THE GAUSS-BONNET THEOREM FOR THE NONCOMMUTATIVE TWO TORUS

ALAIN CONNES AND PAULA TRETKOFF

Abstract. In this paper we shall show that the value at the origin, $\zeta(0)$, of the zeta function of the Laplacian on the non-commutative two torus, endowed with its canonical conformal structure, is independent of the choice of the volume element (Weyl factor) given by a (non-unimodular) state. We had obtained, in the late eighties, in an unpublished computation, a general formula for $\zeta(0)$ involving modified logarithms of the modular operator of the state. We give here the detailed computation and prove that the result is independent of the Weyl factor as in the classical case, thus proving the analogue of the Gauss-Bonnet theorem for the noncommutative two torus.

Contents

1. Introduction

The main result of this paper, namely the analogue of the Gauss-Bonnet theorem for the noncommutative two torus $\mathbb{T}^2_\theta$, follows from the computation of the value at the origin, $\zeta(0)$, of the zeta function of the analogue of the Laplacian for non-unimodular geometric structures on $\mathbb{T}^2_\theta$. The two authors did this computation at the end of the 1980's. The result was mentioned in [6], but it was never published except as an MPI preprint [5]. At the time the computation was done, the result was formulated in terms of the modular operator but its significance was unclear. In the mean time the following two theories have been developed:

- The spectral action
- The non-unimodular spectral triples

The spectral action [2], [4], allows one to interpret gravity coupled to the standard model from spectral invariants of a geometry which encodes a fine structure of space-time. In the two dimensional case the constant term in the spectral action is (up to adding the dimension of the kernel of the operator) given by the value at the origin, $\zeta(0)$, of the zeta function. This value is a topological invariant in classical Riemannian geometry.

The twisted (or non-unimodular) spectral triples appeared very naturally [10] in the study of type III examples of foliation algebras.

Thus, in more modern terminology, our computation was that of the spectral action for spectral triples on the non-commutative two torus $\mathbb{T}^2_\theta$ both in the usual and the twisted cases. Initially, the complexity of the computation and the lack of simplicity of the result made us reticent about publishing it. It is only recently that, by pushing the computation further, we could prove the expected (*cf.* [3]) conformal invariance.

Theorem 1.1. – *Let θ be an irrational number and consider, on the non-commutative two torus $\mathbb{T}^2_\theta$, a translation invariant conformal structure. Let k be an invertible positive element of $C^\infty(\mathbb{T}^2_\theta)$ considered as a Weyl factor rescaling the metric. Then the value at the origin of the zeta function $\zeta(s)$ of the Laplacian of the rescaled metric is independent of k.*

In fact we must retract a bit after stating this theorem since we have only performed the computation for the simplest translation invariant conformal structure but we do not expect that the general case will be different.

There are two main actors in the computation and they display an interesting interplay between two theories which look a priori quite distinct, but use the same notation for their key ingredient:

- The spectral theory of the Laplacian $\triangle$
- The modular operator Δ of states on operator algebras

The second is part of the scene because of the non-unimodularity of the spectral triple, or in simpler terms because the state defining the volume form is no longer assumed to be a trace and hence inherits a modular operator Δ. While both the Laplacian and the modular operator will be denoted by the capital letter delta, we hope the distinction[1] between these two operators will remain clear throughout to the alert reader.

2. Preliminaries

Recall that for a classical Riemann surface Σ with metric g, to the Laplacian $\triangle_g = d^*d$, where d is the de-Rham differential operator acting on functions on the Riemann surface, one associates the zeta function

$$\zeta(s) = \sum \lambda_j^{-s}\,, \quad \mathrm{Re}(s) > 1\,,$$

[1]We use a slightly different typography for the two cases

where the summation is over the non-zero eigenvalues λ_j of $\triangle_g$ (see, for example, [12]). The meromorphic continuation of $\zeta(s)$ to $s = 0$, where it has no pole, gives the important information

$$\zeta(0) + \mathrm{Card}\{j \mid \lambda_j = 0\} = \frac{1}{12\pi}\int_\Sigma R = \frac{1}{6}\,\chi(\Sigma)\,,$$

where R is the scalar curvature and $\chi(\Sigma)$ the Euler-Poincaré characteristic. This vanishes when Σ is the classical 2-torus $\mathbb{R}^2/\mathbb{Z}^2$, for example, and is an invariant within the conformal class of the metric, that is under the transformation $g \to e^f g$ for f a smooth real valued function on Σ.

2.1. The noncommutative two torus $\mathbb{T}^2_\theta$.

Fix a real irrational number θ. We consider the irrational rotation C^*-algebra A_θ, with two unitary generators which satisfy

$$VU = e^{2\pi i\theta}\,UV\,,\quad U^* = U^{-1}\,,\; V^* = V^{-1}\,.$$

We introduce the dynamical system given by the action of $\mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ on A_θ by the 2-parameter group of automorphisms $\{\alpha_s\}$, $s \in \mathbb{R}^2$ determined by

$$\alpha_s(U^n\,V^m) = e^{is.(n,m)}U^n\,V^m\,,\quad (s \in \mathbb{R}^2)\,,$$

We define the sub-algebra A_θ^∞ of smooth elements of A_θ to be those x in A_θ such that the mapping

$$\mathbb{R}^2 \to A_\theta,\quad s \mapsto \alpha_s(x)$$

is smooth. Expressed as a condition on the coefficients of the element $a \in A_\theta$,

$$a = \sum a(n,m)U^nV^m$$

this is the same as saying that they be of rapid decay, namely that $\{|n|^k\,|m|^q\,|a(n,m)|\}$ be bounded for any positive k, q. The derivations associated to the above group of automorphisms are given by the defining relations,

$$\delta_1(U) = U\,,\quad \delta_1(V) = 0\,,$$
$$\delta_2(U) = 0\,,\quad \delta_2(V) = V\,.$$

The derivations δ_1, δ_2 are analogues of the differential operators $\frac{1}{i}\partial/\partial x$, $\frac{1}{i}\partial/\partial y$ on the smooth functions on $\mathbb{R}^2/2\pi\mathbb{Z}^2$.

2.2. Conformal structure on $\mathbb{T}^2_\theta$.

As θ is supposed irrational, there is a unique trace τ on A_θ determined by the orthogonality properties

$$\tau(U^n\,V^m) = 0 \quad \text{if} \quad (n,m) \neq (0,0)\,,\quad \text{and} \quad \tau(1) = 1\,. \tag{1}$$

We can construct a Hilbert space $\mathcal{H}_0$ from A_θ by completing with respect to the inner product

$$\langle a, b\rangle = \tau(b^*a)\,,\quad a, b \in A_\theta\,, \tag{2}$$

and, using the derivations δ_1, δ_2, we introduce a complex structure by defining

$$\partial = \delta_1 + i\delta_2\,,\quad \partial^* = \delta_1 - i\delta_2 \tag{3}$$

where (extending ∂, ∂^* to unbounded operators on $\mathcal{H}_0$) ∂^* is the adjoint of ∂ with respect to the inner product defined by τ. As an appropriate analogue of the space of $(1,0)$-forms on the classical 2-torus, one takes the unitary bi-module $\mathcal{H}^{(1,0)}$ over

A_θ^∞ given by the Hilbert space completion of the space of finite sums $\sum a\ \partial b$, $a, b \in A_\theta^\infty$, with respect to the inner product

$$\langle a\partial b, a'\partial b'\rangle = \tau((a')^*\, a(\partial b)(\partial b')^*)\,, \quad a, a', b, b' \in A_\theta^\infty\,. \tag{4}$$

