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Transcendence of values of transcendental functions at algebraic points.

Paula Tretkoff

Abstract. This paper is a transcript of the Inaugural Monroe H. Martin Lecture and Seminar given at Johns Hopkins University on February 23rd and 24th, 2009. In Part I, we present classical and recent results on the transcendence of values of certain special functions of one variable at algebraic points. In Part II, we describe some new results, joint with Marvin D. Tretkoff, on the transcendence of values at algebraic points of hypergeometric functions of several variables. Note that the author's maiden name is Paula B. Cohen.

Part I: Transcendence of values of some special functions

In this part of our paper, we present some results on the transcendence of values of certain special functions of one variable at algebraic points. Recall that the algebraic numbers are the complex numbers satisfying a non-trivial polynomial relation with rational coefficients. We denote the field of rational numbers by $\mathbb{Q}$, and the field of algebraic numbers by $\overline{\mathbb{Q}}$. A transcendental number, α, is a complex number which is not algebraic. Therefore $P(\alpha) \neq 0$ for every polynomial $P \in \mathbb{Q}[x]$ with at least one non-zero coefficient.

The first examples of explicit transcendental numbers are due to Liouville in 1844 [**Lio1**], but their construction is rather artificial. Liouville showed that an irrational algebraic number cannot be too well approximated by rational numbers with denominators of relatively small size. He then constructed numbers that are so well approximated by rational numbers that they must be transcendental. An example is the number $\xi = \sum_{n=1}^{\infty} 10^{-n!}$. In 1874, Cantor gave another proof of the existence of transcendental numbers. He showed that the set of all algebraic numbers is countable, while the set of real numbers is uncountable. It follows that the set of transcendental numbers is uncountable. Somewhat paradoxically, it is usually very difficult to show that any given number is transcendental.

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The ancient Greeks had asked whether it is possible to square the circle, that is, to construct, with compass and straight edge, a square with area equal to that of a given circle. If we set the radius of the circle equal to 1, this problem reduces to the construction of a segment of length $\sqrt{\pi}$. The impossibility of this construction is implied by the transcendence of π, which was proved by Lindemann in 1882 [**Lin**]. He used a method of Hermite, who proved the transcendence of e in 1873 [**Her**]. These were the first "naturally occurring" numbers proved to be transcendental. Weierstrass called Lindemann's work "one of the most beautiful theorems of arithmetic".

To mark the turn of the 20th century, Hilbert proposed a celebrated list of problems whose solution, he felt, would provide important goals for mathematicians during the following hundred years. These problems have inspired, and continue to inspire, important mathematical ideas. Hilbert said in 1900 that "Hermite's theorem on the arithmetic of the exponential function and their further development by Lindemann will undoubtedly remain an inspiration for mathematicians of future generations". The seventh of Hilbert's problems asked for a proof that α^β is transcendental for α, β algebraic with $\alpha \neq 0, 1$ and β irrational. This implies, for example, that $e^\pi = (e^{i\pi})^{-i} = (-1)^{-i}$ and $2^{\sqrt{2}}$ are transcendental. This generalizes a conjecture made by Euler in 1744 on the logarithms of rational numbers. Euler stated without proof that a number of the form $\log_a b$, where a and b are rational numbers, with b not equal to a rational power of a, must be a transcendental number. Hilbert's seventh problem was solved independently by Gelfond and Schneider in 1934. The Gelfond-Schneider theorem states that for any non-zero algebraic numbers α_1, α_2, β_1, β_2, with $\log \alpha_1$, $\log \alpha_2$ linearly independent over $\mathbb{Q}$, we have $\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$. In 1966, Baker obtained the analogous result for linear forms in arbitrarily many logarithms. This was a major breakthrough for which he was awarded the Fields Medal in 1970. The quantitative versions of Baker's results had a major impact on the study of diophantine equations, implying, in many examples, effectively computable bounds on the number and size of the solutions. For example, in 1909, Thue proved that the equation $F = F(x, y) = m$ has only a finite number of solutions in integers x, y. Here, F is an irreducible binary form of degree at least 3 with integer coefficients, and m is an integer. Thue used an ineffective strengthening of the 1844 theorem of Liouville mentioned earlier. Thue's method yielded an estimate for the number of solutions, but it did not furnish an estimate for the size of the solutions. Therefore, it did not enable one to actually solve the equations explicitly. In 1968, using lower bounds for linear forms in logarithms, Baker gave explicit bounds on the absolute values of the solutions of the Thue equation in terms of m and F. For details, see [**Bak**].

The proofs of these results share a philosophy of method originating in Hermite's work and still evident in many transcendence proofs to the present day. The evolution of this method, in increasingly sophisticated situations, has required breakthroughs and much additional mathematics. Despite this common thread, the description of any given transcendence proof is quite technical. We reproduce here a vague outline of the structure of a typical classical transcendence proof, transcribed from [**BuTu**], page 10. To show a complex number α is transcendental, assume that it is algebraic and show that this leads to a contradiction. If α is algebraic, we have $P(\alpha) = 0$ for some non-zero polynomial $P = P(x)$ with integer coefficients. Using the integer coefficients of P, and some particular properties of α, construct an integer N. This step usually involves some techniques from algebra. Next, show that N is non-zero. This is usually the most difficult part of the proof. Then, show that $|N| < 1$. This step usually involves some analytic methods. To finish, we derive a contradiction, since we have constructed an integer N with $0 < |N| < 1$, and there is no such integer. Therefore our original premise that α is algebraic must be false, so that α must be transcendental.

There are many unsolved problems about the transcendence of everyday numbers that would seem to require new methods. For example, it is suspected, but unproved, that the following numbers are transcendental:

$$e\pi, \quad e+\pi, \quad \gamma, \quad \Gamma(1/5), \quad \zeta(2n+1),\, n \geq 1,\, n \in \mathbb{Z}.$$

Here, γ is Euler's constant, which is given by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right).$$

In 1755, Euler showed that for integers $n \geq 1$ the number $\zeta(2n)$ is a rational multiple of π^{2n}. Therefore, by Lindemann's result on the transcendence of π, the number $\zeta(2n)$ is transcendental. From the classical formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, it follows that $\Gamma(1/2)$ is transcendental. In the 1970's, using the theory of elliptic curves with complex multiplication, the Chudnovskys showed that the numbers $\Gamma(1/3)$ and $\Gamma(1/4)$ are transcendental. More recently, using modular forms, Nesterenko has shown the algebraic independence of π, e^{π} and $\Gamma(1/4)$.

A related challenge is to establish the irrationality of such numbers, a weaker property than transcendence. Again, there are many open problems and ample room for new methods. Euler showed in 1744 that e is irrational and Lambert showed in 1761 that π is irrational. It is not known whether $e\pi$, $e+\pi$, or γ is irrational. The irrationality of $\zeta(3)$ was established in 1978 by Apéry [**Ap**]. The irrationality of $\zeta(5)$ is unproved, although results of Rivoal and others show that infinitely many $\zeta(2n+1)$, $n \geq 1$ are irrational. There are now finer results, in particular Zudilin [**Zu**] has proved that one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

Many results in transcendental number theory can be formulated in terms of the *exceptional set* of a transcendental function, that is, *the set of algebraic arguments at which the function takes algebraic values*. For example, Lindemann showed in 1882 that the exceptional set of $\exp(z)$ is trivial:

$$\mathcal{E} = \{\alpha \in \overline{\mathbb{Q}} : \exp(\alpha) \in \overline{\mathbb{Q}}\} = \{0\}.$$

As $\exp(1) = e$ and $\exp(i\pi) = -1$, it follows that e and π are transcendental. Hilbert's seventh problem can be restated as the assertion that the exceptional set of z^{β} consist only of $\alpha = 0, 1$ when β is an algebraic irrational.

