ILL Article Request

Call #: QA241 .N8663 2003
Location: Valley
AVAILABLE

OSU ILLIAD TN#: 301493

Journal Title: London Mathematical Society lecture note series.
Volume: 303
Issue:
Month/Year: 2004
Pages: 41–62

ILN Number: 42550249

Article Author:
Article Title: An elementary approach to effective diophantine approximation

Borrower: TXA

Odyssey: 165.91.220.135

Borrower: TXA
Lending String: *ORE,COD,KKS,SOI,TJC
Patron: Tretkoff, Paula
Maxcost: $501FM

Ariel: 128.193.163.52

This item will be invoiced at the end of the month

Photocopy/Loan/Invoice/Postage Charges: GWLA - No Charge

Fax: 979-458-2032

OCLC Number: 8088082

NOTICE:
When available, we have included the copyright statement provided in the work from which this copy was made.

If the work from which this copy was made did not include a formal copyright notice, this work may still be protected by copyright law. Uses may be allowed with permission from the rights-holder, or if the copyright on the work has expired, or if the use is "fair use" or within another exemption. The user of this work is responsible for determining lawful use.

Shipping Address:
TEXAS A & M UNIVERSITY
TAMU LIBRARIES
5000 TAMUS
COLLEGE STATION TX 77843-5000
Introduction

To Sir Peter Symington-Deer on His 70th Birthday

Ennio Bompieri and Paula B. Cohen

Diphantine approximation on $\mathbb{C}^n$

An elementary approach to effective
Lemma 2. We have

\[(\sum_{i=1}^{\infty} s_i x_i)^{\infty} \leq (s - (h', y')d')^{\infty}\]

In what follows, for a real number \(t\), we abbreviate \(\max_{x \in [t, \infty)} = t\).

Let \(x, y, z, w, r, s, t\) be positive numbers with the order \(x > y > z > w > r > s > t > 0\).

We define the index of a point \((h', y')d')\) of \((h', y')d')\) as the order of \((h', y')d')\) at a point \((h', y')d')\) of \((h', y')d')\).

1. \(1 = (x, y)\),\(P\) the point of \(1, 0\).

2. \(h' = (h', y')d')\).

\[\sum_{i=1}^{\infty} (h', y')d')^{\infty} = (h', y')d')\]

Define \(c = n^{\infty} d = 0\).

\[\sum_{i=1}^{\infty} (h', y')d')^{\infty} = (h', y')d')^{\infty} \leq (h', y')d')^{\infty}

Then we have

\[\sum_{i=1}^{\infty} (h', y')d')^{\infty} = (h', y')d')^{\infty} \leq (h', y')d')^{\infty}

Theorem 3. Let \(M\) be a member of \(\mathcal{S}\) and \(\alpha, \beta, \gamma, \delta, \epsilon, \zeta\) be positive numbers with the order \(\alpha > \beta > \gamma > \delta > \epsilon > \zeta > 0\).

In the following sections, we abbreviate \(\sum_{i=1}^{\infty} (h', y')d')^{\infty} = (h', y')d')^{\infty} \leq (h', y')d')^{\infty}

2. Equivalent polynomials

In the Appendix we determine the total error to the first order. We shall discuss Diamond's algorithm for a point of the This problem defines the problem of finding a point of the equivalent polynomials. The second author thanks the Institute for Advanced

Acknowledgements

The second author thanks the Institute for Advanced

Endnote 1: Diamond's method is used to solve this problem. The main result is shown in (\(n, t\),\(r\),\(s\),\(t\))d')\).

It is an interesting problem to try to prove the main result.

Effective diagonalization approximation on C.
Consider the polynomial $P(x) = (1 + u)(s - (1 + u)(1 + s))$. By Lemma 2.2, we have $\dim V \geq \dim V/(1 + u)(1 + s)$. Therefore, we have $\dim V \geq \dim V/(1 + u)(1 + s)$.

**Lemma 2.3:** There is a basis of $V$ such that

\[ \frac{\dim V}{1 + u} \geq (1 + u)(1 + s) \]

Our next result gives us a small basis for the vector space $V$.

The proof of Lemma 2.3 consists of the following steps.

1. \( \dim V \geq \dim V/(1 + u)(1 + s) \)
2. Suppose this is not the case. Then, there is a polynomial $f$ not identically 0.
3. Define \( u = (1 + u)(1 + s) \).
4. Then the lemma follows from the statement that
   \[ \{0(1 + u)(1 + s) : v \geq d\} = 1 + u \]

Because \( \frac{\dim V}{1 + u} \geq (1 + u)(1 + s) \), also we have \( \dim V \geq (1 + u)(1 + s) \).