The information on the conformal structure is encoded by the positive Hochschild two cocycle (*cf.* [8] [9]) on A_θ^∞ given by

$$\psi(a, b, c) = -\tau(a\partial b\partial^* c) \tag{5}$$

2.3. Modular automorphism Δ.

In order to vary inside the conformal class of a metric we consider the family of positive linear functionals $\varphi = \varphi_h$, parameterized by $h = h^* \in A_\theta^\infty$, a self-adjoint element of A_θ^∞, and defined on A_θ by

$$\varphi(a) = \tau(ae^{-h})\,, \quad a \in A_\theta\,.$$

Note that, whereas for τ we have the trace relation

$$\tau(b^* a) = \tau(ab^*)\,, \quad a, b \in A_\theta\,,$$

for φ we have

$$\varphi(ab) = \varphi(be^{-h}ae^h) = \varphi(b\sigma_i\,(a))\,, \quad a \in A_\theta\,,$$

which is the KMS condition at $\beta = 1$ for the 1-parameter group σ_t, $t \in \mathbb{R}$, of inner automorphisms

$$\sigma_t(x) = e^{ith}xe^{-ith}.$$

This group is $\sigma_t = \Delta^{-it}$ where the modular operator is

$$\Delta(x) = e^{-h}xe^h$$

which is a positive operator fulfilling

$$\langle \Delta^{1/2}x, \Delta^{1/2}x\rangle_\varphi = \langle x^*, x^*\rangle_\varphi\,,\ \forall x \in A_\theta. \tag{6}$$

where we define the inner product $\langle\ ,\ \rangle_\varphi$ on A_θ by

$$\langle a, b\rangle_\varphi = \varphi(b^* a)\,, \quad a, b \in A_\theta\,.$$

We let $\mathcal{H}_\varphi$ be the Hilbert space completion of A_θ for the inner product $\langle\ ,\ \rangle_\varphi$. It is a unitary left module on A_θ by construction. The 1-parameter group σ_t is generated by the derivation $-\log\Delta$

$$-\log\Delta(x) = [h, x]\,, \quad x \in A_\theta^\infty\,.$$

2.4. Laplacian and Weyl factor on $\mathbb{T}_\theta^2$.

The Laplacian $\triangle$ on functions on $\mathbb{T}_\theta^2$ is given by

$$\triangle = \partial^*\partial = \delta_1^2 + \delta_2^2\,, \tag{7}$$

where ∂ is viewed as an unbounded operator from $\mathcal{H}_0$ to $\mathcal{H}^{(1,0)}$.
When one modifies the volume form on $\mathbb{T}_\theta^2$ by replacing the trace τ by the functional φ the modified Laplacian $\triangle'$ is given by

$$\triangle' = \partial_\varphi^*\partial_\varphi \tag{8}$$

where ∂_φ is the same operator ∂ but viewed as an unbounded operator from $\mathcal{H}_\varphi$ to $\mathcal{H}^{(1,0)}$. By construction $\triangle'$ is a positive unbounded operator in $\mathcal{H}_\varphi$.

Lemma 2.1. *The operator $\triangle'$ is anti-unitarily equivalent to the positive unbounded operator $k\triangle k$ in the Hilbert space $\mathcal{H}_0$, where $k = e^{h/2} \in A_\theta$ acts in $\mathcal{H}_0$ by left multiplication.*

Proof. The right multiplication by k extends to an isometry W from $\mathcal{H}_0$ to $\mathcal{H}_\varphi$,

$$Wa = ak\,,\ \forall a \in A_\theta \tag{9}$$

since

$$\langle Wa, Wb\rangle_\varphi = \tau((bk)^*(ak)k^{-2}) = \tau(b^*a)\,,\ \forall a, b \in A_\theta.$$

The operator $\partial_\varphi \circ W$ from $\mathcal{H}_0$ to $\mathcal{H}^{(1,0)}$ is given by

$$\partial_\varphi \circ W(a) = \partial(ak) = \partial \circ R_k$$

where R_k is the right multiplication by k. Thus $\triangle' = \partial_\varphi^*\partial_\varphi$ is unitarily equivalent to $(\partial \circ R_k)^*\partial \circ R_k = R_k\partial^*\partial R_k$. Now, let J be the anti-unitary involution on $\mathcal{H}_0$ given by the star operation $Ja = a^*$ for all $a \in A_\theta$. The operator J commutes with $\triangle$ and fulfills $JR_kJ = k$ using $k^* = k$. This gives the required equivalence. □

It is the dependence on k in the computations of the behavior of the zeta function of $k\triangle k$ at the origin that will feature in what follows.

2.5. **Spectral Triples.**

With the notations of the previous section, consider the Hilbert space and operator

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}\,, \quad D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix}. \tag{10}$$

We recall that $\mathcal{H}^{(1,0)}$ is naturally a unitary bimodule over A_θ.

Lemma 2.2. *1) The left action of A_θ in $\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}$ and the operator D yield an even spectral triple $(A_\theta, \mathcal{H}, D)$.*
2) Let J_φ be the Tomita antilinear unitary in $\mathcal{H}_\varphi$ and $a \mapsto J_\varphi a^* J_\varphi$ the corresponding unitary right action of A_θ in $\mathcal{H}_\varphi$. Then the right action $a \mapsto a^{\rm op}$ of A_θ in $\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}$ and the operator D yield an even twisted spectral triple $(A_\theta^{\rm op}, \mathcal{H}, D)$, i.e. the following operators are bounded

$$D\,a^{\rm op} - (k^{-1}ak)^{\rm op}D\,,\ \forall a \in A_\theta. \tag{11}$$

3) The zeta function of the operator D i.e. $\zeta_D(s) = \mathrm{Trace}(|D|^{-s})$ is equal to $2\mathrm{Trace}((k\triangle k)^{-s/2})$.

Proof. 1) In order to show that $[D, a]$ is bounded, it is enough to check that $[\partial_\varphi, a]$ is bounded which follows from the derivation property of ∂_φ and the equivalence of the norms $\|.\|_\varphi$ and $\|.\|_0$.
2) Let us first show that the twisted commutator $\partial_\varphi\, a^{\rm op} - (k^{-1}ak)^{\rm op}\,\partial_\varphi$ is bounded or equivalently that

$$\partial_\varphi\,(kak^{-1})^{\rm op} - a^{\rm op}\,\partial_\varphi \tag{12}$$

is bounded as an operator from $\mathcal{H}_\varphi$ to $\mathcal{H}^{(1,0)}$. Using the isometry W from $\mathcal{H}_0$ to $\mathcal{H}_\varphi$ defined in (9), replaces ∂_φ by $\partial \circ R_k$ and replaces $a^{\rm op}$ by the right multiplication R_a by a for any $a \in A_\theta$. Thus it replaces the operator (12) by

$$\partial \circ R_k \circ R_{kak^{-1}} - R_a \circ \partial \circ R_k = R_{\partial a} \circ R_k$$

which is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}^{(1,0)}$. Thus (12) is bounded and so is its adjoint

$$((kak^{-1})^{\rm op})^*\partial_\varphi^* - \partial_\varphi^*(a^{\rm op})^*.$$

Thus the boundedness of (11) follows from the equality

$$((kak^{-1})^{\rm op})^* = ((kak^{-1})^*)^{\rm op} = (k^{-1}a^*k)^{\rm op}$$

so that $\partial_\varphi^*\, a^{\rm op} - (k^{-1}ak)^{\rm op}\, \partial_\varphi^*$ is bounded for all $a \in A_\theta$.
3) We have seen in Lemma 2.1 that the spectrum of $\triangle' = \partial_\varphi^*\partial_\varphi$ is the same as that of $k\triangle k$. Since the non-zero part of the spectrum of a product A^*A is the same as the non-zero part of the spectrum of AA^*, using the unitary equivalence given by the polar decomposition, one gets the required result. □

3. Statement of the theorem

With the notation of §1, we study the Laplacian zeta function defined for $\mathrm{Re}(s) > 1$ by the Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Trace}^+(e^{-t\triangle'})\, t^{s-1}\, dt = \mathrm{Trace}(\triangle'^{-s})\,.$$

Here, $\triangle' \sim k \triangle k$, and

$$\mathrm{Trace}^+(e^{-t\triangle'}) = \mathrm{Trace}\,(e^{-t\triangle'}) - \mathrm{Dim\,Ker}(\triangle')\,,$$

where by $\mathrm{Trace}(\cdot)$ we understand the ordinary trace of the operator. The definition of $\zeta(s)$ can be extended by meromorphic continuation to all values of s, barring $s = 1$ where the function has a simple pole. We now state the main result.