C-L. Siegel made fundamental contributions to the study of the transcendence properties of various classes of functions generalizing the exponential function. In 1929, he introduced a general method for establishing the transcendence and algebraic independence of the values at algebraic points of a class of entire functions satisfying linear differential equations [**Si1**]. Siegel called these functions E-functions since the exponential function is in this class. Another example of an E-function is given by the normalized Bessel function:

$$K_{\lambda}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}x)^{2n}}{n!(\lambda+1)\ldots(\lambda+n)},$$

where $\lambda \in \mathbb{Q}$, $\lambda \neq -1, -2, \ldots$. Siegel showed that if $\lambda \neq n + \frac{1}{2}$, then $K_\lambda(\alpha)$ and $K'_\lambda(\alpha)$ are algebraically independent for all $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$. Siegel's method was later developed by Shidlovsky. At the same time, Siegel proposed studying power series, satisfying linear differential equations, that are reminiscent of a geometric series and have a finite radius of convergence. His E-function methods do not apply to them. He called these new functions G-functions and proposed studying their exceptional set. In particular, he asked whether this set is finite or infinite. Examples of G-functions are algebraic functions over $\mathbb{Q}$, for which the exceptional set is trivially all of $\overline{\mathbb{Q}}$, the k-th polylogarithm $L_k(x)$, and the classical Gauss hypergeometric function $F(a, b, c; x)$ with rational parameters a, b, c. Not much is known about the transcendence properties of the polylogarithm. For example, $L_k(1) = \zeta(k)$, and the transcendence of this number for odd $k \geq 3$ is unproved as mentioned earlier. We return to the example of $F(a, b, c; x)$ later. In 1932, Siegel [**Si2**] established the transcendence of at least one of any fundamental pair ω_1, ω_2 of periods of a Weierstrass elliptic function with algebraic invariants. Weierstrass elliptic functions can be viewed as generalizations of the exponential function. This raised the question as to whether every non-zero period is transcendental. Moreover, Siegel asked if the ratio $z = \omega_1/\omega_2$ is transcendental. This can be reinterpreted as the problem of determining the exceptional set of the classical elliptic modular function. Schneider solved these problems in 1937 [**Sch**].

We now describe Schneider's solution of Siegel's problem to determine the exceptional set of the classical elliptic modular function. This function $j = j(z)$ is defined on the upper half plane $\mathcal{H}$ consisting of complex numbers z with positive imaginary part. It is invariant with respect to the action of $\mathrm{SL}(2, \mathbb{Z})$ given by

$$z \mapsto \frac{az + b}{cz + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Here, $\mathrm{SL}(2, \mathbb{Z})$ is the group of 2×2-matrices with integer entries and determinant equal 1. In particular, $j(z) = j(z+1)$ for all $z \in \mathcal{H}$, and $j(z)$ is uniquely determined by the first two terms in its Fourier expansion:

$$j(z) = \exp(-2\pi i z) + 744 + \sum_{n=1}^{\infty} a_n \exp(2\pi i n z).$$

For all $n \geq 1$, the coefficient a_n turns out to be a positive integer. For example, $a_1 = 196884$, $a_2 = 21493760$. Schneider's celebrated result, proved in 1937, is that:

$$\mathcal{E} = \{z \in \mathcal{H} : z \in \overline{\mathbb{Q}} \text{ and } j(z) \in \overline{\mathbb{Q}}\} = \{z \in \mathcal{H} : [\mathbb{Q}(z) : \mathbb{Q}] = 2\}.$$

We can restate this result in terms of elliptic curves with complex multiplication (CM). To every $z \in \mathcal{H}$ we associate the complex torus of dimension 1:

$$A_z = \mathbb{C}/(\mathbb{Z} + z\mathbb{Z}).$$

This torus has an underlying structure of a projective curve of genus $g = 1$ defined over the field $\mathbb{Q}(j(z))$, called an elliptic curve. Its endomorphism algebra $\mathrm{End}_0(A_z) = \mathrm{End}(A_z) \otimes_{\mathbb{Z}} \mathbb{Q}$ consists of multiplications by the numbers α preserving $\mathbb{Q} + z\mathbb{Q}$. It is not difficult to see that this algebra is either $\mathbb{Q}$ or $\mathbb{Q}(z)$. In the latter case, the number z must be imaginary quadratic, and we say that A_z has *complex multiplication* (CM) and that z is a *CM point*. Therefore, we may reformulate

Schneider's result as saying:

$$\mathcal{E} = \{z \in \mathcal{H} : z \in \overline{\mathbb{Q}} \text{ and } j(z) \in \overline{\mathbb{Q}}\} = \{z \in \mathcal{H} : A_z \text{ has CM}\}.$$

That $j(z)$ is algebraic when A_z has CM was already known from the theory of elliptic curves. The hard part was to show that $j(z)$ is transcendental when z is algebraic and A_z does not have CM. The CM points are very important in number theory. The Kronecker-Weber theorem says that the finite abelian extensions of $\mathbb{Q}$ are obtained by adjoining the roots of unity $\exp(2\pi i\mathbb{Q})$. Hilbert's twelfth problem asks for a description of the abelian extensions of any number field in terms of special values of explicit functions. The theory of complex multiplication realizes this description for any imaginary quadratic field. For example, when z is imaginary quadratic, the field $\mathbb{Q}(z, j(z))$ is an extension of $\mathbb{Q}(z)$ with abelian Galois group. Moreover, we have very explicit information on the action of this Galois group on the number $j(z)$, which is an algebraic integer of degree the class number of $\mathbb{Q}(z)$. The study of abelian extensions of number fields is known as class field theory.

The proof of Schneider's theorem uses properties of doubly periodic functions and relies heavily on the group structure of elliptic curves. Schneider asked whether it is possible to prove his result using only the intrinsic properties of the elliptic modular function. This problem remains unsolved.

The generalization of Schneider's theorem to Siegel modular varieties follows from work of myself, Shiga and Wolfart in 1995. We briefly describe this result. Consider the Siegel upper half space of "genus" $g \geq 1$,

$$\mathcal{H}_g = \left\{z \in M_g(\mathbb{C}) : z = z^t,\ \Im(z) \text{ positive definite}\right\}.$$

