Noncommutative Geometry and Number Theory

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Introduction

In almost every branch of mathematics we use the ring of rational integers, yet in looking beyond the formal structure of this ring we often encounter great gaps in our understanding. The need to find new insights into the ring of integers is, in particular, brought home to us by our inability to decide the validity of the classical Riemann hypothesis, which can be thought of as a question on the distribution of prime numbers. Inspired by ideas from noncommutative geometry, Alain Connes [**8**], [**9**], [**10**] has in recent years proposed a set-up within which to approach the Riemann Hypothesis. The following chapters provide an introduction to these ideas of Alain Connes and are intended to aid in a serious study of his papers and in the analysis of the details of his proofs, which for the most part we do not reproduce here. We also avoid reproducing too much of the classical material, and choose instead to survey, without proofs, basic facts about the Riemann Hypothesis needed directly for understanding Connes's papers. These chapters should be read therefore with a standard textbook on the Riemann zeta function at hand—for example, the book of Harold M. Edwards [**16**], which also includes a translation of Riemann's original paper. For the function field case the reader can consult André Weil's book [**38**]. A good introduction to the Riemann zeta function and the function field case can also be found in Samuel J. Patterson's study [**30**]. A concise and informative survey of the Riemann Hypothesis, from which we quote several times, is given by Enrico Bombieri on the Clay Mathematics Institute website [**3**]. Some advanced notions from number theory are referred to as motivation for Connes's approach, but little knowledge of number theory is assumed for the discussion of the results of his papers. Although Connes's papers apply to arbitrary global fields, we most often restrict our attention to the field of rational numbers, as this still brings out the main points and limits the technicalities.

There are some similarities between Alain Connes's work in [**8**], [**9**], [**10**] and work of Shai Haran in [**22**], [**23**], [**24**]. We do not pursue here the relation to Shai Haran's papers, although we refer to them several times.

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The chapters are as follows:

Chapter 1: The objects of study.
Chapter 2: The Riemann Hypothesis for curves over finite fields.
Chapter 3: The local trace formula and the Polya–Hilbert space.
Chapter 4: The Weil distribution and the global trace formula.
Chapter 5: Related aspects of noncommutative number theory. (with Appendix by Peter Sarnak)

1. The objects of study

1.1. The Riemann zeta function. Riemann formulated his famous hypothesis in 1859 in a foundational paper [**31**], just 8 pages in length, on the number of primes less than a given magnitude. The paper centers on the study of a function $\zeta(s)$, now called the Riemann zeta function, which has the formal expression,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

the right hand side of which converges for $\mathrm{Re}(s) > 1$. This function in fact predates Riemann. In a paper [**18**], published in 1748, Euler observed a connection with primes via the formal product expansion,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \tag{2}$$

valid for $\mathrm{Re}(s) > 1$. Equation (2) is a direct result of the unique factorization, up to permutation of factors, of a positive rational integer into a product of prime numbers. The contents of Euler's paper are described in [**16**]. It seems that Euler was aware of the asymptotic formula,

$$\sum_{p<x} \frac{1}{p} \sim \log(\log x), \qquad (x \to \infty)$$

where the sum on the left side is over the primes p less than the real number x.

The additive structure of the integers leads to considering negative as well as positive integers and to the definition of the usual absolute value $|\cdot|$ on the ring $\mathbb{Z}$ of rational integers, defined by

$$|n| = \mathrm{sg}(n)\, n, \qquad n \in \mathbb{Z}.$$

The p-adic valuations are already implicit in the unique factorization of positive integers into primes. Namely, for every prime p and every integer n one can write

$$n = p^{\mathrm{ord}_p(n)} n'$$

where n' is an integer not divisible by p. The p-adic absolute value of n is then defined to be

$$|n|_p = p^{-\mathrm{ord}_p(n)}.$$

We denote by $\mathbb{Q}$ the field of fractions of $\mathbb{Z}$, namely the field of rational numbers, and by $M_{\mathbb{Q}}$ the set of valuations just introduced, extended to $\mathbb{Q}$ in the obvious way, and indexed by ∞ and by the primes p. We write, for $x \in \mathbb{Q}$,

$$|x|_v = |x|, \qquad v = \infty \in M_{\mathbb{Q}}$$

and

$$|x|_v = |x|_p, \qquad v = p \in M_{\mathbb{Q}}.$$

The following important observation is obvious from the definitions.

PRODUCT FORMULA: *For every $x \in \mathbb{Q}$, $x \neq 0$, we have*

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

Riemann derived a formula for $\sum n^{-s}$ valid for all $s \in \mathbb{C}$. For $\mathrm{Re}(s) > 0$, the Γ-function is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

The function $\Gamma(s)$ has an analytic continuation to all $s \in \mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, with residue $(-1)^m m!$ at $-m$, $m \geq 0$. This can be seen using the formula

$$\Gamma(s) = \lim_{N \to \infty} \frac{1.2.\ldots.N}{s(s+1)\ldots(s+N-1)} (N+1)^{s-1}.$$

Moreover, we have $s\Gamma(s) = \Gamma(s+1)$, and at the positive integers $m > 0$, we have $\Gamma(m) = (m-1)!$. Riemann observed that, for $\mathrm{Re}(s) > 1$,

$$\Gamma(\frac{s}{2}) \pi^{-s/2} \sum_{n=1}^\infty n^{-s} = \int_0^\infty \sum_{n=1}^\infty \exp(-n^2 \pi x) x^{s/2} \frac{dx}{x}. \tag{3}$$

Moreover, he noticed that the function on the right hand side is unchanged by the substitution $s \mapsto 1 - s$ and that one may rewrite the integral in (3) as

$$\int_1^\infty \sum_{n=1}^\infty \exp(-n^2 \pi x) \left(x^{s/2} + x^{(1-s)/2} \right) \frac{dx}{x} \quad - \quad \frac{1}{s(1-s)}, \tag{4}$$

which converges for all $s \in \mathbb{C}$ and has simple poles at $s = 1$ and $s = 0$. This shows that $\zeta(s) = \sum n^{-s}$, $\mathrm{Re}(s) > 1$, can be analytically continued to a function $\zeta(s)$ on all of $s \in \mathbb{C}$ with a simple pole at $s = 1$ (the pole at $s = 0$ in (4) being accounted for by $\Gamma(\frac{s}{2})$).

Riemann defined, for $t \in \mathbb{C}$ given by $s = \frac{1}{2} + it$, the function

$$\xi(t) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-s/2} \zeta(s), \tag{5}$$

for which we have the following important result.

THEOREM 1. (i) *Let Z be the set of zeros of $\xi(t)$. We have a product expansion of the form*

$$\xi(t) = \frac{1}{2} \pi^{-s/2} e^{bs} \prod_{\rho \in Z} (1 - \frac{s}{\rho}) e^{s/\rho}, \qquad s = \frac{1}{2} + it, \tag{6}$$

where $b = \log 2\pi - 1 - \frac{1}{2}\gamma$ and $\gamma = -\Gamma'(1) = 0.577...$ is Euler's constant. Moreover $\xi(t)$ is an entire function, and the set Z is contained in

$$\{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1\} = \{t \in \mathbb{C} \mid -\frac{i}{2} \leq \mathrm{Im}(t) \leq \frac{i}{2}\}$$

(ii) *The function $\xi(t)$ satisfies the* Functional Equation

$$\xi(t) = \xi(-t). \tag{7}$$

Moreover the set Z is closed under complex conjugation.

(iii) *By equation (5), the poles of* $\Gamma(s)$ *at the non-positive integers give rise to zeros of* $\zeta(s)$ *at the negative even integers. These are called the* Trivial Zeros. *The remaining zeros of* $\zeta(s)$ *are at the elements of* Z. *They are called the* Non-trivial Zeros.

The proof of the product formula of part (i) of Theorem 1 was sketched by Riemann and proved rigorously by Hadamard [**20**] in 1893. The rest of Theorem 1 is due to Riemann.

We can now state the central unsolved problem about the zeta function, namely to decide whether the following hypothesis is valid.

Riemann Hypothesis: *The zeros of* $\xi(t)$ *are real. Equivalently, the* non-trivial *zeros of* $\zeta(s)$ *lie on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

Riemann verified this hypothesis by hand for the first zeros and commented that

> "Without doubt it would be desirable to have a rigorous proof of this proposition, however I have left this research aside for the time being after some quick unsuccessful attempts, because it appears unnecessary for the immediate goal of my study."
>
> *(translated from German)*

Riemann's "immediate goal" was to find a formula for the number of primes less than a given positive real number x. However, Riemann's great contribution was not so much his concrete results on this question, but rather the methods of his paper, particularly his realisation of the existence of a relation between the location of the zeros of $\zeta(s)$ and the distribution of the primes. One illustration of this phenomenon is the two types of product formulae, one for $\zeta(s)$ is a product over the primes as in (2) and the other is a product over the zeros Z of the related function $\xi(t)$ as in Theorem 1 (i).

Prior to Riemann, and following some ideas of Euler from 1737, Tchebychev had initiated the study of the distribution of primes by analytic methods around 1850, by studying the function

$$\pi(x) = \mathrm{Card}\{p\,\mathrm{prime}, p \leq x\}.$$

He introduced the function

$$J(x) = \frac{1}{2}\left(\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n}\right) = \pi(x) + \frac{1}{2}\pi(\sqrt[2]{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \dots$$

and showed using the Euler product (2) that

$$\frac{1}{s}\log\zeta(s) = \int_1^\infty J(x)x^{-s}\frac{dx}{x}, \qquad \mathrm{Re}(s) > 1. \tag{8}$$

Riemann later used this formula and the calculus of residues to compute $J(x)$, and hence $\pi(x)$, in terms of the singularities of $\log\zeta(s)$, which occur at the zeros and poles of $\zeta(s)$. Building on work of Tchebychev and Gauss, Riemann made a rigorous study of (8), its inversion, and its relation to $\pi(x)$.

This work was developed after Riemann and culminated in the independent proof in 1896 by Hadamard [**21**] and de la Vallée-Poussin [**11**] of an asymptotic formula for $\pi(x)$.

PRIME NUMBER THEOREM: *As $x \to \infty$ we have,*

$$\pi(x) \quad \sim \quad \mathrm{Li}(x) = \int_0^x \frac{dt}{\log t} \quad \sim \quad \frac{x}{\log x}.$$

where the integral on the right hand side is understood in the sense of a Cauchy principal value, that is

$$\int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right).$$

Moreover, it became clear that the better one understood the location of the zeros of $\zeta(s)$, the better one would understand this approximation to $\pi(x)$. For example, the Prime Number Theorem is equivalent to the statement that there are no zeros of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$ and the Riemann Hypothesis is equivalent to the statement that, for every $\varepsilon > 0$, the relative error in the Prime Number Theorem is less than $x^{-1/2+\varepsilon}$ for all sufficiently large x.

Riemann also studied $N(T)$, $T > 0$—the number of zero of $\xi(t)$ between 0 and T—and sketched a proof of the fact that

$$N(T) \sim \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}.$$

1.2. Local fields. The prime numbers are intimately related to finite fields. If F is a finite field, then it is necessarily of prime characteristic $p > 1$, that is, p is the minimal non-zero integer for which the identity $p1_F = 0$ is true in F, where 1_F is the multiplicative unit element in F. By a result of Wedderburn, a finite field must also be commutative and has the structure of a vector space of dimension $f \geq 1$ over its prime ring $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, the field of p elements. The number of elements of F is then $q = p^f$ and it is isomorphic to the field $\mathbb{F}_q$ of roots of the equation $X^q = X$.

An arbitrary field with the discrete topology is locally compact. (In general, a metric space is called locally compact if every point has a neighborhood which is compact.) From the topological viewpoint, the interesting locally compact fields should not be discrete. This leads to considering, for discrete fields like $\mathbb{Q}$, their embeddings into closely associated locally compact non-discrete fields or rings.

The set $M_{\mathbb{Q}}$ of valuations of the field of fractions $\mathbb{Q}$ of the ring $\mathbb{Z}$ defines a family of metric spaces $(\mathbb{Q}, d_v)$, where $d_v(x, y) = |x - y|_v$, $v \in M_{\mathbb{Q}}$. To each such pair $(\mathbb{Q}, d_v)$, we can associate the corresponding completions with respect to the topologies induced by the metrics. Each of these completions also has the structure of a field. We denote by $\mathbb{Q}_p$ the field given by the completion of $\mathbb{Q}$ with respect to the metric d_p, for p a prime number. The completion of $\mathbb{Q}$ with respect to $d_\infty(x, y) = |x - y|$ is the field $\mathbb{R}$ of real numbers. The fields $\mathbb{Q}_p$ and $\mathbb{R}$ are examples of commutative locally compact non-discrete fields. We can consider $\mathbb{Q}$ as a subfield of its completions $\mathbb{Q}_p$ and $\mathbb{R}$, thereby enriching the ambient topological structure and allowing the application of techniques from classical analysis. Let K^* denote the group of non-zero elements of a field K. As we shall see in Chapter 3, consideration of the actions of $\mathbb{Q}_p^*$ on $\mathbb{Q}_p$ and of $\mathbb{R}^*$ on $\mathbb{R}$ already lead to some interesting trace formulae and can be seen as a first step in analysing "locally" the ring structure of $\mathbb{Z}$ from an operator-theoretic viewpoint.

In general, a local field K is a commutative field which carries a topology with respect to which the field operations are continuous and as a metric space it is complete, non-discrete and locally compact. (Often, one does not assume commutativity in the definition of a local field). Any locally compact Hausdorff topological group has a unique (up to scalars) non-zero left invariant measure which is finite on compact sets. If the group is abelian, this measure is also right invariant. It is called the Haar measure. The action of $K^* = K \setminus \{0\}$ on K by multiplication,

$$(\lambda, x) \mapsto \lambda x, \qquad \lambda \in K^*,\, x \in K,$$

induces a scaling of the Haar measure on the additive group K and hence a homomorphism of multiplicative groups

$$\begin{aligned} K^* \rightarrow \quad & \mathbb{R}^*_{>0} \\ \lambda \mapsto \quad & |\lambda|, \end{aligned}$$

where $\mathbb{R}^*_{>0}$ is the positive real numbers. Let

$$\mathrm{Mod}(K) = \{|\lambda| \in \mathbb{R}^*_{>0}, \lambda \in K^*\}.$$

Then $\mathrm{Mod}(K)$ is a closed subgroup of $\mathbb{R}^*_{>0}$. There are two classes of local fields, as follows (see [**37**], §I-4, Theorem 5 and Theorem 8).

(i) Archimedean local fields: $\mathrm{Mod}(K) = \mathbb{R}^*_{>0}$, in which case $K = \mathbb{R}$ or $\mathbb{C}$.

(ii) Non-archimedean local fields: $\mathrm{Mod}(K) \neq \mathbb{R}^*_{>0}$, in which case one has

$$\mathrm{Mod}(K) = q^{\mathbb{Z}},$$

where $q = p^d$ for some prime p and some positive integer d. Moreover,

$$R = \{x \in K \mid |x| \leq 1\}$$

is the unique maximal compact subring of K. It is a local ring with unique maximal ideal

$$\mathcal{P} = \{x \in K \mid |x| < 1\},$$

with $R/\mathcal{P} \simeq \mathbb{F}_q$, the finite field with q elements. (Notice that if K' is an extension of K of degree d then for $a \in K$ its modulus with respect to K' is the dth power of its modulus with respect to K.) If the non-archimedean local field has characteristic $p > 1$, then it is isomorphic to a field of formal power series in one indeterminate with coefficients in a finite field. If the non-archimedean field is of characteristic zero, then it is a finite algebraic extension of $\mathbb{Q}_p$.

For p prime and $x \in \mathbb{Q}_p$, there exists an integer r such that x can be written in the form

$$x = \sum_{i=0}^{\infty} a_{r+i} p^{r+i},$$

with $0 \leq a_{r+i} \leq p-1$, $i \geq 0$, and $a_r \neq 0$. We then have $|x|_p = p^{-r}$ and $x \in \mathbb{Z}_p$ if and only if $a_j = 0$ for $j < 0$. The maximal compact subring of $\mathbb{Q}_p$ is the ring of p-adic integers $\mathbb{Z}_p$ given by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\},$$

which contain the rational integers $\mathbb{Z}$ as a subring. The unique maximal ideal of $\mathbb{Q}_p$ is

$$\mathcal{P} = \{x \in \mathbb{Z}_p \mid |x|_p < 1\},$$

so that $\mathbb{Z}_p/\mathcal{P} \simeq \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{F}_p$.

1.3. Global fields and their adele rings. A *global field* K can be defined as a discrete cocompact subfield of a (non-discrete) locally compact semi-simple commutative ring $A = A_K$, called its adele ring [**37**]. There are two classes of global fields, as follows.

(i) Global fields of characteristic 0: These are the number fields, that is, the commutative fields which are finite dimensional vector spaces over $\mathbb{Q}$, this dimension being usually referred to as the degree of the field over $\mathbb{Q}$.

(ii) Global fields of characteristic p, where p is prime: These are the finitely generated extensions of $\mathbb{F}_p$ of transcendence degree 1 over $\mathbb{F}_p$. If K is such a field, then there is a $T \in K$, transcendental over $\mathbb{F}_p$, such that K is a finite algebraic extension of $\mathbb{F}_p(T)$. The field $\mathbb{F}_q$, where $q = p^f$ for some f, is included in K as its maximal finite subfield, called the field of constants. The field K may also be realised as a function field of a non-singular algebraic curve over $\mathbb{F}_q$. An analogue of the Riemann Hypothesis exists for such fields and is discussed in Chapter 2.

The field $\mathbb{Q}$ of rational numbers is a global field of characteristic 0 with a set of inequivalent valuations $v \in M_{\mathbb{Q}}$, $v = \infty$ or $v = p$, prime, as in §1.1. As in §1.2, associated local fields are given by the completions

$$\mathbb{Q} \hookrightarrow \mathbb{R}, \quad \text{for} \quad v = \infty$$

and

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_p, \quad \text{for} \quad v = p.$$

The adele ring $A = A_{\mathbb{Q}}$ combines all these local fields in a way that gives them equal status and combines their topologies into an overall locally compact topology. It is given by the restricted product

$$A = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p$$

with respect to $\mathbb{Z}_p$. This means that the elements of A are infinite vectors indexed by $M_{\mathbb{Q}}$,

$$x = (x_\infty, x_2, x_3, \ldots)$$

with $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$ and $x_p \in \mathbb{Z}_p$ for all but finitely many primes p. Addition and multiplication are componentwise. The embeddings of $\mathbb{Q}$ into its completions with respect to the elements of $M_{\mathbb{Q}}$ induce a diagonal embedding of $\mathbb{Q}$ into A whose image is called the principal adeles. Therefore, $a \in \mathbb{Q}$ corresponds to the principal adele

$$a = (a, a, a, \ldots).$$

A basis for the topology on A is given by the restricted products $U_\infty \times \prod_p' U_p$ where U_∞ is open in $\mathbb{R}$ and U_p is open in $\mathbb{Q}_p$ with $U_p = \mathbb{Z}_p$ for all but finitely many primes p. The quotient $A/\mathbb{Q}$ of the adeles by the principal adeles is compact. The reason for the restricted product is precisely to enable the definition of a non-trivial locally compact topology on A extending the locally compact topologies of the factors.

