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Higher Order Ito Product Formula and Generators of Evolutions and Flows

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A simple combinatorial formula is found for the product of two iterated quantum stochastic integrals, and used to find conditions that such an integral represent a unitary-valued or *-algebra homomorphism-valued process.

1. HIGHER ORDER ITO PRODUCT FORMULA

The integrators of multidimensional quantum stochastic calculus can be parametrized by elements of the space $\mathcal{I}$ consisting of linear transformations H in the finite-dimensional Hilbert space $\mathbb{C} \oplus \mathcal{K}$. Such a transformation decomposes naturally (Parthasarathy, 1992) into four components comprising a complex number, a vector in $\mathcal{K}$, a linear form on $\mathcal{K}$, and a linear transformation on $\mathcal{K}$, corresponding to the time, creation, annihilation, and multidimensional gauge components of the integrator. The corresponding integrator process Λ^H consists of operators in the Fock space $\mathcal{H} = \Gamma(L^2(\mathbb{R}_+) \otimes \mathcal{K})$ whose matrix elements between exponential vectors are given by

$$\langle e(f), \Lambda_t^H e(g) \rangle = \int_0^t \langle \tilde{f}(s), H\tilde{g}(s) \rangle \, ds \, \langle e(f), e(g) \rangle$$

where for $u \in \mathcal{K}$, $\tilde{u} = (1, u) \in \mathbb{C} \oplus \mathcal{K}$. We have $(\Lambda_t^H)^\dagger = \Lambda_t^{H^\dagger}$ where $H^\dagger$ is the usual adjoint and $(\Lambda_t^H)^\dagger$ the restriction of the adjoint to the exponential domain. By the quantum Ito formula (Hudson and Parthasarathy, 1984) we have

$$d\Lambda_t^H \, d\Lambda_t^K = d\Lambda_t^{H \circ K} \tag{1.1}$$

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where

$$H \circ K = HEK$$

E being the projector on the canonical embedding of $\mathcal{K}$ in $\mathbb{C} \oplus \mathcal{K}$. When equipped with the usual adjunction and the associative multiplication $\circ$ we call $\mathcal{I}$ the *Ito algebra*. By embedding them in the Fock space (Parthasarathy, 1992), product formulas for other stochastic calculus theories correspond to subalgebras of $\mathcal{I}$.

In this paper we are interested in iterated integrals such as

$$I_t(H_1, \ldots, H_m) = \int_{0<t_1<\cdots<t_m<t} d\Lambda^{H_1}(t_1) \cdots d\Lambda^{H_m}(t_m)$$

Note that

$$I_t(H_1^\dagger, \ldots, H_m^\dagger) = I_t(H_1, \ldots, H_m)^\dagger \tag{1.2}$$

There is a product formula for such integrals expressed in the following theorem, which is proved using (1.1) by induction on $m + n$ [the case of purely gauge integrals was given in Hudson and Parthasarathy (1993)].

Theorem. We have

$$\begin{aligned} &I(H_1, \ldots, H_m)I(H_{m+1}, \ldots, H_{m+n}) \\ &= \sum_{r=\max\{m,n\}}^{m+n} \sum_{P\in \mathrm{P}_r} I(H_{P_1}, \ldots, H_{P_r}) \end{aligned} \tag{1.3}$$

where P_r is the set of ordered partitions $P = (P_1, \ldots, P_r)$ of $(1, \ldots, m + n)$ into r subsets which are either singletons or pairs $\{p, q\}$ with $p \in \{1, \ldots, m\}$ and $q \in \{m + 1, \ldots, m + n\}$ in which $\{1, \ldots, m\}$ and $\{m + 1, \ldots, m + n\}$ occur in their natural orders, and $H_{\{p,q\}} = H_p \circ H_q$.

