

# A simple derivation of an upper bound in the presence of a viscosity gradient in three-layer Hele-Shaw flows

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Received 8 November 2005

Accepted 10 January 2006

Published 30 January 2006

Online at [stacks.iop.org/JSTAT/2006/P01014](http://stacks.iop.org/JSTAT/2006/P01014)

doi:[10.1088/1742-5468/2006/01/P01014](https://doi.org/10.1088/1742-5468/2006/01/P01014)

**Abstract.** An upper bound on the growth rate of disturbances in three-layer Hele-Shaw flows with the middle layer having a smooth viscous profile is obtained using a weak formulation of the disturbance equations. A recently reported approach for the derivation of this bound is tedious, cumbersome, and requires numerical analysis. In contrast, the present approach is very simple, elegant, and requires no numerical analysis. The interpretation and limiting cases of this bound are also addressed in this paper.

**Keywords:** hydrodynamic instabilities, multiphase flow

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**1. Introduction**

In multi-phase flow, one of the problems of fundamental and practical interest is the dynamics of interfaces between different phases. The interfacial dynamics in this context is usually very complex, with rich structures and multiple scales which, if properly understood, can possibly be harnessed to potential industrial advantage by various control mechanisms. In recent years, there has been a significant surge in research in developing fast and accurate numerical methods to solve initial value problems associated with such interfacial dynamics [2, 3, 11, 14, 21, 8]. If an interface that is part of a basic solution of such a problem is unstable, then linear stability analysis can predict the evolution of interfacial disturbances for a short time very accurately but at a larger times nonlinearity usually plays a major role in the evolution process. The time it takes for nonlinearity to come and play a decisive role over linearity depends on the linearized growth rate. In some instances, this linearized growth rate can be obtained analytically and therefore some exact estimate of this timescale can be obtained using linear theory. In other cases, the best one can hope for analytically is an estimate of the upper bound on the growth rate. However, it is not always an easy task to obtain such an estimate, due to analytical difficulties. Nonetheless, such an estimate of the growth rate, if obtained, can be useful in devising some control strategies for interfacial evolution. In this paper, we address one such problem involving three-layer Hele-Shaw flows.

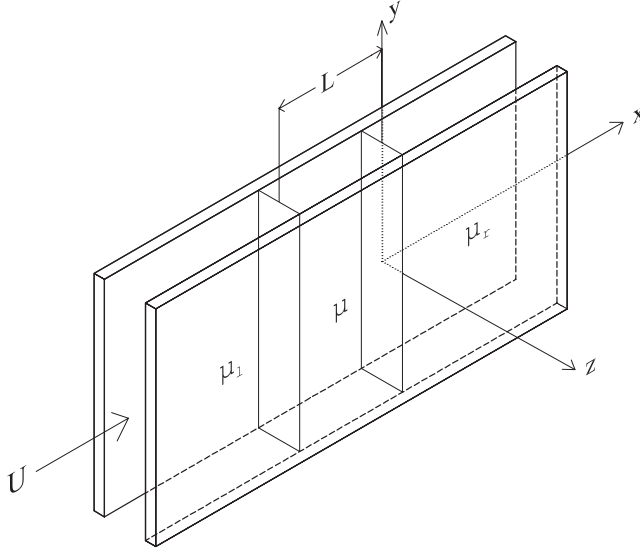
Consider an unstable interface in a rectilinear geometry driven by a positive viscosity jump in the direction of flow. If this interface were followed by another (trailing) stable interface with a stable middle layer, then this stable trailing interface has to move in response to the motion of disturbances on the unstable leading interface due to the incompressibility of the middle layer so that the mass of the middle layer is conserved. Therefore, the presence of an unstable leading interface acts as a catalyst for the instability of an otherwise stable trailing interface. Needless to say, this transfer of instability also involves the transfer of energy which would otherwise feed the instability of the leading interface in the absence of the trailing interface. Thus the trailing interface suppresses the growth activity of disturbances on the leading interface by drawing energy from it. Generalizing this mechanism, it is easy to see that while an otherwise less unstable interface impedes the instability of an otherwise more unstable interface, in turn this more

unstable interface also acts as a catalyst for enhancing the instability of the less unstable interface. This instability transfer mechanism which is consistent with the energy budget of the system is more complicated when the middle layer is also unstable. This would be the case when the middle layer has smooth viscosity profile with unfavourable (positive) viscosity gradient in the direction of flow. A central problem that has been elusive in this connection is an estimate on the maximal growth rate in this three-layer set-up with the middle layer having smooth viscous profile.