The group $J = J_{\mathbb{Q}}$ of ideles of $\mathbb{Q}$ is the restricted product

$$J = \mathbb{R}^* \times \prod_p{}' \mathbb{Q}_p^*$$

with respect to $\mathbb{Z}_p^* := \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$. These are the invertible elements of A. The elements of J are infinite vectors indexed by $M_{\mathbb{Q}}$,

$$u = (u_\infty, u_2, u_3, \ldots)$$

with $u_\infty \in \mathbb{R}, u_\infty \neq 0$, $u_p \in \mathbb{Q}_p, u_p \neq 0$ and $|u_p|_p = 1$ for all but finitely many primes p. Multiplication is componentwise. The topology on J is induced by the inclusion

$$\begin{aligned} J &\hookrightarrow A \times A \\ u &\mapsto (u, u^{-1}) \end{aligned}$$

The group $\mathbb{Q}^*$ of non-zero rational numbers injects diagonally into J and their image is called the principal ideles. Therefore $q \in \mathbb{Q}^*$ corresponds to the principal idele

$$q = (q, q, q, \ldots).$$

The group J carries the norm given by

$$\begin{array}{rcl} |\ \ | : J & \to & \mathbb{R}^*_{>0} \\ u = (u_v)_{v \in M_{\mathbb{Q}}} & \mapsto & \displaystyle\prod_{v \in M_{\mathbb{Q}}} |u_v|_v \ = \ |u_\infty| \prod_{p \text{ prime}} |u_p|_p. \end{array}$$

The Product Formula of §1.1 implies that $\mathbb{Q}^*$ is contained in Ker$|\ \ |$, the elements of J of norm 1, that is for $u \in \mathbb{Q}$, $u \neq 0$ we have

$$|u| = |u|_\infty \prod_p |u|_p = 1.$$

The action of J on A therefore embodies simultaneously the actions of $\mathbb{Q}_p^*$ on $\mathbb{Q}_p$ and $\mathbb{R}^*$ on $\mathbb{R}$. However, the roles of the individual valuations $v \in M_{\mathbb{Q}}$ remain independent, so one cannot hope to gain much additional insight into the structure of the ring $\mathbb{Z}$ in this way.

The *Idele Class Group* is the quotient $C = J/\mathbb{Q}^*$ of the ideles by the principal ideles. By the Product Formula, the norm defined on the ideles induces a well-defined norm on this quotient, which we also denote by

$$|\ | : C \to \mathbb{R}^*_{>0}.$$

Let J^1 be the subgroup of J given by the kernel of the norm map. The principal ideles $\mathbb{Q}^*$ form a discrete subgroup of J^1 and the quotient group $C^1 = J^1/\mathbb{Q}^*$ is compact. The idele class group C is the direct product of C^1 and $\mathbb{R}^*_{>0}$ and as such is called a quasi-compact group. For proofs of these facts, see [**37**], Chapter 4. A (continuous) homomorphism

$$\chi : C \to \mathbb{C}^*$$

is called a quasi-character. The quasi-characters form a group under pointwise multiplication. A quasi-character is called principal if it is trivial on C^1, and the principal quasi-characters form a subgroup. The norm map on C gives a non-trivial homomorphism of C onto $\mathbb{R}^*_{>0}$ and the principal quasi-characters are of the form $u \mapsto |u|^t$, $t \in \mathbb{C}$, $u \in C$. Every quasi-character admits a factorisation

$$\chi(u) = \chi_0(u)|u|^t, \qquad t \in \mathbb{C}, \quad u \in C,$$

with $\chi_0 : C \to U(1)$ a *unitary* character on C, that is, a homomorphism onto the group of complex numbers with absolute value 1. Every quasi-character χ on C can be considered as a homomorphism $\chi : J \to \mathbb{C}$ which is trivial on $\mathbb{Q}^*$. For every

$v \in M_{\mathbb{Q}}$, there is a natural embedding of $\mathbb{Q}_v^*$ (where $\mathbb{Q}_\infty = \mathbb{R}$) into J by sending $x \in \mathbb{Q}_v^*$ to the idele $(u_w)_{w \in M_{\mathbb{Q}}}$ with $u_w = 1$ for all $w \neq v$ and $u_v = x$. Thereby, the quasi-character χ induces a homomorphism χ_v on $\mathbb{Q}_v$ with $\chi_p(\mathbb{Z}_p^*) = \{1\}$ for almost all primes p. The finite set S of p for which $\chi_p(\mathbb{Z}_p^*) \neq \{1\}$ is called the set of ramified primes. We may then write $\chi = \chi_\infty \prod_p \chi_p$. At an unramified prime $p \notin S$, the local factor χ_p is determined by its value at p. The L-function with non-principal quasi-character (or Grossencharacter) $\chi(u) = \chi_0(u)|u|^t$ is defined in the region $\Re(s) > 1 - \tau$, where $\tau = \Re(t)$, by

$$L(\chi, s) = \prod_{p \notin S} \left(1 - \chi_p(p) p^{-s}\right)^{-1}.$$

For $\chi \neq 1$ this function has an analytic continuation to all of $\mathbb{C}$, also denoted $L(\chi, s)$. For more details, see [**37**], Chapter 7. Notice that the Riemann zeta function corresponds to the case where χ is trivial and S is empty. We have the following generalization of the Riemann Hypothesis for unitary characters χ_0 on C (which has a corresponding version for all global fields, not just $\mathbb{Q}$).

Generalized Riemann Hypothesis: *We have* $L(\chi_0, s) = 0$ *for* $\Re(s) \in]0, 1[$ *if and only if* $\Re(s) = \frac{1}{2}$.

1.4. Connes's dynamical system. There are natural symmetry groups which arise for global fields. In the characteristic zero case, given a finite extension K of $\mathbb{Q}$, the associated symmetry group is the group of field automorphisms of K which leave $\mathbb{Q}$ fixed. This symmetry group is called the Galois group of K over $\mathbb{Q}$. In the characteristic p case, the symmetry groups come from Frobenius automorphisms of the corresponding variety over $\mathbb{F}_q$ given by raising coordinates on the variety to the q-th power. For a global field K of finite characteristic p, the corresponding idele class group C_K turns out to be canonically isomorphic to the Weil group W_K generated by all automorphisms, leaving K fixed, and induced on a certain field extension of K by powers of the Frobenius. Therefore the natural symmetry group is in fact the idele class group which in turn has an interpretation as a Galois group.

The following proposal of Weil, made in 1951, is a central motivation for Connes's approach [**35**]

> "The search for an interpretation of C_K when K is a number field, which is in any way analogous to the interpretation as a Galois group when K is a function field, seems to me to constitute one of the fundamental problems of number theory nowadays; it is possible that such an interpretation holds the key to the Riemann hypothesis."
>
> *(translated from French)*

For a local field, the corresponding Weil group W_K is again generated by powers of the Frobenius automorphism of an extension of K. By the main result of local class field theory the group W_K is isomorphic to K^*, which therefore locally plays the role of the idele class group.

Class field theory relates the arithmetic of a number field, or of a local field, to the Galois extensions of the field. For a local field, by the remarks above, class field theory tells us that the group K^* plays a central role in this relation. This group acts naturally on the space consisting of the elements of K, and understanding the action of multiplication on the additive structure of K, that is, the map,

$$\begin{aligned} K^* \times K &\to K \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

is certainly a basic part of understanding the arithmetic of a local field.

In the light of the situation for global fields of characteristic $p > 0$, one can view the analogue of the action of W_K for the field $\mathbb{Q}$ as being the passage to the quotient by $\mathbb{Q}^*$ of the map,

$$\begin{aligned} J \times A &\to A \\ (u, x) &\mapsto ux, \end{aligned}$$

that is

$$\begin{aligned} C \times X &\to X \\ ([u], [x]) &\mapsto [u][x]. \end{aligned}$$

Here C is the idele class group as above and X is the space of cosets $X = A/\mathbb{Q}^*$. The notation [ ] means the class modulo the multiplicative action of $\mathbb{Q}^*$ and will be mainly dropped in future. Therefore $[a] = [b]$ in X for $a, b \in A$ if and only if there is a $q \in \mathbb{Q}^*$ such that $a = qb$.

Connes proposes to study the dynamical system (X, C) using the following guidelines.

- Relate the spectral geometry of the action (X, C) to the zeros of $\zeta(s)$.
- Relate the non-commutative geometry of the orbits of (X, C) to the valuations $M_{\mathbb{Q}}$ of $\mathbb{Q}$.
- Show that the consequent relation of the zeros of $\zeta(s)$ to the primes of $\mathbb{Z}$ is fine enough to prove the Riemann Hypothesis.

1.5. Weil's Explicit Formula. A very crude relation between the zeros of $\zeta(s)$ and the primes appeared already in §1.1 when we compared the product formula over Z for $\xi(t)$ in Theorem 1(i) with the Euler product formula (2) for $\zeta(s)$. Namely, we have, for $\mathrm{Re}(s) > 1$,

$$s(s-1)\Gamma(\frac{s}{2}) \prod_{p \text{ prime}} (1 - p^{-s})^{-1} = \exp(bs) \prod_{\rho \in Z} (1 - \frac{s}{\rho}) \exp(\frac{s}{\rho}).$$

An important refinement of ideas going back to Riemann's paper led Weil to develop his "Explicit Formula". Roughly speaking, the idea is to take logarithms and then Mellin transforms in the last displayed formula. For a function $f : \mathbb{R}_+ \to \mathbb{C}$ we define the (formal) Mellin transform to be

$$M(f, z) = \int_0^\infty f(t) t^z \frac{dt}{t}.$$

Then Weil's formula is (formally) as follows:

$$M(f,0) - \sum_{\rho\in Z} M(f,\rho) + M(f,1) \;=\; (\log 4\pi + \gamma) f(1) \;+\; \sum_{m=1}^{\infty} \sum_{p \text{ prime}} (\log p)\{f(p^m) + p^{-m} f(p^{-m})\} + \int_1^\infty \{f(x) + x^{-1} f(x^{-1}) - \frac{2}{x} f(1)\} \frac{dx}{x - x^{-1}}. \tag{9}$$

Here,

$$\sum_{\rho\in Z} M(f,\rho) := \lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|<T} M(f,\rho).$$

Weil also observed that the Riemann Hypothesis is equivalent to the positivity of

$$R(f) := \sum_{\rho\in Z} M(f,\rho)$$

for functions of the form

$$f(x) = \int_0^\infty g(xy)\overline{g(y)}dy. \tag{10}$$

This translates into the negativity of the left hand side of (9) for such f which also satisfy

$$M(f,0) = \int_0^\infty f(x)\frac{dx}{x} = 0, \quad M(f,1) = \int_0^\infty f(x)dx = 0.$$

Indeed, for f as in (10), we have

$$M(f,\rho) = M(g, \rho - \frac{1}{2})\overline{M(g, -(\overline{\rho} - \frac{1}{2}))},$$

so that RH implies the positivity of $R(f)$. Conversely, we have enough functions $M(f,z)$ to localize the zeros of $\zeta(s)$. To make this rigorous and not just formal, we must impose some conditions on the class of functions $f : \mathbb{R}_+ \to \mathbb{C}$. We require that f be continuous and continuously differentiable except at finitely many points where f and f' have discontinuities of the first kind. At these discontinuities f and f' are defined as the average of their left and right value. Also, there is a $\delta > 0$ such that $f(x) = O(x^\delta)$ as $x \to 0^+$ and $f(x) = O(x^{-1-\delta})$ as $x \to +\infty$. Then $M(f,z)$ is analytic for $-\delta < \mathrm{Re}(z) < 1 + \delta$ (see [**3**]).

As suggested in §1.4, for the action (X, C) the zeros of $\zeta(s)$ should have a spectral interpretation. Inspection of (9) suggests that these eigenvalues should appear with a negative sign to match the term $-\sum_{\rho\in Z} M(f,\rho)$. This is, in fact, a feature of Connes's approach. We discuss this in Section 3.2. For some related comments on the comparison of (X, C) with hamiltonian flows in quantum chaos, see [**10**].

2. The Riemann Hypothesis for curves over finite fields

There is an analogue of the Riemann Hypothesis (RH) for certain zeta functions attached to curves defined over finite fields. This analogue, introduced by E. Artin (1924) and checked by him for a few curves of genus 1, is known as the function field case after the function field of the curve. F. Schmidt (1931) showed that the zeta function for curves is rational and has a functional equation. Hasse (1934), using ideas from algebraic geometry together with some analytic methods, proved the analogue of RH for all genus 1 curves. In the early 1940's, Weil formulated an approach to RH for arbitrary curves defined over finite fields (see [**39**]). Subsequently, Weil developed the methods from algebraic geometry needed to execute his approach, and he published complete proofs in his 1948 book [**38**]. A more elementary proof was developed by Stepanov (1969), and this was further simplified by Bombieri (1972) [**2**]. Weil pioneered the study of zeta functions for arbitrary varieties over finite fields and developed some conjectures about these functions, in particular connecting topological data for these varieties to counting rational points on them over finite extensions of the base field. Included in these Weil Conjectures is a generalization of RH and its interpretation as a statement about the eigenvalues of the Frobenius automorphism acting on the cohomology of a variety. Dwork (1960) used p-adic analysis to establish the rationality of the zeta function for arbitrary varieties. Various cohomology theories relevant to the Weil Conjectures were developed, in particular by M. Artin, Grothendieck, Serre and Verdier. The complete proof of the Weil Conjectures was finally obtained in 1973 by Deligne [**12**], [**13**].

The successful solution of the analogue of RH for function fields provides strong encouragement for believing the validity of RH in the as yet unsolved number field case. It is still, however, an *open problem* to prove RH in the same generality for function fields using the program proposed by Alain Connes: it is anticipated that doing so would give much new information about the program in the number field case.

2.1. The zeta function of a curve over a finite field. Let p be a prime number and $\mathbb{F}_q$ the field of $q = p^d$, $d \geq 1$, elements. The map $\alpha \mapsto \alpha^q$ is the identity on $\mathbb{F}_q$. There are d automorphisms of $\mathbb{F}_q$ leaving $\mathbb{F}_p$ fixed, namely $\alpha \mapsto \alpha^{p^i}$, $\alpha \in \mathbb{F}_q$, $i = 0, \ldots, d-1$. The field $\mathbb{F}_q$ is the finite extension of $\mathbb{F}_p$ of degree d. Let K be a field extension of $\mathbb{F}_q$ with transcendence degree equal to 1. Then K is a finite algebraic extension of $\mathbb{F}_p(T)$ where T is transcendental over $\mathbb{F}_p$, and is a global field of characteristic p, as described in Section 1.3. We can also write $K = \mathbb{F}_q(x,y)$ where x is transcendental over $\mathbb{F}_q$ and there is an irreducible polynomial $F = F(x,y) \in \mathbb{F}_q[x,y]$ such that $F(x,y) = 0$. This equation defines a plane curve defined over $\mathbb{F}_q$ which has a smooth model Σ whose meromorphic function field is K.

Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, and $\overline{\Sigma} = \Sigma(\overline{\mathbb{F}}_q)$ denote the curve over $\overline{\mathbb{F}}_q$ given by the points of Σ rational over $\overline{\mathbb{F}}_q$. The field of functions of $\overline{\Sigma}$ is $\overline{K} = \overline{\mathbb{F}}_q(x,y)$. The Frobenius automorphism of $\overline{K}$ is given by the map $u \mapsto u^q$, $u \in \overline{K}$. This induces the map $(x,y) \mapsto (x^q, y^q)$ on the solutions $(x,y) \in \overline{\mathbb{F}}_q^{\,2}$ of $F(x,y) = 0$, which in turn defines a Frobenius map Fr on $\overline{\Sigma}$. By linearity over the integers, the map Fr extends to the additive group of finite formal sums of points on $\overline{\Sigma}$.

For every integer $j \geq 1$, the automorphism group of $\mathbb{F}_{q^j}$ over $\mathbb{F}_q$ is generated by the Frobenius map whose j-th power fixes the elements of $\mathbb{F}_{q^j}$. In the same way, the fixed points in $\overline{\Sigma}$ of the j-th iterate Fr^j are the points $\Sigma(\mathbb{F}_{q^j})$ of $\overline{\Sigma}$ rational over $\mathbb{F}_{q^j}$.

We introduce some definitions.

Definition 1. The divisor group $\mathrm{Div}(\Sigma)$ of Σ over $\mathbb{F}_q$ is the formal additive group of *finite* sums

$$\mathrm{Div}(\Sigma) = \{\mathcal{A} = \sum_i a_i P_i,\ a_i \in \mathbb{Z},\ P_i \in \Sigma(\mathbb{F}_{q^{d_i}}) \text{ some } d_i \in \mathbb{N},\ \mathrm{Fr}(\mathcal{A}) = \mathcal{A}\}$$

invariant under the Frobenius automorphism Fr on $\overline{\Sigma}$. A divisor $\mathcal{A} = \sum_i a_i P_i$ is said to be effective (written $\mathcal{A} > 0$) if $a_i > 0$ for all i.

For two divisors $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} > \mathcal{B}$ when $\mathcal{A} - \mathcal{B}$ is effective.

Definition 2. If $\mathcal{A} = \sum_i a_i P_i \in \mathrm{Div}(\Sigma)$, then the degree $d(\mathcal{A})$ of $\mathcal{A}$ is $\sum_i a_i$. The norm $N(\mathcal{A})$ of $\mathcal{A}$ is $q^{d(\mathcal{A})}$.

Notice that for $\mathcal{A}, \mathcal{B} \in \mathrm{Div}(\Sigma)$, we have

$$N(\mathcal{A} + \mathcal{B}) = N(\mathcal{A}) N(\mathcal{B}). \tag{11}$$

Definition 3. An effective divisor $\mathcal{A} \in \mathrm{Div}(\Sigma)$ is prime if it cannot be written as the sum of two effective divisors in $\mathrm{Div}(\Sigma)$.

