume $S_R = 0$. Using the representation (2.7) for each layer we can write $\overline{S_k}$ as

$$\overline{S}_k(x,t) = \sum_{i=1}^{k}\sum_{j=1}^{n} h(x - v_j u_i t)\Delta_i m_j, \tag{2.8}$$

where

$$u_i = \frac{f(S_i) - f(S_{i-1})}{S_i - S_{i-1}}, \quad \Delta_i = S_{i-1} - S_i.$$

For further convenience, we denote u_i by $f'(S_i^*)$, where $S_i^* \in [S_i, S_{i-1}]$. Next we will find the homogenized equation $\overline{S}_k$ (see (2.8)). For this purpose we need the following lemma whose proof is in Appendix A.

LEMMA 2.1. *If*

$$\overline{S}(x,t) = \sum_{i=1}^{n} m_i H(x - v_i t), \tag{2.9}$$

then $\overline{S}$ *satisfies*

$$\frac{\partial \overline{S}}{\partial t} + \overline{v}\frac{\partial \overline{S}}{\partial x} = \sum_{i=1}^{n-1}\int_0^t \beta_i \frac{\partial^2}{\partial x^2}\overline{S}(x - u_i(t-\tau), \tau)d\tau, \tag{2.10}$$

where β_i *and* u_i *(*$i = 1, \ldots, n-1$*) satisfy*

$$\sum_{k=1}^{n} \frac{m_k}{u_i - v_k} = 0, \; i = 1, \ldots, n-1 \tag{2.11}$$

$$\sum_{i=1}^{n-1} \frac{\beta_i}{u_i - v_k} = (\overline{v} - v_k), \; k = 1, \ldots, n. \tag{2.12}$$

and $\overline{v} = \sum_{i=1}^{n} m_i v_i$. *Moreover,* β_i *and* u_i *exist, are unique, depend only on one point statistics of the media,* $M_l = \sum_{i=1}^{n} m_i v_i^l$, *and have the following properties:*

i) $v_1 \leq u_1 \leq v_2 \leq \cdots \leq u_{n-1} \leq v_n$

ii) $\sum_{i=1}^{n-1} \beta_i = var(v)$, *where* $var(v)$ *denotes the variance of the velocity field and given by* $var(v) = \sum_{i=1}^{n} m_i v_i^2 - \left(\sum_{i=1}^{n} m_i v_i\right)^2$

Applying the Lemma 2.1 we find the homogenized equation:

$$\frac{\partial \overline{S}_k}{\partial t} + U \frac{\partial \overline{S}_k}{\partial x} = \sum_{i=1}^{nk-1} \int_0^t \beta_i \frac{\partial^2 \overline{S}_k}{\partial x^2}(x - \alpha_i(t-\tau), \tau) d\tau. \tag{2.13}$$

Note that in this case there are nk layers with velocities $v_i f'(S_j^*)$ and volume fractions $m_i \Delta_j$, where $i = 1, \ldots, n$, $j = 1, \ldots, k$.

The relationship that determines β_i and α_i follows from Lemma 2.1,

$$U - v_i f'(S_j^*) = \sum_{l=1}^{nk-1} \frac{\beta_l}{\alpha_l - v_i f'(S_j^*)}, \tag{2.14}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{k} m_i \Delta_j \frac{1}{\alpha_l - v_i f'(S_j^*)} = 0. \tag{2.15}$$

Note that (2.14) holds for each i, j, where $i = 1, \ldots, n$, $j = 1, \ldots, k$, and (2.15) holds for each l, $l = 1, \ldots, nk-1$. Applying $\sum_{i,j}$ to (2.14), and taking into account (2.15) we obtain

$$U = \sum_{i,j} v_i f'(S_j^*) m_i \Delta_j = \sum_i m_i v_i \sum_j f'(S_j^*) \Delta_j. \tag{2.16}$$

It follows from Lemma 2.1 that α_k are between the values of $v_i f'(S_j)$. Further as in the linear case the following identity holds:

$$\left(\sum_{i,j} m_i \Delta_j \frac{1}{z - v_i f'(S_j^*)} \right)^{-1} - z + U = \sum_k \frac{\beta_k}{z - \alpha_k}, \text{ for all } z \in \mathbf{C}. \tag{2.17}$$

Expanding (2.17) around $z = \infty$

$$\left(\sum_{m=0}^{\infty} \sum_{i,j} \frac{m_i \Delta_j v_i^m (f'(S_j^*))^m}{z^{m+1}} \right)^{-1} - z + U = \sum_{m=0}^{\infty} \sum_k \frac{\beta_k \alpha_k^m}{z^{m+1}} \tag{2.18}$$

we see that β_i and α_i depend on one point statistics of the velocity field, $M_l = \sum_{i=1}^{n} m_i v_i^l$ as well as one point statistics of f', $Q_l = \sum_{i=1}^{k} \Delta_i f'(S_i^*)^l$.