Theorem 3.1. – *Let θ be an irrational number and k an invertible positive element of A_θ^∞. Then the value at the origin of the zeta function $\zeta(s)$ of the operator $\triangle' \sim k \triangle k$ is independent of k.*

Our proof relies on a long explicit computation whose main ingredient is the following lemma:

3.1. Main technical Lemma.

Lemma 3.2. – *Let θ be an irrational number and k an invertible positive element of A_θ^∞. Then the value at the origin of the zeta function $\zeta(s)$ of the operator $\triangle' \sim k \triangle k$ is given by*[2]

$$\zeta(0) + 1 = 2\pi\, \varphi(f(\Delta)(\delta_j(k))\delta_j(k)) \tag{13}$$

where φ is the functional $\varphi(x) = \tau(xk^{-2})$, Δ is the modular operator of φ and the function $f(u)$ is given by

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1 + u^{1/2})\mathcal{L}_2(u) + (1 + u^{1/2})^2\, \mathcal{L}_3(u) \tag{14}$$

where $\mathcal{L}_m$, m a positive integer, stands for the modified logarithm

$$\mathcal{L}_m(u) = (-1)^m\, (u-1)^{-(m+1)} \left(\log u - \sum_{j=1}^{m} (-1)^{j+1} \frac{(u-1)^j}{j} \right) .$$

[2]summation of repeated indices over $j = 1, 2$ is understood

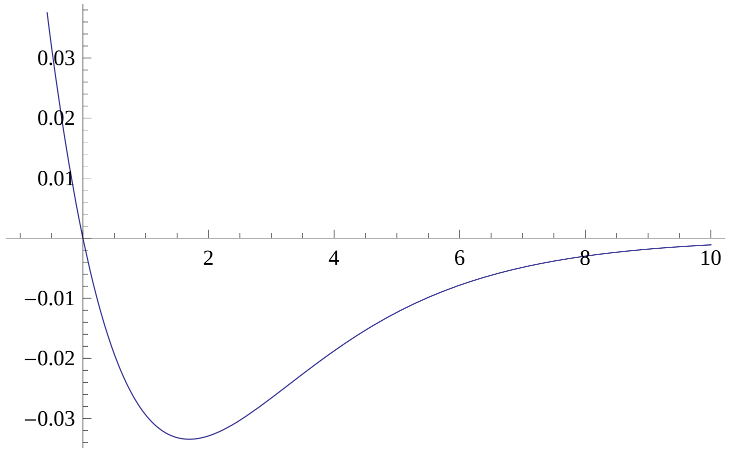

FIGURE 1. Graph of the function h.

The proof of this lemma will occupy the remaining sections of the paper. In the present section, we shall show how it implies Theorem 3.1. The statement of Lemma 3.2 is the same[3] as that of the main Theorem in [5], where the right hand side is written as $\tau(h(\theta,k))$ where

$$h(\theta,k)=\frac{\pi}{3}k^{-1}\delta_j^2(k)-\frac{2\pi}{3}k^{-1}\delta_j(k)\delta_j(k)k^{-1}+2\pi\mathcal{D}_1(k^{-1}\delta_j(k))\delta_j(k)k^{-1} \quad (15)$$

$$-4\pi\mathcal{D}_2(1+\Delta^{1/2})(k^{-1}\delta_j(k))\delta_j(k)k^{-1}+2\pi\mathcal{D}_3(1+2\Delta^{1/2}+\Delta)(k^{-1}\delta_j(k))\delta_j(k)k^{-1}$$

where $\mathcal{D}_n=\mathcal{L}_n(\Delta)$. One has indeed

$$\tau(k^{-1}\delta_j^2(k))=-\tau(\delta_j(k^{-1})\delta_j(k))=\tau(k^{-1}\delta_j(k)k^{-1}\delta_j(k))=\tau(k^{-2}\Delta^{-1/2}(\delta_j(k))\delta_j(k))$$

which accounts for the first term in the right hand side of (14), while the other terms are easily compared since the left multiplication by k^{-1} commutes with any function of Δ.

3.2. **Result in terms of** $\log(k)$**.**

The function $f(u)$ is of the form $h(\log(u))$ where h is the entire function

$$h(x)=-\frac{e^{-x/2}\left(-1+3e^{x/2}+3e^{x}+6e^{3x/2}x-3e^{2x}-3e^{5x/2}+e^{3x}\right)}{6\left(-1+e^{x/2}\right)^4\left(1+e^{x/2}\right)^2}. \quad (16)$$

Thus one can rewrite (13) in the form

$$\zeta(0)+1=2\pi\,\varphi(h(\log\Delta)(\delta_j(k))\delta_j(k)) \quad (17)$$

As pointed out in §1, in the commutative case the corresponding value for the zeta function at the origin is zero. In fact in that case one has $\log\Delta=0$. The function h vanishes at 0 and its Taylor expansion there is

$$h(x)=-\frac{x}{20}+\frac{x^2}{40}-\frac{x^3}{210}+\frac{x^4}{3360}+\frac{x^5}{201600}+O[x]^6.$$

[3] up to adding 1 to $\zeta(0)$

In fact, one obtains a further simplification in general by expressing the result in terms of the element $\psi = \log k$. We introduce the function

$$K(x) = -\frac{x - \mathrm{sh}\left[\frac{x}{2}\right] - \mathrm{sh}[x] + \frac{1}{3}\mathrm{sh}\left[\frac{3x}{2}\right]}{x^2 \mathrm{sh}\left[\frac{x}{2}\right]^2}. \tag{18}$$

Lemma 3.3. – *With the notations of Theorem 3.2, one has*

$$\zeta(0) + 1 = 2\pi\, \tau(K(\log\Delta)(\delta_j(\log k))\delta_j(\log k)). \tag{19}$$

Proof. One uses the following formula

$$k^{-1}\delta_j(k) = \int_0^1 \Delta^{s/2}(\delta_j(\log k))ds \tag{20}$$

which gives

$$k^{-1}\delta_j(k) = 2\,\frac{\Delta^{1/2} - 1}{\log\Delta}(\delta_j(\log k)) \tag{21}$$

and similarly

$$\delta_j(k)k^{-1} = -2\,\frac{\Delta^{-1/2} - 1}{\log\Delta}(\delta_j(\log k)).$$

One then rewrites (13) in the form

$$\zeta(0) + 1 = 2\pi\, \tau(f(\Delta)(k^{-1}\delta_j(k))\delta_j(k)k^{-1})$$

and one uses the following identity, where F is an entire function,

$$\tau(aF(\log\Delta)(b)) = \tau(F(-\log\Delta)(a)b)\,,\ \forall a, b \in A_\theta^\infty \tag{22}$$

which shows that (19) holds with

$$K(x) = 4\left(-1 + e^{x/2}\right)^2\, x^{-2}\, h(x)$$

which simplifies to (18). □

3.3. **Proof of Theorem 3.1.**

By (18) the function $K(x)$ is an odd function. Thus using (22) one has

$$\tau(K(\log\Delta)(\delta_j(\log k))\delta_j(\log k)) = -\tau(K(\log\Delta)(\delta_j(\log k))\delta_j(\log k)) = 0.$$

4. Pseudo-differential calculus

With the notation of the preceding sections, we introduce in the present one the notion of a pseudo-differential operator given the dynamical system $(A_\theta^\infty, \alpha_s)$ as developed in [1] and [7]. First of all, for a non-negative integer n, we define the vector space of differential operators of order at most n to be those polynomial expressions in δ_1, δ_2 of the form

$$P(\delta_1, \delta_2) = \sum_{|j|\le n} a_j\, \delta_1^{j_1}\, \delta_2^{j_2}\,, \quad a_j \in A_\theta^\infty\,, \quad j = (j_1, j_2) \in \mathbb{Z}^2_{\ge 0}\,, \quad |j| = j_1 + j_2\,.$$

To extend this definition, let $\mathbb{R}^2$ be the group dual to $\mathbb{R}^2$ and introduce the class of operator valued distributions given by those complex linear functions $P : C^\infty(\mathbb{R}^2) \to A_\theta^\infty$ which are continuous with respect to the semi-norms p_{i_1,i_2} determined by

$$p_{i_1,i_2}(P(\varphi)) = \|\delta_1^{i_1}\, \delta_2^{i_2}(P(\varphi))\|\,, \quad i_1, i_2 \in \mathbb{Z}_{\ge 0}\,, \quad \varphi \in C^\infty(\mathbb{R}^2)\,.$$

We use the notation $y_1 = e^{2\pi i\xi_1}$, $y_2 = e^{2\pi i\xi_2}$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, for the canonical coordinates of $\mathbb{R}^2$, and $\partial_1 = \partial/\partial\xi_1$, $\partial_2 = \partial/\partial\xi_2$ for the corresponding derivations.

We may now introduce the algebra of pseudo-differential operators via the algebra of operator valued symbols.