The effective divisors are the analogues of the positive integers and the prime divisors are the analogues of the prime integers. Every effective divisor can be uniquely decomposed (up to permutation of the summands) into a sum of prime divisors.

Definition 4. The zeta function of the curve Σ with function field K is given by

$$\zeta_K(s) = \zeta(s, \Sigma) = \sum_{\mathcal{A}>0} \frac{1}{N(\mathcal{A})^s} = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1}, \qquad \mathrm{Re}(s) > 1, \tag{12}$$

where the product is over the prime divisors $\mathcal{P}$ in $\mathrm{Div}(\Sigma)$.

The Euler product decomposition on the right of (12) is a consequence of (11) and the uniqueness of the prime decomposition of effective divisors.

It is useful to use the change of variables $u = q^{-s}$, and write the zeta function as

$$Z(u) = \zeta_K(s) = \prod_{\mathcal{P}} (1 - u^{d(\mathcal{P})})^{-1}. \tag{13}$$

We then have,

$$u \frac{d}{du} \log Z(u) = \sum_{j=1}^{\infty} \left(\sum_{d(\mathcal{P})|j} d(\mathcal{P}) \right) u^j. \tag{14}$$

The quantity $\sum_{d(\mathcal{P})|j} d(\mathcal{P})$ in (14) equals the number of points of $\Sigma(\mathbb{F}_{q^j})$.

Theorem 2. *The zeta function has an analytic continuation to the whole complex plane $\mathbb{C}$ and may be written*

$$\zeta_K(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}, \tag{15}$$

for a certain polynomial P of degree $2g$, where g is the genus of Σ. Moreover, the polynomial P satisfies

$$q^{gs} P(q^{-s}) = q^{g(1-s)} P(q^{s-1}). \tag{16}$$

The genus of $\overline{\Sigma}$ is the dimension over $\overline{\mathbb{F}}_q$ of the space of sections of the canonical sheaf of $\overline{\Sigma}$. From (16) we see that the zeta function satisfies the *functional equation*

$$q^{(g-1)s}\zeta_K(s) = q^{(g-1)(1-s)}\zeta_K(1-s).$$

The proof of Theorem 2 uses the Riemann-Roch Theorem for $\overline{\Sigma}$ (see for example [**17**], Chapter V, §5).

One can write the polynomial P as a product over the 'multiset' Z (points with multiplicities) of reciprocal zeroes of P:

$$P(u) = \prod_{\rho\in Z}(1-\rho u). \tag{17}$$

This gives an analogue of the Hadamard product formula of (6),

$$\zeta_K(s) = \left(\prod_{\rho\in Z}(1-\rho u)\right)(1-u)^{-1}(1-qu)^{-1}. \tag{18}$$

From the Euler product we know that $\zeta_K(s) \neq 0$ for $\mathrm{Re}(s) > 1$ and therefore $1 \leq |\rho| \leq q$. Moreover, (16) implies the symmetry of Z under $\rho \mapsto q/\rho$.

2.2. The Riemann Hypothesis, the explicit formula and positivity. We begin with a statement of the Riemann hypothesis for function fields.

The Riemann Hypothesis for curves over finite fields: *The zeros of $\zeta_K(s)$ lie on $\mathrm{Re}(s) = \frac{1}{2}$, or equivalently each $\rho \in Z$ has $|\rho| = q^{\frac{1}{2}}$.*

The proof of this statement is a theorem due in its full generality to André Weil who proved it in 1942, see [**38**]. It is equivalent to the positivity of a certain functional that we describe below. Following the treatment in [**30**], we note that as in the number field case, we have a formal relation between prime divisors and zeros of ζ_K. Namely, we see from (13) and (18) that

$$\prod_{\mathcal{P}}(1-u^{d(\mathcal{P})})^{-1} = \left(\prod_{\rho\in Z}(1-\rho u)\right)(1-u)^{-1}(1-qu)^{-1}.$$

Taking logarithmic derivatives as in (14) we obtain

$$\sum_{\mathcal{P}} d(\mathcal{P})u^{d(\mathcal{P})}(1-u^{d(\mathcal{P})})^{-1} = -\sum_{\rho\in Z}\rho u(1-\rho u)^{-1} + u(1-u)^{-1} + qu(1-qu)^{-1}.$$

Comparing coefficients of u^j we have

$$\sum_{d(\mathcal{P})|j} d(\mathcal{P}) = 1 - \sum_{\rho\in Z}\rho^j + q^j. \tag{19}$$

The left hand side of this formula is the number of points of $\Sigma(\mathbb{F}_{q^j})$, that is, the number of fixed points of the jth power of the Frobenius map acting on $\Sigma(\overline{\mathbb{F}_q})$. For instance for the projective line $\Sigma = \mathbb{P}_1$, the left hand side is $1 + q^j$.

Multiplying (19) by $q^{-j/2}$ and using the functional equation, we obtain, after some manipulation, the following identity,

$$q^{-|j|/2}\sum_{d(\mathcal{P})|j} d(\mathcal{P}) = q^{j/2} + q^{-j/2} - \sum_{\rho\in Z}(\rho/q^{\frac{1}{2}})^j. \tag{20}$$

Let h be a function $h\colon \mathbb{Z} \to \mathbb{C}$. Define the discrete Mellin transform of h by

$$M^d(h,z) = \sum_{j\in\mathbb{Z}} h(j)z^j.$$

Suppose that h is such that its discrete Mellin transform converges in the interval $q^{-1/2} \le |z| \le q^{1/2}$. Multiplying (20) by $h(j)$ and summing over j gives the following:

Explicit Formula for curves over finite fields:

$$(21)\quad M^d(h,q^{\frac{1}{2}}) - \sum_{\rho\in Z} M^d(h,\rho/q^{\frac{1}{2}}) + M^d(h,q^{-\frac{1}{2}}) = (2-2g)h(0) + \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \sum_{\mathcal{P}} d(\mathcal{P}) q^{-d(\mathcal{P})|\ell|/2} h(d(\mathcal{P})\ell),$$

where g is the genus of Σ.

Just as for the Riemann zeta function (see Section 1.5), the Riemann Hypothesis is equivalent to a positivity statement. Consider the natural involution and convolution on the algebra of functions $h : \mathbb{Z} \to \mathbb{C}$. The involution is given by,

$$(22)\qquad h^*(j) = \overline{h(-j)}, \qquad j \in \mathbb{Z},$$

and the convolution of two functions h_1 and h_2 by

$$h_1 * h_2(j) = \sum_{j_1+j_2=j} h_1(j_1)h_2(j_2), \qquad j \in \mathbb{Z}.$$

The Mellin transform takes this convolution into a product, namely

$$M^d(h_1 * h_2, z) = M^d(h_1,z)M^d(h_2,z).$$

Define a hermitian form $R(h_1,h_2)$ by

$$(23)\qquad R(h_1,h_2) = \sum_{\rho\in Z} M^d(h_1 * h_2^*, \rho/q^{\frac{1}{2}}).$$

By (21) we have

$$(24)\quad R(h_1,h_2) = M^d(h_1 * h_2^*, q^{\frac{1}{2}}) + M^d(h_1 * h_2^*, q^{-\frac{1}{2}}) + (2g-2)\,(h_1 * h_2^*)\,(0) - \sum_{\mathcal{P}} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} d(\mathcal{P}) q^{-d(\mathcal{P})|\ell|/2}\,(h_1 * h_2^*)\,(d(\mathcal{P})\ell).$$

Theorem 3. *The Riemann hypothesis holds for $\zeta_K(s)$ if and only if R is positive semidefinite.*

Proof. Suppose that the Riemann hypothesis holds for $\zeta_K(s)$. Then for any $\rho \in Z$, we have $\mid \rho/q^{\frac{1}{2}} \mid = 1$. Therefore,

$$\begin{aligned} M^d(h * h^*, \rho/q^{\frac{1}{2}}) &= M^d(h,\rho/q^{\frac{1}{2}})M^d(h^*,\rho/q^{\frac{1}{2}}) \\ &= M^d(h,\rho/q^{\frac{1}{2}})\overline{M^d(h,(\bar\rho/q^{\frac{1}{2}})^{-1})} \\ &= |M^d(h,\rho/q^{\frac{1}{2}})|^2. \end{aligned}$$

It follows that the form R is a finite sum of positive semi-definite forms and hence is itself positive semi-definite. The argument can be reversed by making an artful choice of the function h in terms of a presumed zero of ζ away from the critical line. Namely, suppose that R is positive semi-definite and that RH is false for $\zeta_K(s)$.

Then there exists a $\rho_0 \in Z$ with $\rho_1 := q/\bar{\rho}_0 \neq \rho_0$. On the other hand, as the polynomial P in Theorem 2 has real coefficients, we have $\rho_1 \in Z$. Now choose, as we may, a polynomial F with

$$F(\rho_0) = i, \quad F(\rho_1) = -i, \quad F(\rho) = 0, \qquad \rho \in Z \setminus \{\rho_0, \rho_1\}.$$

There exists a function $h : \mathbb{Z} \to \mathbb{C}$ with

$$\begin{aligned} R(h,h) &= \sum_{\rho \in Z} M^d(h, \rho/q^{\frac{1}{2}}) \overline{M^d(h, (\bar{\rho}/q^{\frac{1}{2}})^{-1})} \\ &= \sum_{\rho \in Z} F(\rho) \overline{F(q/\bar{\rho})} \\ &= F(\rho_0)\overline{F(\rho_1)} + F(\rho_1)\overline{F(\rho_0)} = -2. \end{aligned}$$

This is absurd, so that RH must hold. □

To prove the positivity of R, Weil used ideas from algebraic geometry. The corresponding "geometric" ideas are lacking in the number field case, and Connes's papers propose a set-up within which such "non-commutative" geometric ideas may emerge.

The theory of étale l-adic cohomology, a cohomology theory for $\overline{\Sigma}$ with coefficients in $\mathbb{Q}_\ell$ where ℓ is a prime not equal to p, allows the explicit formula of (24) to be interpreted as a trace formula. Recall the formula given in (19),

$$\text{Card}(\Sigma(\mathbb{F}_{q^j})) = 1 - \sum_{\rho \in Z} \rho^j + q^j. \tag{25}$$

This can be viewed as a Lefschetz fixed point formula in the context of finite fields. The classical Lefschetz fixed point formula applies to a complex variety V. Let F be an action on V. Then the number of fixed points of F is, by this formula,

$$\text{Card}(\text{fixed points of } F) = \sum_j (-1)^j \text{Tr}(F^* H^j(V)),$$

where F^* is the induced action of F on cohomology.

The dimensions of the cohomology groups $H^*(\overline{\Sigma}, \mathbb{Q}_\ell)$ are given by

$$\dim(H^j(\overline{\Sigma}, \mathbb{Q}_l)) = \begin{cases} 0, & j > 2, \\ 1, & j = 0, 2, \\ 2g, & j = 1. \end{cases}$$

Recall that the Frobenius gives a map on $\overline{\Sigma}$, and that the points of $\Sigma(\mathbb{F}_{q^j})$ are the fixed points of Fr^j acting on the curve over the algebraic closure. By a version of the Lefschetz fixed point theorem, one identifies this with an alternating sum of traces on cohomology groups,

$$\begin{aligned} &\text{Card}(\Sigma(\mathbb{F}_{q^j})) = \text{Card}(\text{fixed points of } \text{F}^j) = 1 - \sum_{\rho \in Z} \rho^j + q^j \\ &\quad = \text{Tr}((\text{Fr}^*)^j \mid H^0(\overline{\Sigma}, \mathbb{Q}_\ell)) - \text{Tr}((\text{Fr}^*)^j \mid H^1(\overline{\Sigma}, \mathbb{Q}_\ell)) + \text{Tr}((\text{Fr}^*)^j \mid H^2(\overline{\Sigma}, \mathbb{Q}_\ell)). \end{aligned}$$

The first and last terms of the right hand side together sum to $1 + q^j$, so that the zeros of the zeta function appear as the eigenvalues of Fr^* acting on H^1 and we have,

$$\mathrm{Tr}((\mathrm{Fr}^*)^j \mid H^1(\overline{\Sigma}, \mathbb{Q}_\ell)) = \sum_{\rho \in Z} \rho^j.$$

To show the positivity of the form in (23), Weil essentially works with $\overline{\Sigma}/S_g$, the symmetric product of $\overline{\Sigma}$ with itself g times. This is closely related to the Jacobian J of the curve $\overline{\Sigma}$. The Frobenius map on $\overline{\Sigma}$ induces an endomorphism Fr of J which is invertible over $\mathbb{Q}$. There is a standard involution $e \to e'$ on the endomorphisms e of J, called the Rosati involution, for which $\mathrm{Fr}\mathrm{Fr}' = q$. Working over $\mathbb{Q}(q^{1/2})$ in this endomorphism algebra, one can reinterpret Theorem 3 as a statement about the positivity of the Rosati involution, which in turn follows from the Castelnuovo–Severi inequality for surfaces, see [**30**] §5.17 and [**33**].

2.3. Weil's proof by the theory of correspondences. In this section we briefly discuss Weil's proof of the Riemann Hypothesis for curves over finite fields using the theory of correspondences. Full details are given in his book [**38**]. Useful additional references are [**17**], [**33**]. Once again the Riemann Hypothesis is reduced to a positivity statement which follows from the Castelnuovo–Severi inequality for surfaces.

A correspondence on a curve is the graph of a multi-valued map of a curve Σ to itself, and it may also be viewed as a divisor on the surface $\Sigma \times \Sigma$. Recall the formula given in (19), which yields the explicit formula, and is given by

$$\sum_{\rho \in Z} \rho^j = 1 + q^j - \mathrm{Card}(\Sigma(\mathbb{F}_{q^j})), \qquad j \geq 1. \tag{26}$$

If $\mathrm{Fr} : \overline{\Sigma} \to \overline{\Sigma}$ is the Frobenius map, then for every integer $j \geq 1$, the map Fr^j is single-valued and of degree q^j. Let F^j be the graph of Fr^j on $\overline{\Sigma} \times \overline{\Sigma}$. Then, for any point P on $\overline{\Sigma}$, we have

$$m(F^j) = \mathrm{Card}\left(F^j \cap \left(\{P\} \times \overline{\Sigma}\right)\right) = 1, \tag{27}$$

and

$$d(F^j) = \mathrm{Card}\left(F^j \cap \left(\overline{\Sigma} \times \{P\}\right)\right) = q^j. \tag{28}$$

Moreover, the graph F^j intersects the diagonal Δ of $\overline{\Sigma} \times \overline{\Sigma}$ in $N_j = \mathrm{Card}(\Sigma(\mathbb{F}_{q^j}))$ points.

There is an intersection form $(D_1, D_2) \to D_1 \cdot D_2$ defined on divisors on the surface $\overline{\Sigma} \times \overline{\Sigma}$, or correspondences on the curve $\overline{\Sigma}$, which remains well-defined on divisor classes, two divisors being in the same class if their difference is the divisor of a function on $\overline{\Sigma} \times \overline{\Sigma}$. The divisors form a ring with multiplication $\circ$ induced by composition of maps, and has a transpose $D \to D^t$ given by composition with permutation of the two factors of $\overline{\Sigma} \times \overline{\Sigma}$. For a divisor D on $\overline{\Sigma} \times \overline{\Sigma}$, and a point P on $\overline{\Sigma}$, we generalize (27) and (28) to arbitrary divisors by setting

$$m(D) = D \cdot \left(\{P\} \times \overline{\Sigma}\right) \quad \text{and} \quad d(D) = D \cdot \left(\overline{\Sigma} \times \{P\}\right).$$

Weil defines the trace on divisors, invariant on a given class, by

$$\mathrm{Trace}(D) = m(D) + d(D) - D \cdot \Delta,$$

where as above Δ is the diagonal of $\overline{\Sigma} \times \overline{\Sigma}$. The Riemann hypothesis is then equivalent to the positivity statement

$$\mathrm{Trace}(D \circ D^t) > 0, \tag{29}$$

for D not equivalent to a divisor of the form $\{P\} \times \overline{\Sigma}$ or $\overline{\Sigma} \times \{P\}$, with P a point of $\overline{\Sigma}$. Indeed, if $D = m\Delta + nF$ for integers m and n, we have

$$D \circ D^t = \left(m^2 + qn^2\right)\Delta + mn\left(F + F^t\right),$$

using $\Delta = \Delta^t$ and $F \circ F^t = q\Delta$. As the self-intersection number of Δ is $2 - 2g$, the trace of Δ equals $2g$ and, as we saw,

$$\mathrm{Trace}(F) = \mathrm{Trace}(F^t) = 1 + q - N_1.$$

The positivity statement in (29) implies that the form

$$\mathrm{Trace}(D \circ D^t)(m, n) = 2gm^2 + 2\left(1 + q - N_1\right)mn + 2gqn^2,$$

is positive definite for $g \neq 0$ (it is identically zero for $g = 0$). Therefore, it has negative discriminant, so that

$$\left(1 + q - N_1\right)^2 - (2g)(2gq) < 0,$$

that is

$$|N_1 - 1 - q| < 2gq^{1/2}.$$

The same argument applied to $\mathbb{F}_{q^j}$ yields, for all $j \geq 1$,

$$|N_j - 1 - q^j| < 2gq^{j/2}.$$

Applying (25), we deduce that for all $j \geq 1$,

$$|\sum_{\rho \in Z} \rho^j| \leq 2gq^{j/2}. \tag{30}$$

From (17) we have,

$$\log P(u) = -\sum_{\rho \in Z}\sum_{j=1}^{\infty} j\rho^j u^j = -\sum_{j=1}^{\infty} j\left(\sum_{\rho \in Z}\rho^j\right)u^j. \tag{31}$$

Because of (30), we see that the series in (31) converges absolutely for $|u| < q^{-1/2}$, and this means that we have $|\rho| \leq q^{1/2}$ for all $\rho \in Z$. Moreover, (16) shows if $\rho \in Z$ then $q\rho^{-1} \in Z$, so that we deduce that $|\rho| = q^{1/2}$ for all $\rho \in Z$. Therefore the Riemann Hypothesis is proven.

Weil showed that the positivity of (29) is a consequence of the negative semidefiniteness of the intersection form on divisors of degree zero of a projective embedding of the surface $\overline{\Sigma} \times \overline{\Sigma}$, which in turn follows from the Castelnuovo–Severi inequality.