Moreover, from Lemma 2.1 it follows that

$$\sum_k \beta_k = \sum_{i=1}^{n} m_i v_i^2 \sum_{j=1}^{k} (f'(S_j))^2 \Delta_j - \left(\sum_{i=1}^{n} m_i v_i \sum_{j=1}^{k} (f'(S_j)) \Delta_j \right)^2. \tag{2.19}$$

Finalizing the above results we have the following theorem.

THEOREM 2.1. *Assume S^ϵ is the solution of*

$$S_t^\epsilon + v(\frac{y}{\epsilon})f(S_x^\epsilon) = 0,$$

where $S^\epsilon(t=0) = \{S_L,\ \text{if } x \le 0;\ S_R,\ \text{if } x > 0\}$, the velocity is a periodic discrete value function, $v(z) = v_i$, $m_i = meas(v(z) = v_i)$, $i = 1, \ldots, n$, and $f(S)$ is Lipschitz continuous and piecewise C^2 function. Then $\overline{S}(x,t)$ can be approximated in L_1 norm on finite time interval by $\overline{S_k}(x,t)$ that is the solution of

$$\frac{\partial \overline{S}_k}{\partial t} + U\frac{\partial \overline{S}_k}{\partial x} = \sum_i \int_0^t \beta_i \frac{\partial^2 \overline{S}_k}{\partial x^2}(x - \alpha_i(t-\tau), \tau)d\tau, \tag{2.20}$$

where U, α_i and β_i are defined by (2.16), (2.14), and (2.15). Moreover,

$$\|\overline{S_k}(x,t) - \overline{S}(x,t)\|_{L_1(\mathbf{R})} \le \frac{Ct}{k},$$

where k defines piecewise linear approximation of f and can be chosen an arbitrarily large, and C only depends on $|S^\epsilon(t=0)|_{BV}$ and $\max v_i$. Moreover, β_i and α_i depend on one point statistics of the velocity field, $M_l = \sum_{i=1}^n m_i v_i^l$ as well as one point statistics of f', $Q_l = \sum_{i=1}^k \Delta_i f'(S_i^*)^l$.

REMARK 2.1. The above results can be extended to the continuous case. Assume that the velocity field is given by $\mathbf{v} = (v^\epsilon(y), 0)$. We write (2.17) in the following form:

$$\left(\int\int d\nu_n(\lambda)d\mu_k(\eta)\frac{1}{z-\lambda\eta}\right)^{-1} - z + U = \int \frac{d\beta_{kn}(u)}{z-u}, \tag{2.21}$$

where $d\nu_n(\lambda) = \sum_{i=1}^n \delta(\lambda - v_i)m_i$, $d\mu_k(\eta) = \sum_{j=1}^k \delta(\eta - f'(S_j^*))\Delta_j$, and $d\beta_{kn}(u) = \sum_{i=1}^{nk-1}\delta(u-\alpha_i)\beta_i$. In the continuum limit, as $n \to \infty$, $d\nu_n$ converges to the Young measure, $d\nu$, associated with $v^\epsilon(y)$ and, in the limit as $k \to \infty$, $d\mu_k(\eta)$ converges to $d\mu(\eta)$ weakly. Moreover, it is easy to check that $\mu_k(\eta)$ is given by $1 - w_k(\eta)$, where $w_k(\eta)$ is the self-similar solution of Riemann problem given by (2.6). Consequently, the limiting measure will be $d\mu(\eta) = 1 - w(\eta)$, where $w(\eta)$ is the self-similar solution of Riemann problem. Taking the continuum limit of the right hand side (2.21) ($n \to \infty$, $k \to \infty$) we have

$$\left(\int\int d\nu(\lambda)d\mu(\eta)\frac{1}{z-\lambda\eta}\right)^{-1} - z + U. \tag{2.22}$$

Here λ changes from $v_{\min}$ to $v_{\max}$, and η changes from $f'_{\min}$ to $f'_{\max}$. Further, it can be shown that there exists a measure $\beta(u)$ defined outside $[(vf')_{\min}, (vf')_{\max}]$, such that (2.22) is equal to

$$\left(\int\int d\nu(\lambda)d\mu(\eta)\frac{1}{z-\lambda\eta}\right)^{-1} - z + U = \int \frac{d\beta(u)}{z-u}. \tag{2.23}$$