Definition 4.1. – *An element $\rho = \rho(\xi) = \rho(\xi_1, \xi_2)$ of $C^\infty(\mathbb{R}^2, A_\theta^\infty)$ is a symbol of order the integer n if and only if for all non-negative integers i_1, i_2, j_1, j_2*

$$p_{i_1,i_2}(\partial_1^{j_1}\,\partial_2^{j_2}\,\rho(\xi)) \leq c\,(1+|\xi|)^{n-|j|}\,,$$

where c is a constant depending only on ρ, and if there exists an element $k = k(\xi_1, \xi_2)$ of $C^\infty(\mathbb{R}^2 - \{0,0\}, A_\theta^\infty)$ such that

$$\lim_{\lambda\to\infty} \lambda^{-n}\,\rho(\lambda\,\xi_1, \lambda\,\xi_2) = k(\xi_1, \xi_2)\,.$$

We denote the space of symbols of order n by S_n, the union $S = \cup_{n\in\mathbb{Z}}\, S_n$ forming an algebra. Symbols of non-integral order are not required for this paper. An example of a symbol of order n a positive integer is provided by the polynomial $\rho(\xi) = \sum_{|j|\leq n} a_j\,\xi_1^{j_1}\,\xi_2^{j_2}$, $a_j \in A_\theta^\infty$, and one has $\rho(n,m) = \sum_{|j|\leq n} a_j\,n^{j_1}\,m^{j_2}$ so that $\rho(n,m)\,U^n\,V^m = \sum_{|j|\leq n} a_j\,\delta_1^{j_1}\,\delta_2^{j_2}(U^n\,V^m)$. For an element $a = \sum_{n,m} a(n,m)\,U^n\,V^m$ of A_θ^∞ one therefore has $\sum_{n,m}\rho(n,m)\,a(n,m)\,U^n\,V^n = \sum_{|j|\leq n} a_j\,\delta_1^{j_1}\,\delta_2^{j_2}(a)$, associating to the symbol ρ the differential operator $P_\rho = P(\delta_1, \delta_2) = \sum_{|j|\leq n} a_j\,\delta_1^{j_1}\,\delta_2^{j_2}$ on A_θ^∞.
For every integer n, a symbol ρ of that order determines an operator on A_θ^∞ via the map $\psi : \rho \mapsto P_\rho$ given by the general formula

$$P_\rho(a) = (2\pi)^{-2}\int\int e^{-is.\xi}\rho(\xi)\,\alpha_s(a)ds d\xi\,. \tag{23}$$

In our case this gives, using

$$\alpha_s(U^n\,V^m) = e^{is.(n,m)}U^n\,V^m$$

the simpler formula

$$P_\rho(a) = \sum_{n,m\in\mathbb{Z}} \rho(n,m)\,a(n,m)\,U^n\,V^m\,, \quad a = \sum_{n,m} a(n,m)\,U^n\,V^m\,. \tag{24}$$

For example, the image under ψ of the symbol $(1+|\xi|^2)^{-k}$, $k \geq 1$, of order $-2k$ acts on A_θ^∞.

Definition 4.2. – *The space ψ of pseudo-differential operators is given by the image of the algebra S under the map ψ.*

Definition 4.3. – *The equivalence $\rho \sim \rho'$ between two symbols ρ, ρ' in S_k, $k \in \mathbb{Z}$, holds if and only if $\rho - \rho'$ is a symbol of order n for all integers n.*

Definition 4.4. – *The class of pseudo-differential operators is the space ψ modulo addition by an element of $\psi(Z)$, where Z is the sub-algebra of S with elements equivalent to the zero symbol.*

It is possible to invert the map ψ to obtain for each element P of ψ a unique symbol $\sigma(P)$ up to equivalence. Recall from §1 that the trace τ on A_θ^∞ enables one to define the adjoint of operators acting on A_θ^∞ via their extension to $\mathcal{H}_0$. By direct analogy with [11], Chapter 1, Theorem, p. 16, one may deduce the following result.

Proposition 4.5. – *For an element P of ψ with symbol $\sigma(P) = \rho = \rho(\xi)$, the symbol of the adjoint P^* satisfies*

$$\sigma(P^*) \sim \sum_{(\ell_1,\ell_2)\in(\mathbb{Z}_{\geq 0})^2} (1/(\ell_1)!\,(\ell_2)!)\,[\partial_1^{\ell_1}\,\partial_2^{\ell_2}\,\delta_1^{\ell_1}\,\delta_2^{\ell_2}(\rho(\xi))^*]\,.$$

If Q is an element of ψ with symbol $\sigma(Q) = \rho' = \rho'(\xi)$, then the product PQ is also in ψ and has symbol

$$\sigma(PQ) \sim \sum_{(\ell_1,\ell_2)\in(\mathbb{Z}_{\geq 0})^2} (1/(\ell_1)!\,\ell_2)!)\,[\partial_1^{\ell_1}\,\partial_2^{\ell_2}(\rho(\xi))\,\delta_1^{\ell_1}\,\delta_2^{\ell_2}(\rho'(\xi))]\,.$$

Notice that in the above Proposition as throughout the present paper, given symbols $\{\rho_j\}_{j=0}^\infty$ the relation $\rho \sim \sum_{j=0}^\infty \rho_j$ signifies that for any given k there exists a positive integer h such that for all $n > h$, the difference $\rho - \sum_{j=0}^n \rho_j$ is in S_k. The elliptic pseudo-differential operators are those whose symbols fulfil the criterion which follows.

Definition 4.6. – *Let n be an integer and ρ a symbol of order n. Then $\rho = \rho(\xi)$ is elliptic if it is invertible within the algebra $C^\infty(\mathbb{R}^2, A_\theta^\infty)$ and if its inverse satisfies*

$$\|\rho(\xi)^{-1}\| \leq c(1 + |\xi|)^{-n}$$

for a constant c depending only on ρ and for $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$ sufficiently large.

An example of an elliptic operator is provided by the Laplacian $\triangle = \delta_1^2 + \delta_2^2$ on A_θ^∞ introduced in §1 which has the corresponding invertible symbol $\sigma(\triangle) = |\xi|^2$.

5. Understanding the computations

The arguments of this section are kept brief, being direct analogues of standard ones. Bearing in mind the definition of the zeta function given in §3, we observe that by Cauchy's formula we have

$$e^{-t\triangle'} = (1/2\pi i)\int_C e^{-t\lambda}(\triangle' - \lambda\mathbf{1})^{-1}\,d\lambda$$

where λ is a complex number but not real non-negative, and C encircles the non-negative real axis in the anti-clockwise direction without touching it. One then obtains a workable estimate of $(\triangle' - \lambda\mathbf{1})^{-1}$ by passing to the algebra of symbols with the important nuance that the scalar λ is now treated as a symbol of order two in all calculations. Using the definition of a symbol, one can replace the trace in the formula for the zeta function by an integration in the symbol space (argument along the diagonal), namely,

$$\zeta(s) = (1/\Gamma(s))\int_0^\infty\int \tau(\sigma(e^{-t\triangle'})(\xi))\,t^{s-1}\,d\xi\,dt\,.$$

The function $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 so that,

$$\zeta(0) = \mathrm{Res}_{s=0} \int_0^\infty \int \tau(\sigma(e^{-t\triangle'})(\xi))\, t^{s-1}\, d\xi\, dt\,.$$

Just as in the arguments employed in the derivation of the asymptotic formula (see for example [11]),

$$\int \tau(\sigma(e^{-t\triangle'})(\xi))\, d\xi \sim t^{-1} \sum_{n=0}^{\infty} B_{2n}(\triangle')\, t^n\,, \quad t \to 0_+\,,$$

one may appeal to the Cauchy formula quoted above. In particular, if B_λ denotes (a chosen approximation) to the inverse operator of $(\lambda\mathbf{1} - \triangle')$, its symbol has an expansion of the form

$$\sigma(B_\lambda) = \sigma(B_\lambda)(\xi) = b_0(\xi) + b_1(\xi) + b_2(\xi) + \dots$$

where j ranges over the non-negative integers and $b_j(\xi) = b_j(\xi, \lambda)$ is a symbol of order $-2 - j$. As we shall explain at more length in §6, these symbols may be calculated inductively using the symbol algebra formulae beginning with $b_0(\xi) = (\lambda - k^2|\xi|^2)^{-1}$ which is the principal (highest homogeneous degree in ξ) symbol of $(\lambda\mathbf{1} - \triangle')^{-1}$. It turns out that $\zeta(0)$ equals the coefficient of λ^{-1} in $\int \tau(b_2(\xi))\, d\xi$. By a homogeneity argument one has in fact

$$\zeta(0) = \int \tau(b_2(\xi))\, d\xi\,. \tag{25}$$

6. Computational proof of the main Lemma

Following on from the arguments of §5, by homogeneity there is no loss of generality in placing $\lambda = -1$ throughout the computation of $\zeta(0)$ and multiplying the final answer by -1. The problem is then to derive in the symbol algebra a recursive solution of the form $\sigma = b_0(\xi) + b_1(\xi) + b_2(\xi) + \dots$ to the equation

$$\sigma \cdot (\sigma(\triangle' - \lambda)) = 1 + 0(|\xi|^{-3})\,.$$

The accuracy to order -3 in ξ on the right hand side is in practice sufficient as we are only interested in evaluating σ up to $b_2(\xi)$. Throughout this section the convention of summation over repeated indices in the range $i, j = 1, 2$ is observed.

