Injective, Surjective, and Bijective Functions

Paul Skoufranis

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**Definition** Let $X$ and $Y$ be sets and let $f : X \to Y$ be a function. We say $f$ is injective or one-to-one if whenever $x_1, x_2 \in X$ are such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ (that is, $f$ maps any two distinct elements of $X$ to distinct elements of $Y$).

**Example** Consider the function $f : \{1, 2, 3\} \to \{3, 5, 6, 9\}$ define by $f(1) = 6$, $f(2) = 5$, and $f(3) = 9$. Then $f$ is injective since $f(1)$, $f(2)$, and $f(3)$ are all distinct elements.

**Example** Consider the function $f : \{1, 2, 3\} \to \{3, 5, 6, 9\}$ define by $f(1) = 9$, $f(2) = 5$, and $f(3) = 9$. Then $f$ is not injective since $f(2) = f(3)$.

**Example** Consider the function $f : [-1, 1] \to [0, 1]$ define by $f(x) = x^2$. Then $f$ is not injective since $1 \neq -1$ yet $f(1) = f(-1)$. However, the function $g : [0, 1] \to [0, 1]$ defined by $g(x) = x^2$ is injective.

The following result is simple to prove and can be proven by simple results in logic.

**Lemma 1** Let $X$ and $Y$ be two sets and let $f : X \to Y$ be a function. Then $f$ is injective if and only if whenever $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$ then $x_1 = x_2$.

**Proof:** Suppose that $f$ is injective. We desire to show that $f$ has the property listed above. Suppose $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$. Then either $x_1 = x_2$ or $x_1 \neq x_2$. Suppose that $x_1 \neq x_2$. Then, since $f$ is injective, $f(x_1) \neq f(x_2)$ which contradicts the fact that $f(x_1) = f(x_2)$. Hence $x_1 = x_2$ as desired.

Suppose that $f$ has the property that whenever $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$ then $x_1 = x_2$. We desire to show that $f$ is injective. Suppose $x_1, x_2 \in X$ are such that $x_1 \neq x_2$. Then either $f(x_1) = f(x_2)$ or $f(x_1) \neq f(x_2)$. Suppose that $f(x_1) = f(x_2)$. Then, since $f$ has the above property, $x_1 = x_2$ which contradicts the fact that $x_1 \neq x_2$. Hence $f(x_1) = f(x_2)$ and thus $f$ must be injective. 

**Example** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x - y)$. Then $T$ is an injective map. To see this, we will apply the above lemma. Suppose that $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ are such that $T(x_1, y_1) = T(x_2, y_2)$. Then $(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$. Thus $x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$. By adding the two equations together, we obtain that $2x_1 = 2x_2$ so $x_1 = x_2$. By subtracting the second equation from the first, we obtain that $2y_1 = 2y_2$ so $y_1 = y_2$. Hence $(x_1, y_1) = (x_2, y_2)$. Hence $T$ is injective by the lemma.

Next we will briefly discuss surjective functions. To begin, we recall the definition of the range of a function.

**Definition** Let $X$ and $Y$ be sets and let $f : X \to Y$ be a function. The range of $f$, denoted $Ran(f)$ or $f(X)$, is $\{y \in Y \mid \exists x \in X$ such that $f(x) = y\}$.

**Definition** Let $X$ and $Y$ be sets and let $f : X \to Y$ be a function. We say that $f$ is surjective or onto if $Ran(f) = Y$ (that is, for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$).
Example) Consider the function \( f : \{1, 2, 3, 4\} \to \{3, 5, 9\} \) define by \( f(1) = 3, f(2) = 5, f(3) = 9, \) and \( f(4) = 5. \) Then \( f \) is surjective since the range of \( f \) is the entire set \( \{3, 5, 9\}. \)

Example) Consider the function \( f : \{1, 2, 3, 4\} \to \{3, 5, 9\} \) define by \( f(1) = 9, f(2) = 5, f(3) = 9, \) and \( f(4) = 5. \) Then \( f \) is not surjective since \( 3 \) is not in the range of \( f. \)

Example) Consider the function \( f : [0, 1] \to [0, 2] \) define by \( f(x) = x^2. \) Then \( f \) is not surjective since \( 2 \notin \text{Ran}(f). \) However, the function \( g : [0, 1] \to [0, 1] \) defined by \( g(x) = x^2 \) is surjective.