This can be proved following [21] by using the following representation theorem. If 1) $F(z)$ is an analytic function outside the interval I of the real axis, 2) $Im\ F(z) > 0$ for $Im\ z > 0$, and 3) $\lim_{y\to\infty} yF(iy) < \infty$, then the function $F(z)$ admits a representation

$$F(z) = \int_I \frac{d\beta(u)}{u-z}.$$

Denote $G(z)$ by

$$G(z) = U - z + \left(\int_0^\infty \int \frac{1}{z - \lambda\alpha} d\eta(\alpha) d\nu_y(\lambda) \right)^{-1}.$$

It is easy to show that $G(z)$ is analytic outside the interval $[f'_{min}v_-, f'_{max}v_+]$, and $Im\ G(z) > 0$ if $Im\ z > 0$. Moreover, expanding the r.h.s. around $z = \infty$ it can be readily checked that $\lim_{y\to\infty} yG(iy) < \infty$. Thus, the representation theorem holds. Consequently, the homogenized equation has the form

$$\frac{\partial \overline{S}}{\partial t} + U \frac{\partial \overline{S}}{\partial x} = \int_0^t \int \frac{\partial^2 \overline{S}}{\partial x^2}(x - u(t-\tau), \tau) d\beta(u) d\tau.$$

Expanding (2.23) at $z \to \infty$ we see that $\beta(u)$ depends on one point correlations $\int \lambda^l d\nu(\lambda)$ and $\int \eta^l d\mu(\eta)$.

2.2. **Remark on Cauchy problem.** Next we attempt to generalize the results to Cauchy problem. First we discuss the form of the solution for

$$S_t + \lambda f(S)_x = 0.$$

With this example we will try to separate the heterogeneities from nonlinearities as much as possible. For general Cauchy problem the time discretization is also needed. Following [3] we discretize both the initial condition and the flux function by piecewise constant and piecewise linear functions respectively. In particular, assume that the initial condition $S(t = 0, x)$ is discretized at discrete points x_i and the flux function is discretized in between the points $S(t = 0, x_i)$ and $S(t = 0, x_{i+1})$ with a number of points S_{ij} (e.g. $S_i = S(x_i, t = 0) = S_{i1} < \dots S_{il} = S_{i+1} = S(x_{i+1}, t = 0)$ if $S_i < S_{i+1}$). For further convenience one index will be used describing locations and the values of S. When we have a finite number of S at the point x_i we assume that $x_{ij} = x_i$. It is known that [3] the solution will be a J-valued function, where $J = \{S_i\}$. Further, we discretize the time interval $[0, T]$ by $0 \le t_1 \le \dots t_N = T$, where t_i are time instants when shock collisions occur. Note that each shock interaction lowers the number of shocks or the number of oscillations (see [4, 1] for details). The discrete approximation of the solution in this case for $t \in [t_k, t_{k+1}]$ is given by

$$S_k(x, t) = \sum h(x - x_i^k(\lambda) - \lambda p_i^k(\lambda) t) \Delta_i^k(\lambda). \tag{2.24}$$

Here $p_i^k(\lambda)$ are shock speeds, $\Delta_i^k(\lambda)$ are shock strengths and $x_i^k(\lambda)$ are used to define shock locations. The scaling factor of the velocity (λ) affects the collision times, consequently the shock speeds and locations. Note that here we assume that the initial condition is a function of bounded variation. The latter allows us to state that there exists a finite number of points x_i such that $S_k(t = 0, x)$ can be approximated with

$$\sum h(x - x_i)\Delta_i.$$

For our analysis we need a slightly different formulation of (2.24). In particular, we will put all λ dependence to $\Delta_i^k(\lambda)$. For this reason we define a large finite set (see [4, 1]) of x_i and v_i that contains all $x_i^k(\lambda)$ and $v_i^k(\lambda)$. This set can be defined independent of λ and is the same for all λ, consequently for $\lambda = 1$ (this follows from the fact that entropy solutions for different λ's can be obtained from the entropy solution for $\lambda = 1$ by rescaling (see also [1])). Then the solution can be written as

$$S_k(x, t) = \sum h(x - x_i - \lambda p_i t) \Delta_i^k(\lambda). \tag{2.25}$$

REMARK 2.2. The independence of the set of x_i and p_i of λ can be easily illustrated when no new shocks arise. Assume that two shocks with velocities u_1 and u_2 and with strengths Δ_1 and Δ_2 and initial locations x_1 and x_2 collide, i.e., $h(x - x_1 - u_1 t)\Delta_1 + h(x - x_2 - u_2 t)\Delta_2$. Then after collision new shock will be formed that has form $h(x - x_0 - ut)\Delta$, where $\Delta = \Delta_1 + \Delta_2$, $u = (v_1\Delta_1 + v_2\Delta_2)/(\Delta_1 + \Delta_2)$, $x_0 = (x_1\Delta_1 + x_2\Delta_2)/(\Delta_1 + \Delta_2)$.