Notice that $\mathcal{H}_1 = \mathcal{H}$. The symplectic group with integer entries is given by

$$\mathrm{Sp}(2g, \mathbb{Z}) = \left\{\gamma \in M_{2g}(\mathbb{Z}) : \gamma \begin{pmatrix} O_g & -I_g \\ I_g & O_g \end{pmatrix} \gamma^t = \begin{pmatrix} O_g & -I_g \\ I_g & O_g \end{pmatrix}\right\},$$

where I_g and O_g are the $g \times g$ identity and zero matrix respectively. We have $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$. The group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the space $\mathcal{H}_g$ by

$$z \mapsto (Az + B)(Cz + D)^{-1}, \qquad z \in \mathcal{H}_g, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}),$$

where A, B, C, D are in $M_g(\mathbb{Z})$. The quotient space for this action,

$$\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$$

is an analytic space parameterizing the complex isomorphism classes of (principally polarized) abelian varieties of dimension g. Indeed, to $z \in \mathcal{H}_g$, we associate the g-dimensional complex torus

$$A_z = \mathbb{C}^g / (\mathbb{Z}^g + z\mathbb{Z}^g),$$

which has the underlying structure of a projective commutative group variety, and is called an abelian variety. The space $\mathcal{A}_g$ has the underlying structure of a quasi-projective variety defined over $\overline{\mathbb{Q}}$, the Siegel modular variety. Two abelian varieties A, B related by a surjection $A \mapsto B$ with finite kernel are said to be isogenous, written $A \hat{=} B$. By the Poincaré irreducibility theorem, the abelian variety A_z is isogenous to a product of powers of simple mutually non-isogenous abelian varieties:

$$A_z \hat{=} A_1^{n_1} \times \ldots \times A_k^{n_k}, \qquad A_i \text{ simple},\ A_i \not\hat{=} A_j,\ i \neq j.$$

The endomorphism algebra $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of a simple abelian variety A is a division algebra over $\mathbb{Q}$ with positive involution. For $g > 1$, there are more possibilities for such division algebras than in the case $g = 1$. When A is simple and $\mathrm{End}_0(A)$ is a number field $\mathbb{L}$ with $[\mathbb{L} : \mathbb{Q}] = 2\dim(A)$, we say that A has complex multiplication (CM). In this case, the field $\mathbb{L}$ will be a totally imaginary quadratic extension of a totally real field. We say that A_z has CM when all its simple factors (in the above decomposition up to isogeny) have CM. The corresponding point $z \in \mathcal{H}_g$, or its $\mathrm{Sp}(2g,\mathbb{Z})$-orbit in $\mathcal{A}_g$, is said to be a CM or special point. The abelian varieties defined over $\overline{\mathbb{Q}}$ correspond to points of $\mathcal{A}_g(\overline{\mathbb{Q}})$, and include all CM abelian varieties. Moreover, the isomorphism classes of CM abelian varieties correspond to $\mathrm{Sp}(2g,\mathbb{Z})$-orbits of points in $\mathcal{H}_g \cap M_g(\overline{\mathbb{Q}})$. Conversely, we have the following generalization of Schneider's theorem, proved jointly by myself (using my maiden name Paula B. Cohen) [**Co1**], Shiga and Wolfart [**ShWo**], using modern transcendence techniques, especially results of Wüstholz [**Wu2**], [**Wu3**].

Theorem 1. *Let $J : \mathcal{H}_g \to \mathcal{A}_g(\mathbb{C})$ be a holomorphic $Sp(2g,\mathbb{Z})$-invariant map such that, for all CM points z, we have $J(z) \in \mathcal{A}_g(\overline{\mathbb{Q}})$. Then, the exceptional set of J, given by*

$$\mathcal{E} = \left\{ z \in \mathcal{H}_g \cap M_g(\overline{\mathbb{Q}}) : J(z) \in \mathcal{A}_g(\overline{\mathbb{Q}}) \right\}$$

consists exactly of the CM points.

Therefore, the special values $J(z)$, $z \in \mathcal{H}_g \cap M_g(\overline{\mathbb{Q}})$, are "transcendental", that is, not in $\mathcal{A}_g(\overline{\mathbb{Q}})$, whenever z is not a CM point. Some refinements of this result were obtained by Derome [**De**]. As in the case $g = 1$, the CM points are important in number theory. One reason is that the action of the absolute Galois group on the torsion points of CM abelian varieties, and on the values of $J(z)$ at CM points, is well understood. This is a basic tool for studying the arithmetic of abelian varieties and modular forms.

A natural question to ask is how the image $\overline{\mathcal{E}}$ in $\mathcal{A}_g$ of the exceptional set $\mathcal{E}$ of J is distributed. We make the following conjecture, which by Theorem 1 is just a restatement of the André–Oort conjecture [**An**], [**Oo**]. A proof of this conjecture, assuming the Riemann hypothesis, has been announced by Klingler, Ullmo and Yafaev [**KY**], [**UY**].

Conjecture. *Let Z be an irreducible closed algebraic subvariety of $\mathcal{A}_g$. Then $Z \cap \overline{\mathcal{E}}$ is a Zariski dense subset of Z if and only if Z is a special subvariety of $\mathcal{A}_g$.*

The Siegel modular variety $\mathcal{A}_g$ is a moduli space for principally polarized abelian varieties of dimension g. If we impose some extra structure on these polarized abelian varieties, for example on their endomorphism ring, the corresponding moduli space is called a Shimura subvariety of $\mathcal{A}_g$. A special subvariety is an irreducible component of a Shimura subvariety or of its image under the correspondences coming from the action of $\mathrm{Sp}(2g,\mathbb{Q})$ on $\mathcal{H}_g$ (the Hecke correspondences).

We now turn to a special case of the problem on G-functions proposed by Siegel in 1929. Namely, we study the exceptional set of the classical Gauss hypergeometric function with rational parameters, focusing on criteria for this set to be finite or infinite. The methods we use are only relevant to this function, there being as yet no general method that applies to all G-functions. The Gauss hypergeometric differential equation with parameters a, b, c, $c \neq 0, -1, -2, \ldots$ is given by

$$x(1-x)\frac{d^2F}{dx^2} + (c - (a+b+1)x)\frac{dF}{dx} - abF = 0,$$

and has three regular singular points at $x = 0, 1, \infty$. The Gauss hypergeometric function $F = F(x) = F(a, b, c; x)$ is the solution of this differential equation given by the multi-valued function on $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ with a branch defined for $|x| < 1$ by the series

$$F(x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} x^n, \qquad |x| < 1,$$

where $(w, 0) = 1$ and $(w, n) = w(w + 1) \dots (w + n - 1)$, for $w \in \mathbb{C}$ and $n \geq 1$.

The solution space of the differential equation is two dimensional. A basis of the solution space at any point x_0 in $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ changes into another such basis upon analytic continuation along closed loops in $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ starting and ending at x_0. This gives rise to a representation of the fundamental group of $\mathbb{P}_1 \setminus \{0, 1, \infty\}$, with base point x_0, in the 2×2-matrices, whose image $\Gamma(a, b, c)$ in $\mathrm{PGL}(2, \mathbb{C})$ is called the monodromy group. If we change the choice of base point and solution space basis, we replace $\Gamma(a, b, c)$ by a conjugate group. If $\Gamma(a, b, c)$ is finite, then F is an algebraic function. The list of finite monodromy groups was found by Schwarz in 1873 [**Schw**]. In what follows, we assume that a, b, c, $c \neq 0, -1, -2, \dots,$ are rational and that $\Gamma(a, b, c)$ is infinite. To simplify the present discussion, from now on we make the additional implicit assumption that $0 < a < c$, $0 < b < c$, and $c < 1$. This ensures that the differential forms we introduce shortly are all holomorphic and that the monodromy group acts (not necessarily discontinuously) on $\mathcal{H}$. These assumptions are not very restrictive (see [**Wo**], [**CoWu**]). Under our assumptions, the series $F(x)$, $|x| < 1$, above has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$ given by the Euler integral formula:

$$F = \frac{\int_1^{\infty} u^{b-c}(u-1)^{c-a-1}(u-x)^{-b} du}{\int_1^{\infty} u^{-c}(u-1)^{c-a-1} du}.$$

We assume that F is given by this analytic continuation in the discussion of the proof of Theorem 2 below. Our arguments are valid for any branch of F, since this only affects the choice of path of integration in the numerator of the above expression. In Theorem 2 below, by $F(a, b, c; x) \in \overline{\mathbb{Q}}$ we mean that some branch of the multi-valued function F takes an algebraic value at x. We have the following result.