2.4. Finding a theory over the complex numbers. Connes aims to construct a theory over $\mathbb{C}$ rather than over $\mathbb{Q}_\ell$. As before, let K be the function field of a smooth curve Σ over a finite field $\mathbb{F}_q$, and K_{un} the field generated over K by the roots of unity of order prime to p. Let K_{ab} be the maximal abelian extension of K. Then the elements of $\mathrm{Gal}(K_{\mathrm{ab}}/K)$ inducing powers of the Frobenius on K_{un} form a group called the Weil group W_K. The *Main Theorem of Class Field Theory* for function fields tells us that W_K is isomorphic to the idele class group C_K of K. Therefore, in this set-up one can view the idele class group as capturing the Frobenius action. In a formulation using operator algebras over $\mathbb{C}$, the elements of the zero set Z of the zeta function should ideally appear as eigenvalues of $C = C_K$ acting via a representation

$$W : C \to \mathcal{L}(H)$$

where $\mathcal{L}(H)$ is the algebra of bounded operators in a complex Hilbert space H. Moreover, guided by the Lefschetz formula (alternating sum), the Hilbert space H should appear via its negative $\ominus H$. The problem in the function field case is to replace the ℓ-adic cohomology groups by an object defined over $\mathbb{C}$ which exists also for number fields.

We summarise these goals in the following dictionary:

Zeta Function	**Classical Geometry**	**Noncommutative Geometry**
Spectral interpretation of zeros	Spectrum of Fr^* on $H^1(\Sigma, \mathbb{Q}_\ell)$	Spectrum of C on H
Functional equation	Riemann Roch Theorem	Appropriate symmetry
Explicit formula	Lefschetz formula	Geometric trace formula
Riemann hypothesis	Castelnuovo positivity	Positivity of Weil functional

3. The local trace formula and the Polya-Hilbert space

We now turn to a more direct study of Connes's approach. In the previous chapters, we motivated the study of the dynamical system (X, C) of Connes, where $C = J/K^*$ is the idele class group of a global field K and $X = A/K^*$ is the space of adele cosets. Once again we, for simplicity, restrict mainly to the case $K = \mathbb{Q}$, although the discussion goes through for arbitrary global fields.

Classical spaces, their topology, their differentiable and conformal structure, can be understood from the study of their associated (commutative!) algebras. A main goal of noncommutative geometry is to extend these structures to a wider class of examples by developing their analogues on noncommutative algebras whose corresponding spaces are non-existent or hard to study. Connes proposes that, although global fields, their adele rings and idele class groups are commutative, the Riemann Hypothesis (RH) itself should be viewed as part of "noncommutative number theory".

We saw in Chapter 1 that a natural first step in this program is to study, in the case where K is a local field, the action on K of the group K^* of its non-zero elements. Connes develops a rigorous trace formula for the action (K^*, K). This trace formula, which we discuss in §3.1, turns out to provide positive support for his approach to the Riemann Hypothesis for global fields.

Indeed, Connes conjectures a trace formula for the action (C, X) for global fields which is a sum of the contributions of the local trace formulae and this conjecture turns out to be equivalent to RH. In §3.2, we discuss Connes's interesting and rigorous interpretation of the non-trivial zeros of the L-functions with Grossencharacter for a global field (of which the Riemann zeta function is a special case) in terms of the action of the idele class group on a certain Hilbert space. We call this the Polya-Hilbert space after earlier suggestions by Polya and Hilbert that there should be a spectral interpretation of the non-trivial zeros of the Riemann zeta function. This is not sufficient to prove Connes's conjectured global trace formula, but it provides evidence for the rich information on RH contained in the action (C, X). Although we do not give proofs of the results of §3.2, we end in §3.3 with an explanation of why the non-trivial zeros of the L-functions turn up in the spectral formula of §3.2. Full proofs are given in [**10**].

3.1. The local trace formula. For the time being we assume that K is a local field and we look at the action of K^* by multiplication on K. For simplicity, we work with the case $K = \mathbb{R}$ or $K = \mathbb{Q}_p$ for p a prime number. These are the local fields obtained by completing the field $\mathbb{Q}$ at the elements of $M_{\mathbb{Q}}$, as in Chapter 1.

Let $H = L^2(K)$ where the L^2-norm is with respect to the additive Haar measure dx on K. Let U be the regular representation of K^*, the non-zero elements of K, on H. Therefore, for $\lambda \in K^*$ and $x \in K$, we have

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x).$$

The operator $U(\lambda)$ is not of trace class and, in order to associate to it a trace class operator, one averages it over a test function $h \in \mathcal{S}(K^*)$ with compact support. If $h(1) \neq 0$, it is necessary to further modify the operator, as we shall see.

Define the operator in H given by,

$$U(h) = \int_{K^*} h(\lambda)U(\lambda)d^*\lambda,$$

where $d^*\lambda$ denotes the multiplicative Haar measure on K^*, normalized by requiring that

$$\int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log\Lambda, \qquad \Lambda \to \infty.$$

To the operator $U(h)$ we associate the Schwartz kernel $k(x, y)$ on K^2 for which

$$(U(h)\xi)(x) = \int_K k(x, y)\xi(y)dy.$$

Let $\delta = \delta(x)$ denote the Dirac delta distribution on K. We have

$$k(x, y) = \int_{K^*} h(\lambda^{-1})\delta(y - \lambda x)d^*\lambda.$$

The associated distributional trace is given by,

$$\mathrm{Tr}_D(U(h)) = \int_K k(x, x)dx = \int_{K^*} h(\lambda^{-1}) \left(\int_K \int_K \delta(x - y)\delta(y - \lambda x)\, dx\, dy\right) d^*\lambda.$$

When $\lambda \neq 1$, the distribution $\delta(x - y)$ has support on the line $x = y$ on K^2, whereas $\delta(x - \lambda y)$ has support on the transverse line $x = \lambda y$. As explained in [**10**], the integral

$$\int_K \int_K \delta(x - y)\delta(x - \lambda y)\, dx\, dy = |1 - \lambda|^{-1}$$

is well-defined and equals, as expected, the displayed value. When $h(1) = 0$, so that the non-transverse case $\lambda = 1$ is cancelled out, we deduce that

$$\mathrm{Tr}_D(U(h)) = \int_{K^*} \frac{h(\lambda^{-1})}{|1 - \lambda|} d^*\lambda.$$

In [**10**], the case $h(1) \neq 1$ is dealt with by introducing a cut-off. For $\Lambda > 0$, let P_Λ be the projection onto those functions $\xi = \xi(x) \in H$ supported on $|x| \leq \Lambda$. We define the corresponding projection in Fourier space by

$$\widehat{P}_\Lambda = F P_\Lambda F^{-1},$$

where

$$(F\xi)(x) = \widehat{\xi}(x) = \int_K \xi(y)\alpha(xy)dy,$$

for α a fixed nontrivial character of the additive group K. The cut-off at Λ is defined by the trace-class operator

$$R_\Lambda = \widehat{P}_\Lambda P_\Lambda.$$

As R_Λ is trace-class, the operator $R_\Lambda U(h)$ is also. By using standard formulae from symbol calculus, Connes derives the identity

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) &= \int_{K^*} h(\lambda^{-1}) \int_K \int_{|x|\leq\Lambda;|\xi|\leq\Lambda} \delta(x+u-\lambda x)\alpha(u\xi)\, dx\, d\xi\, du\, d^*\lambda \\ &= \int_{K^*} h(\lambda^{-1}) \int_{|x|\leq\Lambda;|\xi|\leq\Lambda} \alpha((\lambda-1)x\xi)\, dx\, d\xi\, d^*\lambda.\end{aligned}$$

Having fixed the character α, we normalize the additive Haar measure on K to be self-dual. Then, there is a constant $\rho > 0$ such that

$$\int_{1\leq|\lambda|\leq\Lambda} \frac{d\lambda}{|\lambda|} \sim \rho \log \Lambda, \qquad \Lambda \to \infty,$$

so that

$$d^*\lambda = \rho^{-1} \frac{d\lambda}{|\lambda|}.$$

We have therefore

$$\int_{K^*} h(\lambda^{-1})\alpha((\lambda-1)x\xi)d^*\lambda = \rho^{-1} \int_K \frac{h((u+1)^{-1})}{|u+1|}\alpha(ux\xi)du.$$

We deduce that

$$\mathrm{Tr}(R_\Lambda U(h)) = \rho^{-1} \int_{|x|,|\xi|\leq\Lambda} \widehat{g}(x\xi)\, dx\, d\xi,$$

where $g \in C_c^\infty(K)$ is given by

$$g(u) = \frac{h((u+1)^{-1})}{|u+1|}.$$

Let $v = x\xi$. Then $dx\, d\xi = dv \frac{dx}{|x|}$ and, for $|v| \leq \Lambda^2$, we have

$$\rho^{-1} \int_{\frac{|v|}{\Lambda}\leq|x|\leq\Lambda} \frac{dx}{|x|} = \rho^{-1} \int_{1\leq|y|\leq\frac{\Lambda^2}{|v|}} \frac{dy}{|y|} = 2\log'\Lambda - \log|v|,$$

where we define

$$2\log'\Lambda = \int_{|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda.$$

It follows that

$$\mathrm{Tr}(R_\Lambda U(h)) = \int_{|v|\leq\Lambda^2} \widehat{g}(v)(2\log'\Lambda - \log|v|)dv.$$

Using the fact that $g \in C_c^\infty(K)$ we deduce from this the asymptotic formula, as $\Lambda \to \infty$,

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) =& 2g(0)\log'\Lambda - \int_K \widehat{g}(v)\log|v|dv + o(1)\\ =& 2h(1)\log'\Lambda - \int_K \frac{h((x+1)^{-1})}{|x+1|}\left(\int_K \log|v|\alpha(xv)dv\right)dx + o(1).\end{aligned}$$

In [**10**] it is shown that the distribution given by pairing with

$$-\int_K \log|v|\alpha(xv)dv$$

differs by a multiple $c_v\delta(x)$ of the delta function at $x=0$ from the distribution on K defined by

$$P(f) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathrm{Mod}(K)}}\left(\rho^{-1}\int_{|x|\geq\varepsilon} f(x)\frac{dx}{|x|} + f(0)\log\varepsilon\right).$$

As $\log|v|$ vanishes at $v=1$, on replacing x by $x-1$, the constant c_v is determined by the condition that

$$L(f) = c_v f(0) + P(f)$$

is the unique distribution on K extending $\rho^{-1}\frac{1}{|u-1|}$, for $u \neq 1$, whose Fourier transform vanishes at 1, that is, $\widehat{L}(1) = 0$. If $k(u) = |u|^{-1}h(u^{-1})$, we have

$$\begin{aligned}L(k) =& -\int_K \frac{h((x+1)^{-1})}{|x+1|}\left(\int_K \log|v|\alpha(xv)dv\right)dx\\ =& -\int_K \frac{h(u^{-1})}{|u|}\left(\int_K \log|v|\alpha((u-1)v)dv\right)du.\end{aligned}$$

Therefore, outside of $x=0$ the distribution

$$-\int_K \log|v|\alpha(xv)dv$$

agrees with $\rho^{-1}\frac{1}{|x|}$. If $h(1)=0$, we find, as expected from our computation above of the distributional trace in this case, that

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) =& \rho^{-1}\int_K \frac{h((x+1)^{-1})}{|x+1|}\frac{dx}{|x|} + o(1)\\ =& \int_{K^*}\frac{h(\lambda^{-1})}{|1-\lambda|}d^*\lambda + o(1), \qquad (h(1)=0),\end{aligned}$$

and taking the limit as $\Lambda\to\infty$ we have

$$\mathrm{Tr}(U(h)) = \int_{K^*}\frac{h(\lambda^{-1})}{|1-\lambda|}d^*\lambda, \qquad (h(1)=0).$$

Returning to the general case, $h(1)\neq 0$, we see that

$$\begin{aligned}\mathrm{Tr}(R_\Lambda U(h)) =& h(1)(2\log'\Lambda + c_v) + \lim_{\varepsilon\to 0}\left(\rho^{-1}\int_{|u-1|\geq\varepsilon}\frac{h(u^{-1})}{|u-1|}\frac{du}{|u|} + h(1)\log\varepsilon\right)\\ =& h(1)(2\log'\Lambda + c_v) + \lim_{\varepsilon\to 0}\left(\int_{|u-1|\geq\varepsilon}\frac{h(u^{-1})}{|u-1|}d^*u + h(1)\log\varepsilon\right)\end{aligned}$$

In [**10**], it is shown, for example, that for $v = \infty$ we have $c_v = \log(2\pi) + \gamma$, where γ is Euler's constant. We have therefore the following result.

THEOREM 4. *If $h \in \mathcal{S}_c(K^*)$, then $R_\Lambda U(h)$ is a trace class operator in $L^2(K)$ and as $\Lambda \to \infty$ we have the asymptotic formula*

$$\mathrm{Tr}(R_\Lambda U(h)) = 2h(1)\log'\Lambda + \int' \frac{h(\lambda^{-1})}{|1-\lambda|} d^*\lambda + o(1),$$

where $2\log'\Lambda = \int_{\lambda \in K^*, |\lambda| \in [\Lambda^{-1},\Lambda]} d^*\lambda$ and $\int'$ is the pairing of $h(u^{-1})/|u|$ with the unique distribution extending $\rho^{-1}du/|1-u|$ whose Fourier transform vanishes at 1.

For $K = \mathbb{R}$ we have,

$$\begin{aligned}\text{(32)}\quad \mathrm{Tr}(R_\Lambda U(h)) &= h(1)(2\log\Lambda + \log(2\pi) + \gamma)\\ &\quad + \lim_{\varepsilon \to 0}\left(\int_{|u-1|\geq\varepsilon} \frac{h(u^{-1})}{|u-1|} d^*u + h(1)(\log\varepsilon)\right) + o(1).\end{aligned}$$

3.2. The global case and the Polya-Hilbert space. Connes conjectures that an analogue of the formula in Theorem 4 holds for the action of C on X. However, a major obstacle is that there are no nonconstant Lebesgue measurable functions on X. By analogy with functions on a manifold, one may try to think of A as a universal cover of X. One then views functions on X as averages of functions on A over the "universal covering group" $\mathbb{Q}^*$. (This is the map E below). Connes shows nonetheless that a "Polya-Hilbert space" related to the action (C, X) allows a spectral interpretation of the non-trivial zeros of the Riemann zeta function.

We work with the case $K = \mathbb{Q}$. Let $\mathcal{S}(A)_0$ denote the subspace of $\mathcal{S}(A)$ given by

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A) : f(0) = \int f dx = 0\}. \tag{33}$$

Let E be the "averaging over $\mathbb{Q}^*$" operator which to $f \in \mathcal{S}(A)_0$ associates the element of $\mathcal{S}(C)$ given by

$$E(f)(u) = |u|^{1/2} \sum_{q \in \mathbb{Q}^*} f(qu). \tag{34}$$

For $\delta \geq 0$, let $L^2(X)_{0,\delta}$ be the completion of $\mathcal{S}(A)_0$ with respect to the norm given by

$$\begin{aligned}\text{(35)}\quad ||f||_\delta^2 &= \int_C |E(f)(u)|^2 (1 + \log^2|u|)^{\delta/2} d^*u,\\ &\qquad \text{for} \int_{|u| \in [1,\Lambda]} d^*u \simeq \log\Lambda, \qquad \Lambda \to \infty.\end{aligned}$$

If $g(x) = f(qx)$ for some fixed $q \in \mathbb{Q}^*$, then $||g||_\delta = ||f||_\delta$ and so one sees that this norm respects, in this sense, the passage to the quotient $A/\mathbb{Q}^*$. We define $L^2(X)_\delta$ by the short exact sequence

$$0 \to L^2(X)_{0,\delta} \to L^2(X)_\delta \to \mathbb{C} \oplus \mathbb{C}(1) \to 0. \tag{36}$$

When $\delta = 0$ we write $L^2(X)_0$ and $L^2(X)$ for the first two terms. Here $\mathbb{C}$ is the trivial C-module and $\mathbb{C}(1)$ is the C-module for which $u \in C$ acts by $|u|$, where $|\cdot|$ is the norm on C. Indeed, by the definition of $\mathcal{S}(A)_0$ in (33), we see that its two-dimensional supplement in $\mathcal{S}(A)$ is the C-module $\mathbb{C} \oplus \mathbb{C}(1)$.

Multiplication of C on A induces a representation of C on $L^2(X)_\delta$ given by,

$$(U(\lambda)\xi)(x) = \xi(\lambda^{-1}x). \tag{37}$$

We introduce a Hilbert space H_δ via another short exact sequence,

$$0 \to L^2(X)_{0,\delta} \to L^2(C)_\delta \to H_\delta \to 0, \tag{38}$$

where the inclusion of $L^2(X)_{0,\delta}$ into $L^2(C)_\delta$ is effected by the isometry E. Here $L^2(C)_\delta$ is the completion with respect to the weighted Haar measure as in (35), where we write $L^2(C)$ when $\delta = 0$. The spectral interpretation on H_δ of the critical zeros of the L-functions in [**8**] relies on taking $\delta > 0$. Indeed, this is needed to control the growth of the functions on the non-compact quotient X; ultimately this parameter is eliminated from the conjectural trace formula by using cut-offs. It is important here to use the measure $|u|d^*u$ (implicit in (35)) instead of the additive Haar measure dx, this difference being a veritable one for global fields, where one has $dx = \lim_{\varepsilon\to 0} \varepsilon|x|^{1+\varepsilon}d^*x$.

The regular representation V of C on $L^2_\delta(C)$ descends to H_δ (it commutes with E up to a phase as an easy calculation shows) and we denote it by W. Connes describes (H_δ, W) as the Polya-Hilbert space with group action for his approach to the Riemann hypothesis. He proves in [**8**] and [**10**] the remarkable result given in Theorem 5 relating the trace of this action to the zeros on the critical line of the L-functions with Grossencharacter discussed in Chapter 1.