Using the representation (2.25) for the solution in each layer we can obtain the homogenized solution for the multi-layered system averaged across the layers. Denoting by v_l the velocities of the layers and m_l their relative weights the homogenized solution in each time interval $[t_k, t_{k+1}]$ is given by

$$\overline{S}_k(x,t) = \sum_{i,l} h(x - x_i - v_l p_i t)\Delta_{il}^k m_l. \tag{2.26}$$

Later on we present an example to illustrate the nature of (2.26). The above formulation assumes that the initial condition is independent of the vertical variable y and has bounded variation. (2.26) holds also when the initial condition slowly changes with respect to y. In this case the finite set of x_i and p_i are the union of the sets of x_i and p_i for each layer.

Using the discussion from previous sections we can look for the homogenized equation for each i and k in the following form:

$$\frac{\partial \overline{S}_k}{\partial t} + U\frac{\partial \overline{S}_k}{\partial x} = \sum_q \int_0^t \beta_q \frac{\partial^2 \overline{S}_k}{\partial x^2}(x - \alpha_q(t-\tau), \tau)d\tau. \tag{2.27}$$

REMARK 2.3. β_q and α_q depend on i and k. Note that x_i and time intervals do not depend on heterogeneities and they are determined from the nonlinear Cauchy problem for $\lambda = 1$.

The relationship that determines β_q and α_q for each k and i follows from Lemma 2.1,

$$U - v_l p_i = \sum_q \frac{\beta_q}{\alpha_q - v_l p_i}, \ \forall l, \tag{2.28}$$

$$\sum_l m_l \Delta_{il}^k \frac{1}{\alpha_q - v_l p_i} = 0, \ \forall q. \tag{2.29}$$

Here q changes from 1 to $n-1$, where n is the total number of layers as in the previous discussions. Multiplying first equation (2.28) by $m_l\Delta_{il}^k$ and summing over all l and taking into account (2.29) we can easily obtain that

$$U = \sum_l v_l p_i \Delta_{il}^k m_l.$$

Using our previous results, equations (2.28) and (2.29) can be written for each i and k as

$$\left(\sum_l m_l \Delta_{il}^k \frac{1}{z - v_l p_i}\right)^{-1} - z + U_i^k = \sum_q \frac{\beta_q}{z - \alpha_q}. \tag{2.30}$$

Expanding (2.30) around $z = \infty$ we see that β_q and u_q depends on $M_{vf'} = \sum v_l^d p_i^d \Delta_{il}^k m_l$ for each i and l. This indicates that nonlinearities and heterogeneities interact on the coarse level. This is in contrast to the Riemann problem. The reason for this is that in the case of the Riemann problem we convexify the flux and the latter gets rid of wave interaction. If the separation of variables is possible for Δ_{il} then nonlinearities and heterogeneities can be separated. Next we present a simple example, where this does not hold.

Next we illustrate the nature of (2.26) on a simple example with two layers where each layer initially contains two shock waves that will collide. Assume the velocities of the layers to be v_1 and v_2. Further, we assume the solution in a layer i (i=1,2) at time $t = 0$ is given by

$$h(x - x_1 - v_i p_1 t)\delta_1 + h(x - x_2 - v_i p_2 t)\delta_2. \tag{2.31}$$

From here assuming for simplicity $m_1 = m_2 = 1$ the averaged solution $\overline{S}$ before the first shock interaction is given by

$$\overline{S}_1(x,t) = \sum_{i,j=1}^{2} h(x - x_j - v_i p_j t)\delta_j.$$

Assuming $v_2 > v_1$ we have that first shock interaction will occur in the second layer. The solution then will have the form (2.31) in the first layer, but in the second layer the solution will have the form $h(x - x_3 - v_2 p_3 t)(\Delta_1 + \Delta_2)$. Consequently, the average solution has the form

$$\overline{S}_2(x,t) = \sum_{i=1}^{2} h(x - x_1 - v_1 p_i t)\delta_i + h(x - x_3 - v_2 p_3 t)(\delta_1 + \delta_2),$$

where $p_3 = (p_1\delta_1 + p_2\delta_2)/(\delta_1 + \delta_2)$ and $x_3 = (x_1\delta_1 + x_2\delta_2)/(\delta_1 + \delta_2)$. In the next time step the average solution will have the form

$$\overline{S}_3(x,t) = h(x - x_3 - v_1 p_3 t)(\delta_1 + \delta_2) + h(x - x_3 - v_2 p_3 t)(\delta_1 + \delta_2).$$