THEOREM 2. *The exceptional set*

$$\mathcal{E} = \{x \in \overline{\mathbb{Q}} : F(a, b, c; x) \in \overline{\mathbb{Q}}\}$$

is infinite if and only if $\Gamma(a, b, c)$ *is an arithmetic lattice in* $PSL(2, \mathbb{R})$. *Moreover, every element of the set* $\mathcal{E}$ *corresponds to a CM point in a certain moduli space* $V(a, b, c)$ *for abelian varieties.*

An arithmetic group is a group commensurable with the integer points $G(\mathcal{O})$ of a linear algebraic group G defined over a number field F, where $\mathcal{O}$ is the ring of integers of F. A linear algebraic group is a group of matrices defined by algebraic conditions. Takeuchi [**Ta**] computed the complete list of arithmetic $\Gamma(a, b, c)$ in 1977. There are 85 such groups, up to conjugacy, and therefore infinitely many non-arithmetic $\Gamma(a, b, c)$. For example, the group $\mathrm{PSL}(2, \mathbb{Z})$ is arithmetic. It is the monodromy group of $F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; x)$, which by Theorem 2 has infinite exceptional set. The subgroup of $\mathrm{PSL}(2, \mathbb{R})$ generated by the fractional linear transformations given by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & \frac{1}{2}(1+\sqrt{5}) \\ 0 & 1 \end{pmatrix}$$

is non-arithmetic. It is the monodromy group of $F(\frac{3}{20}, \frac{7}{20}, \frac{1}{2}; x)$, which has finite exceptional set by Theorem 2. Wolfart [**Wo**] proved in 1988 that $\mathcal{E}$ is infinite when $\Gamma(a,b,c)$ is arithmetic. He was the first person to realize that there is a relation between the size of the exceptional set and the arithmetic properties of the monodromy group. Siegel seems to have been unaware of this relation. In the same paper, Wolfart claimed that $\mathcal{E}$ is finite when $\Gamma(a,b,c)$ is not arithmetic, but his proof contained a serious error discovered by Gubler. In 2002, Wüstholz and I (using my maiden name Cohen) [**CoWu**] showed that Wolfart's claim follows from a special case of the André–Oort conjecture in the theory of moduli varieties for abelian varieties (Shimura varieties). This special case was subsequently proved by Edixhoven and Yafaev [**EdYa**] in 2003. Together, these results give Theorem 2.

We briefly indicate some of the main ingredients of the proof of Theorem 2. Some of this will be revisited in Part 2 of the present paper. We mentioned above the Euler integral representation of $F(x)$. We can rewrite this as

$$F(x) = C \, \frac{\int_\gamma u^{b-c}(u-1)^{c-a-1}(u-x)^{-b} du}{\int_\gamma u^{-c}(u-1)^{c-a-1} du},$$

where $C \in \overline{\mathbb{Q}}^*$ and γ is a Pochhammer cycle (closed double loop) around $1, \infty$. The numerator is a period of a holomorphic differential form on the smooth projective curve $X(x)$ with affine model

$$w^N = u^{N(c-b)}(u-1)^{N(a+1-c)}(u-x)^{Nb},$$

where N is the least common denominator of $c-b$, $a+1-c$, b. This differential form and the curve $X(x)$ are defined over $\mathbb{Q}(x)$. To simplify our discussion, we assume N is a prime number (in Part II, we do not assume this). The denominator is a period of a holomorphic differential form on $X(0)$, both being defined over $\overline{\mathbb{Q}}$. When $x \in \mathcal{E}$, and $F(x) \neq 0$, the ratio of these non-zero periods is in $\overline{\mathbb{Q}}^*$ and the relevant differential forms and curves are defined over $\overline{\mathbb{Q}}$. (If $F(x) = 0$ then $x \notin \overline{\mathbb{Q}}$ [**Wo**].) We can associate to a smooth projective curve X an abelian variety, called its Jacobian, whose complex points are given by

$$\mathrm{Jac}(X) = H^{1,0}(X, \mathbb{C})^* / H_1(X, \mathbb{Z}).$$

Transcendence techniques imply that the only linear dependence relations between periods on abelian varieties are those induced by isogenies (assuming the base field $\overline{\mathbb{Q}}$ throughout). From this we deduce that, if $x \in \mathcal{E}$, then we have an isogeny of the form

$$\mathrm{Jac}(X(x)) \widehat{=} \mathrm{Jac}(X(0)) \times A.$$

The abelian variety A turns out to be independent of x. Moreover, both $\mathrm{Jac}(X(0))$ and A have CM, coming from the action of $\zeta = \exp(2\pi i/N)$ by the automorphism $(u,w) \mapsto (u, \zeta^{-1}w)$ of $X(x)$. The Shimura variety $V(a,b,c)$ of Theorem 2 is the smallest moduli space (Shimura variety) for the $\mathrm{Jac}(X(x))$ and this 1-parameter family determines a curve Z in $V(a,b,c)$. The exceptional set $\mathcal{E}$ is given by the intersection of Z with the moduli in $V(a,b,c)$ of abelian varieties isogenous to the fixed abelian variety $\mathrm{Jac}(X(0)) \times A$. By the special case of the André–Oort conjecture proved in [**EdYa**], this intersection is Zariski dense in Z if and only if Z is a special subvariety of $V(a,b,c)$. Finally, we observe that Z is a special subvariety of $V(a,b,c)$ if and only if $\Gamma(a,b,c)$ is arithmetic.

The monodromy groups of the higher dimensional hypergeometric functions, given by the Appell-Lauricella functions, were studied by Picard [**Pi**], Terada [**Te**], Deligne and Mostow [**DeMo**], [**Mo1**], [**Mo2**]. In particular, they determined the monodromy groups that act discontinuously on the complex n-ball for all $n \geq 1$, and, of those, which ones are arithmetic. The generalization of Theorem 2 to the exceptional set of these functions is due to Desrousseaux, M.D. Tretkoff, and myself [**DTT2**]. There, infinitude of the exceptional set is instead its Zariski density in the space of regular points of the system of partial differential equations of the Appell-Lauricella function. In the arithmetic case, the exceptional set again corresponds to certain CM points and we have Zariski density. In the non-arithmetic (which includes the non-discontinuous) case, when $n \geq 2$, the elements of the exceptional set no longer necessarily correspond to CM points, but rather may correspond to Shimura varieties of positive dimension. We can show that, in this situation, the exceptional set is not Zariski dense modulo a condition that arises naturally from the transcendence techniques and is related to unsolved problems on subvarieties of Shimura varieties. Recently, with M.D. Tretkoff, we improved the results of [**DTT2**] in the discontinuous case. Namely, the results are now unconditional, as we show in Part II of the present paper.