In our recent paper [7], we solved this problem. In particular, we obtained an upper bound of this growth rate using Gerschgorin's localization theorem on a discrete version of the continuous flow problem and assuming convergence of the discrete problem under mesh refinement. This method is tedious and cumbersome. In this paper, we provide a simple derivation of the same bound using a more elegant approach without any recourse to discretization. The approach is based on an exact analysis of the weak formulation of an ordinary differential equation which is derived from the flow equations.

Before closing this section, we cite some works on three-layer flows. The interfacial instability in two-layer porous media and Hele-Shaw flows ([10]) has been studied by Saffman and Taylor (see [17]) and Chouke *et al* (see [1]). When a less mobile phase is displaced by a more mobile phase, the initial interfacial disturbances grow in a complex manner developing spikes and fingers in the nonlinear regime. There are instances when such fingers are not desirable and their growth needs to be controlled. Some of the ideas on control of such interfacial instabilities are embedded in the linearized dispersion relation (see also [15, 18, 19, 22]). According to the linearized dispersion relation, all modes are linearly unstable in the absence of surface tension when a less viscous phase is displacing a more viscous phase in a Hele-Shaw cell. Surface tension merely stabilizes the short waves but the long waves remain unstable. The growth rates of interfacial disturbances depend on the viscosity jump across the interface. This fact has been harnessed technologically (cf [18, 19, 22]) using a three-layer set-up in which the middle layer's viscosity profile is controlled appropriately in order to stabilize the interface sweeping the less mobile phase.

An industrial setting of this problem involves enhanced oil recovery (EOR henceforth) processes by polymer flooding. This is a tertiary oil recovery process in which oil is produced by the injection of polymer-thickened water which in turn is displaced by water. The polymer-thickened water has more viscosity than water depending on the concentration of polymer. The rarefaction waves behind the leading front in such processes create a graded viscosity behind the leading interface. The three-layer Hele-Shaw model we consider here is a very good approximation to this EOR process in porous media and provides insight into the interaction mechanism of interfaces. Details of the physical processes taking place can be found in, for example, Shah and Schechter [18] and the mathematics of EOR by polymer flooding is detailed Daripa *et al* in [5]. For recent articles including review articles on the use of variable viscosity fluid in enhanced oil recovery, see [13, 12, 20, 23]. Most of the research in this direction does not provide any *a priori* knowledge of the most dangerous growth rate due to analytical difficulties associated with underlying equations for three-layer flows within the Hele-Shaw model or saturation model. In this paper, we make some progress in this direction by providing an upper bound on this growth rate when the middle layer has a variable viscosity profile.



**Figure 1.** Three-layer fluid flow in a Hele-Shaw cell.

The paper is laid out as follows. In section 2, we introduce various notations, formulate the problem, and derive the disturbance equations within linearized theory. The upper bound on the growth rate is derived in section 3. Finally we conclude in section 4.

## 2. Problem formulation

We consider two-dimensional fluid flows in a three-layer Hele-Shaw cell. The set-up is shown in figure 1. Because of the symmetry of the flow in the  $z$ -direction, the flow is essentially two-dimensional. Therefore, the domain of interest is  $\Omega := (x, y) = \mathbb{R}^2$  (with a periodic extension of the set-up in the  $y$ -direction). The fluid upstream (i.e., as  $x \rightarrow -\infty$ ) has a velocity  $\mathbf{u} = (U, 0)$ . The fluid in the left layer with viscosity  $\mu_1$  extends up to  $x = -\infty$ , the fluid in the right layer with viscosity  $\mu_r$  extends up to  $x = \infty$ , and the fluid in the middle layer of length  $L$  has a smooth viscous profile. The underlying equations of this problem are

$$\nabla \cdot \mathbf{u} = 0, \quad (1a)$$

$$\nabla p = -\mu \mathbf{u}, \quad (1b)$$

$$\frac{D\mu}{Dt} = 0, \quad (1c)$$

where  $\nabla = (\partial/\partial x, \partial/\partial y)$  and  $D/Dt$  is the material derivative. The first equation (1a) is the continuity equation for incompressible flow, the second equation (1b) is Darcy's law [4], and the third equation (1c) is the advection equation for viscosity [9, 6].