Example) Define \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x, y) = (x + y, x - y). \) Then \( T \) is a surjective map. To see this, suppose \((z, w) \in \mathbb{R}^2. \) Then, since
\[
T\left(\frac{z+w}{2}, \frac{z-w}{2}\right) = (z, w)
\]
\((z, w) \in \text{Ran}(T) \) (the range of a linear map is also called the image). Hence, as \((z, w) \in \mathbb{R}^2 \) was arbitrary, \( \text{Ran}(T) = \mathbb{R}^2 \) so \( T \) is surjective.

Finally, we may begin our discussion of bijective functions.

Definition) Let \( X \) and \( Y \) be sets and let \( f : X \to Y \) be a function. We say that \( f \) is bijective if it is injective and surjective.

Example) Define \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x, y) = (x + y, x - y). \) We have shown that \( T \) is injective and surjective. Hence \( T \) is bijective.

Example) Consider the function \( f : \mathbb{N} \to \mathbb{N} \cup \{0\} \) defined by \( f(n) = n - 1. \) We claim \( f \) is a bijection.

To see that \( f \) is injective, we must show that if \( n, m \in \mathbb{N} \) are such that \( f(n) = f(m) \) then \( n = m. \) Suppose \( n, m \in \mathbb{N} \) are such that \( f(n) = f(m). \) Then \( n - 1 = f(n) = f(m) = m - 1. \) Therefore \( n - 1 = m - 1 \) so \( n = m. \) Hence \( f \) is injective.

To see that \( f \) is surjective, we must show that for every \( k \in \mathbb{N} \cup \{0\} \) there exists a \( n \in \mathbb{N} \) (depending on \( k \)) such that \( f(n) = k. \) Suppose \( k \in \mathbb{N} \cup \{0\}. \) Then \( k + 1 \in \mathbb{N}. \) Moreover \( f(k + 1) = (k + 1) - 1 = k. \) Whence \( f \) is surjective. Hence \( f \) is a bijection.

Bijective maps have some very nice properties. First we will show that if \( f : X \to Y \) is a bijection then there is a function \( g : Y \to X \) that 'undoes' what \( f \) does.

Theorem 2) Let \( X \) and \( Y \) be two sets and suppose \( f : X \to Y \) is a bijection. Then there exists a unique function \( g : Y \to X \) such that
\[
\begin{align*}
1. \ g(f(x)) & = x \text{ for all } x \in X. \\
2. \ f(g(y)) & = y \text{ for all } y \in Y.
\end{align*}
\]
The function \( g \) is called the inverse of \( f \) and is denoted \( f^{-1}. \)

Proof: Let \( X \) and \( Y \) be two sets and suppose \( f : X \to Y \) is a bijection. There are two parts to this proof: show that there exists a function \( g : Y \to X \) with the above properties, and show that there exists only one function \( g \) with these properties. Thus we will split the proof into two parts.

First we desire to show that there exists a \( g \) with the two properties above. Therefore, for each \( y \in Y \) we need to define \( g(y) \) to be an element of \( X. \) Since \( f \) is bijective, \( \text{Ran}(f) = Y. \) Therefore for each \( y \in Y \) there exists at least one point \( x_y \in X \) such that \( f(x_y) = y. \) We then define \( g(y) = x_y \) for all \( y \in Y. \) Hence \( g : Y \to X \) is a function.

It remains to check that \( g \) has the above two properties. To see that \( g \) has the first property, suppose \( z \in X. \) Since \( z \in X, f(z) \in Y \) and thus, by the definition of \( x_{f(z)}, g(f(z)) = x_{f(z)}. \) We desire to show that
\(g(f(z)) = z\). We claim that \(x_{f(z)} = z\). To see this, we notice that \(f(x_{f(z)}) = f(z)\) by the definition of \(x_{f(z)}\). Since \(f\) is injective, \(f(x_{f(z)}) = f(z)\) implies \(x_{f(z)} = z\) as claimed. Therefore \(g(f(z)) = x_{f(z)} = z\) as desired.