Since $I(H_1, \ldots, H_m) = I(H_1 \otimes \cdots \otimes H_m)$ is linear in $H_1, \ldots, H_m$, we may extend I to a linear map from elements of the tensor space

$$\mathcal{T} = \mathbb{C} \oplus \mathcal{I} \oplus \mathcal{I} \otimes \mathcal{I} \oplus \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} \oplus \cdots$$

over $\mathcal{I}$ to processes in Fock space, in such a way that

$$I(A)I(B) = I(A * B), \qquad I(A^\dagger) = I(A)^\dagger \tag{1.4}$$

where the associative multiplication $*$ is determined by (1.3) and the involution on tensors is inherited from that on $\mathcal{I}$. Note also that $*$ is well defined on the extended tensor space (in which infinitely many homogeneous components may be nonzero) even though the integral I may no longer be defined. We denote the extended tensor space equipped with the product $*$ and the involution $\dagger$ inherited from $\mathcal{I}$ by $\Gamma(\mathcal{I})$.

In most applications of quantum stochastic calculus there is given an *initial* unital $\dagger$-algebra $\mathcal{A}$. We define the multiplication $*$ in the tensor product $\tilde{\Gamma}(\mathcal{I}) = \mathcal{A} \otimes \Gamma(\mathcal{I})$ by the natural product rule $(a \otimes A) * (b \otimes B) = ab \otimes A * B$, and equip it with the product involution. The integration map I ampliates to the tensor product with $\mathcal{A}$, so that (1.4) remains true for integrands in $\mathcal{A} \otimes \Gamma(\mathcal{I})$ when the integrals are defined.

2. EVOLUTION GENERATORS

Let $R = \{k_1 < \cdots < k_r\}$ and $S = \{l_1 < \cdots < l_s\}$ be possibly empty sets whose union is $\{1, \ldots, n\}$, with respectively $r = |R|$ and $s = |S|$ elements. For a nonnegative integer n let $\mathcal{A} \otimes \mathcal{I}^{\otimes n} = \mathcal{A}$ if $n = 0$ and $\mathcal{A} \otimes \mathcal{I} \otimes \cdots \otimes \mathcal{I}$ (n factors) if $n > 0$. Let $a \in \mathcal{A} \otimes \mathcal{I}^{\otimes r}$ and $b \in \mathcal{A} \otimes \mathcal{I}^{\otimes s}$, respectively, and define an element $a^R b^S$ of $\mathcal{A} \otimes \mathcal{I}^{\otimes n}$ by bilinear extension of the rule for product elements:

$$(a^0 \otimes a^1 \otimes \cdots \otimes a^r)^R (b^0 \otimes b^1 \otimes \cdots \otimes b^s)^S = c^0 \otimes c^1 \otimes \cdots \otimes c^n$$

where $c^0 = a^0 b^0$, and for $j = 1, \ldots, n$, c^j is a^i if $j_i = k_i \in R \cap S'$, b^m if $j = l_m \in R' \cap S$, and $a^i \circ b^m$ if $j = k_i = l_m \in R \cap S$ (complements are in $\{1, \ldots, n\}$). Note that by taking either R or S empty we obtain a bimodule action of $\mathcal{A}$ on each $\mathcal{A} \otimes \mathcal{I}^{\otimes n}$. Using (1.3), the product $a * b = c = (c_0, c_1, \ldots)$ of elements of $\tilde{\Gamma}(\mathcal{I})$ can then be expressed in component form as

$$c_n = \sum a_r^R b_s^S \tag{2.1}$$

where, for $n = 0, 1, 2, \ldots$, the sum is over the 3^n choices of ordered pairs of subsets R and S whose union is $\{1, \ldots, n\}$.

Whether or not $I(u)$ exists, conditions on $u \in \tilde{\Gamma}(\mathcal{I})$ formally equivalent to the unitarity of the process $I(u)$ are that

$$u * u^\dagger = u^\dagger * u = (\mathbf{1}, 0, 0, \ldots)$$

where $\mathbf{1}$ is the identity element of $\mathcal{A}$. Evidently such elements form a group G under $*$. Moreover, $u \in G$ if and only if its components satisfy

$$\sum_{R\cup S=\{1,\ldots,n\}} u_r^R u_s^{\dagger S} = \sum_{R\cup S=\{1,\ldots,n\}} u_r^{\dagger R} u_s^S = \delta_{n,0}\mathbf{1}, \qquad n = 0, 1, \ldots \tag{2.2}$$

For $N = 0, 1, \ldots$ let us denote by G_N the set of sequences $(u_0, \ldots, u_N)$, with each $u_n \in \mathcal{A} \otimes \mathcal{I}^{\otimes n}$, and such that (2.2) holds for $n = 0, \ldots, N$. Then G_N is a group under the composition defined by (2.1); we call its elements *Nth-order evolution generators*. Can such a generator $(u_0, \ldots, u_N)$ be extended to an element of G?