The above system admits a simple basic solution, namely the whole fluid set-up moves with speed  $U$  in the  $x$  direction and the two interfaces, namely the one separating the left layer from the middle layer and the other separating the right layer from the middle layer, are planar, i.e. parallel to the  $y$ -axis. The pressure corresponding to this basic solution is

obtained by integrating (1b). In a frame moving with velocity  $(U, 0)$ , the above system is stationary along with two planar interfaces separating these three fluid layers, and the smooth viscous profile  $\mu(x)$  of the middle-layer fluid satisfies  $\mu_l < \mu(x) < \mu_r$ . Here and below, with slight abuse of notation, the same variable  $x$  is used in the moving reference frame. One of the ways to have a smooth viscous profile of the middle-layer fluid is to use polymer-thickened water (as the middle-layer fluid) in which the polymer concentration along the width of the middle layer is properly adjusted to obtain the desired viscous profile of the middle layer.

In linearized stability analysis by normal modes, disturbances in the moving reference frame are written in the form  $(\tilde{u}, \tilde{p}, \tilde{\mu}) = (f(x), \psi(x), \phi(x))e^{iky+\sigma t}$  and then inserted into the linearized disturbance equations obtained from (1) and also into the linearized dynamic and kinematic interfacial conditions (see [6]). After some algebraic manipulation of the resulting equations in  $f(x), \psi(x)$  and  $\phi(x)$ , we obtain the following problem for the eigenfunction  $f$ :

$$\begin{aligned} -(\mu f_x)_x + k^2 \mu f &= \lambda k^2 U \mu_x f, & x \in (-L, 0), \\ f_x(0) &= (\lambda p + q)f(0), & f_x(-L) = (\lambda r + s)f(-L), \end{aligned} \tag{2}$$

where  $\lambda = 1/\sigma$ , and  $p, q, r, s$  are defined by

$$\begin{aligned} p &= \{[\mu]_r U k^2 - T k^4\} / \mu(0), & q &= -\mu_r k / \mu(0) \leq 0, \\ r &= \{-[\mu]_l U k^2 + S k^4\} / \mu(-L), & s &= \mu_l k / \mu(-L) \geq 0. \end{aligned} \tag{3}$$

We have used the notations  $[\mu]_l$  and  $[\mu]_r$  for jumps in viscosities at the left and right interfaces respectively, i.e.,  $[\mu]_l = (\mu(-L) - \mu_l)$  and  $[\mu]_r = (\mu_r - \mu(0))$ . It is worth noting that

$$p \geq 0 \quad \text{for } k^2 \leq k_1^2 = [\mu]_r U / T, \quad r \leq 0 \quad \text{for } k^2 \leq k_2^2 = [\mu]_l U / S. \tag{4}$$

All these equations are in dimensional form. In (3),  $T$  is the surface tension at the interface  $x = 0$  and  $S$  is the surface tension at the interface  $x = -L$ . The solution  $f$  of the problem (2) will be a function of  $x$  as well as  $k$ . Below we write both of these dependences of  $f$  explicitly where needed and later, starting with (8), we revert back to using the notation  $f(x)$  for  $f(x, k)$ .

