Example short paper topics

Here are some examples of suitable topics for a short paper such as the one you need to write for 50% of your grade. These are suggestions only. You are free to choose any topic as long as it is not covered in class and it is related to algebraic number theory. The level of difficulty of the paper should be a little higher than the material in MATH627, and may be quite higher. The following examples are of mixed difficulty. The more difficult examples are marked with a *****: if you choose a more difficult topic then you will be graded more leniently and you will learn more. In all cases, it will be more important to show that you understand the structure of the mathematical arguments involved rather than their technical details. This will be especially true of the more technical proofs used in the more difficult topics.

Example 1(*): Write an account of a proof of the Kronecker-Weber Theorem. This theorem says that every abelian extension of the rational numbers is contained in a cyclotomic field. To understand the statement of the theorem, note that a Galois extension K of the rational numbers is a field generated by the roots of a polynomial with rational coefficients. The Galois group G of K is the group of automorphisms of K fixing every rational number. The field K is called an abelian extension of the rational numbers if the group G is abelian.

Example 2 (*): Expand on the principal ideal theorem given by Theorem 9.12 of the text, in the section "How to make an ideal principal". This involves showing that the field L in which the ideals of a number field K become principal is the so-called Hilbert class field of K, that is the maximal abelian unramified extension of K. Here, the word "unramified" refers to both the finite and infinite "places" of K. Part of your paper will be identifying and understanding such definitions. Moreover, the degree of L over K equals the class number of K and the Galois group of L over K is isomorphic to the ideal class group of K. Give examples.

Example 3: Read Chapter 13, on Elliptic Curves, of the textbook. Then, write an account of the proof of Theorem 13.19 (Mordell's Theorem).

Example 4: Read Appendix A, on Quadratic Residues, of the textbook. Write about the history of Gauss's Quadratic Reciprocity Law (Theorem A.20), discussing in particular the origins of the law, the different proofs, and generalizations of the law to other rings such as the Gaussian and Eisenstein integers.

Example 5: A complex number which is not algebraic is called transcendental. The first explicit transcendental numbers were found in 1844 by Liouville, of which $\xi = \sum_{n=1}^{\infty} 10^{-n!}$ is an example. In 1873, Hermite showed that the more "familiar" number e is transcendental. He introduced a method of proof which is still influential today. Discuss, in detail, proofs of the transcendence of the numbers ξ and e. Comment on further developments of transcendence results in the 19th century.

Example 6: Mahler's measure of a polynomial P is defined to be the absolute value of the product of those roots of P which lie outside the unit disk, multiplied by the absolute value of the coefficient of the leading term of P. We denote it $M(P)$. Lehmer's problem, sometimes called Lehmer's question, or Lehmer's conjecture, asks if there exists a constant $C > 1$ such that every polynomial P with integer coefficients and $M(P) > 1$ has $M(P) \geq C$. This problem is still open. Give a historical survey of the major results inspired by Lehmer's problem, being careful to explain all the mathematical definitions that occur.

Example 7: The field of rational numbers is completed to yield the field of real numbers by the using the absolute value. This generalizes to the theory of valuations and completions of a number field. Describe the basic notions and results of this theory. Discuss several examples in detail. Show how one may use valuations to define the height of an algebraic number. Discuss the basic properties of the height.

Example 8: Define and discuss the basic properties of the Dedekind zeta function of an algebraic number field K, such as product representation, functional equation, meromorphic continuation. Derive the formula for the residue of this function at $s = 1$ that involves the class number of K.

Example 9 (*): Describe the construction of the adeles and ideles of a number field K and the definition of the idele class group. For this, you will need some basics of the theory of valuations described in Example 7. Why are the notions of adele and idele useful? Justify your answer with some examples.

Example 10: Fermat introduced the method of infinite descent, of which he was extremely proud! Briefly, the method proves that certain properties are impossible for positive integers by proving that if they held for any numbers they would hold for some smaller numbers; then, by the same argument, they would hold for some numbers that are smaller still, and so forth *ad infinitum*, which is impossible because a sequence of positive integers cannot decrease indefinitely. Give a brief account of the history and impact of the method of infinite descent. Give interesting examples of its application to diophantine equations, including to some cases of Fermat's Last Theorem.