Above we presented the solution that has different form in each time interval. This solution has the form (2.26)

$$\overline{S}_k(x,t) = \sum h(x - x_j - v_i p_j t)\Delta_{ij}^k,$$

where

$$\Delta^1 = \begin{bmatrix} \delta_1 & \delta_2 & 0 \\ \delta_1 & \delta_2 & 0 \end{bmatrix},$$
$$\Delta^2 = \begin{bmatrix} \delta_1 & \delta_2 & 0 \\ 0 & 0 & \delta_1 + \delta_2 \end{bmatrix},$$
$$\Delta^3 = \begin{bmatrix} 0 & 0 & \delta_1 + \delta_2 \\ 0 & 0 & \delta_1 + \delta_2 \end{bmatrix}.$$

We would like to note that using estimate (2.3) one can obtain that $S_k(x,t)$ given by (2.27) approximates the homogenized solution on a finite time interval when time and space intervals are sufficiently small.

3. **Homogenization in a Divergence-free Field.** In this section we study the homogenization of the Riemann problem for general divergence free velocity fields,

$$\frac{\partial S^\epsilon}{\partial t} + \mathbf{v}^\epsilon(\mathbf{x}) \cdot \nabla f(S^\epsilon) = 0, \tag{3.1}$$

where $\nabla \cdot \mathbf{v}^\epsilon = 0$.

The main idea is the following. First, we reduce the problem into the Riemann problems along each streamline, defined by $d\mathbf{x}^\epsilon/dt = \mathbf{v}^\epsilon$. Second, along each streamline we reduce the problem into the superposition of linear Riemann problems by discretizing the flux with piecewise linear functions. This renders linear problems for "fixed value" of S^ϵ. Next, we consider the average motion of the linear problems under the assumptions that the velocity possesses strong mixing properties. Using known results for the homogenization of linear transport equation we derive weak limit of S^ϵ.

As in the previous section, we assume $f_k(S)$ to be a piecewise linear functions that approximates the boundary of the convex hull $\{(w,v)|w_R \leq w \leq w_L, v \leq f(w)\}$ in Lipschitz norm. In this section we consider the Riemann problem. The results of this section hold in general if the transport along each streamline is a Riemann problem.

Next we discuss the decomposition of the solution along each streamlines. For these purposes, we omit the index ϵ. Along each streamline $d\mathbf{x}/dt = \mathbf{v}$ we have

$$\frac{\partial S_k}{\partial t} + \mathbf{v} \cdot \nabla f_k(S_k) = 0. \tag{3.2}$$

For further simplicity we assume that $\mathbf{v} = (v_x, v_y)$ and v_x is positive at all times. Then each streamline is defined by $y(x) = y(x, y_0)$, where y_0 is the starting point for the streamline and $y(x)$ solves

$$\frac{dy}{dx} = \frac{v_y(x,y)}{v_x(x,y)}.$$

Along a streamline we introduce a new variable (known as a travel time, [2]) by

$$\frac{d\tau}{dx} = \frac{1}{v_x(x,y)}. \tag{3.3}$$

The equation (3.2) along each streamline then becomes

$$\frac{\partial S_k}{\partial t} + \frac{\partial f(S_k)}{\partial \tau} = 0. \tag{3.4}$$

Indeed, it can be easily checked that

$$\mathbf{v} \cdot \nabla = \frac{\partial}{\partial \tau}.$$

For the equation (3.4) $S_k(x,t)$ is constant between the points moving according

$$\frac{d\tau}{dt} = \frac{f(S_i) - f(S_{i-1})}{S_i - S_{i-1}}.$$

Denoting $f'(S_i^*) = \frac{f(S_i)-f(S_{i-1})}{S_i - S_{i-1}}$, one can find that $S_k(x,t)$ is constant between the points that travel according

$$\frac{d\mathbf{x}_i}{dt} = f'(S_i^*)\mathbf{v}$$

on a fixed streamline. Note that f' from now on will refer to the derivative of the modified flux. Thus, the solution of the equation (3.2) along each streamline of