### 3. An upper bound

We multiply the ordinary differential equation in (2) by  $f^*$  and rewrite the resulting equation as

$$-(\mu f^* f_x)_x + \mu |f_x|^2 + k^2 (\mu - \lambda U \mu_x) |f|^2 = 0. \tag{5}$$

Upon integrating over  $(-L, 0)$  and using the boundary conditions defined in problem (2), we obtain

$$\begin{aligned} |f(-L; k)|^2 (\lambda r + s) \mu(-L) - |f(0; k)|^2 (\lambda p + q) \mu(0) + \int_{-L}^0 \mu |f_x|^2 dx \\ + k^2 \int_{-L}^0 (\mu - \lambda U \mu_x) |f|^2 dx = 0. \end{aligned} \tag{6}$$

After substituting the expressions for  $p, q, r, s$  from (4) and rearranging, we obtain

$$|f(-L; k)|^2 \{ \sigma \mu_1 k - U k^2 [\mu]_1 + S k^4 \} - |f(0; k)|^2 \{ -\sigma \mu_r k + U k^2 [\mu]_r - T k^4 \} + \sigma \int_{-L}^0 \mu |f_x|^2 dx + k^2 \int_{-L}^0 (\sigma \mu - U \mu_x) |f|^2 dx = 0. \tag{7}$$

From the imaginary part of this equation, it is easy to see that  $\sigma$  is real. In normal mode analysis, the real perturbations such as the perturbation  $\tilde{u}$  of the constant velocity  $U$  are written as superposition of modes  $f(x; k)e^{ikx + \sigma t}$ , and since  $\tilde{u}$  is real, the coefficients  $f(x; k)$  bear the relationship  $f(x; k) = f^*(x; -k)$  for each mode  $k$  for all time because  $\sigma$  is real. Since only an even power in  $k$  (only  $k^2$ ) appears in (2), the solution  $f(x)$  must also satisfy  $f(x; k) = f(x; -k)$ . Therefore it follows from these two relations that  $f(x, k) = f^*(x, -k) = f(x, -k)$ , which means that the eigenfunction  $f$  is also real. Thus, we can take  $|f(x; k)|^2 = f^2(x; k) \forall x \in (-L, 0)$  and  $\sigma$  real in equation (7) as well as in equation (6). Henceforth, we use  $f(x)$  for  $f(x; k)$ . Then we have from (7) or equivalently from (6)

$$\sigma = \frac{p\mu(0)f^2(0) - r\mu(-L)f^2(-L) + k^2 U \int_{-L}^0 \mu_x f^2 dx}{-q\mu(0)f^2(0) + s\mu(-L)f^2(-L) + k^2 \int_{-L}^0 \mu(x)f^2 dx + \int_{-L}^0 \mu(x)f_x^2 dx}. \tag{8}$$

Since the variable viscous profile  $\mu(x)$  with the property  $\mu_1 < \mu(x) < \mu_r, \forall x \in (-L, 0)$  is smooth,

$$\exists D > 0, \quad D = \sup\{\mu_x, x \in [-L, 0]\}. \tag{9}$$

Moreover, from (4) we have  $p > 0$ , and  $-r > 0$  for modes with wavenumbers  $k$  in the range

$$k^2 < \min \{ U[\mu]_r/T, U[\mu]_1/S \}, \tag{10}$$

and such modes always exist for the viscous profiles  $\mu(x)$  provided there are finite viscosity jumps at both the interfaces. Therefore, it follows from (8) that these modes for which  $p > 0$  and  $(-r) > 0$  (also recall that  $(-q) > 0$ , and  $s > 0$ ) have  $\sigma > 0$  and hence are unstable. For waves outside the range, one or both of  $p$  and  $(-r)$  can be negative and therefore such waves can also be unstable, but the most unstable mode will be in the band given by (10) because each of the terms in the numerator of (10) is positive for waves in this band. Therefore, we need to analyse the growth rates of only these modes for an estimate of the upper bound on the growth rate, which we do next. To do this, in the denominator of (8) we neglect the last term and use the obvious inequality  $\int_{-L}^0 \mu(x)f_x^2 dx > \mu_1 k^2 \int_{-L}^0 f^2 dx$  since  $0 < \mu_1 < \mu(x) < \mu_r, \forall x \in (-L, 0)$ . Then we have

$$\sigma \leq \frac{p\mu(0)f^2(0) - r\mu(-L)f^2(-L) + DUk^2 \int_{-L}^0 f^2 dx}{-q\mu(0)f^2(0) + s\mu(-L)f^2(-L) + \mu_1 k^2 \int_{-L}^0 f^2 dx} = \sigma_1, \tag{11}$$

and we get

$$0 \leq \sigma_1 = \frac{a + b + m}{c + d + n}, \tag{12}$$

where

$$a = p \mu(0) f^2(0) > 0, \quad b = -r \mu(-L) f^2(-L) > 0, \quad c = -q \mu(0) f^2(0) > 0, \quad (13)$$

$$d = s \mu(-L) f^2(-L) > 0, \quad m = DUk^2 \int_{-L}^0 f^2 dx > 0, \quad n = \mu_1 k^2 \int_{-L}^0 f^2 dx > 0. \quad (14)$$