Part II: Transcendence properties of Appell–Lauricella functions

In this part we prove a new result, joint with Marvin D. Tretkoff, and stated in Theorem 3, which is a partial improvement of our results with Desrousseaux in [**DTT2**]. Namely, we generalize Theorem 2 of Part I of the present paper to Appell-Lauricella hypergeometric functions of $n \geq 2$ variables whose monodromy groups act discontinuously on the complex n-ball, denoted $\mathbb{B}_n$. Recall that $\mathbb{B}_n$ is the set of points $(x_0 : x_1 : \ldots : x_n) \in \mathbb{P}_n(\mathbb{C})$ satisfying $|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 < |x_0|^2$ with automorphism group $\mathrm{Aut}(\mathbb{B}_n) = \mathrm{PU}(1,n)$. When $n = 1$, we have a bijection $z \mapsto (z-i)/(z+i)$ from $\mathcal{H}$ to $\mathbb{B}_1$. This enables us to associate a subgroup of $\mathrm{PU}(1,1)$ to each monodromy group $\Gamma(a,b,c)$ of $F(a,b,c;x)$ acting on $\mathcal{H}$. At the end of this paper, we show that we do indeed obtain new examples not appearing in [**DTT2**] for which Theorem 3 is valid. One of these examples corrects an error in the list of discontinuous monodromy groups for $n \geq 2$ given in [**Mo2**].

The Appell-Lauricella hypergeometric functions are multi-valued functions of $n \geq 1$ complex variables defined on the weighted configuration space of $n+3$ distinct points in $\mathbb{P}_1$, given by

$$Q_n = \{(x_0, x_1, \ldots, x_{n+2}) \in \mathbb{P}_1^{n+3} : x_k \neq x_\ell, k \neq \ell\}/\mathrm{Aut}(\mathbb{P}_1),$$

where $\mathrm{Aut}(\mathbb{P}_1)$ acts diagonally. Using $\mathrm{Aut}(\mathbb{P}_1)$ to normalize the coordinates x_0, x_1, x_{n+2} to 0, 1, ∞, we can replace Q_n by

$$\mathcal{Q}_n = \{x = (x_2, \ldots, x_{n+1}) \in \mathbb{C}^n : x_i \neq 0, 1, x_i \neq x_j, i \neq j\}.$$

The space $\mathcal{Q}_n$ has a natural underlying quasi-projective variety structure. The weights are given by $n+3$ numbers $\mu = \{\mu_i\}_{i=0}^{n+2}$. We assume throughout that the μ_i are all rational numbers satisfying the so-called "ball $(n+3)$-tuple" condition (in [**Mo2**], it is called the "disc $(n+3)$-tuple" condition) given by

$$\sum_{i=0}^{n+2} \mu_i = 2, \qquad 0 < \mu_i < 1, \; i = 0, \dots, n+2.$$

For each choice of μ, there is a system $\mathcal{H}_\mu$ of linear partial differential equations in the n complex variables x_2, ..., x_{n+1}, whose $(n+1)$-dimensional solution space gives rise to functions generalizing the classical hypergeometric functions of Part I. These functions are named after Gauss (when $n = 1$), after Appell (when $n = 2$) and after Lauricella (when $n \geq 3$). The space of regular points for $\mathcal{H}_\mu$ is $\mathcal{Q}_n$. When $n = 1$, we recover the discussion of Part 1 once we set $\mu_0 = c - b$, $\mu_1 = a + 1 - c$, $\mu_2 = b$, $\mu_3 = 1 - a$. The ball 4-tuple condition is implied by the inequalities $0 < a < c$, $0 < b < c$, $c < 1$ assumed in Part I. When $n = 2$, let $x = x_2$, $y = x_3$ and let a, b, b', c be such that $\mu_0 = c - b - b'$, $\mu_1 = a + 1 - c$, $\mu_2 = b$, $\mu_3 = b'$, $\mu_4 = 1 - a$. Let $F_\mu = F_\mu(x, y) = F(a, b, b', c; x, y)$ be the multi-valued function on $\mathcal{Q}_2$ given by the solution of $\mathcal{H}_\mu$ with a branch defined for $|x|, |y| < 1$ by the series,

$$F(a, b, b', c; x, y) = \sum_{m,n} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n)(1, m)(1, n)} x^m y^n, \qquad |x|, |y| < 1.$$

Notice that $F(a, b, b', c; x, 0) = F(a, b, c; x)$, $|x| < 1$. The system $\mathcal{H}_\mu$ is given, in this case, by the three equations

$$x(1-x)\frac{\partial^2 F}{\partial x^2} + y(1-x)\frac{\partial^2 F}{\partial x \partial y} + (c - (a+b+1)x)\frac{\partial F}{\partial x} - by\frac{\partial F}{\partial y} - abF = 0,$$

$$y(1-y)\frac{\partial^2 F}{\partial y^2} + x(1-y)\frac{\partial^2 F}{\partial y \partial x} + (c - (a+b'+1)y)\frac{\partial F}{\partial y} - b'x\frac{\partial F}{\partial x} - ab'F = 0,$$

$$(x-y)\frac{\partial^2 F}{\partial x \partial y} - b'\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} = 0.$$

Returning to the general case $n \geq 1$, for $x \in \mathbb{C}^n$ consider the differential form

$$\omega(\mu; x) = u^{-\mu_0}(u-1)^{-\mu_1} \prod_{i=2}^{n+1} (u - x_i)^{-\mu_i} du.$$

Let $F_\mu = F_\mu(x)$ be the multi-valued function on $\mathcal{Q}_n$ given by the solution of $\mathcal{H}_\mu$ with a branch defined for $|x_i| < 1$, $i = 2, \dots, n+1$, by a holomorphic series with constant term equal 1, and analytic continuation given by the Euler integral representation

$$F_\mu(x) = \int_1^\infty \omega(\mu; x) \Big/ \int_1^\infty \omega(\mu; 0), \qquad x \in \mathcal{Q}_n.$$

We will mainly work with this branch of F_μ in what follows. As in the case $n = 1$ of Part I, our arguments are valid for any branch of the multi-valued function F_μ since this only affects the choice of path of integration in the numerator of the above expression. Notice that we may write the denominator of this last expression in terms of the classical beta function, namely

$$\int_1^\infty \omega(\mu; 0) = B(1 - \mu_{n+2}, 1 - \mu_1) = \int_0^1 u^{-\mu_{n+2}}(1-u)^{-\mu_1} du.$$

If $\mu_1 + \mu_{n+2} = 1$, then $B(1 - \mu_{n+2}, 1 - \mu_1)$ is the product of a non-zero algebraic number and π. If $\mu_1 + \mu_{n+2} < 1$, then, up to multiplication by a non-zero algebraic number, $B(1-\mu_{n+2}, 1-\mu_1)$ is a non-zero period of a differential form of the second kind defined over $\overline{\mathbb{Q}}$. As $\omega(\mu; x)$ is holomorphic, by [**Wu2**], Theorem 5, for $\mu_1+\mu_{n+2} \leq 1$, the number $F_\mu(x)$ is either zero or transcendental for all $x \in \mathcal{Q}_n \cap \overline{\mathbb{Q}}^n$. We suppose from now on that $\mu_1 + \mu_{n+2} > 1$. Notice that, when $n = 1$, this corresponds to the condition $c < 1$, which was one of our assumptions in Part I.