Lemma 6.1. – *The operator $\triangle'$ has symbol $\sigma(\triangle') = a_2(\xi) + a_1(\xi) + a_0(\xi)$ where, with summation over repeated indices in the range $i = 1, 2$, one has*

$$a_2 = a_2(\xi) = k^2 \xi_i\, \xi_i$$

$$a_1 = a_1(\xi) = 2\, \xi_i (k\, \delta_i(k))$$

$$a_0 = a_0(\xi) = k\, \delta_i\, \delta_i(k)\,.$$

These expressions are derived by applying the product formula within the algebra of symbols given in Proposition §4 to $\sigma_1(\xi) = \xi_i\, \xi_i$ and $\sigma_2(\xi) = k$ and then multiplying on the left by k.

To begin the inductive calculation of the inverse of the symbol of $\triangle' - \lambda$, set

$$b_0 = b_0(\xi) = (k^2\, |\xi|^2 + 1)^{-1} \tag{26}$$

and compute to order -3 in ξ the product $b_0 \cdot ((a_2+1)+a_1+a_0)$. By singling out terms of the appropriate degree -1 in ξ and using the Proposition of §4, one obtains

$$b_1 = -(b_0\, a_1\, b_0 + \partial_i(b_0)\, \delta_i(a_2)\, b_0)\,. \tag{27}$$

Note that b_0 appears on the right in this formula, unlike what is given in the formula of page 52 of [11] which is valid for scalar principal symbol only. In a similar fashion, collecting terms of degree -2 in ξ and using (27) one obtains

$$\begin{aligned} b_2 \quad &= \quad -(b_0\, a_0\, b_0 + b_1\, a_1\, b_0 + \partial_i(b_0)\, \delta_i(a_1)\, b_0 \\ &+ \quad \partial_i(b_1)\, \delta_i(a_2)\, b_0 + (1/2)\, \partial_i\, \partial_j(b_0)\, \delta_i\, \delta_j(a_2)\, b_0)\,. \end{aligned} \tag{28}$$

A direct computation of these terms (up to the last multiplication by b_0) gives, ignoring the terms which are odd in ξ,

$$\begin{gathered}
-b_0k\delta_1^2(k) - b_0k\delta_2^2(k) + \left(\xi_2^2+5\xi_1^2\right)\left(k^2b_0^2\right)k\delta_1^2(k) + \left(5\xi_2^2+\xi_1^2\right)\left(k^2b_0^2\right)k\delta_2^2(k) + \\
2\xi_2^2\left(k^2b_0^2\right)\delta_1(k)\delta_1(k) + 6\xi_1^2\left(k^2b_0^2\right)\delta_1(k)\delta_1(k) + \xi_2^2\left(k^2b_0^2\right)\delta_1^2(k)k + \xi_1^2\left(k^2b_0^2\right)\delta_1^2(k)k + \\
6\xi_2^2\left(k^2b_0^2\right)\delta_2(k)\delta_2(k) + 2\xi_1^2\left(k^2b_0^2\right)\delta_2(k)\delta_2(k) + \xi_2^2\left(k^2b_0^2\right)\delta_2^2(k)k + \xi_1^2\left(k^2b_0^2\right)\delta_2^2(k)k - \\
4\xi_2^2\xi_1^2\left(k^4b_0^3\right)k\delta_1^2(k) - 4\xi_1^4\left(k^4b_0^3\right)k\delta_1^2(k) - 4\xi_2^4\left(k^4b_0^3\right)k\delta_2^2(k) - 4\xi_2^2\xi_1^2\left(k^4b_0^3\right)k\delta_2^2(k) - \\
8\xi_2^2\xi_1^2\left(k^4b_0^3\right)\delta_1(k)\delta_1(k) - 8\xi_1^4\left(k^4b_0^3\right)\delta_1(k)\delta_1(k) - 4\xi_2^2\xi_1^2\left(k^4b_0^3\right)\delta_1^2(k)k - 4\xi_1^4\left(k^4b_0^3\right)\delta_1^2(k)k - \\
8\xi_2^4\left(k^4b_0^3\right)\delta_2(k)\delta_2(k) - 8\xi_2^2\xi_1^2\left(k^4b_0^3\right)\delta_2(k)\delta_2(k) - 4\xi_2^4\left(k^4b_0^3\right)\delta_2^2(k)k - 4\xi_2^2\xi_1^2\left(k^4b_0^3\right)\delta_2^2(k)k + \\
2\xi_2^2b_0k\delta_1(k)b_0k\delta_1(k) + 6\xi_1^2b_0k\delta_1(k)b_0k\delta_1(k) + 2\xi_2^2b_0k\delta_1(k)b_0\delta_1(k)k + 2\xi_1^2b_0k\delta_1(k)b_0\delta_1(k)k - \\
4\xi_2^2\xi_1^2b_0k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k) - 4\xi_1^4b_0k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k) - 4\xi_2^2\xi_1^2b_0k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k - \\
4\xi_1^4b_0k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k + 6\xi_2^2b_0k\delta_2(k)b_0k\delta_2(k) + 2\xi_1^2b_0k\delta_2(k)b_0k\delta_2(k) + \\
2\xi_2^2b_0k\delta_2(k)b_0\delta_2(k)k + 2\xi_1^2b_0k\delta_2(k)b_0\delta_2(k)k - 4\xi_2^4b_0k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k) - \\
4\xi_2^2\xi_1^2b_0k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k) - 4\xi_2^4b_0k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k - 4\xi_2^2\xi_1^2b_0k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k - \\
2\xi_2^4\left(k^2b_0^2\right)k\delta_1(k)b_0k\delta_1(k) - 16\xi_2^2\xi_1^2\left(k^2b_0^2\right)k\delta_1(k)b_0k\delta_1(k) - 14\xi_1^4\left(k^2b_0^2\right)k\delta_1(k)b_0k\delta_1(k) - \\
2\xi_2^4\left(k^2b_0^2\right)k\delta_1(k)b_0\delta_1(k)k - 12\xi_2^2\xi_1^2\left(k^2b_0^2\right)k\delta_1(k)b_0\delta_1(k)k - 10\xi_1^4\left(k^2b_0^2\right)k\delta_1(k)b_0\delta_1(k)k + \\
4\xi_2^4\xi_1^2\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k) + 8\xi_2^2\xi_1^4\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k) + \\
4\xi_1^6\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k) + 4\xi_2^4\xi_1^2\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k + \\
8\xi_2^2\xi_1^4\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k + 4\xi_1^6\left(k^2b_0^2\right)k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k - \\
14\xi_2^4\left(k^2b_0^2\right)k\delta_2(k)b_0k\delta_2(k) - 16\xi_2^2\xi_1^2\left(k^2b_0^2\right)k\delta_2(k)b_0k\delta_2(k) - 2\xi_1^4\left(k^2b_0^2\right)k\delta_2(k)b_0k\delta_2(k) - \\
10\xi_2^4\left(k^2b_0^2\right)k\delta_2(k)b_0\delta_2(k)k - 12\xi_2^2\xi_1^2\left(k^2b_0^2\right)k\delta_2(k)b_0\delta_2(k)k - 2\xi_1^4\left(k^2b_0^2\right)k\delta_2(k)b_0\delta_2(k)k + \\
4\xi_2^6\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k) + 8\xi_2^4\xi_1^2\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k) + \\
4\xi_2^2\xi_1^4\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k) + 4\xi_2^6\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k + \\
8\xi_2^4\xi_1^2\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k + 4\xi_2^2\xi_1^4\left(k^2b_0^2\right)k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k - \\
2\xi_2^4\left(k^2b_0^2\right)\delta_1(k)kb_0k\delta_1(k) - 12\xi_2^2\xi_1^2\left(k^2b_0^2\right)\delta_1(k)kb_0k\delta_1(k) - 10\xi_1^4\left(k^2b_0^2\right)\delta_1(k)kb_0k\delta_1(k) - \\
2\xi_2^4\left(k^2b_0^2\right)\delta_1(k)kb_0\delta_1(k)k - 8\xi_2^2\xi_1^2\left(k^2b_0^2\right)\delta_1(k)kb_0\delta_1(k)k - 6\xi_1^4\left(k^2b_0^2\right)\delta_1(k)kb_0\delta_1(k)k + \\
4\xi_2^4\xi_1^2\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)k\delta_1(k) + 8\xi_2^2\xi_1^4\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)k\delta_1(k) + \\
4\xi_1^6\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)k\delta_1(k) + 4\xi_2^4\xi_1^2\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)\delta_1(k)k + \\
8\xi_2^2\xi_1^4\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)\delta_1(k)k + 4\xi_1^6\left(k^2b_0^2\right)\delta_1(k)k\left(k^2b_0^2\right)\delta_1(k)k - \\
10\xi_2^4\left(k^2b_0^2\right)\delta_2(k)kb_0k\delta_2(k) - 12\xi_2^2\xi_1^2\left(k^2b_0^2\right)\delta_2(k)kb_0k\delta_2(k) - 2\xi_1^4\left(k^2b_0^2\right)\delta_2(k)kb_0k\delta_2(k) - \\
6\xi_2^4\left(k^2b_0^2\right)\delta_2(k)kb_0\delta_2(k)k - 8\xi_2^2\xi_1^2\left(k^2b_0^2\right)\delta_2(k)kb_0\delta_2(k)k - 2\xi_1^4\left(k^2b_0^2\right)\delta_2(k)kb_0\delta_2(k)k + \\
4\xi_2^6\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)k\delta_2(k) + 8\xi_2^4\xi_1^2\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)k\delta_2(k) + \\
4\xi_2^2\xi_1^4\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)k\delta_2(k) + 4\xi_2^6\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)\delta_2(k)k + \\
8\xi_2^4\xi_1^2\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)\delta_2(k)k + 4\xi_2^2\xi_1^4\left(k^2b_0^2\right)\delta_2(k)k\left(k^2b_0^2\right)\delta_2(k)k + \\
8\xi_2^4\xi_1^2\left(k^4b_0^3\right)k\delta_1(k)b_0k\delta_1(k) + 16\xi_2^2\xi_1^4\left(k^4b_0^3\right)k\delta_1(k)b_0k\delta_1(k) + 8\xi_1^6\left(k^4b_0^3\right)k\delta_1(k)b_0k\delta_1(k) + \\
8\xi_2^4\xi_1^2\left(k^4b_0^3\right)k\delta_1(k)b_0\delta_1(k)k + 16\xi_2^2\xi_1^4\left(k^4b_0^3\right)k\delta_1(k)b_0\delta_1(k)k + 8\xi_1^6\left(k^4b_0^3\right)k\delta_1(k)b_0\delta_1(k)k + \\
8\xi_2^6\left(k^4b_0^3\right)k\delta_2(k)b_0k\delta_2(k) + 16\xi_2^4\xi_1^2\left(k^4b_0^3\right)k\delta_2(k)b_0k\delta_2(k) + 8\xi_2^2\xi_1^4\left(k^4b_0^3\right)k\delta_2(k)b_0k\delta_2(k) + \\
8\xi_2^6\left(k^4b_0^3\right)k\delta_2(k)b_0\delta_2(k)k + 16\xi_2^4\xi_1^2\left(k^4b_0^3\right)k\delta_2(k)b_0\delta_2(k)k + 8\xi_2^2\xi_1^4\left(k^4b_0^3\right)k\delta_2(k)b_0\delta_2(k)k + \\
8\xi_2^4\xi_1^2\left(k^4b_0^3\right)\delta_1(k)kb_0k\delta_1(k) + 16\xi_2^2\xi_1^4\left(k^4b_0^3\right)\delta_1(k)kb_0k\delta_1(k) + 8\xi_1^6\left(k^4b_0^3\right)\delta_1(k)kb_0k\delta_1(k) +
\end{gathered}$$