It follows from the above definitions that ratios  $m/n$ ,  $\max_k(a/c)$  and  $\max_k(b/d)$  are positive numbers (see also (3) and (18) below). Bound (11) is not very useful since it depends on the eigenfunction  $f$  and the wavenumber  $k$ . Below we reduce this to an upper bound only in terms of the parameters of the problem which is more relevant and useful. For this purpose, we consider the following two possibilities, namely:

- (i)  $a/c \leq m/n$  and  $b/d \leq m/n$ ;
- (ii)  $m/n \leq \max\{\max_k(a/c), \max_k(b/d)\}$ .

Case (i): In this case it easily follows that

$$\sigma_1 = \frac{a + b + m}{c + d + n} \leq \frac{m}{n} = \left( \frac{DU}{\mu_1} \right). \quad (15)$$

Case (ii): Similarly it easily follows that in this case

$$\sigma_1 = \frac{a + b + m}{c + d + n} \leq \max \left\{ \max_k \left( \frac{p}{-q} \right), \max_k \left( \frac{-r}{s} \right) \right\}. \quad (16)$$

From (15) and (16), finally we obtain

$$\sigma_1 \leq \max \left\{ \max_k \left( \frac{p}{-q} \right), \max_k \left( \frac{-r}{s} \right), \left( \frac{DU}{\mu_1} \right) \right\}. \quad (17)$$

This last bound does not depend on the wavenumber  $k$ . Using the definitions given in (3), the maximum values of  $p/(-q)$  and  $(-r)/s$  are found over the range (10) of unstable modes and we arrive at

$$\max_k \left( \frac{p}{-q} \right) = \frac{2T}{\mu_r} \left( \frac{U[\mu]_r}{3T} \right)^{3/2}, \quad \max_k \left( \frac{-r}{s} \right) = \frac{2S}{\mu_l} \left( \frac{U[\mu]_l}{3S} \right)^{3/2}. \quad (18)$$

We use (9) and (18) in (17). Then, the upper bound of the growth rate in terms of parameters of the problem is given by

$$\sigma \leq \sigma_1 \leq \max \left\{ \frac{2T}{\mu_r} \left( \frac{U[\mu]_r}{3T} \right)^{3/2}, \frac{2S}{\mu_l} \left( \frac{U[\mu]_l}{3S} \right)^{3/2}, \frac{U}{\mu_l} \sup_x \{\mu_x\} \right\}. \quad (19)$$

This bound is based on an analysis of (8) for waves in the wavenumber band given by (10) which is non-empty iff there is a finite viscosity jump at each of the two interfaces. As the viscosity jump at one or both of the interfaces approaches zero, so does the width of the wavenumber band given by (10). But, however small this band is in this limiting case, the most unstable wavenumber is still in this band as it approaches zero, and therefore the above bound (19) should still be valid. Therefore, we can use (19) when the viscosity is continuous at one or both of the interfaces. In fact, the resulting formula from (18) when the viscosity is continuous at both the interfaces can also be derived directly from (8) in the following manner. In this case, we obtain from (19)

$$\sigma \leq \sigma_1 \leq \frac{U}{\mu_1} \sup_x \{\mu_x\}. \quad (20)$$

Note that we can derive this same bound directly from (8). Since, in this case,  $p = -Tk^4/\mu_r$  and  $r = Sk^4/\mu_l$  (see (3)), the first two terms of the numerator in (8) are negative for all  $k$  and hence

$$\sigma \leq \frac{k^2 U \int_{-L}^0 \mu_x f^2 dx}{\int_{-L}^0 \mu(x) f_x^2 dx + k^2 \int_{-L}^0 \mu(x) f^2 dx} = \sigma_1 \leq \frac{U}{\mu_1} \sup_x \{\mu_x\}, \quad (21)$$

which is the same as (20) as it should be.