**Proof:** We approximate $f$ for any $u \neq 0$.

\[ \frac{\dim V}{1 + u} \geq (1 + u)(1 + s) \]

**Lemma 2.2:** The vector space $V$ has dimension

\[ \{a \geq (1 + u)(1 + s) : \exists v \geq d : d \} = 1 + u \]

Consider now the $\mathbb{C}$-vector space $V$ of $\mathbb{C}$-valued functions defined by

\[ \{f(x) x \{(1 + u)(1 + s) : \exists v \geq d : d \} = 0 \}

**Proof:** Construct the proof.
\[
\begin{align*}
\begin{array}{c}
\text{Lemma 2.3,}\hspace{1cm} & \text{There is an integer polynomial } \phi \in \mathbb{Z}[x], \text{ such that}\hspace{1cm} \\
& \phi (x, y) = (d) \eta \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Proof: }
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Lemma 2.4.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Theorem 2.5.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Lemma 2.6.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Theorem 2.7.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Lemma 2.8.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Theorem 2.9.}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Lemma 2.10.}
\end{array}
\end{align*}
\]
\[
\left(1 + \frac{u_2}{1 + u_1}\right) \leq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

With these estimates, we find
\[
\left(\frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right) \leq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

By definition, we have
\[
\forall \alpha \in \mathbb{R}, \ |\alpha| \geq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

Therefore, \(0 \neq (u, y) \in (d_{\alpha})\) if and only if \(\alpha \neq 0\).

**Lemma 3.1**: Let \(\mathcal{M} \equiv \mu_{\alpha} X \equiv X \mu_{\alpha} \in \mathcal{D}^\omega \mathcal{D}^\omega\)

**Theorem 3.2**: The last line of the above expansion is essential for our next section.

\[
\forall \alpha \in \mathbb{R}, \ |\alpha| \geq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

\[
\forall \alpha \in \mathbb{R}, \ |\alpha| \geq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

\[
\forall \alpha \in \mathbb{R}, \ |\alpha| \geq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

**Proof**:

1. \(X \leq \mathcal{M} X + \mathcal{M} X \mathcal{M} X \leq \mathcal{M} X + \mathcal{M} X \mathcal{M} X \leq \mathcal{M} X + \mathcal{M} X \mathcal{M} X \)

We have

\[
X \geq \mathcal{M} X + \mathcal{M} X \mathcal{M} X \leq \mathcal{M} X + \mathcal{M} X \mathcal{M} X \leq \mathcal{M} X + \mathcal{M} X \mathcal{M} X
\]

Now we estimate each term in \(X\) as follows.

\[
\left(1 + \frac{u_2}{1 + u_1}\right) \leq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

and since the left-hand side of this inequality is satisfied by \(\rho_1\) and \(\rho_2\), we have

\[
\left| (1 + u)(1 + u_1) \right| \geq \left| |u| |u_1| \right|
\]

\[
\left| (1 + u)(1 + u_1) \right| \geq \left| |u| |u_1| \right|
\]

We have

\[
\left(1 + \frac{u_2}{1 + u_1}\right) \leq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

and since the left-hand side of this inequality is satisfied by \(\rho_1\) and \(\rho_2\), we have

\[
\left(1 + \frac{u_2}{1 + u_1}\right) \leq \max \left\{ \frac{1 - \rho_1}{1}, \frac{1 - \rho_2}{1}, \frac{1}{1 + |\alpha|} \right\}
\]

**Effective diagonal approximation on \(\mathcal{D}\)**

\[
0 = \frac{a}{g} \Rightarrow \frac{a}{g} \Rightarrow \frac{a}{g} \Rightarrow \frac{a}{g}
\]

**Therefore** \(0 \neq (u, y) \in (d_{\alpha})\)
4 Simplification of the main inequality

Proof of Lemma 3.1. This completes the proof of the lemma. Since, for the terms in the statement of the lemma, we have $A = I + (s - w_i)$, the inequality becomes

$$V(s - w_i) \leq V(s)$$

where $V(s)$ is the maximum of $V(s)$ over all $s$. In order to bound the right-hand side of this inequality, we replace $V(s)$ by $V(s - w_i)$.