This question can be answered affirmatively in the case of purely gauge stochastic integrals (Hudson and Parthasarathy, 1993). Indeed, let $\mathcal{I}_0$ be the subalgebra of $\mathcal{I}$ consisting of linear transformations on $\mathcal{H}$. Then in this case $u_j \in \mathcal{A} \otimes \mathcal{I}_0^{\otimes j}$, $j = 1, \ldots, N$. The projector E of Section 1 is just the identity of $\mathcal{I}_0$, so that $H \circ K = HK$ for $H, K \in \mathcal{I}_0$. We may write the condition on the additional element u_{N+1} that $(u_0, \ldots, u_N, u_{N+1}) \in G_{N+1}$ in the form

$$\left[\sum u_r^R(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + u_{N+1}\right]\left[\sum u_r^{\dagger R}(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + u_{N+1}^\dagger\right] = \mathbf{1} \otimes E^{\otimes N+1} \tag{2.3}$$

together with the corresponding relations with all u's and $u^\dagger$ exchanged. In (2.3) the summation is over all proper subsets R of $\{1, \ldots, N+1\}$ and complements are in the latter set. $E^{\otimes j}$ means $E \otimes \cdots \otimes E$ (j factors). Equation (2.3) says that u_{N+1} differs from a unitary element of $\mathcal{A} \otimes \mathcal{I}_0^{\otimes N+1}$ by a linear combination of elements formed from $(u_2, \ldots, u_N)$. Provided that each $\mathcal{A} \otimes \mathcal{I}_0^{\otimes N+1}$ contains unitary elements, which will be so if $\mathcal{A}$ does, extension is always possible.

3. FLOW GENERATORS

For the linear map $j: \mathcal{A} \to \tilde{\Gamma}(\mathcal{I})$ to satisfy the relations

$$j(xy) = j(x) * j(y), \qquad j(x^\dagger) = j(x)^\dagger, \qquad x, y \in \mathcal{A}$$

corresponding to $*$-algebra morphism (flow) properties of $J = I(j)$ (if it exists), its components must satisfy

$$j_n(xy) = \sum_{R \cup S = \{1, \ldots, n\}} j_r^R(x) j_s^S(y) \tag{3.1}$$

and

$$j_n(x^\dagger) = j_n(x)^\dagger \tag{3.2}$$

for arbitrary $x, y \in \mathcal{A}$. Denote by Z the space of such maps j and by Z_N the space of *Nth-order flow generators* $(j_0, \ldots, j_N)$, where $j_n: \mathcal{A} \to \mathcal{A} \otimes \mathcal{I}^{\otimes n}$ is linear and satisfies (3.1) and (3.2) for all $n \le N$. The extension question for such generators can again be answered affirmatively in the purely gauge case. If $(j_0, \ldots, j_N) \in Z_N$, then $(j_0, \ldots, j_{N+1}) \in Z_{N+1}$ if and only if the linear $\dagger$-map j_{N+1} satisfies

$$\begin{aligned}&\left[\sum j_r^R(x)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(x)\right]\\ &\quad\times \left[\sum j_r^R(y)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(y)\right]\\ &\quad= \sum j_r^R(xy)(\mathbf{1} \otimes E^{\otimes N+1-r})^{R'} + j_{N+1}(xy)\end{aligned}$$

for arbitrary $x, y \in \mathcal{A}$, where the sum is over all proper subsets of $\{1, \ldots, N+1\}$. Evidently there is a plentiful supply of such maps j_{N+1}, differing from unital $\dagger$-algebra morphisms by maps already determined.