**Remark 1.** It is important to note that the first and the second terms in formula (19) for the bound are **NOT** the individual Saffman–Taylor (ST) growth rates of the two interfaces because the viscosity terms in the denominators are different from what they would otherwise be according to the Saffman–Taylor formula (see (22)). If the first term had  $(\mu_r - \mu(0))$  instead of  $\mu_r$  in its denominator, then this modified term would represent the individual maximal ST growth rate of the leading interface in the absence of the other interface and the variable viscosity middle layer. Similarly, if the second term had  $(\mu(-L) - \mu_l)$  instead of  $\mu_l$  in its denominator, then this modified term would represent a similar individual maximal ST growth rate of the trailing interface. However, this is not the case. The fact that each of these terms in (19) is more than its individual ST growth rate in the above sense is very comforting because of the reasons already alluded to in the introduction, partly having to do with the interfacial interactions already built into the linear model through the boundary conditions (see the problem defined in (2)). Therefore, perhaps it is appropriate to attribute the first and second terms in (19) as upper bounds on the ‘effective’ growth rate of the disturbances on leading and trailing interfaces respectively. The third term in formula (19) is an upper bound of the growth rate of the disturbances within the central fluid layer, which is an upper bound since it is scaled with the smallest value of the viscosity of the central fluid layer. Such precise results cannot possibly be obvious intuitively since the bounds on the ‘effective’ growth rates of the disturbances on the two interfaces that appear in formula (19) for the upper bound in the combined set-up are **NOT** the same as the individual ST growth rates of disturbances.

In summary, the upper bound identifies that the growth rate of the disturbances in the combined three-layer set-up is smaller than the larger of (i) the upper bounds on the ‘effective’ growth rates of the two interacting interfaces, each of which exceeds its own individual ST growth rate, and (ii) an upper bound of the growth rate of the disturbances

within the central fluid layer, which is an upper bound since it is scaled with the smallest value of the viscosity of the central fluid layer.

**Remark 2.** The Saffman–Taylor formula for the growth rate of the interfacial disturbance in the absence of the middle layer is given by

$$\sigma_{\text{ST}} = \frac{Uk^2(\mu_r - \mu_l) - k^3T}{\mu_r + \mu_l}, \quad (22)$$

from which follows the growth rate,  $\sigma_{\text{ST}}^*$ , of the most dangerous wave, which is

$$\sigma_{\text{ST}}^* = \frac{2T}{(\mu_r + \mu_l)} \left( \frac{U(\mu_r - \mu_l)}{3T} \right)^{3/2}. \quad (23)$$

This  $\sigma_{\text{ST}}^*$  depends only on the steady state speed  $U$ , interfacial surface tension  $T$ , and the viscosities of the two fluids across the interface.

In the three-layer case, consider, as an example, a linear viscosity profile for the middle layer,

$$\mu(x) = \frac{\mu(0) - \mu(-L)}{L}x + \mu(0) \quad \mu_x(x) = \frac{\mu(0) - \mu(-L)}{L}, \quad (24)$$

and

$$\mu(0) \rightarrow \mu_r, \quad \mu(-L) \rightarrow \mu_l, \quad L \rightarrow \infty, \quad (25)$$

then the upper bound (19) goes to zero in the above limit (25). Therefore, a linear viscosity profile in the middle layer can provide an arbitrary small (positive) maximal value of the growth rate, much less than  $\sigma_{\text{ST}}^*$ . Similarly, other viscous profiles can be chosen carefully to contain the interfacial growth rate well below  $\sigma_{\text{ST}}^*$ .

#### 4. Discussion

From the bound (19), one can only conclude that growth rate is bounded from above by infinity when there is no surface tension at an interface with a finite viscosity jump. The bound can be improved in these cases, which we present next.

In the case of continuous viscosity at both the interfaces and zero interfacial surface tension,  $p = r = 0$  from (3). (Note that this case is basically a one-layer case with variable viscosity profile). Therefore it follows from (8) that all modes are unstable with the upper bound on the growth rate of these modes still given by the last inequality in (21). This shows that the upper bound (19) in this case should be estimated by first taking the limits of zero viscosity jumps at the interfaces.