$d\mathbf{x}/dt = \mathbf{v}$ can be written as

$$\begin{aligned} &S_k(x,t) = S_L, \quad \text{for } -\infty < \mathbf{x} \le \mathbf{x_1}(t), \\ &S_k(x,t) = S_1, \quad \text{for } \mathbf{x_1}(t) < \mathbf{x} \le \mathbf{x_2}(t), \\ &\dots \\ &S_k(x,t) = S_{k-1}, \quad \text{for } \mathbf{x}_{k-1}(t) < \mathbf{x} \le \mathbf{x_R}(t), \\ &S_k(x,t) = S_R, \quad \text{for } \mathbf{x_R}(t) < \mathbf{x} < \infty. \end{aligned} \tag{3.5}$$

Using the above decomposition, the solution of (3.1) can be represented as

$$S_k^\epsilon(\mathbf{x},t) = \sum h(\mathbf{x} - \mathbf{x}_i^\epsilon(t))\Delta_i,$$

where $(d/dt)\mathbf{x}_i^\epsilon(t) = \mathbf{v}^\epsilon(\mathbf{x}) f'(S_i^*)$ and $h(\mathbf{x}) = \{1 \text{ if } \mathbf{x} \le 0,\ 0 \text{ otherwise}\}$. Introducing $d\mu_k(\eta) = \sum_{i=1}^k \delta(\eta - f'(S_i))\Delta_i$ we have

$$S_k^\epsilon(\mathbf{x},t) = \int h(\mathbf{x} - \eta\mathbf{x}^\epsilon(t)) d\mu_k(\eta). \tag{3.6}$$

Here the integral is taken over $[f'_{\min}, f'_{\max}]$. Note that the measure $d\mu_k(\eta)$ is independent of ϵ (heterogeneities) and only depends on the nonlinear flux. Further as $k \to \infty$ $\mu_k(\eta) \to \mu(\eta)$ in L_1, where $\mu(\eta)$ depends on the self-similar solution of the Riemann problem. Indeed, it is easy to check that $\mu_k(\eta)$ is given by $1 - w_k$, where w_k is defined by (2.6). As $k \to \infty$ $\mu_k(\eta) \to \mu(\eta)$, where $\mu(\eta) = 1 - w(\eta)$ and $w(\eta)$ is the self similar solution of 1-D Riemann problem. Taking limit as $k \to \infty$ of (3.6) we obtain

$$S^\epsilon(\mathbf{x},t) = \int h(\mathbf{x} - \eta\mathbf{x}^\epsilon(t)) d\mu(\eta). \tag{3.7}$$

Note that since $\mu(\eta)$ is a BV function, (3.7) is well defined for a.e. $(\mathbf{x},t)$.

Next we note that $R^\epsilon(x,t) = h(\mathbf{x} - \eta\mathbf{x}^\epsilon(t))$ is the solution of the linear transport equation with velocity $\eta\mathbf{v}^\epsilon(\mathbf{x})$,

$$\frac{\partial R^\epsilon}{\partial t} + \eta\mathbf{v}^\epsilon(\mathbf{x}) \cdot \nabla R^\epsilon = 0.$$

Moreover, the information about heterogeneities is in $h(\mathbf{x} - \eta\mathbf{x}^\epsilon(t))$ in the representation (3.7). The homogenization of the linear equation, for the velocity field that has strong mixing property has been studied (see e.g. [14, 16, 15, 19] for mathematical theory, also see [17, 5, 10, 22]). These results [14, 16, 15] are obtained using probabilistic framework and involve some additional technical assumptions. Assuming $\mathbf{v}^\epsilon = \overline{\mathbf{v}} + \epsilon\mathbf{u}(\mathbf{x},\omega)$ it can be shown that under strong mixing and strict stationarity condition on the velocity field, $R^\epsilon(\mathbf{x} - \eta\overline{\mathbf{v}}t, t)$ converges weakly (in a probabilistic sense, see [14, 16, 15, 19]) in a time interval of order $O(\frac{1}{\epsilon^2})$ to a Markov diffusion process with generator $L = \eta^2 \sum_{i,j=1}^d b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \eta^2 \sum_{i=1}^d c_i \frac{\partial}{\partial x_i}$, where b_{ij} and c_i are functions of two point correlations of $\mathbf{u}(\mathbf{x},\omega)$ (see [14, 16, 15, 19]). To keep the presentation more concise we will simply assume that $\overline{R}_\eta(\mathbf{x},t) = E(R^\epsilon(\mathbf{x},t,\omega))$ satisfies

$$\frac{\partial \overline{R}_\eta}{\partial t} + \eta\overline{\mathbf{v}} \cdot \nabla \overline{R}_\eta = \nabla_i \eta^2 a_{ij} \nabla_j \overline{R}_\eta, \tag{3.8}$$

where a_{ij} are determined from two point correlations of $\epsilon\mathbf{u}$. We would like to note that (3.8) can be obtained for deterministic flows using a perturbation argument (see [8]), where a macrodiffusion term appears as a two-point correlation of the velocity field.