We define the exceptional set $\mathcal{E}_\mu$ of F_μ to be

$$\mathcal{E}_\mu = \{x \in \mathcal{Q}_n \cap \overline{\mathbb{Q}}^n : F_\mu(x) \in \overline{\mathbb{Q}}^*\},$$

where, by $F_\mu(x) \in \overline{\mathbb{Q}}^*$, we mean that the value of some branch of F_μ at x is a non-zero algebraic number. For every $x \in \mathcal{Q}_n$, there will always be a branch of F_μ which does not vanish at x. However, if some branch vanishes at $x \in \mathcal{Q}_n$, we cannot assume some other branch takes a non-zero algebraic value at x. Wolfart's arguments in [**Wo**] showing that, when $n = 1$, the zeros of any branch of F_μ are transcendental, do not seem to readily extend to the case $n > 1$.

Let Γ_μ be the monodromy group of the system $\mathcal{H}_\mu$. The "ball $(n+3)$-tuple" condition ensures that we may assume Γ_μ acts on $\mathbb{B}_n$. Our new result is the following generalization of Theorem 2 in the case where Γ_μ acts discontinuously.

THEOREM 3. *Suppose that $\mu_1 + \mu_{n+2} > 1$ and that Γ_μ acts discontinuously on $\mathbb{B}_n$. Then the exceptional set $\mathcal{E}_\mu$ of F_μ is Zariski dense in $\mathcal{Q}_n$ if and only if Γ_μ is an arithmetic lattice in $PU(1, n)$.*

A lattice Γ in $\mathrm{PU}(1, n)$ is a discrete subgroup of $\mathrm{PU}(1, n)$ such that the quotient group $\Gamma\backslash\mathrm{PU}(1, n)$ has finite Haar measure. A subgroup of $\mathrm{PU}(1, n)$ acting discontinuously on $\mathbb{B}_n$ is a discrete subgroup of $\mathrm{PU}(1, n)$. Conversely, as $\mathrm{PU}(1, n)$ acts transitively on $\mathbb{B}_n$ with compact isotropy group, a discrete subgroup of $\mathrm{PU}(1, n)$ acts discontinuously on $\mathbb{B}_n$. By [**Mo2**], Proposition 5.3, if the monodromy group Γ_μ associated to a ball $(n+3)$-tuple μ is a discrete subgroup of $\mathrm{PU}(1, n)$, then it is a lattice in $\mathrm{PU}(1, n)$. Therefore the Γ_μ acting discontinuously on $\mathbb{B}_n$ are precisely the Γ_μ that are lattices in $\mathrm{PU}(1, n)$, and we shall use both descriptions in what follows.

When $n = 1$, there are infinitely many Γ_μ acting discontinuously on the unit disk. In fact, the Schwarz triangle groups $\Delta(p, q, r)$ with p, q, r either integers at least 2, or infinity, and $p^{-1} + q^{-1} + r^{-1} < 1$ provide examples (the numbers $|1 - \mu_i - \mu_j|^{-1}$, $i \neq j$, of the ball 4-tuple μ equal p, q, or r). Up to conjugation, the subgroup $\Delta(p, q, r)$ of $\mathrm{PU}(1, 1)$ is determined by the following presentation in terms of generators and relations,

$$\langle M_1, M_2, M_3 : M_1^p = M_2^q = M_3^r = M_1 M_2 M_3 = 1\rangle,$$

with the corresponding relation being omitted if p, q or r is infinite. When $n = 1$, the property of being Zariski dense in $\mathcal{Q}_1 = \mathbb{P}_1 \setminus \{0, 1, \infty\}$ means being of infinite cardinality, and Theorem 2 implies Theorem 3 in that case. Picard studied the monodromy groups in the case $n = 2$. His work was made rigorous and extended to the case $n > 2$ by Terada [**Te**] and Deligne–Mostow [**DeMo**], who were mainly interested in finding examples of non-arithmetic Γ_μ acting discontinuously on higher dimensional spaces. When $n = 2$, there are 63 groups Γ_μ acting discontinuously of which 19 are non-arithmetic. For $3 \leq n \leq 9$, there are 41 groups Γ_μ acting discontinuously of which 1 is not arithmetic. For $n \geq 10$, no Γ_μ acts discontinuously.

The explicit list of all μ corresponding to these groups is given in [**Mo2**], p.579 and pp.584–586. (The entry 82 of the table on p.586 of that same reference is incorrect. It should be (3/18,3/18,3/18,12/18,15/18). Moreover, the corresponding monodromy group is non-arithmetic, not arithmetic as claimed there. This gives an additional non-trivial non-arithmetic example covered by Theorem 3, as remarked before the case by case list at the end of this paper.)

We now prove Theorem 3. Those parts of the proof already found in [**DTT2**] are only summarized in what follows. We refer the reader to that same reference for more details. For two non-zero complex numbers a and b, we write $a \sim b$ if $a/b \in \overline{\mathbb{Q}}^*$. We then say that a and b are proportional over $\overline{\mathbb{Q}}^*$.

Let $\mu = \{\mu_i\}_{i=0}^{n+2}$ be a ball $(n+3)$-tuple of rational numbers with $\mu_1 + \mu_{n+2} > 1$, and let N be the least common denominator of the μ_i. Let $K = \mathbb{Q}(\zeta)$, where $\zeta = \exp(2\pi i/N)$. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$, let σ_s be the Galois embedding of K which maps ζ to ζ^s.

For $x \in \mathcal{Q}_n \cup \{0\}$, and f a divisor of N, let $X(f, \mu, x)$ be the projective non-singular curve with affine model

$$w^f = u^{N\mu_0}(u-1)^{N\mu_1} \prod_{i=2}^{n+1} (u - x_i)^{N\mu_i}.$$

The ball $(n+3)$-tuple condition ensures that $\omega(\mu; x)$, $x \in \mathcal{Q}_n$, is a holomorphic differential form on $X(N, \mu, x)$, and the condition $\mu_1 + \mu_{n+2} > 1$ ensures that $\omega(\mu; 0)$ is a holomorphic differential on $X(N, \mu, 0)$. For $x \in \mathcal{Q}_n \cup \{0\}$, let $T(N, \mu, x)$ be the connected component of the origin in the intersection over the proper divisors f of N of the kernel of the natural map between Jacobians: $\mathrm{Jac}(X(N, \mu, x)) \to \mathrm{Jac}(X(f, \mu, x))$.

The automorphism $(u, w) \mapsto (u, \zeta^{-1} w)$ of $X(N, \mu, x)$ induces an action of K on the holomorphic differential forms $H^0(T(N, \mu, x), \Omega)$ of $T(N, \mu, x)$. The principally polarized abelian variety $T_0 = T(N, \mu, 0)$ has dimension $\varphi(N)/2$. It is a power of a simple abelian variety with complex multiplication by a subfield of K. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$, the dimension of the eigenspace of $H^0(T_0, \Omega)$ on which ζ acts by ζ^s is given by

$$r_s^{(0)} = 1 - \langle s(1-\mu_1) \rangle - \langle s(1-\mu_{n+2}) \rangle + \langle s(2 - \mu_1 - \mu_{n+2}) \rangle,$$

where $0 \le \langle x \rangle < 1$ denotes the fractional part of a real number x and $r_s^{(0)} + r_{N-s}^{(0)} = 1$. The number $B(1-\mu_{n+2}, 1-\mu_1)$ is proportional over $\overline{\mathbb{Q}}^*$ to a non-zero period of $\omega(\mu; 0)$. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$, and $x \in \mathcal{Q}_n$, the dimension of the eigenspace of $H^0(T(N, \mu, x), \Omega)$ on which ζ acts by ζ^s is given by

$$r_s = -1 + \sum_{i=0}^{n+2} \langle s\mu_i \rangle,$$

and we have $r_s + r_{N-s} = n+1$. Notice that $r_1 = r_1^{(0)} = 1$. The dimension of $T(N, \mu, x)$ is $(n+1)\varphi(N)/2$. When the number $\int_1^\infty \omega(\mu; x)$ is non-zero, it is proportional over $\overline{\mathbb{Q}}^*$ to a non-zero period of $\omega(\mu; x)$. For details, see [**DTT2**], §3.