If surface tension is non-zero at one or both of these interfaces (with the viscosity continuous at both the interfaces), then correspondingly one or both of the functions  $p$  and  $r$  defined in (3) is (are) of the order  $O(-k^4)$  as  $k \rightarrow \infty$ . From (8), it is easily seen that long waves are unstable with growth rate approaching zero as  $k \rightarrow 0$ . Similarly, it is seen that short waves are stable in this case. Again, the bound (19) should be estimated by first taking the zero viscosity jump limit. Thus, short waves at the leading interface can be stabilized even if there is no surface tension at this interface so long as the trailing interface has finite surface tension.

If the surface tension is zero at both interfaces with a finite jump in viscosity at one or both the interfaces, then the above bound (19) can still be used, from which one can only conclude that the upper bound is infinity. Actually, this particular case has unbounded growth rate due to the short wave instability, similar to the one seen in a pure two-layer zero surface tension Saffman–Taylor instability [17]. Consider the growth rate (8) for large  $k$ . As  $k \rightarrow \infty$ , the wavelength associated with the disturbance approaches zero and hence it is much smaller than length scale associated with the viscosity gradient. Thus, the limit of high wavenumber is equivalent to the limit of zero viscosity gradient and therefore the corresponding eigenfunction  $f(x)$  should behave similarly to the constant viscosity case, i.e.,  $f(x; k) = O(\exp(-kx))$ ,  $x \in (-L, 0)$  for large  $k$  (see equation (2) for  $\mu_x = 0$ ). Using this in (8) along with  $S = T = 0$  shows after some simple algebraic manipulation that  $\sigma = O(k)$  as  $k \rightarrow \infty$ . Thus we recover the usual ST-type short wave instability for the zero surface tension case.

The growth rate given by (8) shows that each of the three terms in the numerator is positive and proportional to  $k^2$  in the absence of interfacial surface tension, the first and second terms containing jumps in viscosity at the leading and trailing interfaces respectively, and the last term containing the viscosity gradient. In this sense, the viscosity gradient plays a role similar to interfacial viscosity jumps and contributes equally to modal instabilities on the average. Thus, in the absence of viscosity jumps at the interfaces, we have a pure viscosity-gradient-driven instability with a finite upper bound for the growth rate (see (8)). In the absence of surface tension at both the interfaces, all modes will be unstable. The presence of surface tension even at one of these interfaces stabilizes the short waves, with the growth rate of the remaining unstable modes still bounded by the last inequality in (21).

We should mention in closing that the problem considered here involves two interacting interfaces. The upper bound (19) based on linear theory shows that, even within linearized theory, the interaction between the interfaces prevails. Thus, there is a transfer of instability between the interfaces regardless of how weak the interfacial disturbances are. As the disturbances grow and the shapes of the interfaces change, nonlinearity comes into play and the bound (19) based on linear theory may not hold in the nonlinear regime. Nonetheless, it will be worthwhile to test this using numerical as well as physical experiments, which falls outside the scope of this paper. Another interesting aspect that is worthwhile investigating is the asymptotic shapes of the interfaces in a long narrow rectilinear channel, analogous to the Saffman–Taylor finger case for a single interface problem which was investigated in great detail during the 1980s (see [16]). However, there has been hardly any investigation of similar issues involving two-interfacial Hele-Shaw flows. The problem here, in general, is much more complex. It is not difficult to envisage that unless the initial gap between the interfaces and the design parameters of the environment in which the interfaces are immersed are right, the interfaces as they evolve will collide at one or more point(s) with subsequent break-up and merging of segments of interfaces. However, if the initial gap between interfaces is just right with correct values of the parameters, then it is possible for each of these interfaces to slowly develop into finger-type shapes before the trailing interface tries to catch up the leading interface. Some of this we have alluded to in section 1 (see second paragraph there). It will be interesting using the set-up discussed before (see figure 1) to investigate these issues numerically as well as

experimentally. These are topics of future research and certainly fall outside the scope of this paper.

## Acknowledgment

It is a pleasure to thank one of the referees for helpful and constructive remarks.

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