Now, let

$$V(s) = \sum_{i} \frac{\mu_i (s - w_i)}{s - w_i}$$

and set

$$\mathcal{D} = I + u.$$

Next, we choose $b$ such that

$$\sum_{i} \frac{\mu_i (s - w_i)}{s - w_i} \geq V(s - w_i)$$

and

$$b + \frac{b}{d} \geq \frac{b}{d}.$$ 

Finally, combining these conditions, we obtain

$$\sum_{i} \frac{\mu_i (s - w_i)}{s - w_i} \geq V(s - w_i)$$

and

$$V(s - w_i) \leq V(s)$$

where

$$V(s) = \sum_{i} \frac{\mu_i (s - w_i)}{s - w_i}.$$
as a corollary of Proposition 1, we derive in this section important results.

Multiplicative Group

in a number field Q a finitely generated

Application to Diophantine Approximation

\[ d \frac{\partial \lambda}{\partial \nu} \leq \delta \left( \frac{\partial \lambda}{\partial \nu} + \frac{1}{\nu \log \nu} \right) \]

Then if we have

\[ \left| \frac{d \partial \lambda}{\partial \nu} \right| \left| \frac{\partial \lambda}{\partial \nu} \right| < \varphi \log \nu - 1 \]

\[ d \frac{\partial \lambda}{\partial \nu} \leq \delta \left( \frac{\partial \lambda}{\partial \nu} + \frac{1}{\varphi \log \nu} \right) \]

In order to prove further that

\[ V/V = \delta \left( \frac{\partial \lambda}{\partial \nu} + \frac{1}{\varphi \log \nu} \right) \]

and \( d \frac{\partial \lambda}{\partial \nu} \leq \delta \left( \frac{\partial \lambda}{\partial \nu} + \frac{1}{\varphi \log \nu} \right) \)

Let's say we have the following assumptions that

\[ \varphi \log \nu \geq \frac{1}{\delta} \]

Then, we can conclude that

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

Therefore, we conclude that the second half of (4.1) does not hold. Note also that by

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

we have the following assumptions that

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

This is a contradiction, and we have the above.

\[ \varphi \log \nu \geq \frac{1}{\delta} \]

and after multiplication by \( \varphi \log \nu \) and an easy simplification we find

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

We now choose

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

This is a contradiction, and we have the above.

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

We now choose

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

This is a contradiction, and we have the above.

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

We now choose

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

This is a contradiction, and we have the above.

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

We now choose

\[ \varphi \log \nu \leq \frac{1}{\delta} \]

This is a contradiction, and we have the above.
and can be further improved by replacing \( C \) with the smaller constant
\[
1 > |a| - |b| > 0
\]
Moreover, if \( a + b \) and \( a - b \) are both odd, then
\[
\text{min}(\nu, H) \geq |a - b|
\]
Then
\[
(\nu, H) \left( \frac{\nu}{\text{ord} d} \right) d \geq \text{ord} d
\]
and suppose that
\[
\nu > 0 \quad \text{and} \quad \text{ord} d = C > 0
\]
and suppose that
\[
\nu > 0 \quad \text{with a non trivial field of degree} \quad d \quad \text{and a non trivial root of width} \quad d
\]
We now summarize our results as follows: The now condition in Proposition 1 is satisfied
\[
1 + \nu' \theta', \nu' \theta' > 0 \quad \text{in} \quad \text{ord} d
\]
before the condition of \( \nu' \theta' > 0 \) is satisfied which is always the case and \( \nu' \theta' > 0 \)
\[
1 + \nu' \theta' > 0 \quad \text{in} \quad \text{ord} d
\]
However, \( \nu' \theta' > 0 \) if and only if
\[
\nu' \theta' > 0 \quad \text{in} \quad \text{ord} d
\]
This immediately leads to the conclusion that the variable \( \mu \) should be
\[
(\nu' \theta')\left( \frac{\nu}{\text{ord} d} \right) \geq \text{ord} d
\]
This in turn satisfies the condition that \( \nu' \theta' > 0 \) is satisfied which is always the case and \( \nu' \theta' > 0 \)
\[
1 + \nu' \theta' > 0 \quad \text{in} \quad \text{ord} d
\]
Then we apply Proposition 1 and choose
\[
\nu' \theta' > 0 \quad \text{in} \quad \text{ord} d
\]
Continuing with the notation of \( \nu' \), we suppose that \( \nu' < 1 \) and choose
\[
\frac{a}{1} \leq \frac{\nu' \theta'}{1} < \frac{a}{\text{ord} d} \frac{p}{1} < \frac{a}{\mu - 1} \frac{p}{1} \]
From Lemma 1 of \( k \), we may suppose that
\[
1 > |a - b| > 0 \quad \text{in} \quad \text{ord} d
\]
We assume that \( k = 1 \) so that we choose
\[
(\nu' \theta') \text{max} = \frac{a}{\mu - 1} \frac{p}{1} = \frac{a}{p}
\]
and we abbreviate
\[
\text{effective multiplicative approximation of} \quad r^{\nu' \theta'}
\]

\[ (\forall \varepsilon > 0) \exists N \in \mathbb{N} : |\mathbb{N} - \varepsilon| < N \]

Theorem 3.2 Let \( L \) be a number field of degree \( d \) and \( E \) a place of \( L \).