It is easy to see that $\overline{R}_\eta$ is invariant under the change of variable $x \to x/\eta$, i.e.,

$$\overline{R}_\eta(\mathbf{x}, t) = g(\frac{\mathbf{x}}{\eta}, t), \tag{3.9}$$

where g is the solution of

$$\frac{\partial g}{\partial t} + \overline{\mathbf{v}} \cdot \nabla g = \nabla_i a_{ij} \nabla_j g. \tag{3.10}$$

g is subject to the Riemann initial condition, $g(\mathbf{x}, 0) = \{S_L, \text{if } x \leq 0;\ S_R,\ \text{otherwise}\}$ because of (3.9). From (3.7) we have

$$\overline{S}(\mathbf{x}, t) = \int R_\eta(\mathbf{x}, t) d\mu(\eta). \tag{3.11}$$

Using equation (3.8) one can derive the following equation for $\overline{S}$,

$$\frac{\partial \overline{S}}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \int \overline{R}_\eta(\mathbf{x}, t) \eta d\mu(\eta) = \nabla_i a_{ij} \nabla_j \int \overline{R}_\eta(\mathbf{x}, t) \eta^2 d\mu(\eta). \tag{3.12}$$

It is not difficult to see that the terms involving $\overline{R}_\eta$ can be written in terms of $\overline{S}$ and some other functions. Using mean value theorem and assuming that $\overline{R}_\eta(\mathbf{x}, t)$ is a positive function (e.g., assuming $S_L, S_R \geq 0$) we can obtain

$$\int \overline{R}_\eta(\mathbf{x}, t) \eta d\mu(\eta) \left(= \sum \overline{R}_{\eta = f'(S_i^*)}(\mathbf{x}, t) f'(S_i) \Delta_i\right) = f'(S^*(\mathbf{x}, t)) \overline{S}(\mathbf{x}, t),$$

for some function $S^*(\mathbf{x}, t)$. Similarly,

$$\int \overline{R}_\eta(\mathbf{x}, t) \eta^2 d\mu(\eta) = f'(S^{**})^2 \overline{S}(\mathbf{x}, t),$$

for some function $S^{**}(\mathbf{x}, t)$. Note that the functions $S^*(\mathbf{x}, t)$ and $S^{**}(\mathbf{x}, t)$ depend only g and μ, i.e., they depend on both nonlinearities and the averaged properties of linear transport equations. With these notations the homogenized equation becomes:

$$\frac{\partial \overline{S}}{\partial t} + \overline{\mathbf{v}} \cdot \nabla f'(S^*(\mathbf{x}, t)) \overline{S} = \nabla_i a_{ij} \nabla_j f'(S^{**}(\mathbf{x}, t))^2 \overline{S}. \tag{3.13}$$

Remark 3.1. (3.13) holds in general if the transport along each streamline is a Riemann problem. In this case $\overline{R}_\eta$ is subject to the initial condition $H(x, y)$. Using (3.11) we can readily obtain (3.13) in a same way as we did above.

Remark 3.2. The interaction of the heterogeneities and nonlinearities in the case of general Cauchy problem does not allow us to carry out the homogenization for general heterogeneous velocity fields.

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Appendix A. **Proof of Lemma 2.1.** Substituting the expression (2.9) for $\overline{S}$ into (2.10) yields

$$\sum_{k=1}^{n} m_k(\overline{v}-v_k)H^{'}(x-v_kt) = \sum_{k=1}^{n}\sum_{i=1}^{n-1} m_k\beta_i \int_0^t \frac{\partial^2}{\partial x^2}H(x-u_i(t-\tau)-v_k\tau)d\tau. \quad \text{(A.1)}$$

The right hand side (r.h.s.) of (A.1) can be simplified in the following manner

$$\sum_{k=1}^{n}\sum_{i=1}^{n-1} m_k\beta_i \int_0^t \frac{1}{u_i-v_k}\frac{d}{d\tau}H^{'}(x-u_i(t-\tau)-v_k\tau)d\tau$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n-1} m_k\beta_i\frac{1}{u_i-v_k}\right)H^{'}(x-v_k\tau) - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n} m_k\beta_i\frac{1}{u_i-v_k}\right)H^{'}(x-u_i\tau). \quad \text{(A.2)}$$

Further, equating the l.h.s. of (A.1) and the r.h.s. of (A.2) we have the following relations for our unknown parameters $\overline{v}$, β_i, and u_i $(i=1,\ldots,n-1)$

$$\sum_{k=1}^{n} m_k\beta_i\frac{1}{u_i-v_k} = 0,\ i=1,\ldots,n-1$$

$$\sum_{i=1}^{n-1} m_k\beta_i\frac{1}{u_i-v_k} = m_k(\overline{v}-v_k),\ k=1,\ldots,n.$$

Note that there are $2n-1$ equations for $2n-1$ unknowns $\overline{v}$, β_i, and u_i $(i = 1,\ldots,n-1)$. The above expressions are equivalent to (2.11) and (2.12).