To see that \(g\) has the second property, we notice that if \(y \in Y\) then \(f(g(y)) = f(x_y) = y\) by definitions. Therefore \(g\) has the second property. Hence \(g\) has the two properties.

It remains only to show that \(g\) is the only function with the above properties. Suppose \(g_1 : Y \to X\) is another function such that \(g_1(f(x)) = x\) for all \(x \in X\) and \(f(g_1(y)) = y\) for all \(y \in Y\). Fix \(y \in Y\). Then \(f(g_1(y)) = y = f(g(y))\). Since \(f\) is injective, \(f(g_1(y)) = f(g(y))\) implies \(g_1(y) = g(y)\). However, this equation holds for all \(y \in Y\). Hence \(g = g_1\). Thus \(g\) is the only function with the above properties. \(\square\)

Next we will show properties 1) and 2) of the above theorem implies that the functions under consideration are bijections.

**Theorem 3** Let \(X\) and \(Y\) be two sets and suppose \(f : X \to Y\) and \(g : Y \to X\) are functions such that

1. \(g(f(x)) = x\) for all \(x \in X\).
2. \(f(g(y)) = y\) for all \(y \in Y\).

Then \(f\) and \(g\) are bijections.

**Proof:** Let \(X\) and \(Y\) be two sets and suppose \(f : X \to Y\) and \(g : Y \to X\) are functions with the above properties. First we will show that \(f\) is a bijection. To see that \(f\) is injective, suppose \(x_1, x_2 \in X\) are such that \(f(x_1) = f(x_2)\). We desire to show that \(x_1 = x_2\). Since \(f(x_1) = f(x_2), x_1 = g(f(x_1)) = g(f(x_2)) = x_2\). Whence \(f\) is injective. To see that \(f\) is surjective, let \(y \in Y\). Then \(g(y) \in X\) and \(f(g(y)) = y\). Whence \(f\) is surjective. Therefore \(f\) is bijective. The proof that \(g\) is bijective is identical (and can be obtained by interchanging the roles of \(f\) and \(g\) with \(g\) and \(f\)). \(\square\)

To summarize:

**Theorem 4** Let \(X\) and \(Y\) be two sets and suppose \(f : X \to Y\) is a bijection. Then \(f^{-1} : Y \to X\) is also a bijection. Moreover \((f^{-1})^{-1} = f\) (that is, the inverse of \(f^{-1}\) is \(f\)).

**Proof:** Suppose \(f\) is bijective. Then \(f^{-1} : Y \to X\) satisfies the properties that \(f^{-1}(f(x)) = x\) and \(f(f^{-1}(y)) = y\) for all \(x \in X\) and \(y \in Y\). Hence \(f^{-1}\) is bijective by the above theorem.

To see that \((f^{-1})^{-1} = f\) we notice that, since \(f^{-1} : Y \to X\) is bijective, \((f^{-1})^{-1}\) exists and (by a previous theorem) is the unique function such that \((f^{-1})^{-1}(f^{-1}(y)) = y\) for all \(y \in Y\) and \(f^{-1}((f^{-1})^{-1}(x)) = x\) for all \(x \in X\). Since \(f(f^{-1}(y)) = y\) and \(f^{-1}(f(x)) = x\) for all \(x \in X\) and \(y \in Y\), the uniqueness of the function \((f^{-1})^{-1}\) implies \((f^{-1})^{-1} = f\) as desired. \(\square\)

**Challenging Exercises**

1. Show that there exists a bijective function \(f : \mathbb{N} \to \mathbb{Q}\).
2. Show that if \(f : \mathbb{N} \to \mathbb{R}\), then \(f\) is not a bijection.

**Extremely Challenging Exercise**

Let \(A\) and \(B\) be two sets. Suppose there exists subsets \(A_1 \subseteq A\) and \(B_1 \subseteq B\) and bijective functions \(f : A \to B_1\) and \(g : B \to A_1\). Prove there exists a bijective function \(h : A \to B\).