The condition (IV) (of \( g \)) is now emended.

The condition (IV) is no consequence for the verification of Theorem 3.2.

This shows that the condition of Theorem 3.2 is not vacuous as soon as \( \varepsilon > 0 \).

Define \( (\forall \varepsilon > 0) \exists N \in \mathbb{N} : |\mathbb{N} - \varepsilon| < N \)

On the other hand, for any \( \varepsilon > 0 \), there exists a \( \varepsilon > 0 \) such that \( \varepsilon > 0 \), and for \( \varepsilon > 0 \), this condition

Proof By (5.6) it suffices that \( r \) be the result in Theorem 3.2, which is

In the method of proof above, in the result that

where is better than Theorem 3.2. Thus the interest of Theorem 3.2 is more

\[ (\forall \varepsilon > 0) \exists N \in \mathbb{N} : |\mathbb{N} - \varepsilon| < N \]

On the other hand, from (9) we may show that

\[ (\forall \varepsilon > 0) \exists N \in \mathbb{N} : |\mathbb{N} - \varepsilon| < N \]

Proof. In order to complete the proof of Theorem 3.2, a contradiction

Remark 3.1 Before completing the proof of Theorem 3.2, a contradiction

Define

\[ (\forall \varepsilon > 0) \exists N \in \mathbb{N} : |\mathbb{N} - \varepsilon| < N \]

Effective deformation approximation on the
(9.a) \[
\frac{x^2}{\theta} \left( \frac{\theta}{\ell} \right) \frac{f}{\ell} = (h'x)_{f} \\
\text{subject to} \quad \theta 
\]

where \( h'x \) is the best estimate of \( h \) using the observed data. For the results presented in this section, the data points are considered to be independent, and the estimates are obtained using maximum likelihood estimation.

In this appendix, we reproduce material from a letter of David Massey to the editor of the *Publications of the Astronomical Society of the Pacific*.

**Appendix from a Private Communication**

By David Massey
Lemma 1. For each \( k \geq 2 \), there exists a nonzero polynomial \( P(x, y) \) in \( \mathbb{Z}[x, y] \) of degree at most \( k \) in \( x \) and at most \( k \) in \( y \), with coefficients of at most \( C_k \), such that \( \delta \), \( \epsilon \), bounded by an integer \( d \), and \( \lambda \), the usual effective arguments.

Lemma 2. Suppose \( k \geq 2 \), let \( \xi \), \( \eta \), and \( \zeta \), \( \eta \) be arbitrary numbers with \( \xi \) independent of \( \eta \). Then there exists an integer \( d \) such that

\[
P(A_d(\xi, \eta)) = 0.
\]

The point is that \( B_0(\xi, \eta) = \cdots = B_k(\xi, \eta) = 0 \) if possible by the last sentence of Lemma 1. This is the step usually done by Gauss’s Lemma. It is interesting that the Dyson Lemma appears to give only \( \delta \geq 0 \) in place of \( c_0 = 1 \).

References

\[ (5q + 1)(1 + i)u = (5q + 1)(1 - i)u \]

where \( \mu \neq 0 \), so that (2) above becomes

\[ u = \frac{5q + 1}{u} \]
\[ u = \frac{1}{u} \]
\[ u + \bar{u} = 0 \]
\[ u - \bar{u} = 0 \]
\[ u + \bar{u} = 0 \]
\[ u - \bar{u} = 0 \]

(3)

First, make the substitution

\[ i = 1 \]
\[ i = 1 \]
\[ i = 1 \]
\[ i = 1 \]
\[ i = 1 \]
\[ i = 1 \]

For \( i = 1 \), there are only infinitely many solutions on the section with \( i = 1 \).

\[ (5q + 1)(1 + i) = 2x + 1 \]

Note: we investigate the section of the surface (1) cut by the plane

and we observe that this point also satisfies \( x + 1 \) for \( x + 1 \).

\[ x = \frac{5q + 1}{u} \]

As a notional solution,

\[ x = \frac{5q + 1}{u} \]

This is the Diophantine system

This note concerns the Diophantine system

Dedicated to Peter Samelson-Dyer on the occasion of his seventy-first birthday

Andrew Bremner