$$8\xi_2^4\xi_1^2\left(k^4b_0^3\right)\delta_1(k)kb_0\delta_1(k)k+16\xi_2^2\xi_1^4\left(k^4b_0^3\right)\delta_1(k)kb_0\delta_1(k)k+8\xi_1^6\left(k^4b_0^3\right)\delta_1(k)kb_0\delta_1(k)k+$$
$$8\xi_2^6\left(k^4b_0^3\right)\delta_2(k)kb_0k\delta_2(k)+16\xi_2^4\xi_1^2\left(k^4b_0^3\right)\delta_2(k)kb_0k\delta_2(k)+8\xi_2^2\xi_1^4\left(k^4b_0^3\right)\delta_2(k)kb_0k\delta_2(k)+$$
$$8\xi_2^6\left(k^4b_0^3\right)\delta_2(k)kb_0\delta_2(k)k+16\xi_2^4\xi_1^2\left(k^4b_0^3\right)\delta_2(k)kb_0\delta_2(k)k+8\xi_2^2\xi_1^4\left(k^4b_0^3\right)\delta_2(k)kb_0\delta_2(k)k$$

It is extremely useful during the computation to exploit the fact that in the target formula for $\zeta(0)$ given in (25), §6, one invokes the trace, so that members of the factors of the individual summands may be permuted cyclically without loss of generality for the answer. In our case this means that instead of multiplying the above sum by b_0 on the right we can just multiply it on the left, thus simply adding one to the exponent of the first occurrence of b_0. By formula (25), one then has to sum the integrals of each of these terms over the whole ξ-plane. After multiplying by b_0 on the left, passing in polar coordinates and integrating the angular variable one gets, up to an overall factor of 2π,

$$-b_0^2k\delta_1^2(k)-b_0^2k\delta_2^2(k)+3r^2\left(k^2b_0^3\right)k\delta_1^2(k)+3r^2\left(k^2b_0^3\right)k\delta_2^2(k)+4r^2\left(k^2b_0^3\right)\delta_1(k)\delta_1(k)+$$
$$r^2\left(k^2b_0^3\right)\delta_1^2(k)k+4r^2\left(k^2b_0^3\right)\delta_2(k)\delta_2(k)+r^2\left(k^2b_0^3\right)\delta_2^2(k)k-2r^4\left(k^4b_0^4\right)k\delta_1^2(k)-$$
$$2r^4\left(k^4b_0^4\right)k\delta_2^2(k)-4r^4\left(k^4b_0^4\right)\delta_1(k)\delta_1(k)-2r^4\left(k^4b_0^4\right)\delta_1^2(k)k-4r^4\left(k^4b_0^4\right)\delta_2(k)\delta_2(k)-$$
$$2r^4\left(k^4b_0^4\right)\delta_2^2(k)k+4r^2b_0^2k\delta_1(k)b_0k\delta_1(k)+2r^2b_0^2k\delta_1(k)b_0\delta_1(k)k-$$
$$2r^4b_0^2k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k)-2r^4b_0^2k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k+4r^2b_0^2k\delta_2(k)b_0k\delta_2(k)+$$
$$2r^2b_0^2k\delta_2(k)b_0\delta_2(k)k-2r^4b_0^2k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k)-2r^4b_0^2k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k-$$
$$8r^4\left(k^2b_0^3\right)k\delta_1(k)b_0k\delta_1(k)-6r^4\left(k^2b_0^3\right)k\delta_1(k)b_0\delta_1(k)k+2r^6\left(k^2b_0^3\right)k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k)+$$
$$2r^6\left(k^2b_0^3\right)k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k-8r^4\left(k^2b_0^3\right)k\delta_2(k)b_0k\delta_2(k)-6r^4\left(k^2b_0^3\right)k\delta_2(k)b_0\delta_2(k)k+$$
$$2r^6\left(k^2b_0^3\right)k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k)+2r^6\left(k^2b_0^3\right)k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k-$$
$$6r^4\left(k^2b_0^3\right)\delta_1(k)kb_0k\delta_1(k)-4r^4\left(k^2b_0^3\right)\delta_1(k)kb_0\delta_1(k)k+2r^6\left(k^2b_0^3\right)\delta_1(k)k\left(k^2b_0^2\right)k\delta_1(k)+$$
$$2r^6\left(k^2b_0^3\right)\delta_1(k)k\left(k^2b_0^2\right)\delta_1(k)k-6r^4\left(k^2b_0^3\right)\delta_2(k)kb_0k\delta_2(k)-4r^4\left(k^2b_0^3\right)\delta_2(k)kb_0\delta_2(k)k+$$
$$2r^6\left(k^2b_0^3\right)\delta_2(k)k\left(k^2b_0^2\right)k\delta_2(k)+2r^6\left(k^2b_0^3\right)\delta_2(k)k\left(k^2b_0^2\right)\delta_2(k)k+$$
$$4r^6\left(k^4b_0^4\right)k\delta_1(k)b_0k\delta_1(k)+4r^6\left(k^4b_0^4\right)k\delta_1(k)b_0\delta_1(k)k+4r^6\left(k^4b_0^4\right)k\delta_2(k)b_0k\delta_2(k)+$$
$$4r^6\left(k^4b_0^4\right)k\delta_2(k)b_0\delta_2(k)k+4r^6\left(k^4b_0^4\right)\delta_1(k)kb_0k\delta_1(k)+4r^6\left(k^4b_0^4\right)\delta_1(k)kb_0\delta_1(k)k+$$
$$4r^6\left(k^4b_0^4\right)\delta_2(k)kb_0k\delta_2(k)+4r^6\left(k^4b_0^4\right)\delta_2(k)kb_0\delta_2(k)k$$

where $b_0=(r^2k^2-\lambda)^{-1}$ and where the integration is in rdr and from 0 to ∞.