For $x \in \mathcal{Q}_n \cap \overline{\mathbb{Q}}^n \cup \{0\}$, the principally polarized abelian variety $T(N, \mu, x)$ is defined over $\overline{\mathbb{Q}}$ as is the differential form $\omega(\mu; x)$. Moreover, when $x \in \mathcal{E}_\mu$, we have the relation between non-zero complex numbers

$$B(1-\mu_1, 1-\mu_{n+2}) \sim \int_1^\infty \omega(\mu; x),$$

which, as already remarked, implies the relation between non-zero periods

$$\int_\gamma \omega(\mu; 0) \sim \int_\gamma \omega(\mu; x),$$

where γ is the cycle on each curve induced by the Pochhammer cycle between 1 and ∞.

As explained in [**ShWo**], Prop.1, p.6, and [**Co1**], it follows from [**Wu3**] that as these non-zero periods are proportional over $\overline{\mathbb{Q}}^*$, the abelian varieties $T(N, \mu, x)$ and T_0 must share a common simple factor B up to isogeny. By the arguments of [**Be**], §1, Exemple 3, the abelian variety T_0 is isogenous to B^s, the smallest power of B whose endomorphism algebra contains K. Moreover, $T(N, \mu, x)$ is isogenous to $B^s \times C \widehat{=} T_0 \times C$ for some abelian variety C whose endomorphism algebra contains K. The dimension of the eigenspace of $H^0(C, \Omega)$ on which ζ acts by ζ^s is given by $r_s^{(1)} = r_s - r_s^{(0)}$, where $r_s^{(1)} + r_{N-s}^{(1)} = n$. In particular, $r_1^{(1)} = 0$. We have the following generalization of Proposition 4.2 of [**DTT2**] to now include non-arithmetic lattices.

PROPOSITION 4. *Suppose that* $\mu_1 + \mu_{n+2} > 1$ *and that* Γ_μ *is a lattice in* $PU(1, n)$. *Then there is an abelian variety* A_μ *with complex multiplication, whose isogeny class depends only on* μ*, such that* $x \in \mathcal{E}_\mu$ *if and only if* $T(N, \mu, x) \widehat{=} T_0 \times A_\mu^n$.

PROOF. Suppose that $x \in \mathcal{E}_\mu$. We first treat the case where Γ_μ is arithmetic. By [**Mo2**], Proposition 5.4, the group Γ_μ is arithmetic if and only if $r_s = 0$ or $n+1$ for all $1 < s < N-1$ coprime to N. By the ball $(n+3)$-tuple condition, we have $r_1 = 1$, $r_{N-1} = n$. By the discussion preceding the statement of the proposition, we have $r_1 = r_1^{(0)} = 1$ and $r_1^{(1)} r_{N-1}^{(1)} = 0$. It follows that $r_s^{(1)} r_{N-s}^{(1)} = 0$ for all $1 \le s \le N-1$ coprime to N and therefore that the abelian variety C is isogenous to A_μ^n, where A_μ has complex multiplication and is, up to isogeny, independent of x and determined soley by μ (see [**Shi**], [**DTT2**]). Suppose now that Γ_μ is a non-arithmetic lattice in $\mathrm{PU}(1, n)$. We first discuss the case $n = 2$, so that μ is a ball quintuple of rational numbers with $\mu_1 + \mu_4 > 1$. By [**CoWo2**], Lemme 1 and 2, p.676, with μ_0, μ_2 replaced by μ_1, μ_4, we know that for all μ satisfying the ΣINT condition of Mostow [**Mo1**], we have $\langle s\mu_1 \rangle + \langle s\mu_4 \rangle > 1$ for all $1 \le s \le N-1$ coprime to N with $r_s = 1$, which implies for these s that $r_s^{(0)} = 1$ and $r_s^{(1)} = 0$. (Note that the references [**CoWo1**] and [**CoWo2**] again feature my maiden name Cohen.) Of course, one can also check this directly for the finite list of all Γ_μ satisfying ΣINT given in [**Mo2**]. The non-arithmetic lattices Γ_μ in $\mathrm{PU}(1, n)$ not satisfying ΣINT are listed in [**Mo2**], §5.1, and were also studied by Sauter [**Sa**]. For $n = 2$, there are four such μ which give rise to non-arithmetic groups. One checks directly for these four μ that we always have $\mu_1 + \mu_4 < 1$ for all permutations of the indices of the μ, except in the case $\mu = (\frac{4}{18}, \frac{11}{18}, \frac{5}{18}, \frac{5}{18}, \frac{11}{18})$. For this μ one sees directly that $\langle s\mu_1 \rangle + \langle s\mu_4 \rangle > 1$ for all $1 \le s \le N-1$ coprime to N with $r_s = 1$. Note that, when $n = 2$ and $r_s r_{N-s} \neq 0$, we must have $r_s = 1$ or 2 and $r_{N-s} = 3 - r_s$. When $n = 2$, for all μ with Γ_μ non-arithmetic, it therefore follows that if $s \in (\mathbb{Z}/N\mathbb{Z})^*$ has $r_s = 1$, then $r_s^{(0)} = 1$ and therefore $r_s^{(1)} = 0$, $r_{N-s}^{(1)} = 2$. Therefore, as in the arithmetic case with $n = 2$, the abelian variety C must always be isogenous to the square A_μ^2 of an abelian variety with complex multiplication whose isogeny class is independent of $x \in \mathcal{E}_\mu$ and depends only on μ. There is only one remaining non-arithmetic lattice when $n \geq 3$, namely the sextuple $\mu = (\frac{7}{12}, \frac{5}{12}, \frac{3}{12}, \frac{3}{12}, \frac{3}{12}, \frac{3}{12})$. In this case we have $\mu_i + \mu_j \leq 1$ for all $i \neq j$, which we have excluded.

Conversely, for a lattice Γ_μ in $\mathrm{PU}(1,n)$ with $\mu_1+\mu_{n+2} > 1$, let A_μ be the abelian variety with complex multiplication whose isogeny class depends only on μ and is determined as above. Suppose that $T(N,\mu,x) \widehat{=} T_0 \times A_\mu^n$. Therefore $x \in \mathcal{Q}_n \cap \overline{\mathbb{Q}}^n$ and $T(N,\mu,x)$ has complex multiplication. As $r_1 = r_1^{(0)} = 1$, the eigendifferentials $\omega(\mu;x)$ and $\omega(\mu;0)$ generate, on $T(N,\mu,x)$ and T_0 respectively, the 1-dimensional eigenspaces for the action of K on the differentials of the first kind via the identity Galois embedding. Furthermore, as $T(N,\mu,x)$ has complex multiplication, the non-zero periods of $\omega(\mu;x)$ are all proportional to each other over $\overline{\mathbb{Q}}^*$ [**Shi**]. Similarly, the non-zero periods of $\omega(\mu;0)$ are all proportional to each other over $\overline{\mathbb{Q}}^*$. In each case, the 1-dimensional $\overline{\mathbb{Q}}$-vector space generated by these periods is an isogeny invariant. Therefore

$$\int_\gamma \omega(\mu;0) \sim \int_\gamma \omega(\mu;x),$$

and $F_\mu(x) \in \overline{\mathbb{Q}}^*$, so that $x \in \mathcal{E}_\mu$.