4. ACTION OF EVOLUTION GENERATORS ON FLOW GENERATORS

The group G acts on the space Z by the action

$$u(j)(x) = u * j(x) * u^\dagger$$

In terms of components

$$u(j)_n(x) = \sum_{R \cup S \cup T = \{1, \ldots, n\}} u_r^R j_s^S(x) u_t^{\dagger T}$$

Evidently the same formula gives an action of each G_N on Z_N.

5. EXAMPLES

Consider the first-order evolution generator $(1, u_1)$, where

$$u_1 + u_1^\dagger + u_1 u_1^\dagger = u_1^\dagger + u_1 + u_1^\dagger u_1 = 0$$

This may be extended to the element $u = (1, u_1, u_1^{\{1\}} u_1^{\{2\}}, u_1^{\{1\}} u_1^{\{2\}} u_1^{\{3\}}, \ldots)$ of G for which $U = I(u)$ is the iterative solution of the stochastic differential equation

$$dU = U u_1, \qquad U(0) = 1 \tag{5.1}$$

Existence, uniqueness, and unitarity of the solution of (5.1) were proved in Hudson and Parthasarathy (1984) in the case when the components of u_1 in $\mathcal{A}$ are norm bounded.

Similarly the first-order flow generator (id, j_1), where the $\dagger$-map j_1 satisfies

$$j_1(xy) = j_1(x)y + x j_1(y) + j_1(x) j_1(y)$$

extends to the element $j = (id, j_1, (j_1 \otimes id) j_1, (j_1 \otimes id \otimes id)(j_1 \otimes id) j_1, \ldots)$ of Z for which $J = I(j)$ is the quantum stochastic flow satisfying

$$dJ(x) = J \otimes id j_1(x), \qquad J_0(x) = x$$

for which an existence theorem was proved in Evans (1989) in the bounded case.

An amusing example of a different kind is found by taking the trivial initial algebra C and seeking a flow generator $j = (j_0, j_1, \ldots)$ of the form

$$j_n(x) = x e_n \qquad (x \in \mathrm{C})$$

where $e_n = e_n^\dagger$ is an element of $\mathcal{I} \otimes \cdots \otimes \mathcal{I}^n$ (n factors). In the pure gauge case, (3.1) is satisfied if

$$e_n = (-1)^n e \otimes \cdots \otimes e$$

where e is an idempotent in $\mathcal{J}$, as is easily seen using the identity

$$(-1)^n = \sum_{R \cup S = \{1,\ldots,n\}} (-1)^r(-1)^s$$

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Symmetries of Physical Theories

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A physical theory is, by definition, a complete orthomodular atomic lattice having the covering property. Given L a quantum logic and S_L the set of all its states, a theorem is proved which asserts that, if certain reasonable assumptions concerning S_L are satisfied, then for any bijective convex mapping $U: S_L \to S_L$, satisfying also certain physically meaningful conditions, there exists a unique automorphism $V: L \to L$ such that $U(p) = p \circ V^{-1}$ for all $p \in S_L$.

1. INTRODUCTION

In this work complete orthomodular atomic lattices having the covering property (COMALC) will be called physical theories. Given L a physical theory, we will denote by S_L the set of all its states and by $\mathbb{O}_L$ the set of all its observables. In our language, any observable of L is a Boolean subalgebra of L (Ivanov, 1992).

We intend to prove a theorem concerning automorphism of L, which will be formulated in the next paragraph. Since this theorem is strongly connected with the symmetries of physical theories, we will discuss first the general problem of symmetries. It will be easily seen that this discussion gives also a quite transparent physical interpretation of some mathematical conditions required for proving the above-mentioned theorem.

One of the most general definitions of a physical theory is the following: a physical theory is a pair $T = (\mathbb{O}, S)$ of two nonempty sets, which are called the set of observables and the set of states of the theory T. For the sake of convenience we may consider that the set of all possible values of any given observable $\omega \in \mathbb{O}$ is a subset of R. In this case the result of the measurement of a given observable ω in a given state σ is considered to be a probability $P(\omega, \sigma)$: $\mathcal{B}(R) \to [0, 1]$, where $\mathcal{B}(R)$ is the set of all Borel subsets of the

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