6.1. **Terms with all b_0 on the left.** Using the trace property these terms give the following:

$$-b_0^2k\delta_1^2(k)-b_0^2k\delta_2^2(k)+r^2(3k^2b_0^3k\delta_1^2(k)-2k^4r^2b_0^4k\delta_1^2(k)+3k^2b_0^3k\delta_2^2(k)-$$
$$2k^4r^2b_0^4k\delta_2^2(k)+4k^2b_0^3\delta_1(k)\delta_1(k)-4k^4r^2b_0^4\delta_1(k)\delta_1(k)+k^2b_0^3\delta_1^2(k)k-$$
$$2k^4r^2b_0^4\delta_1^2(k)k+4k^2b_0^3\delta_2(k)\delta_2(k)-4k^4r^2b_0^4\delta_2(k)\delta_2(k)+k^2b_0^3\delta_2^2(k)k-2k^4r^2b_0^4\delta_2^2(k)k)$$

which gives the same result as

$$\left(4k^2r^2b_0^3-4k^4r^4b_0^4\right)\left(\delta_1(k)^2+\delta_2(k)^2\right)+$$
$$\left(-kb_0^2+4k^3r^2b_0^3-4k^5r^4b_0^4\right)\left(\delta_1^2(k)+\delta_2^2(k)\right)$$

and the coefficient of $1/\lambda$ in the integral $\int\bullet\, rdr$ gives, up to an overall coefficient of 2π,

$$-\frac{1}{3}k^{-2}\left(\delta_1(k)^2+\delta_2(k)^2\right)+\frac{1}{6}k^{-1}\left(\delta_1^2(k)+\delta_2^2(k)\right)\tag{29}$$

which corresponds to the first two terms of the formula for $h(\theta,k)$ in the statement of the Lemma in the form (15).

6.2. **Terms with b_0^2 in the middle.** They are the following

$$-2r^4b_0^2k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k)-2r^4b_0^2k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k-$$
$$2r^4b_0^2k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k)-2r^4b_0^2k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k+$$
$$2r^6\left(k^2b_0^3\right)k\delta_1(k)\left(k^2b_0^2\right)k\delta_1(k)+2r^6\left(k^2b_0^3\right)k\delta_1(k)\left(k^2b_0^2\right)\delta_1(k)k+$$
$$2r^6\left(k^2b_0^3\right)k\delta_2(k)\left(k^2b_0^2\right)k\delta_2(k)+2r^6\left(k^2b_0^3\right)k\delta_2(k)\left(k^2b_0^2\right)\delta_2(k)k+$$
$$2r^6\left(k^2b_0^3\right)\delta_1(k)k\left(k^2b_0^2\right)k\delta_1(k)+2r^6\left(k^2b_0^3\right)\delta_1(k)k\left(k^2b_0^2\right)\delta_1(k)k+$$
$$2r^6\left(k^2b_0^3\right)\delta_2(k)k\left(k^2b_0^2\right)k\delta_2(k)+2r^6\left(k^2b_0^3\right)\delta_2(k)k\left(k^2b_0^2\right)\delta_2(k)k$$

One has

$$\partial_r(b_0)=-2k^2rb_0^2$$

and one can use integration by parts in r to transform terms such as

$$\int_0^\infty r^6(k^2b_0^3)\delta_1(k)k(k^2b_0^2)\delta_1(k)krdr$$

and replace them by

$$\frac{1}{2}\int_0^\infty \partial_r\left(r^6(k^2b_0^3)\right)\delta_1(k)kb_0\delta_1(k)kdr$$

where now b_0 only appears at the first power on the second occurrence.
After performing this operation, and combining with those which have b_0 in the middle, namely

$$4r^2b_0^2k\delta_1(k)b_0k\delta_1(k)+2r^2b_0^2k\delta_1(k)b_0\delta_1(k)k+4r^2b_0^2k\delta_2(k)b_0k\delta_2(k)+$$
$$2r^2b_0^2k\delta_2(k)b_0\delta_2(k)k-8r^4\left(k^2b_0^3\right)k\delta_1(k)b_0k\delta_1(k)-6r^4\left(k^2b_0^3\right)k\delta_1(k)b_0\delta_1(k)k-$$
$$8r^4\left(k^2b_0^3\right)k\delta_2(k)b_0k\delta_2(k)-6r^4\left(k^2b_0^3\right)k\delta_2(k)b_0\delta_2(k)k-6r^4\left(k^2b_0^3\right)\delta_1(k)kb_0k\delta_1(k)-$$
$$4r^4\left(k^2b_0^3\right)\delta_1(k)kb_0\delta_1(k)k-6r^4\left(k^2b_0^3\right)\delta_2(k)kb_0k\delta_2(k)-4r^4\left(k^2b_0^3\right)\delta_2(k)kb_0\delta_2(k)k+$$
$$4r^6\left(k^4b_0^4\right)k\delta_1(k)b_0k\delta_1(k)+4r^6\left(k^4b_0^4\right)k\delta_1(k)b_0\delta_1(k)k+4r^6\left(k^4b_0^4\right)k\delta_2(k)b_0k\delta_2(k)+$$
$$4r^6\left(k^4b_0^4\right)k\delta_2(k)b_0\delta_2(k)k+4r^6\left(k^4b_0^4\right)\delta_1(k)kb_0k\delta_1(k)+$$
$$4r^6\left(k^4b_0^4\right)\delta_1(k)kb_0\delta_1(k)k+4r^6\left(k^4b_0^4\right)\delta_2(k)kb_0k\delta_2(k)+4r^6\left(k^4b_0^4\right)\delta_2(k)kb_0\delta_2(k)k$$

one obtains the following terms

$$T=-2\left(r^2b_0^2\right)\left(k\delta_1(k)b_0\delta_1(k)k+k\delta_2(k)b_0\delta_2(k)k\right)+2\left(k^2r^4b_0^3\right)\left(k\delta_1(k)b_0k\delta_1(k)+\right.$$
$$2k\delta_1(k)b_0\delta_1(k)k+k\delta_2(k)b_0k\delta_2(k)+2k\delta_2(k)b_0\delta_2(k)k+\delta_1(k)kb_0\delta_1(k)k+$$
$$\left.\delta_2(k)kb_0\delta_2(k)k\right)-2\left(k^4r^6b_0^4\right)\left(k\delta_1(k)b_0k\delta_1(k)+k\delta_1(k)b_0\delta_1(k)k+k\delta_2(k)b_0k\delta_2(k)+\right.$$
$$\left.k\delta_2(k)b_0\delta_2(k)k+\delta_1(k)kb_0k\delta_1(k)+\delta_1(k)kb_0\delta_1(k)k+\delta_2(k)kb_0k\delta_2(k)+\delta_2(k)kb_0\delta_2(k)k\right)$$

which all have b_0 in the middle.

6.3. **Terms with b_0 in the middle.** Since we are in the non-commutative case, when in particular k and $\delta_i(k)$, $i=1,2$, do not commute, the computation of such terms requires us to permute k with elements of A_θ^∞. This is achieved using the following lemma.