The norm $|\ |$ on the abelian locally compact group C has kernel

$$C^1 = \{u \in C : |u| = 1\}$$

and, as $K = \mathbb{Q}$, it has image $\mathbb{R}^*_{>0}$. Now the compact group C^1 acts on H_δ, which splits with respect to this action into a canonical direct sum of pairwise orthogonal subspaces H_{δ,χ^1} where χ^1 runs through the discrete abelian group $\widehat{C^1}$ of characters of C^1. One can restrict the action (H_δ, W) to an action (H_{δ,χ^1}, W) for any $\chi^1 \in \widehat{C^1}$ and we have

$$H_{\delta,\chi^1} = \{\xi \in H_\delta : W(u)\xi = \chi^1(u)\xi \quad \text{for all } u \in C^1\}.$$

Choose a (non-canonical) decomposition $C = C^1 \times N$ with $N \simeq \mathbb{R}^*_{>0}$. For $\chi^1 \in \widehat{C^1}$, there is a unique extension to a quasi-character χ of C, vanishing on N. The choice of χ is unimportant in what follows, as if $\chi'(u) = \chi(u)|u|^{i\tau}$, $u \in C$, we have $L(\chi', s) = L(\chi, s + i\tau)$.

THEOREM 5. *For any Schwartz function $h \in \mathcal{S}(C)$ the operator*

$$W(h) = \int_C W(u)h(u)d^*u$$

in H_δ is trace class, and its trace is given by

$$\mathrm{Trace}(W(h)) = \sum_{\substack{L(\chi,\frac{1}{2}+i\sigma)=0,\\ \sigma\in\mathbb{R}}} \widehat{h}(\chi, i\sigma)$$

where the sum is over the characters of C^1 with χ being the unique extension to a quasi-character on C vanishing on N. The multiplicity of the zero is counted as *the largest integer* $n < \frac{1}{2}(1+\delta)$ *with* n *at most the multiplicity of* $\frac{1}{2}+i\sigma$ *as a zero of* $L(\chi, s)$. *Also, we define,*

$$\widehat{h}(\chi, s) := \int_C h(u)\chi(u)|u|^s d^*u.$$

Now, the action of C is free on $L^2(C)_\delta$, so that the short exact sequence (38) tells us that the trace of the action of C on H_δ should be, up to a correction due to a regularisation, the negative of the trace of the action of C on $L^2(X)_{0,\delta}$. From (36), we see that the regularised trace of the action of C on $L^2(X)_\delta$ should involve the sum of the corresponding trace on $L^2(X)_{0,\delta}$ and the trace on $\mathbb{C}\oplus\mathbb{C}(1)$. Therefore the regularised trace of the action of C on $L^2(X)_\delta$ should involve the trace of the action on $\mathbb{C}\oplus\mathbb{C}(1)$ *minus* the trace of this action on H_δ. This *minus* sign is crucial for the comparison with the Weil distribution.

When $\chi = 1$ in Theorem 5, the corresponding L-function is the Riemann zeta function $\zeta(s)$ and the proof of the theorem shows that

$$\mathrm{Trace}(W(h)_{|H_{\delta,1}}) = \sum_{\substack{\zeta(\frac{1}{2}+is)=0,\\ s\in\mathbb{R}}} \widehat{h}(is),$$

where $\widehat{h}(is) = \widehat{h}(1, is)$, with the same convention for the multiplicity. There is a conjecture that the zeros of the Riemann zeta function are all simple, in which case we could take $n = 1$ and any $\delta > 1$.

3.3. A computation. The following result formulated by Weil in [**36**], and appearing in Tate's thesis, shows how the operation E in (34) brings in the zeros of the L functions in the critical strip and provides the key to proof of Theorem 5 and the appearance of the non-trivial zeros of the $L(\chi, s)$. It indicates that the non-trivial zeros of the L-functions should "span" H_δ as they are "orthogonal" to the image of E.

PROPOSITION 1. *Let* χ *be a unitary character on* C. *For any* $\rho \in \mathbb{C}$ *with* $\Re(\rho) \in \left]-\frac{1}{2}, \frac{1}{2}\right[$, *we have*

$$\int_C E(\xi)(u)\chi(u)|u|^\rho d^*u = 0, \qquad \text{for all } \xi \in \mathcal{S}(A)_0,$$

precisely when $L(\chi, \frac{1}{2}+\rho) = 0$.

We discuss the classical zeta function, the other cases being similar. Let p be a finite prime and let $f \in \mathcal{S}(\mathbb{Q}_p)$. Let

$$\Delta_{p,s}(f) = \int_{\mathbb{Q}_p^*} f(x)|x|^s d^*x, \qquad \Re(s) > 0,$$

and also

$$\Delta'_{p,s}(f) = \int_{\mathbb{Q}_p^*} (f(x) - f(p^{-1}x)|x|^s d^*x,$$

where

$$< 1_{\mathbb{Z}_p}, \Delta'_{p,s} >= \int_{\mathbb{Z}_p^*} d^*x.$$

We have

$$\Delta'_{p,s} = (1-p^{-s})\Delta_{p,s} \tag{39}$$

For $p = \infty$ define

$$\Delta_{\infty,s}(f) = \Delta'_{\infty,s}(f) = \int_{\mathbb{R}^*} f(x)|x|^s ds.$$

One can define these functions globally by multiplying over all places. For $\sigma = \Re(s) > 1$ and $f \in \mathcal{S}(A)$, the following integral converges absolutely:

$$\int_J f(u)|u|^s d^*u.$$

Moreover, from (39), for $\sigma = \Re(s) > 1$,

$$\int_J f(u)|u|^s d^*u = \Delta_s(f) = \zeta(s)\Delta'_s(f) \neq 0,$$

where

$$\Delta_s = \Delta_{\infty,s} \times \prod_p \Delta_{p,s}$$

and

$$\Delta'_s = \Delta'_{\infty,s} \times \prod_p \Delta'_{p,s}.$$

In fact, for a somewhat larger range of convergence we have the following.

Lemma 1. *If $\sigma = \Re(s) > 0$ then there is a constant $c \neq 0$ with, for all $f \in \mathcal{S}(A)_0$,*

$$\int_J f(x)|x|^s d^*x = c\int_C E(f)(u)|u|^{s-\frac{1}{2}} d^*u = c\zeta(s)\Delta'_s(f).$$

4. The Weil distribution and the global trace formula

In this chapter, we discuss the conjectural part of Connes's approach and in particular a new (G)RH equivalent. Roughly speaking, the (generalized) Riemann Hypothesis for a global field K would follow if a global asymptotic trace formula for the action (C_K, X_K), where $X_K = A_K/K^*$, can be shown to be a sum of local contributions of trace formulae as in Theorem 4 for the actions (K_v^*, K_v), where v ranges over a set M_K of inequivalent valuations of K, and K_v is the completion of the field K with respect to the metric induced by the valuation v . If one restricts to a finite set S of places this works, but the error term depends on S in a way that has not yet been controlled.

4.1. Global trace formula. Unless stated otherwise, we let $K = \mathbb{Q}$ from now on, and drop the K-indices, although our discussion goes through for arbitrary global fields. In Chapter 3, we introduced for $\delta \geq 0$ the Hilbert spaces $L^2(X)_{0,\delta}$, $L^2(X)_\delta$ and $L^2(C)_\delta$, and an inclusion of $L^2(X)_{0,\delta}$ in $L^2(C)_\delta$, affected by the isometry,

$$(Ef)(u) = |u|^{\frac{1}{2}} \sum_{q \in \mathbb{Q}^*} f(qu), \qquad u \in C.$$

We also saw that for $\sigma = \Re(s) > 0$, there is a constant $c \neq 0$ with, for all $f \in \mathcal{S}(A)_0$,

$$\int_C (Ef)(u)|u|^{s-\frac{1}{2}} d^*u = c\zeta(s)\Delta'_s(f).$$

As $\Delta'_s \neq 0$, if for $\Re(s) \in]0,1[$ the left hand side of this formula vanishes for all $f \in \mathcal{S}(A)_0$, then $\zeta(s) = 0$. We therefore see a link between orthogonality to the image of E and the non-trivial zeros of $\zeta(s)$. As in Chapter 1, let χ be a unitary quasi-character on C and $L(\chi, s)$ the associated L-function. Adapting the argument of Chapter 3 for the case $\chi = 1$ to arbitrary χ, we have

$$\int_C (Ef)(u)\chi(u)|u|^{s-\frac{1}{2}} d^*u = cL(\chi, s)\Delta'_s(f).$$

To study certain spectral aspects of the general L functions of $\mathbb{Q}$, of which the Riemann zeta function is a special case, Connes works mostly in the Hilbert space $L^2(C)$ (so that $\delta = 0$). As in the local case, one has again to regularise using a cut-off at $\Lambda > 0$. Let S_Λ be the orthogonal projection onto

$$S_\Lambda = \{\xi \in L^2(C) : \xi(u) = 0, \quad |u| \notin [\Lambda^{-1}, \Lambda]\}.$$

Let

$$V : C \to \mathcal{L}(L^2(C))$$

be the regular representation

$$(V(\lambda)\xi)(u) = \xi(\lambda^{-1}u), \qquad \lambda, u \in C.$$

Let the measure $d^*\lambda$ on C be normalized so that

$$\int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log \Lambda, \qquad \Lambda \to \infty,$$

and let

$$2\log' \Lambda = \int_{|\lambda|\in[\Lambda^{-1},\Lambda]} d^*\lambda.$$

For $h \in \mathcal{S}_c(C)$ with compact support, let

$$V(h) = \int_C h(\lambda)V(\lambda)d^*\lambda.$$

Then, the operator $S_\Lambda V(h)$ is trace class and

$$\mathrm{Trace}(S_\Lambda V(h)) = 2h(1)\log' \Lambda$$

as the action of C on C has no fixed points. The cut-off in $L^2_0(X)$ is much less straightforward.

In the function field case, Connes proposes working with a family of closed subspaces $B_{\Lambda,0}$ of $L^2_0(X)$ such that $E(B_{\Lambda,0}) \subset S_\Lambda$. These subspaces are given by

$$B_{\Lambda,0} = \{f \in \mathcal{S}(A)_0 : f(x) = 0,\ \widehat{f}(x) = 0,\ |x| > \Lambda\},$$

where $|\ |$ is the obvious extension to A of the norm on J, and this set is non-empty. Therefore, if $Q_{\Lambda,0}$ is the orthogonal projection onto $B_{\Lambda,0}$ and

$$Q'_{\Lambda,0} = EQ_{\Lambda,0}E^{-1}$$

then we have the inequality of projections

$$Q'_{\Lambda,0} \leq S_\Lambda.$$

For all $\Lambda > 0$, the following distribution, Δ_Λ, is therefore positive:

$$\Delta_\Lambda = \mathrm{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h)), \qquad h \in \mathcal{S}(C).$$

The positivity of Δ_Λ signifies that for $h \in \mathcal{S}(C)$,

$$\Delta_\Lambda(h * h^*) \geq 0$$

where $h^*(u) = \overline{h(u^{-1})}$. Therefore, the limiting distribution

$$\Lambda_\infty = \lim_{\Lambda\to\infty} \Delta_\Lambda$$

is also positive.

However, in the number field case, the analogous set to $B_{\Lambda,0}$ is empty. To save this situation, one is guided by the physical fact that there exist signals with finite support in the time variable and also in the dual frequency variable, such as a musical signal. That is because the relative position of P_Λ and $\widehat{P}_\Lambda$, as defined in the local case, can be analyzed. To do this, Connes appeals to the work of Landau, Pollak and Slepian [**28**], [**29**], [**32**]. We will also often quote in what follows the related and useful discussion in [**24**, §3].

Consider the case $K = \mathbb{Q}$. Recall that $\mathbb{Q}$ has one infinite place given by the usual euclidean absolute value and that the corresponding completion is the field of real numbers. As in Chapter 3, at the infinite place we define P_Λ to be the orthogonal projection onto the subspace,

$$P_\Lambda = \{\xi \in L^2(\mathbb{R}) : \xi(x) = 0, \text{ for all } x, |x| > \Lambda\}$$

and $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$, where F is the Fourier transform associated to the character $\alpha(x) = \exp(-2\pi i x)$.

For $\xi \in P_\Lambda$, the function $F\xi(y)$ is an analytic function, and is therefore never supported on $|y| \le \Lambda$. Therefore the spaces P_Λ and $\widehat{P}_\Lambda$ have zero intersection, which explains our claim above that, for $K = \mathbb{Q}$, the analogous space to $B_{\Lambda,0}$ would be trivial.

The projections P_Λ and $\widehat{P}_\Lambda$ commute with the second order differential operator on $\mathbb{R}$ given by

$$H_\Lambda \psi(x) = -\partial((\Lambda^2 - x^2)\partial)\psi(x) + (2\pi\Lambda x)^2\psi(x),$$

where $\partial = \frac{d}{dx}$, as can be checked by a straightforward calculation. The operator H_Λ has a discrete simple spectrum, which we may index by integers $n \ge 0$, on functions with support in $[-\Lambda, \Lambda]$. The corresponding eigenfunctions ψ_n are called the prolate spheroidal wavefunctions which can be taken as real-valued.

Since $P_\Lambda \widehat{P}_\Lambda P_\Lambda$ (that is, the operator $\widehat{P}_\Lambda$ restricted to $[-\Lambda, \Lambda]$) commutes with H_Λ, the prolate spheroidal functions are also eigenfunctions of $P_\Lambda \widehat{P}_\Lambda P_\Lambda$. By [**28**], [**29**], [**32**] (see also [**24**]), one knows also that their eigenvalues are close to zero for $n \ge 4\Lambda^2 + O(\log \Lambda)$. Moreover, the projections P_Λ and $\widehat{P}_\Lambda$ in $L^2(\mathbb{R})$ have relative angles close to zero outside the space spanned by the eigenfunctions ψ_n, $n \le 4\Lambda^2$ (see [**10**]). These results motivate Connes to substitute for the zero space $P_\Lambda \cap \widehat{P}_\Lambda$ the subspace $B_{\infty,\Lambda}$ of P_Λ given by the linear span in $L^2(\mathbb{R})$ of the ψ_n, $n \le 4\Lambda^2$.

We may write the adeles A over $\mathbb{Q}$ as the direct product $A = \mathbb{R} \times A_f$ where A_f, the finite adeles, is the restricted product over the finite primes, p, of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. Let $R = \prod_p \mathbb{Z}_p$ and let 1_R be the characteristic function on R. Let $W_f = \prod_p \mathbb{Z}_p^*$ be the units of R and $W = \{\pm 1\} \times W_f$. Consider the elements $f \in \mathcal{S}(A) = \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(A_f)$ satisfying

$$f(wa) = f(a), \qquad a \in A, w \in W.$$

Then f may be written as a finite linear combination

$$f(a) = \sum_{j=1}^{m} f_j(q_j a_\infty) \otimes 1_R(q_j a_f), \qquad a = (a_\infty, a_f), \quad a_\infty \in \mathbb{R}, a_f \in A_f,$$

where the $f_j \in \mathcal{S}(\mathbb{R})$ are even and $q_j \in \mathbb{Q}^*$, $q_j > 0$. Let

$$B^1_{\Lambda,0} = \{f \in \mathcal{S}(A)_0 : f = f_\infty \otimes 1_R, f_\infty \in B_{\infty,\Lambda}, f_\infty \text{ even}\}.$$

As in Section 3.2, let C^1 be the compact subgroup of the idele class group C of $\mathbb{Q}$ given by the kernel of the norm map on C. We have a (non-canonical) isomorphism between C and $C^1 \times N$ where $N \simeq \mathbb{R}^*_{>0}$. We can extend any character χ^1 of C^1 to a quasi-character χ of C, vanishing on N. As in [**10**], §8, Lemma 1 we may, in a similar way to the case $\chi = 1$ above, define subspaces $B^\chi_{\Lambda,0}$, where χ is the extension of a character χ^1 of C^1, by writing $A = \mathbb{R} \times \prod_{p\in S} \mathbb{Q}_p \times A_{f,S}$ where S is the finite set of ramified primes for χ (see Section 1.3) and $A_{f,S}$ is the finite adeles with the restricted product over the finite primes not in S. Let $B_{\Lambda,0} = \oplus_{\chi^1\in\widehat{C}^1} B^\chi_{\Lambda,0}$ and $Q_{\Lambda,0}$ is the corresponding orthogonal projection, with $Q'_{\Lambda,0} = EQ_{\Lambda,0}E^{-1}$.

Again, for all $\Lambda > 0$, we introduce the positive distribution

$$\Delta_\Lambda = \text{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h)), \qquad h \in \mathcal{S}(C).$$

Connes observes that the above considerations show that the limiting distribution

$$\Lambda_\infty = \lim_{\Lambda\to\infty} \Delta_\Lambda$$

is positive, just as it was in the function field case.

If we let Q_Λ be the projection in $L^2(X)$ onto $B_{\Lambda,0} \oplus \mathbb{C} \oplus \mathbb{C}(1)$, then Connes conjectures the following *global* analogue of the local geometric trace formula. We state it only for the global field $K = \mathbb{Q}$, but the statement for an arbitrary global field is analogous.

CONJECTURE 1. *For $h \in \mathcal{S}_c(C)$, we have as $\Lambda \to \infty$,*

$$\text{(40)}\quad \text{Trace}(Q_\Lambda U(h)) = 2h(1)\log'\Lambda + \sum_{p \text{ prime}} \int'_{\mathbb{Q}^*_p} \frac{h(u^{-1})}{|1-u|} d^*u + \int'_{\mathbb{R}^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1),$$

where

$$2\log'\Lambda = \int_{|\lambda|\in[\Lambda^{-1},\Lambda]} d^*u \sim 2\log\Lambda.$$

This means that one *conjectures* that the global trace formula in $L^2(X)$ behaves like a sum of local formulae. Observe that,

$$\text{(41)}\qquad \text{Trace}(Q_\Lambda U(h)) = \int_C h(u)(1+|u|)d^*u + \text{Trace}(Q_{\Lambda,0}U(h)).$$

Using (41), and noting the phase shift by 1/2 in E, we have as a consequence of Conjecture 1 that, as $\Lambda \to \infty$,

$$\Delta_\Lambda(h) = \int_C h(u)(|u|^{1/2} + |u|^{-1/2})d^*u - \sum_{p \text{ prime}} \int'_{\mathbb{Q}^*_p} \frac{h(u)}{|1-u|}|u|^{1/2} d^*u - \int'_{\mathbb{R}^*} \frac{h(u)}{|1-u|}|u|^{1/2} d^*u + o(1).$$

Therefore,

$$\Delta_\infty(h) = \int_C h(u)(|u|^{1/2} + |u|^{-1/2})d^*u$$
$$- \sum_{p \text{ prime}} \int'_{\mathbb{Q}_p^*} \frac{h(u)}{|1-u|}|u|^{1/2}d^*u - \int'_{\mathbb{R}^*} \frac{h(u)}{|1-u|}|u|^{1/2}d^*u,$$

In Chapter 1 we gave a version of Weil's explicit formula relating the zeros of $\zeta(s)$ to the rational primes p by comparing the Hadamard product formula with Euler's product formula. An adelic version of this explicit formula was developed by Weil which invokes a relation between the zeros of all the $L(\chi, s)$, where χ ranges over the Grossencharacters of $\mathbb{Q}$, and the rational primes p. In Appendix II of [**10**], Connes reworks (for an arbitrary global field) this version of Weil's explicit formula and shows thereby that the right hand side of the above formula gives

$$\Delta_\infty(h) = \sum_{\substack{L(\chi,\frac{1}{2}+\rho)=0,\\ \Re(\rho)\in\,]-\frac{1}{2},\frac{1}{2}[}} \int_C h(u)\chi(u)|u|^\rho d^*u.$$

Conjecture 1 would show Δ_∞ to be positive by construction. To prove GRH one would then apply the following result of Weil.