It is easy to see that $n-1$ equations in (2.11) define the values of u_i $(i = 1,\ldots,n-1)$ and n equations in (2.12) define the values of β_i, $(i=1,\ldots,n-1)$, and $\overline{v}$. The value of $\overline{v}$ can be readily calculated by multiplying (2.12) to m_k and summing over k,

$$\sum_{k=1}^{n} m_k(\overline{v}-v_k) = \sum_{k=1}^{n}\sum_{i=1}^{n-1} m_k\frac{\beta_i}{u_i-v_k}.$$

The r.h.s of this equality is zero because of (2.11), therefore ,

$$\overline{v} = \sum_{k=1}^{n} m_kv_k.$$

From (2.11) one can find that u_i $(i=1,\ldots,n-1)$ satisfy

$$\sum_{k=1}^{n} m_k \prod_{p\neq k,p=1}^{n}(u_i-v_p) = 0.$$

This indicates that u_i $(i=1,\ldots,n-1)$ are the roots of the following polynomial of the degree $n-1$

$$R(z) = \sum_{k=1}^{n} m_k \prod_{p\neq k,p=1}^{n}(z-v_p) = 0.$$

It is easy to check that $R(z)$ changes sign from v_i to v_{i+1}, i.e., $R(v_i)R(v_{i+1}) \leq 0$. Consequently, the roots of the polynomial $R(z)$ are between the values of v_i. Therefore,

$$v_1 \leq u_1 \leq v_2 \leq u_2 \leq \cdots \leq u_{n-1} \leq v_n.$$

The latter also proves the existence and uniqueness of u_i's.

It is more difficult to understand the behavior of individual β_i $(i = 1, \dots, n-1)$, though their mean properties can be computed. The equations (2.11) and (2.12) can be combined in the following identity:

$$\left(\sum_{k=1}^{n} \frac{m_k}{z - v_k} \right)^{-1} - z + \overline{v} = \sum_{i=1}^{n-1} \frac{\beta_i}{z - u_i}. \tag{A.3}$$

This is an identity for all $z \in \mathbf{R}$. To prove this identity we note that (A.3) can be written as a polynomial of the degree less than $2n-1$. The polynomial has degree $2n-1$ when all v_i's $(i = 1, \dots, n)$ are distinct. In the case of distinct v_i's the identity (A.3) holds since it holds at $2n-1$ distinct points, $z = v_i$, $(i = 1, \dots, n)$ and $z = u_i$, $(i = 1, \dots, n-1)$. Indeed, for the values of $z = v_i$ $(i = 1, \dots, n)$ the identity (A.3) becomes (2.12) and for the values of $z = u_i$ $(i = 1, \dots, n-1)$ the identity (A.3) becomes (2.11). In the event that some of the values of v_i are coincide the degree of the polynomial of (A.3) decreases. In a similar manner it can be verified that the identity (A.3) holds in this case.

The existence and uniqueness of β_i follow from (2.29) noting that the left hand side has the form

$$\frac{P_{n-2}(z)}{\prod_{i=1}^{n-1}(z - u_i)},$$

where $P_{n-2}(z)$ is a polynomial of degree $n-2$. It can be readily checked that the leading term of $P_{n-2}(z)$ has the form $var(v)z^{n-2}$, where $var(v)$ refers to the variance of the v. This implies that $\sum_i \beta_i = var(v)$. The coefficient in front of z^{n-3} is $skew(v) = \sum_{i=1}^{n} m_i (v_i - \overline{v})^3$.

Expanding (A.3) around $z = \infty$,

$$\left(\sum_{m=0}^{\infty} \sum_{i=1}^{n} \frac{m_i v_i^m}{z^{m+1}} \right)^{-1} - z + \overline{v} = \sum_{m=0}^{\infty} \sum_{k=1}^{n-1} \frac{\beta_k u_k^m}{z^{m+1}}, \tag{A.4}$$

we see that β_i and u_i depend only on $M_l = \sum_{i=1}^{n} m_i v_i^l$.

The homogenization results presented in Lemma 2.1 also can be derived from the results of Tartar, [21].

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E-mail address: efendiev@math.tamu.edu
E-mail address: popov@math.tamu.edu