Lemma 6.2. – *For every element ρ of A_θ^∞ and every non-negative integer m one has,*

$$\int_0^\infty \frac{k^{2m+2}\,u^m}{(k^2\,u+1)^{m+1}}\,\rho\,\frac{1}{(k^2\,u+1)}\,du=\mathcal{D}_m(\rho)\,, \tag{30}$$

where the modified logarithm function is $\mathcal{D}_m = \mathcal{L}_m(\Delta)$, Δ *is the operator introduced in §1, and*

$$\mathcal{L}_m(u) = \int_0^\infty \frac{x^m}{(x+1)^{m+1}} \frac{1}{(xu+1)}\, dx\,. \tag{31}$$

Proof. On effecting the change of variables $u = e^s$ one obtains, with $k = e^{f/2}$,

$$\begin{aligned}
&\int_0^\infty \frac{k^{2m+2}\, u^m}{(k^2\, u+1)^{m+1}}\, \rho\, \frac{1}{(k^2\, u+1)}\, du \\
=\ &\int_{-\infty}^\infty \frac{e^{(m+1)f+ms}}{(e^{(s+f)}+1)^{m+1}}\, \rho\, \frac{e^s}{(e^{(s+f)}+1)}\, ds \\
=\ &\int_{-\infty}^\infty \frac{e^{(m+1/2)(s+f)}}{(e^{(s+f)}+1)^{m+1}}\, \Delta^{-1/2}(\rho)\, \frac{e^{(s+f)/2}}{(e^{(s+f)}+1)}\, ds \\
=\ &\int_{-\infty}^\infty \frac{e^{(m+1/2)(s+f)}}{(e^{(s+f)}+1)^{m+1}}\, \Delta^{-1/2}(\rho) \int_{-\infty}^\infty \frac{e^{it(s+f)}}{e^{\pi t}+e^{-\pi t}}\, dt\, ds \\
=\ &\int_{-\infty}^\infty \frac{e^{(m+1/2)(s+f)}}{(e^{(s+f)}+1)^{m+1}} \int_{-\infty}^\infty \frac{e^{it(s+f)}}{e^{\pi t}+e^{-\pi t}}\, \Delta^{-1/2+it}(\rho)\, dt\, ds \\
=\ &\int_{-\infty}^\infty \frac{e^{itf}}{e^{\pi t}+e^{-\pi t}} \left(\int_{-\infty}^\infty \frac{e^{(m+1/2)(s+f)} e^{its}}{(e^{(s+f)}+1)^{m+1}}\, ds \right) \Delta^{-1/2+it}(\rho)\, dt
\end{aligned}$$

Now one has

$$\int_{-\infty}^\infty \frac{e^{(m+1/2)(s+f)} e^{its}}{(e^{(s+f)}+1)^{m+1}}\, ds = e^{-itf} F_m(t)$$

where F_m is the Fourier transform of the function

$$h_m(s) = \frac{e^{(m+1/2)s}}{(e^s+1)^{m+1}}$$

We thus get

$$\begin{aligned}
&\int_0^\infty \frac{k^{2m+2}\, u^m}{(k^2\, u+1)^{m+1}}\, \rho\, \frac{1}{(k^2\, u+1)}\, du \\
=\ &\int_{-\infty}^\infty \frac{F_m(t)}{e^{\pi t}+e^{-\pi t}}\, \Delta^{-1/2+it}(\rho)\, dt\,.
\end{aligned}$$

Moreover one has

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{F_m(t)}{e^{\pi t}+e^{-\pi t}}\, u^{-it}\, dt \\
=\ &\int_0^\infty \frac{x^{(m+1/2)}}{(x+1)^{m+1}} \frac{x^{-1/2}u^{1/2}}{x^{-1}u+1}\, \frac{dx}{x} = u^{-1/2}\, \mathcal{L}_m(u^{-1})\,.
\end{aligned}$$

Replacing u by Δ^{-1} one gets the required equality (30). One can check the normalization taking $u = 1$. Then the integral $\mathcal{L}_m$ equals I_m

$$I_m = \int_0^\infty v^m/(v+1)^{m+2}\, dv = 1/(m+1)\,,$$

for every positive integer m. On the other hand, by inspection one sees that $\mathcal{L}_m$ is of the form

$$\mathcal{L}_m(u) = c_m\,(\log(u) - P(u))\,/(u-1)^{m+1}\,,$$

where P is a polynomial of degree at most m. In the neighborhood of $u = 1$ one has

$$\log(u) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(u-1)^j \,,$$

and from its value at $u = 1$, where $\mathcal{L}_m$ is non-singular, one sees from this last expression that $\mathcal{L}_m$ is the modified logarithm $\mathcal{D}_m$ introduced in §3 where,

$$\mathcal{D}_m(\Delta) = ((-1)^m/(\Delta-1)^{m+1})\left\{\log(\Delta) - \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j}(\Delta-1)^j\right\}.$$

This completes the proof of the Lemma. □

We now split the sum T of §6.2 as a sum of three terms $T = T_1 + T_2 + T_3$ and compute the coefficient of $1/\lambda$ in the integral with respect to rdr in each of them using the above Lemma.

6.3.1. *Terms involving* $\mathcal{D}_1$. These terms come from

$$T_1 = -2\left(r^2 b_0^2\right)(k\delta_1(k)b_0\delta_1(k)k + k\delta_2(k)b_0\delta_2(k)k)$$

With $u = r^2$ the integrand is $rdr = \frac{1}{2}du$ and thus (up to the overall factor of 2π) these terms give, by setting $\lambda = -1$ and changing the overall sign,

$$\tau(k^{-2}\int_0^\infty \frac{k^4\,u}{(k^2\,u+1)^2}\,\delta_i(k)\,\frac{1}{(k^2\,u+1)}\,du\,\delta_i(k)) = \tau(\mathcal{D}_1(\delta_i(k))\,\delta_i(k)\,k^{-2}) \qquad (32)$$

6.3.2. *Terms involving* $\mathcal{D}_2$. These terms come from

$$T_2 = 2\left(k^2 r^4 b_0^3\right)(k\delta_1(k)b_0k\delta_1(k) + 2k\delta_1(k)b_0\delta_1(k)k + k\delta_2(k)b_0k\delta_2(k) + 2k\delta_2(k)b_0\delta_2(k)k + \delta_1(k)kb_0\delta_1(k)k + \delta_2(k)kb_0\delta_2(k)k)$$

Since k commutes with b_0 and one works under the trace, they give the same as

$$4\left(k^2 r^4 b_0^3\right)(k\delta_j(k)kb_0\delta_j(k) + k^2\delta_j(k)b_0\delta_j(k))$$

One has $k\delta_j(k)k = k^2\Delta^{1/2}(\delta_j(k))$. Thus after setting $\lambda = -1$ and changing the overall sign, one gets

$$-2\tau(k^{-2}\int_0^\infty \frac{k^6\,u^2}{(k^2\,u+1)^3}\,(\Delta^{1/2}(\delta_i(k)) + \delta_i(k))\,\frac{1}{(k^2\,u+1)}\,du\,\delta_i(k))$$
$$= -2\tau((\mathcal{D}_2\,\Delta^{1/2})(\delta_i(k))\,\delta_i(k)\,k^{-2}) - 2\tau(\mathcal{D}_2(\delta_i(k))\,\delta_i(k)\,k^{-2}) \qquad (33)$$

6.3.3. *Terms involving* $\mathcal{D}_3$. These terms come from

$$T_3 = -2\left(k^4 r^6 b_0^4\right)(k\delta_1(k)b_0k\delta_1(k) + k\delta_1(k)b_0\delta_1(k)k + k\delta_2(k)b_0k\delta_2(k) + k\delta_2(k)b_0\delta_2(k)k + \delta_1(k)kb_0k\delta_1(k) + \delta_1(k)kb_0\delta_1(k)k + \delta_2(k)kb_0k\delta_2(k) + \delta_2(k)kb_0\delta_2(k)k)$$

Since k commutes with b_0 and one works under the trace, they give the same as

$$-2\left(k^4 r^6 b_0^4\right)\left(k^2\delta_j(k)b_0\delta_j(k) + 2k\delta_j(k)kb_0\delta_j(k) + \delta_j(k)k^2 b_0\delta_j(k)\right)$$

One has $k\delta_j(k)k = k^2\Delta^{1/2}(\delta_j(k))$ and $\delta_j(k)k^2 = k^2\Delta(\delta_j(k))$. Thus after setting $\lambda = -1$ and changing the overall sign, one gets

$$\tau\left(\left(\mathcal{D}_3(\delta_i(k))\,\delta_i(k) + 2(\mathcal{D}_3\,\Delta^{1/2})(\delta_i(k))\,\delta_i(k) + (\mathcal{D}_3\,\Delta)(\delta_i(k))\,\delta_i(k)\right)k^{-2}\right) \qquad (34)$$

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