Theorem 6. *The Generalized Riemann Hypothesis (GRH) is equivalent to the positivity of* Δ_∞.

This is an adelic version of RH and of Weil's result on the equivalence with the positivity of R discussed in Section 1.5. Replacing $h(u)$ by $f(u) = |u|^{-1/2}h(u^{-1})$, we may write Weil's explicit formula as,

$$\int'_{\mathbb{R}^*} \frac{h(u^{-1})}{|1-u|}d^*u + \sum_{p \text{ prime}} \int'_{\mathbb{Q}_p^*} \frac{h(u^{-1})}{|1-u|}d^*u =$$
$$\int_C h(u)d^*u \; - \sum_{\substack{L(\chi,\rho)=0,\\ 0<\Re(\rho)<1}} \int_C h(u)\chi(u)|u|^\rho d^*u + \int_C h(u)|u|d^*u.$$

Therefore, Connes *conjectures* that,

$$\Delta_\infty(f) = \int_C h(u)d^*u - \sum_{v\in M_{\mathbb{Q}}} \int'_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|}d^*u + \int_C h(u)|u|d^*u,$$

and this *would imply* by the explicit formula that

$$\Delta_\infty(f) = \sum_{\substack{L(\chi,\rho)=0,\\ 0<\Re(\rho)<1}} \int_C h(u)\chi(u)|u|^\rho d^*u,$$

which *would be* positive.

In some sense the formulae, for $\Lambda > 0$,

$$\Delta_\Lambda(h) = \text{Trace}((S_\Lambda - Q'_{\Lambda,0})V(h))$$

are "calculating the trace in H". Connes is able to prove the following version of the above explicit formula which shows that the action of C on H by W picks up the zeros of the L-functions as an absorption spectrum in $L^2(X)$ with the non-critical zeros as resonances. A full proof in the function field case is given in Part

VIII, Lemma 3 of [**10**] and the necessary modifications for the number field case are indicated in the subsequent discussion in that paper of the analysis of the relative position of the projections P_Λ and $\widehat{P}_\Lambda$.

THEOREM 7. *Let $h \in \mathcal{S}_c(C)$, then*

$$\Delta_\infty(h) = \sum_{\chi,\rho} N(\chi, \frac{1}{2} + \rho) \int_{z \in i\mathbb{R}} \widehat{h}(\chi, z) d\mu_\rho(z),$$

where the sum is over the pairs (χ^1, ρ) of characters χ^1 of C^1, with χ being the unique quasi-character on C vanishing on N, and over the zeros ρ of $L(\chi, \frac{1}{2} + \rho)$ with $\Re(\rho) \in] - \frac{1}{2}, \frac{1}{2}[$. The number $N(\chi, \frac{1}{2} + \rho)$ is the multiplicity of the zero, the measure $d\mu_\rho(z)$ is the harmonic measure with respect to $i\mathbb{R} \subset \mathbb{C}$ and

$$\widehat{h}(\chi, z) = \int_C h(u)\chi(u)|u|^z d^*u.$$

The measure $d\mu_\rho(z)$ is a probability measure on the line $i\mathbb{R}$ which coincides with the Dirac mass at $\rho \in i\mathbb{R}$. Transforming the area to the right of $i\mathbb{R} \subset \mathbb{C}$ to the interior $|z| < 1$ of the unit circle, so that ρ is mapped to u with $|u| = 1$, we may write this measure as $P_z(u)du$ where P_z is the Poisson kernel,

$$P_z(u) = \frac{1 - |z|^2}{|u - z|^2}.$$

4.2. Analogy with the Guillemin trace formula. One can summarize Connes's approach to the Riemann Hypothesis (RH) as a program for the derivation of a conjectured explicit formula à la Weil. Whereas in Weil's set-up, the explicit formula is known and the open problem is to prove a positivity result, in Connes's set-up the positivity is part of the Polya-Hilbert space (H, W) construction and the problem is to prove the corresponding explicit formula. In his original paper [**8**], Connes found a striking analogy between his conjectured global trace formula for the action U of C on $L^2(X)$ and the distributional trace formula à la Guillemin for flows on manifolds.

Let M be a C^∞ manifold and v a smooth vector field on M with isolated zeros. We have the associated flow $F_t = \exp(tv)$, $t \in \mathbb{R}$, with its action on smooth functions,

$$(U(t)\xi)(x) = \xi(F_t(x)), \qquad \xi \in C^\infty(M), \quad x \in M, \quad t \in \mathbb{R}.$$

For $h \in C_c^\infty(\mathbb{R})$ with $h(0) = 0$, let

$$U(h) = \int_{\mathbb{R}} h(t)U(t)dt.$$

If $U(h)$ has kernel $k(x, y)$, that is,

$$U(h) = \int_{\mathbb{R}} k(\,\cdot\,, y)dy,$$

then the distributional trace of $U(h)$ is defined as

$$\mathrm{Trace}_D(U(h)) = \int_{\mathbb{R}} k(x, x)dx = \rho(h).$$

That is, ρ is viewed as a distribution. The Guillemin trace formula tells us that

$$\begin{aligned}\mathrm{Trace}_D(U(h)) &= \sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1-(F_u)_*|} d^*u \\ &= \sum_{x, v_x=0} \int \frac{h(t)}{|1-(F_t)_*|} dt + \sum_{\gamma \text{ periodic}} \sum_{T=T_\gamma^m} T_\gamma^* \frac{1}{|1-(F_T)_*|} h(T).\end{aligned}$$

Here $(F)_*$ is the Poincaré return map: it is the restriction of $d(\exp(Tv))$, where T is a period, to the normal of the orbit, and at a zero of the vector field it is the map induced on the tangent space by the flow. Therefore $(F_u)_*$ is the restriction of the tangent map to F_u to the transverse space of the orbits. In the formula (42), one considers the zeros as periodic orbits, while $I_\gamma \subset \mathbb{R}$ is the isotropy group of any $x \in \gamma$ and d^*u is the unique Haar measure $d\mu$ of total mass 1. Notice the resemblance to Connes's global trace formula when $h(0) = 0$. Assume $1 - (F_u)_*$ is invertible. Then $|1-(F_u)_*| = \det(1-(F_u)_*)$. With the assumption $h(1) = 0$, we can write the global trace formula for $U : C \to \mathcal{L}^2(X)$, $h \in \mathcal{S}_c(C)$ as

$$\mathrm{Trace}_D(U(h)) = \sum_{v \in M_{\mathbb{Q}}} \int_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|_v} d^*u,$$

where for the evaluation of h, we can embed $u \in \mathbb{Q}_v$ in J in the obvious way with 1's in every place but the v-th place. Notice that the action of J on A has fixed points coming from the elements of A with zero components. In [**8**], Connes shows the following.

Lemma 2. *For $x \in X = A/\mathbb{Q}^*$, $x \neq 0$, the isotropy group I_x of x in $C = J/\mathbb{Q}^*$ is cocompact if and only if there exists exactly one $v \in M_{\mathbb{Q}}$ with $\widetilde{x}_v = 0$ where $\widetilde{x} = (\widetilde{x}_v)_{v \in M_{\mathbb{Q}}}$ is a lift of x to J.*

Proof. For $v_1 \neq v_2$ in $M_{\mathbb{Q}}$, the map $|\ | : \mathbb{Q}_{v_1}^* \times \mathbb{Q}_{v_2}^* \to \mathbb{R}_+^*$ is not proper. □

Assume that only fixed points of the C-action as in the Lemma contribute to the trace formula. Then Connes has the following *heuristic.* For $v \in M_{\mathbb{Q}}$, let

$$\widetilde{H}_v = \{\widetilde{x} \in A : \widetilde{x} = (\widetilde{x}_v), \widetilde{x}_v = 0, \widetilde{x}_u \neq 0, u \neq v\},$$

and

$$H_v = \{[\widetilde{x}] \in X : \widetilde{x} \in \widetilde{H}_v\}.$$

Let N_x be the "normal space" to $x \in H_v$, that is,

$$N_x \simeq X/H_v \simeq A/\widetilde{H}_v \simeq \mathbb{Q}_v,$$

so that $\mathbb{Q}_v$ can be viewed as the "transverse space" to H_v. Let $j \in I_x$, the isotropy group of $x \in H_v$, and let $\widetilde{j}$ be a lift of j to J. Then $\widetilde{j}$ acts on A linearly and fixes $\widetilde{x}$. The induced action on the transverse space N_x, the Poincaré return map in this situation, is just the multiplication map,

$$\begin{aligned}\mathbb{Q}_v^* \times \mathbb{Q}_v &\to \mathbb{Q}_v \\ (\lambda, a) &\mapsto \lambda a.\end{aligned}$$

By *analogy* with the Guillemin trace formula, the corresponding contribution to the trace formula should be $\int_{\mathbb{Q}_v^*} \frac{h(\lambda^{-1})}{|1-\lambda|_v} d^*\lambda$, for $h \in \mathcal{S}(C)$ with $h(1) = 0$. This is the contribution form the *local* trace formula. The cut-offs were giving the regularisations. When $v = p$, prime, the period of the orbit is the covolume of the isotropy group $\mathbb{Q}_p^*$, and this equals $\log p$, since the image of the p-adic norm on $\mathbb{Q}_p^*$ is $p^{\mathbb{Z}}$.

Connes proposes that Weil's explicit formula incorporates a noncommutative number theoretic analogue for (X, C) of Guillemin's trace formula for flows on manifolds.

5. Related aspects of noncommutative number theory (With Appendix by Peter Sarnak)

In the previous chapters, we have tried to present in a direct and elementary way the essentials of Connes's proposed approach to the Riemann Hypothesis. Connes has presented in his papers [**8**], [**9**], [**10**] more sophisticated motivations which are of interest in a broader context as they invite a new interaction between operator algebras and number theory. They all relate in some measure to older work of Bost–Connes [**5**], an overview of which is the focus of the present chapter.

We have added, with his permission, an Appendix authored by Peter Sarnak, which is a reproduction of a letter he wrote to Enrico Bombieri regarding the appearance in Connes's set-up of symplectic symmetry, in the sense of work of Katz–Sarnak [**27**], for families of Dirichlet L-functions with quadratic characters.

5.1. Von Neumann Algebras and Galois Theory. An additional motivation for Connes's approach, that is described in detail in [**9**], arises from his observation that certain features of Galois theory related to the idele class group resemble those of the classification of factors of von Neumann algebras. For *local fields*, the role of the idele class group is played by the group of non-zero elements of the field, which by local class field theory has a Galois interpretation as the Weil group. For *global fields K of characteristic $p > 0$* we have an isomorphism between the idele class group and the Weil group W_K for the global field K (see Section 2.4). The subfields K' of K_{un} with $[K' : K] < \infty$ are classified by the subgroups

$$\{1\} \neq \Gamma \subset \mathrm{Mod}(K) = q^{\mathbb{Z}} \subset \mathbb{R}_+^*.$$

Define

$$\theta_\lambda(\mu) = \mu^\lambda, \quad \lambda \in \Gamma,$$

for μ an ℓ-th root of unity $(\ell, p) = 1$. Then,

$$K' = \{x \in K_{\mathrm{un}} : \theta(x) = x, \text{ for all } \lambda \in \Gamma\}.$$

The Galois groups of infinite extensions are constructed as projective limits of the finite groups attached to finite extensions. When K is a global field of characteristic 0, the main result of class field theory says that there is an isomorphism between $\mathrm{Gal}(K_{\mathrm{ab}}/K)$, where K_{ab} is the maximal abelian extension of K, and the quotient C/D of the idele class group of K by the connected component D of the identity in C.

When $K = \mathbb{Q}$, this translates into an isomorphism between C and $\mathbb{R}_+^* \times \prod_p \mathbb{Z}_p^*$, and moreover $D = \mathbb{R}_+^*$, so that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}/\mathbb{Q})$ is isomorphic to $\prod_p \mathbb{Z}_p^*$. Indeed, for a global field of characteristic 0, the connected component D is always non-trivial due to the archimedean places.

Can operator algebras enable us to do Galois theory "with the infinite place", as proposed by Weil (see Section 1.4)? Von Neumann algebras appear as the commutants of unitary representations in Hilbert space, the central simple ones are called factors and the approximately finite dimensional ones are the weak closure of the union of increasing sequences of finite dimensional algebras. As in Galois theory, one has a correspondence between virtual subgroups Γ of $\mathbb{R}_+^*$ (ergodic actions of $\mathbb{R}_+^*$) and the factors M. The non-simple approximately finite dimensional factors are $M_\infty(\mathbb{C})$, the operators in Hilbert space, which is of Type I_∞ and $R_{0,1} = R \otimes M_\infty(\mathbb{C})$, which is of Type II_∞ and has trace $\tau = \tau_0 \otimes \mathrm{Tr}_{M_\infty(\mathbb{C})}$, where R is the unique approximately finite factor with finite trace τ_0. By the theory of von Neumann algebras, there exists up to conjugacy a unique 1-parameter group $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$, $\lambda \in \mathbb{R}_+^*$ with

$$\tau(\theta_\lambda(a)) = \lambda\tau(a), \quad a \in \mathrm{Dom}(\tau), \lambda \in \mathbb{R}_+^*$$

If Γ is a virtual subgroup of $\mathbb{R}_+^*$ and α the corresponding ergodic action of $\mathbb{R}_+^*$ on an abelian algebra A, then

$$R_\Gamma = \{x \in R_{0,1} \otimes A : (\theta_\lambda \otimes \alpha_\lambda)x = x, \text{ for all } \lambda \in \mathbb{R}_+^*\}$$

is the corresponding factor. For the background on the material from the theory of von Neumann algebras, see [**7**]. A direction for further research, proposed by Connes, is to develop this analogy with Galois theory.

We remark the following corollary of the work of Bost-Connes [**5**], see also §5.4.

Theorem 8. *Let A be the adele ring of $\mathbb{Q}$ and $L^\infty(A)$ the essentially bounded functions on A with the supremum norm. Then $L^\infty(A) \rtimes \mathbb{Q}^*$, the crossed product with $\mathbb{Q}^*$ for multiplication on A, is isomorphic to $R_{0,1}$. Moreover, the restriction of the action of C on $A/\mathbb{Q}^*$ corresponds to the action θ_λ on $R_{0,1}$.*

The space $X = A/\mathbb{Q}^*$ is the orbit space associated to $L^\infty(A) \rtimes \mathbb{Q}^*$. The main result of [**5**] was to construct a dynamical system, with natural symmetry group $W = \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}}/\mathbb{Q})$, and partition function the Riemann zeta function at whose pole at $s = 1$ there was a phase transition. This phase transition corresponded to a passage from a family of Type I_∞ factor equilibrium states indexed by W, in the region $s > 1$, to a unique Type III_1 factor equilibrium state in the region $0 < s \leq 1$. Bost and Connes show that the corresponding Type III_1 factor for the critical strip $0 < s \leq 1$ has a Type II_∞ factor in its continuous decomposition given by $L^\infty(A) \rtimes \mathbb{Q}^*$. This provides a motivation for studying the action (X, C). The II_∞ nature of the von Neumann algebra associated to X points away from a study of this space using classical measure theory. Extensions of these results to arbitrary global fields are due to Harari-Leichtnam [**25**], Arledge-Laca-Raeburn [**1**] and the author [**6**]. This work is also related to earlier work of Julia [**26**] and others, which aims at enriching our knowledge of the Riemann zeta function by creating a dictionary between its properties and phenomena in statistical mechanics. The starting point of these approaches is the observation that, just as the zeta functions encode arithmetic information, the partition functions of quantum statistical mechanical systems encode their large-scale thermodynamical properties. The first step is therefore to construct a quantum dynamical system with partition function the Riemann zeta function. In order for the quantum dynamical system to reflect the arithmetic of the primes, it must also capture some sort of interaction between them. This last feature translates in the statistical mechanical language into the phenomenon of spontaneous symmetry breaking at a critical temperature with respect to a natural symmetry group. In the region of high temperature there is a unique equilibrium state, as the system is in disorder and is symmetric with respect to the action of the symmetry group. In the region of low temperature, a phase transition occurs and the symmetry is broken. This symmetry group acts transitively on a family of possible extremal equilibrium states.

In the following sections of this chapter, we give an overview of the construction of [**5**], emphasizing even more than in that paper the intervention of adeles and ideles (see also [**6**]). The symmetry group of the system is a Galois group, in fact the Galois group over the rational number field of its maximal abelian extension.

5.2. The problem studied by Bost–Connes. We recall a few basic notions from the C^*-algebraic formulation of quantum statistical mechanics. For the background, see [**7**]. Recall that a C^*-algebra B is an algebra over the complex numbers $\mathbb{C}$ with an adjoint $x \mapsto x^*$, $x \in B$, that is, an anti-linear map with $x^{**} = x$, $(xy)^* = y^*x^*$, $x, y \in B$, and a norm $\| . \|$ with respect to which B is complete and addition and multiplication are continuous operations. One requires in addition that $\|xx^*\| = \|x\|^2$ for all $x \in B$. All our C^*-algebras will be assumed unital. The most basic example of a noncommutative C^*-algebra is $B = M_N(\mathbb{C})$ for $N \geq 2$ an integer. The C^*-algebra plays the role of the "space" on which the system evolves, the evolution itself being described by a 1-parameter group of C^*-automorphisms $\sigma : \mathbb{R} \mapsto \mathrm{Aut}(B)$. The quantum dynamical system is therefore the pair (B, σ_t). It is customary to use the inverse temperature $\beta = 1/kT$ rather than the temperature T, where k is Boltzmann's constant. One has a notion due to Kubo-Martin-Schwinger (KMS) of an equilibrium state at inverse temperature β. Recall that a state φ on a C^*-algebra B is a positive linear functional on B satisfying $\varphi(1) = 1$. It is the generalization of a probability distribution.

Definition 5. Let (B, σ_t) be a dynamical system, and φ a state on B. Then φ is an equilibrium state at inverse temperature β, or KMS$_\beta$-state, if for each $x, y \in B$ there is a function $F_{x,y}(z)$, bounded and holomorphic in the band $0 < \mathrm{Im}(z) < \beta$ and continuous on its closure, such that for all $t \in \mathbb{R}$,

$$F_{x,y}(t) = \varphi(x\sigma_t(y)), \qquad F_{x,y}(t + \sqrt{-1}\beta) = \varphi(\sigma_t(y)x). \tag{42}$$

In the case where $B = M_N(\mathbb{C})$, every 1-parameter group σ_t of automorphisms of B can be written in the form,

$$\sigma_t(x) = e^{itH} x e^{-itH}, \qquad x \in B, \qquad t \in \mathbb{R},$$

for a self-adjoint matrix $H = H^*$. For $H \geq 0$ and for all $\beta > 0$, there is a unique KMS$_\beta$ equilibrium state for (B, σ_t) given by

$$\phi_\beta(x) = \mathrm{Trace}(xe^{-\beta H})/\mathrm{Trace}(e^{-\beta H}), \qquad x \in M_N(\mathbb{C}). \tag{43}$$

This has the form of a classical "Gibbs state" and is easily seen to satisfy the KMS$_\beta$ condition of Definition 5. The KMS$_\beta$ states can therefore be seen as generalisations of Gibbs states. The normalisation constant $\mathrm{Trace}(e^{-\beta H})$ is known as the partition function of the system. A symmetry group G of the dynamical system (B, σ_t) is a subgroup of $\mathrm{Aut}(B)$ commuting with σ:

$$g \circ \sigma_t = \sigma_t \circ g, \qquad g \in G, t \in \mathbb{R}.$$

Consider now a system (B, σ_t) with interaction. Guided by quantum statistical mechanics, one hopes to see the following features. When the temperature is high, so that β is small, the system is in disorder, there is no interaction between its constituents and the state of the system does not see the action of the symmetry group G: the KMS$_\beta$-state is unique. As the temperature is lowered, the constituents of the system begin to interact. At a critical temperature β_0 a phase transition occurs and the symmetry is broken. The symmetry group G then permutes transitively a family of extremal KMS$_\beta$- states generating the possible states of the system after phase transition: the KMS$_\beta$-state is no longer unique. This phase transition phenomenon is known as spontaneous symmetry breaking at the critical inverse temperature β_0. The partition function should have a pole at β_0. For a fuller explanation, see [**5**]. The problem solved by Bost and Connes was the following.

Problem 1. Construct a dynamical system (B, σ_t) with partition function the zeta function $\zeta(\beta)$ of Riemann, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function with respect to a natural symmetry group.

As mentioned in §5.1, the symmetry group is the unit group of the ideles, given by $W = \prod_p \mathbb{Z}_p^*$ where the product is over the primes p and $\mathbb{Z}_p^* = \{u_p \in \mathbb{Q}_p : |u_p|_p = 1\}$. We use, as before, the normalization $|p|_p = p^{-1}$. This is the same as the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Here $\mathbb{Q}^{ab}$ is the maximal abelian extension of the rational number field $\mathbb{Q}$, which in turn is isomorphic to its maximal cyclotomic extension, that is, the extension obtained by adjoining to $\mathbb{Q}$ all the roots of unity. The interaction detected in the phase transition comes about from the interaction between the primes coming from considering at once all the embeddings of the non-zero rational numbers $\mathbb{Q}^*$ into the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the prime valuations $|.|_p$. The natural generalisation of this problem to the number field case was solved in [**6**] and is the following.

Problem 2. Given a number field K, construct a dynamical system (B, σ_t) with partition function the Dedekind zeta function $\zeta_K(\beta)$, where $\beta > 0$ is the inverse temperature, having spontaneous symmetry breaking at the pole $\beta = 1$ of the Dedekind function with respect to a natural symmetry group.

Recall that the Dedekind zeta function is given by

$$\zeta_K(s) = \sum_{\mathcal{C} \subset \mathcal{O}} \frac{1}{N(\mathcal{C})^s}, \qquad \mathrm{Re}(s) > 1. \tag{44}$$

Here $\mathcal{O}$ is the ring of integers of K and the summation is over the ideals $\mathcal{C}$ of K contained in $\mathcal{O}$. The symmetry group is the unit group of the finite ideles of K.

For a generalization to the function field case see [**25**]. We restrict ourselves in what follows to the case of the rational numbers, that is, to a discussion of Problem 1.

5.3. Construction of the C^*-algebra. We give a different construction of the C^*-algebra of [**5**] to that found in their original paper. It is essentially equivalent to the construction of [**1**], except that we work with adeles and ideles, and turns out to be especially useful for the generalization to the number field case in [**6**] . Let A_f denote the finite adeles of $\mathbb{Q}$, that is the restricted product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. Recall that this restricted product consists of the infinite vectors $(a_p)_p$, indexed by the primes p, such that $a_p \in \mathbb{Q}_p$ with $a_p \in \mathbb{Z}_p$ for almost all primes p. The (finite) adeles form a ring under componentwise addition and multiplication.

The (finite) ideles $\mathcal{J}$ are the invertible elements of the adeles. They form a group under componentwise multiplication. Let $\mathbb{Z}_p^*$ be those elements of $u_p \in \mathbb{Z}_p$ with $|u_p|_p = 1$. Notice that an idele $(u_p)_p$ has $u_p \in \mathbb{Q}_p^*$ with $u_p \in \mathbb{Z}_p^*$ for almost all primes p. Let

$$R = \prod_p \mathbb{Z}_p, \qquad I = \mathcal{J} \cap R, \qquad W = \prod_p \mathbb{Z}_p^*.$$

Further, let $\mathcal{I}$ denote the semigroup of integral ideals of $\mathbb{Z}$. It is the semigroup of $\mathbb{Z}$-modules of the form $m\mathbb{Z}$ where $m \in \mathbb{Z}$. Notice that I as above is also a semigroup. We have a natural short exact sequence,

$$1 \to W \to I \to \mathcal{I} \to 1. \tag{45}$$

The map $I \to \mathcal{I}$ in this short exact sequence is given as follows. To $(u_p)_p \in I$ associate the ideal $\prod_p p^{\mathrm{ord}_p(u_p)}$ where $\mathrm{ord}_p(u_p)$ is determined by the formula $|u_p|_p = p^{-\mathrm{ord}_p(u_p)}$. It is clear that this map is surjective with kernel W, that is that the above sequence is indeed short exact. By the Strong Approximation Theorem we have

$$\mathbb{Q}/\mathbb{Z} \simeq A_f/R \simeq \oplus_p \mathbb{Q}_p/\mathbb{Z}_p \tag{46}$$

and we have therefore a natural action of I on $\mathbb{Q}/\mathbb{Z}$ by multiplication in A_f/R and transport of structure. We use here that $I \cdot R \subset R$. Mostly we shall work in A_f/R rather than $\mathbb{Q}/\mathbb{Z}$. We have the following straightforward Lemma (see [**6**]).

LEMMA 3. *For $a = (a_p)_p \in I$ and $y \in A_f/R$, the equation*

$$ax = y$$

has $n(a) := \prod_p p^{\mathrm{ord}_p(a_p)}$ solutions in $x \in A_f/R$. Denote these solutions by

$$[x : ax = y].$$

In the above lemma it is important to bear in mind that we are computing modulo R. Now, let $\mathbb{C}[A_f/R] := \mathrm{span}\{\delta_x : x \in A_f/R\}$ be the group algebra of A_f/R over $\mathbb{C}$, so that $\delta_x\delta_{x'} = \delta_{x+x'}$ for $x, x' \in A_f/R$. We have (see for comparison [**1**]),

LEMMA 4. *The formula*

$$\alpha_a(\delta_y) = \frac{1}{n(a)} \sum_{[x:ax=y]} \delta_x,$$

for $a \in I$, defines an action of I by endomorphisms of $C^*(A_f/R)$.

The endomorphism α_a for $a \in I$ is a one-sided inverse of the map $\delta_x \mapsto \delta_{ax}$ for $x \in A_f/R$, so it is like a semigroup "division". The C^*-algebra can be thought of as the operator norm closure of $\mathbb{C}[A_f/R]$ in its natural left regular representation in $l^2(A_f/R)$. We now appeal to the notion of semigroup crossed product developed by Laca and Raeburn and used in [**1**], applying it to our situation. A covariant representation of $(C^*(A_f/R), I, \alpha)$ is a pair (π, V) where

$$\pi : C^*(A_f/R) \to B(\mathcal{H})$$

is a unital representation and

$$V : I \to B(\mathcal{H})$$

is an isometric representation in the bounded operators in a Hilbert space $\mathcal{H}$. The pair (π, V) is required to satisfy

$$\pi(\alpha_a(f)) = V_a \pi(f) V_a^*, \qquad a \in I, \quad f \in C^*(A_f/R).$$

Notice that the V_a are not in general unitary. Such a representation is given by (λ, L) on $l^2(A_f/R)$ with orthonormal basis $\{e_x : x \in A_f/R\}$ where λ is the left regular representation of $C^*(A_f/R)$ on $l^2(A_f/R)$ and

$$L_a e_y = \frac{1}{\sqrt{n(a)}} \sum_{[x:ax=y]} e_x.$$

The universal covariant representation, through which all other covariant representations factor, is called the (semigroup) crossed product $C^*(A_f/R) \rtimes_\alpha I$. This algebra is the universal C^*-algebra generated by the symbols $\{e(x) : x \in A_f/R\}$ and $\{\mu_a : a \in I\}$ subject to the relations

$$\mu_a^* \mu_a = 1, \quad \mu_a \mu_b = \mu_{ab}, \qquad a, b \in I, \tag{47}$$

$$e(0) = 1, \quad e(x)^* = e(-x), \quad e(x)e(y) = e(x+y), \qquad x, y \in A_f/R, \tag{48}$$

$$\frac{1}{n(a)} \sum_{[x:ax=y]} e(x) = \mu_a e(y) \mu_a^*, \qquad a \in I, y \in A_f/R. \tag{49}$$

The relations in (47) reflect a multiplicative structure, those in (48) an additive structure and those in (49) how these multiplicative and additive structures are related via the crossed product action. Julia [**26**] observed that by using only the multiplicative structure of the integers one cannot hope to capture an interaction between the different primes. When $u \in W$ then μ_u is a unitary, so that $\mu_u^* \mu_u = \mu_u \mu_u^* = 1$ and we have for all $x \in A_f/R$,

$$\mu_u e(x) \mu_u^* = e(u^{-1}x), \qquad \mu_u^* e(x) \mu_u = e(ux). \tag{50}$$

Therefore we have a natural action of W as inner automorphisms of $C^*(A_f/R) \rtimes_\alpha I$ using (50).

To recover the C^*-algebra of [**5**] we must split the short exact sequence (45). The ideals in $\mathcal{I}$ are all of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$. This generator m is determined up to sign. Consider the image of $|m|$ in I under the diagonal embedding $q \mapsto (q)_p$ of $\mathbb{Q}^*$ into I, where the p-th component of $(q)_p$ is the image of q in $\mathbb{Q}_p^*$ under the natural embedding of $\mathbb{Q}^*$ in $\mathbb{Q}_p^*$. The map

$$+ : m\mathbb{Z} \mapsto (|m|)_p \tag{51}$$

defines a splitting of (45). Let I_+ denote the image and define B to be the semigroup crossed product $C^*(A_f/R) \rtimes_\alpha I_+$ with the restricted action α from I to I_+. By transport of structure using (46), this algebra is easily seen to be isomorphic to a semigroup crossed product of $C^*(\mathbb{Q}/\mathbb{Z})$ by $\mathbb{N}_+$, where $\mathbb{N}_+$ denotes the positive natural numbers. This is the algebra constructed in [**5**] (see also [**1**]). From now on, we use the symbols $\{e(x) : x \in \mathbb{Q}/\mathbb{Z}\}$ and $\{\mu_a : a \in \mathbb{N}_+\}$. It is essential to split the short exact sequence in this way in order to obtain the symmetry breaking phenomenon. In particular, this replacement of I by I_+ now means that the group W acts by outer automorphisms. For $x \in B$, one has that $\mu_u^* x \mu_u$ is still in B (computing in the larger algebra $C^*(A_f/R) \rtimes_\alpha I$), but now this defines an outer action of W. This coincides with the definition of W as the symmetry group as in [**5**].

5.4. The Theorem of Bost–Connes. Using the abstract description of the C^*-algebra B of §5.3, to define the time evolution σ of our dynamical system (B, σ) it suffices to define it on the symbols $\{e(x) : x \in \mathbb{Q}/\mathbb{Z}\}$ and $\{\mu_a : a \in \mathbb{N}_+\}$. For $t \in \mathbb{R}$, let σ_t be the automorphism of B defined by

$$\sigma_t(\mu_m) = m^{it}\mu_m, \quad m \in \mathbb{N}_+, \qquad \sigma_t(e(x)) = e(x), \quad x \in \mathbb{Q}/\mathbb{Z}. \tag{52}$$

By (47) and (50) we clearly have that the action of W commutes with this 1-parameter group σ_t. Hence W will permute the extremal KMS$_\beta$-states of (B, σ_t). To describe the KMS$_\beta$-states for $\beta > 1$, we shall represent (B, σ_t) on a Hilbert space. Namely, following [**5**], let $\mathcal{H}$ be the Hilbert space $l^2(\mathbb{N}_+)$ with canonical orthonormal basis $\{\varepsilon_m, m \in \mathbb{N}_+\}$. For each $u \in W$, one has a representation π_u of B in $B(\mathcal{H})$ given by,

$$\pi_u(\mu_m)\varepsilon_n = \varepsilon_{mn}, \; m, n \in \mathbb{N}_+$$

$$\pi_u(e(x))\varepsilon_n = \exp(2i\pi nu \circ x)\varepsilon_n, \; n \in \mathbb{N}_+, \; x \in \mathbb{Q}/\mathbb{Z}. \tag{53}$$

Here $u \circ x$ for $u \in W$ and $x \in \mathbb{Q}/\mathbb{Z}$ is the multiplication induced by transport of structure using (46). One verifies easily that (53) does indeed give a C^*-algebra representation of B. Let H be the unbounded operator in $\mathcal{H}$ whose action on the canonical basis is given by

$$H\varepsilon_n = (\log n)\varepsilon_n, \quad n \in \mathbb{N}_+. \tag{54}$$

Then clearly, for each $u \in W$, we have

$$\pi_u(\sigma_t(x)) = e^{itH}\pi_u(x)e^{-itH}, \qquad t \in \mathbb{R}, \; x \in B.$$

Notice that, for $\beta > 1$,

$$\mathrm{Trace}(e^{-\beta H}) = \sum_{n=1}^{\infty}\langle e^{-\beta H}\varepsilon_n, \varepsilon_n\rangle = \sum_{n=1}^{\infty} n^{-\beta}\langle \varepsilon_n, \varepsilon_n\rangle = \sum_{n=1}^{\infty} n^{-\beta},$$

so that the Riemann zeta function appears as a partition function of Gibbs state type. We can now state the main result of [**5**].

THEOREM 9 (Bost-Connes). *The dynamical system (B, σ_t) has symmetry group W. The action of $u \in W$ is given by $[u] \in \mathrm{Aut}(B)$ where*

$$[u] : e(y) \mapsto e(u \circ y), \quad y \in \mathbb{Q}/\mathbb{Z}, \qquad [u] : \mu_a \mapsto \mu_a, \quad a \in \mathbb{N}.$$

This action commutes with σ,

$$[u] \circ \sigma_t = \sigma_t \circ [u], \qquad u \in W, \quad t \in \mathbb{R}.$$

Moreover,

1. *for $0 < \beta \le 1$, there is a unique* KMS$_\beta$ *state. (It is a factor state of Type* III$_1$ *with associated factor the Araki-Woods factor R_∞.)*
2. *for $\beta > 1$ and $u \in W$, the state*

$$\phi_{\beta,u}(x) = \zeta(\beta)^{-1}\mathrm{Trace}(\pi_u(x)e^{-\beta H}), \qquad x \in B$$

is a KMS$_\beta$ *state for (B, σ_t). (It is a factor state of Type* I$_\infty$*). The action of W on B induces an action on these* KMS$_\beta$ *states which permutes them transitively and the map $u \mapsto \phi_{\beta,u}$ is a homomorphism of the compact group W onto the space $\mathcal{E}_\beta$ of extremal points of the simplex of* KMS$_\beta$ *states for (B, σ_t).*

3. *the ζ function of Riemann is the partition function of (B, σ_t).*

Part (1) of the above theorem is difficult and the reader is referred to [**5**] for complete details, as for a full proof of (2). That for $\beta > 1$ the KMS$_\beta$-states given in part (2) fulfil Definition 5 of §5.2 is a straightforward exercise. Notice that they have the form of Gibbs equilibrium states.

Theorem 9 solves Problem 1 of §5.2. More information is contained in its proof however. As mentioned already, given the existence of the Artin isomorphism in class field theory for the rationals, one can recover the Galois action of W explicitly. Despite the progress in [**6**], it is still an open problem to exhibit this Galois action in terms of an analogue of (B, σ_t) in a completely satisfactory way for general number fields. Another feature occurs in the analysis of the proof of part (1) of Theorem 9. One can treat the infinite places in a similar way to that already described for the finite places, so working with the (full) adeles A and (full) ideles J. The ring of adeles A of $\mathbb{Q}$ consists of the infinite vectors $(a_\infty, a_p)_p$ indexed by the archimedean place and the primes p of $\mathbb{Q}$ with $a_p \in \mathbb{Z}_p$ for all but finitely many p. The group J of ideles consists of the infinite vectors $(u_\infty, u_p)_p$ with $u_\infty \in \mathbb{R}, u_\infty \neq 0$ and $u_p \in \mathbb{Q}_p, u_p \neq 0$ and $|u_p|_p = 1$ for all but finitely many primes p. There is a norm $|\cdot|$ defined on J given by $|u| = |u_\infty|_\infty \prod_p |u_p|_p$. We have natural diagonal embeddings of $\mathbb{Q}$ in A and $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ in J induced by the embeddings of $\mathbb{Q}$ into its completions. Notice that by the product formula $\mathbb{Q}^* \subset \mathrm{Ker}|\cdot|$. We define an equivalence relation on A by $a \equiv b$ if and only if there exists a $q \in \mathbb{Q}^*$ with $a = qb$. With respect to this equivalence, we form the coset space $X = A/\mathbb{Q}^*$. The ideles J act on A by componentwise multiplication, which induces an action of $C = J/\mathbb{Q}^*$ on X. Notice that this action has fixed points. For example, whenever an adele a has $a_p = 0$ it is a fixed point of the embedding of $\mathbb{Q}_p^*$ into J (to $q_p \in \mathbb{Q}_p^*$ one assigns the idele with 1 in every place except the pth place.) On the other hand, every Type III$_1$ factor has a continuous decomposition, that is it can be written as a cross-product of $\mathbb{R}$ with a Type II$_\infty$ factor. Connes has observed that the von-Neumann algebra of Type III$_1$ in the region $0 < \beta \leq 1$ of Theorem 9 has in its continuous decomposition the Type II$_\infty$ factor given by the crossed product of $L^\infty(A)$ by the action of $\mathbb{Q}^*$ by multiplication. The associated von Neumann algebra has orbit space $X = A/\mathbb{Q}^*$. As we have seen, the pair (X, C) plays a fundamental role in Connes's proposed approach to the Riemann hypothesis in [**8**] and can be thought of as playing the role for number fields of the curve and Frobenius for the proof of the Riemann hypothesis in the case of curves over finite fields.

5.5. Appendix by Peter Sarnak. In this appendix we reproduce, with his permission, the text of a letter written by Peter Sarnak in June, 2001 and addressed to Enrico Bombieri.

Below is the symplectic pairing that I mentioned to you in Zurich. There is nothing deep about it or the analysis that goes with it. Still, its existence is consistent with various themes. To put things in context, recall that the phenomenological and analytic results on the high order zeroes of a given L-function and the low zeroes for families of L-functions suggest that there is a natural spectral interpretation for the zeroes as well as a symmetry group associated with a family [KaSa] (N. Katz and P. Sarnak, Bulletin of the AMS, **36** (1999), 1–26). In particular, for Dirichlet L-functions $L(s, \chi)$, $\chi^2 = 1$, the symmetry predicted in [KaSa] is a symplectic one, *i.e.* $\mathrm{Sp}(\infty)$. So we expect that there is a suitable spectral interpretation, the linear transformation whose spectrum corresponds to the zeroes of $L(s, \chi)$ should correspond to a symplectic form.

It should be emphasized that this by itself does not put the zeroes on the line. Such a symplectic pairing is a symmetry which is central to understanding this family of L-functions. On the other hand, the existence of an invariant unitary (or hermitian) pairing for the operator, as suggested by Hilbert and Polya would of course put zeroes on the line. However, I think the existence of the latter is not very likely. In the analogous function field settings there are spectral interpretations of the zeroes and invariant bilinear pairings due to Grothendieck. The known proofs of the Riemann Hypothesis (that is, the Weil Conjectures in this setting) do not proceed with any magical unitary structures but rather with families and their monodromy, high tensor power representations of the latter and positivity [De] (P. Deligne, Publ, IHES, **48** (1974), 273–308).

One can look for symplectic pairing in the well-known spectral interpretation of the zeroes of $\zeta(2s)$ in the eigenvalue problem for $X = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Indeed the resonances (or scattering frequencies) through the theory of Eisenstein series for X are at the zeroes of $\xi(2s)$, ξ being the completed zeta function of Riemann. Lax and Phillips have constructed an operator B (see [LaPh] (P. Lax, R. Phillips, Bulletin of the AMS, **2** (1980), 261–295)) whose spectrum consists of the Maass cusp forms on X together with the zeroes of $\xi(2s)$. The problem with finding a symplectic pairing for the part of the spectrum corresponding to $\xi(2s)$ is that I don't know of any geometric way of isolating this part of the spectrum of B. The space $L^2(X)$ as it stands is too big. Nevertheless, this spectral interpretation of the zeroes of $\xi(2s)$ is important since it can be used to give reasonable zero-free regions for $\zeta(s)$, see [Sa](P. Sarnak, Letter to F. Shahidi and S. Gelbart, January 2001). Moreover, this spectral proof of the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$ extends to much more general L-functions where the method of Hadamard and de la Vallee Poussin does not work (at least with our present knowledge)[Sh](F. Shahidi, Perspect. Math., **10** (1990), 415–437, Academic Press).

Assuming the Riemann hypothesis for $L(s,\chi)$, Connes [Co] (A. Connes, Selecta Math. (N.S.), **5** (1999), No. 1, 29–106) gives a spectral interpretation of the zeroes. I recall this construction below. Like the spectral interpretation as resonances, Connes' space is defined very indirectly – as the annihilator of the complicated space of functions (it is very close to the space considered by Beurling [Be] (A. Beurling, Proc. Nat. Acad. Sci., USA, **41** (1955), 312–314)). We can look for invariant pairings for his operator. Note that for an even dimensional space, a necessary condition that a transformation A, of determinant equal to 1, preserve a standard symplectic form or orthogonal pairing is that its eigenvalues (as a set) be invariant under $\lambda \to \lambda^{-1}$. In fact, if the eigenvalues are also distinct, then this is also a sufficient condition. On the other hand, if A is not diagonalizable then there are other obstructions (besides the "functional equation" $\lambda \to \lambda^{-1}$) for preserving such pairings.

The set-up in [Co] is as follows: Let χ be a non-trivial Dirichlet character of conductor q and with $\chi(-1) = 1$ (one can easily include all χ). For $f \in \mathcal{S}(\mathbb{R})$ and even and $x > 0$ set

$$\Theta_f(x) := \left(\frac{x}{\sqrt{q}}\right)^{1/2} \sum_{n=1}^{\infty} f\left(\frac{nx}{q}\right)\chi(n). \tag{1}$$

According to Poisson Summation and Gauss sums we have

$$\Theta_f\left(\frac{1}{x}\right) = \Theta_{\hat{f}}(x). \tag{2}$$

Hence $\Theta_f(x)$ is rapidly decreasing as $x \to 0$ or $x \to \infty$. Consider the vector space W of distributions D on $(0, \infty)$ (with respect to the multiplicative group) with suitable growth conditions at 0 and ∞ for which

$$D(\Theta_f) = \int_0^\infty D(x)\Theta_f(x)\frac{dx}{x} = 0, \tag{3}$$

for all f as above.

For $y > 0$, let U_y be the translation on the space of distributions, $U_y D(x) = D(yx)$. Clearly, U_y leaves the subspace W invariant and yields a representation of $\mathbb{R}^*_{>0}$. By (2), if $D \in W$ then so is $RD := D(1/x)$. This R acts as an involution on W. To see which characters (*i.e.* eigenvalues of U_y) x^s, $s \in \mathbb{C}$, of $\mathbb{R}^*$ are in W, consider

$$\int_0^\infty \Theta_f(x)x^s\frac{dx}{x} = q^{s/2}L(s+\frac{1}{2},\chi)\int_0^\infty f(y)y^{s+1/2}\frac{dy}{y}. \tag{4}$$

Now $f \in \mathcal{S}(\mathbb{R})$ and is even, hence

$$I = \int_0^\infty f(x)x^{s+\frac{1}{2}}\frac{dx}{x} = \int_0^1 f(x)x^{s+\frac{1}{2}}\frac{dx}{x} + g(s),$$

where $g(s)$ is entire. Moreover, for $N \geq 0$,

$$I = \int_0^1 \sum_{n=0}^{N} a_{2n}x^{2n}x^{s+\frac{1}{2}}\frac{dx}{x} + \text{a holomorphic function in } \mathrm{Re}(s) > -N+1,$$

$$= \sum_{n=0}^{\infty} \frac{a_{2n}}{2n - \frac{1}{2} + s} + \text{a holomorphic function in } \mathrm{Re}(s) > -N+1. \tag{5}$$

Thus, for general such f, I has a simple pole at $s = \frac{1}{2} - 2n$.

According to (4) and (5) we have that $x^s(\Theta_f) = 0$ for all f iff

$$s = i\gamma \text{ where } \rho = \frac{1}{2} + i\gamma \text{ is a } \textbf{nontrivial} \text{ zero of } L(s,\chi). \tag{6}$$

If the multiplicity of the zero of $L(s,\chi)$ at $\rho = \frac{1}{2} + i\gamma$ is $m_\gamma \geq 1$, then differentiating (4) $m_\gamma - 1$ times shows that

$$x^{i\gamma},\ (\log x)^{i\gamma}, \ldots, (\log x)^{m_\gamma - 1} x^{i\gamma} \tag{7}$$

are in W.

The involution R of W ensures that $x^{i\gamma} \in W$ iff $x^{-i\gamma} \in W$ (and similarly with multiplicities). Of special interest is $\gamma = 0$. We have from (2) that for j odd,

$$\int_0^\infty (\log x)^j\, \Theta_f(x)\frac{dx}{x} = -\int_0^\infty (\log x)^j\, \Theta_{\widehat{f}}(x)\frac{dx}{x}. \tag{8}$$

Hence, if $f = \widehat{f}$ and j is odd,

$$\int_0^\infty (\log x)^j\, \Theta_f(x)\frac{dx}{x} = 0. \tag{9}$$

If $f = -\widehat{f}$ then

$$\int_0^\infty f(x)x^{1/2}\frac{dx}{x} = 0. \tag{10}$$

So from (4) we see that if $f = -\widehat{f}$ then

$$\int_0^\infty \Theta_f(x)\,(\log x)^j\,\frac{dx}{x} = 0, \qquad \text{for } j = 0, \ldots, m_0. \tag{11}$$

Combining (9) and (11) we see that

$$\begin{aligned} W_0 &= \operatorname{span}\{1, \log x, (\log x)^2, \ldots\} \cap W \\ &= \operatorname{span}\{1, \log x, \ldots, (\log x)^{m_0-1}\} \end{aligned} \tag{12}$$

is even dimensional.

Hence m_0 is even (of course this also follows from the functional equation for $L(s,\chi)$).

In order to continue, we need to specify the precise space of distributions that we are working with. To allow for zeroes ρ of $L(s,\chi)$ with $\mathrm{Re}(\rho) \neq \frac{1}{2}$, one needs to allow spaces of distributions which have exponential growth at infinity. This can be done and one can proceed as we do here, however to avoid such definitions we will assume the Riemann Hypothesis for $L(s,\chi)$ (anyway, this is not the issue as far as the symplectic pairing goes). This way we can work with the familiar tempered distributions. We change variable, setting $x = e^t$ so that our distributions $D(t)$ satisfy

$$\int_{-\infty}^{\infty} D(t)\Theta_f(e^t)dt = 0. \tag{13}$$

The group U_y now acts by translations $\tau \in \mathbb{R}$,

$$U_\tau D(t) = D(t+\tau). \tag{14}$$

If now V is the space of such tempered distributions satisfying the annihilation condition (13) for all f (which is a topological vector space), then for $D \in V$, its Fourier transform $\widehat{D}(\xi)$ is supported in $\{\gamma \mid \xi(\frac{1}{2} + i\gamma, \chi) = 0\}$. Since $\widehat{D}$ is also tempered, it is easy to describe $\widehat{D}$ and hence the space V. It consists of all tempered distributions D of the form

$$D(t) = \sum_\gamma \sum_{j=0}^{m_\gamma - 1} a_{j,\gamma}(D) t^j e^{i\gamma t}. \tag{15}$$

The representation (15) is unique and the series converges as a tempered distribution, *i.e.*

$$\sum_{|\gamma| \leq T} \sum_{j=0}^{m_\gamma - 1} |a_{j,\gamma}(D)| \ll T^A \tag{16}$$

for some A depending on D.

The action (14) on V gives a group of transformations whose spectrum consists of the numbers $e^{i\gamma\tau}$ with multiplicity m_γ. The subspaces V_γ of V given by

$$V_\gamma = \operatorname{span}\{e^{i\gamma t}, te^{i\gamma t}, \ldots, t^{m_\gamma - 1} e^{i\gamma t}\} \tag{17}$$

are U_τ invariant, the action taking the form

$$e^{i\gamma\tau}\begin{pmatrix} 1 & \tau & \tau^2 & \dots & \tau^{m_\gamma-1} \\ & 1 & 2\tau & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{pmatrix}, \tag{18}$$

in the apparent basis. Thus U_τ is not diagonalizable if $m_\gamma > 1$ for some γ. The span of the subspaces V_γ is dense in V. So the action U_τ on V gives a spectral interpretation of the nontrivial zeroes of $L(s,\chi)$. While one expects for these $L(s,\chi)$'s that all their zeroes are simple, there are more general L-functions (*e.g.* this of elliptic curves of rank bigger than 1) which have multiple zeroes. Thus the possibility of multiple zeroes especially at $s = \frac{1}{2}$ must be entertained and it is instructive to do so. In any case, since multiple zeroes mean that this action U_τ is not diagonalizable, we infer that there cannot be any direct unitarity that goes along with it.

There is however a symplectic pairing on V preserved by U_τ. It is borrowed from $\mathrm{sym}^\nu\rho$, where ρ is the standard two dimensional representation of SL_2 (via (18) above) when ν is odd. We pair V_γ with $V_{-\gamma}$ for $\gamma > 0$ and separate the even dimensional space V_0.

For $D, E \in V$ set

$$[D,E] := \sum_{j=0}^{m_0-1} \frac{(-1)^j a_{j,0}(D) a_{m_0-1-j,0}(E)}{\binom{m_0-1}{j}} + \sum_{\gamma>0} \gamma e^{-\gamma^2} \sum_{j=0} m_\gamma - 1 \frac{(-1)^j}{\binom{m_\gamma-1}{j}} \cdot \left(a_{j,\gamma}(D) a_{m_\gamma-1-j,-\gamma}(E) - a_{m_\gamma-1-j,-\gamma}(D) a_{j,\gamma}(E)\right). \tag{19}$$

There is nothing special about the factor $\gamma e^{-\gamma^2}$ —it is put there for convergence.

The bilinear pairing $[\ ,\ ]$ on $V \times V$ is symplectic and U_τ invariant. That is

(1) $[D,E] = -[E,D]$
(2) It is non-degenerate: for $D \neq 0$ there is an E such that $[D,E] \neq 0$
(3) $[U_\tau D, U_\tau E] = [D,E]$ for $\tau \in \mathbb{R}$.

The verification of these is straightforward. Note that if $m_0 > 0$ and being even, one checks that the transformations (18) cannot preserve a symmetric pairing. Thus the symplectic feature is intrinsic to this spectral interpretation of the zeroes of $L(s,\chi)$.

It would be of some interest to carry out the above adelically as in Connes' papers and also for other (say GL_2) L-functions, especially where, for example, an orthogonal rather than symplectic invariance is expected [KaSa]. Another point is that it would be nice to define the pairing $[\ ,\ ]$ directly without the Fourier Transform (*i.e.* without first diagonalizing to Jordan form). If we assume RH as we have done as well as that the zeroes of $L(s,\chi)$ are simple, then such a definition is possible. Set $H(t) = e^{-t^2/2}$; then for D and E in V, we have that $H * D(t)$ and $\frac{d}{dt}(H * E)(t)$ are almost periodic functions on $\mathbb{R}$. Up to a constant factor we have that

$$[D,E] = M\left((H * D)(t) \frac{d}{dt}(H * E)(t)\right). \tag{20}$$

Here for an almost periodic function $f(t)$ on $\mathbb{R}$, $M(f)$ is its mean-value given by

$$M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)dx.$$

5.5.1. *Section added by Peter Sarnak on February, 2002.* The Hilbert-Polya idea that there is a naturally defined *self-adjoint* operator whose eigenvalues are simply related to the zeroes of an L-function seems far-fetched. However, Luo and I [LuSa] (W. Luo and P. Sarnak, Israel, Quantum ergodicity on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, II, in preparation) have recently constructed a self-adjoint *non-negative* operator A on

$$L^2_0(X) = \{\psi \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}) \mid \int_X \psi(z)dv(z) = 0\}$$

whose eigenvalues are essentially the central critical values $L(\frac{1}{2},\varphi)$ as φ varies over the (Hecke-eigen) Maass cusp forms for X. In particular, this gives a spectral proof that $L(\frac{1}{2},\varphi) \geq 0$. The fact that $L(\frac{1}{2},\varphi)$ cannot be negative (which is an immediate consequence of RH for $L(s,\varphi)$) is known and was proved by theta function methods (see [KatSa], S. Katok and P. Sarnak, Israel Math. Jnl., **84** (1993) 193–227, and also [Wal], L. Waldspurger, J. Math. Pures et Appl., **60** (1981), 365–384). The operator A comes from polarizing the quadratic form $B(\psi)$ on $L^2_0(X)$ which appears as the main term in the Shnirelman sums for the measures $\varphi_j^2(z)dv(z)$, where φ_j is an orthonormal basis of Maass cusp forms for $L^2(X)$. Denote by λ_j the (Laplace) eigenvalue of φ_j. It is known, see page 688 in [Se](A. Selberg, Collected Papers, Vol. I (1989), Springer-Verlag), that

$$\sum_{\lambda_j\leq\lambda} 1 \sim \frac{\lambda}{12}, \qquad \lambda\to\infty.$$

The quadratic form $B(\psi)$ comes from the following: For $\psi \in L^2_0(X)$ fixed,

$$\sum_{\lambda_j\leq\lambda} |\langle\varphi_j^2,\psi\rangle|^2 \sim B(\psi)\sqrt{\lambda}$$

as $\lambda\to\infty$.

Incidentally, the family of L-functions $L(s,\varphi)$ as φ varies as above, has an orthogonal $O(\infty)$ symmetry in the sense of [KaSa], see also [Ke-Sn] (J. Keating and N. Snaith, Comm. Math. Phys., **214** (2000), 91–110).

Notes added in proof: (1) In his paper "On a representation of the idele class group related to primes and zeros of L-functions" (to appear in Duke Math. J. and available on arXiv:math.NT/0311468 v2 15 Dec 2003), Ralf Meyer gives another approach to a spectral interpretation for the poles and zeros of the L-function of a global field K. His construction is motivated by the work of Alain Connes. Meyer uses natural spaces of functions on the adele ring and the idele class group of K, to construct a virtual representation of the idele class group of K whose character is equal to a variant of the Weil distribution that occurs in Weil's explicit formula.

(2) For further progress on the analogue of Problem 1, §5.2, for imaginary quadratic fields see: A. Connes, M. Marcolli: "From Physics to Number Theory via Noncommutative Geometry" available on arXiv:math.NT/0404128 v1 6 Apr 2004.

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