We now complete the proof of Theorem 3. As already remarked, the abelian varieties $T(N,\mu,x)$, $x \in \mathcal{Q}_n$, have generalized complex multiplication by K of so-called "type" $\Phi_\mu = \sum_{s\in(\mathbb{Z}/N\mathbb{Z})^*} r_s\sigma_s$, which encodes the representation of K on the holomorphic 1-forms of the $T(N,\mu,x)$. The data (K,Φ_μ) determines a complex symmetric domain

$$\mathcal{H}(K,\Phi_\mu) = \prod_{s\in(\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}} \mathcal{H}_s,$$

where $\mathcal{H}_s$ is a point if $r_s r_{N-s} = 0$ and, otherwise,

$$\mathcal{H}_s = \{z \in M_{r_s,r_{N-s}}(\mathbb{C}) : 1 - z^t\overline{z} \text{ positive definite}\}.$$

As we saw during the course of the proof of Proposition 4, when Γ_μ is an arithmetic lattice, we have $r_s r_{N-s} = 0$, when $s \neq 1, N-1$, $(s,N) = 1$, and $(r_1, r_{N-1}) = (1,n)$, so that $\mathcal{H}(K,\Phi_\mu) = \mathbb{B}_n$. Those non-arithmetic lattices not excluded by the extra condition $\mu_1+\mu_{n+2} > 1$ all occur for $n = 1,2$, so that $r_s = 0,1,2$ and $r_{N-s} = 2-r_s$ when $n = 1$, and $r_s = 0,1,2,3$ and $r_{N-s} = 3-r_s$ when $n = 2$. Therefore, in both the arithmetic and non-arithmetic cases we have $\mathcal{H}(K,\Phi_\mu) = \mathbb{B}_n^m$, with $m > 1$ when Γ_μ is non-arithmetic. The abelian varieties $T(N,\mu,x)$, $x \in \mathcal{Q}_n$, are principally polarized and we can assume they have lattices isomorphic to $M = \mathbb{Z}[\zeta]^{(n+1)}$. The data (K,Φ_μ,M) determines a Shimura variety S which is the quotient $\Gamma'\backslash\mathcal{H}(K,\Phi_\mu)$ of $\mathcal{H}(K,\Phi_\mu)$ by an arithmetic group Γ'.

When Γ_μ is arithmetic, it is of finite index in Γ' (see, for example, the discussion in [**CoWo2**], §3, which generalizes easily to our case). Therefore, there is a Zariski dense subset of $x \in \mathcal{Q}_n$ with $T(N,\mu,x)$ isogenous to $T_0 \times A_\mu^n$.

In the non-arithmetic case, the group Γ_μ is of infinite index in Γ', and by [**CoWo1**], [**CoWo2**], [**DTT2**] there is an embedding of $\mathcal{Q}_n$ into S as a quasi-projective subvariety whose Zariski closure Z is neither a Shimura subvariety of S nor a component of the Hecke image of a Shimura subvariety of S. By the methods and results of [**KY**], [**UY**] on the André–Oort conjecture, the points of

Z corresponding to abelian varieties in the same isogeny class as a fixed abelian variety with complex multiplication are not Zariski dense in Z.

Combining these facts with the result of Proposition 4 gives Theorem 3.

Notice that the hypergeometric function F_μ depends on the *ordered* $(n+3)$-tuple μ, whereas the monodromy group Γ_μ depends only on the *unordered* $(n+3)$-tuple μ. In other words, we could have stated Theorem 3 as follows: *Suppose* $\mu_i + \mu_j > 1$ *for some* $i \neq j$ *and that* Γ_μ *acts discontinuously on* $\mathbb{B}_n$. *Let* ν *be any permutation of* μ *such that* $\nu_1 + \nu_{n+2} > 1$. *Then, the exceptional set* $\mathcal{E}_\nu$ *of* F_ν *is Zariski dense in* $\mathcal{Q}_n$ *if and only if* Γ_μ *is an arithmetic lattice in* $PU(1,n)$. When $n = 1$, we have a 4-tuple μ_0, μ_1, μ_2, μ_3 with sum equal 2. Except in the case when all the μ_i equal $1/2$ (so that Γ_μ is the triangle group $\Delta(\infty,\infty,\infty)$, which is arithmetic, and $\mathcal{E}_\mu$ is empty) we always have $\mu_i + \mu_j > 1$ for some $i \neq j$.

The list of ball $(n+3)$-tuples, $n \geq 2$, such that Γ_μ acts discontinuously on $\mathbb{B}_n$ is given in [**Mo2**]. In light of our results, we have three possibilities for these μ:

Case (I): If $\mu_i + \mu_j \leq 1$ for all $i \neq j$, then the exceptional set $\mathcal{E}_\nu$ of F_ν is empty for any permutation ν of μ.
Case (II): If $\mu_i + \mu_j > 1$ for some $i \neq j$, and Γ_μ is an arithmetic lattice in $\mathrm{PU}(1,n)$, then $\mathcal{E}_\nu$ is Zariski dense in $\mathcal{Q}_n$ for all permutations ν of μ with $\nu_1 + \nu_{n+2} > 1$. However $\mathcal{E}_\nu$ is empty for all permutations ν of μ with $\nu_1 + \nu_{n+2} \leq 1$.
Case (III): If $\mu_i + \mu_j > 1$ for some $i \neq j$, and Γ_μ is a non-arithmetic lattice in $\mathrm{PU}(1,n)$, then $\mathcal{E}_\nu$ is not Zariski dense in $\mathcal{Q}_n$ for any permutation ν of μ. What's more $\mathcal{E}_\nu$ is empty for all permutations ν of μ with $\nu_1 + \nu_{n+2} \leq 1$.

As remarked previously, case (I) follows from [**Wu2**], Theorem 5, its statement remaining true even in the non-discontinuous case. Case (II) is already contained in [**DTT2**] and was revisited here in our proof of Theorem 3. The new advance of Part II of the present paper is the unconditional treatment of case (III) (in [**DTT2**] we needed a conjecture of Pink that is still open). Of course, combining case (I) and case (III) we have: *if* Γ_μ *is a non-arithmetic lattice in* $PU(1,n)$ *then* $\mathcal{E}_\nu$ *is not Zariski dense in* $\mathcal{Q}_n$ *for any permutation* ν *of* μ.

We show below how the list, given in [**Mo2**], of ball $(n+3)$-tuples μ with corresponding monodromy group Γ_μ a lattice in $\mathrm{PU}(1,n)$ is distributed between these three cases, in particular showing that case (III) is non-empty. In case (I), we write (NA) when Γ_μ is not arithmetic, the rest being arithmetic. Instead of writing μ, we write $(n; N; N\mu_0, \ldots, N\mu_{n+2})$, where, as before, N is the least common denominator of the μ_i and $n = n + 3 - 3$ corresponds to the number of variables in the associated hypergeometric functions. Notice that the non-trivial cases, namely (II) and (III), all occur in dimension at most 4. The non-arithmetic entry (2;18;3,3,3,12,15) in case (III) below is incorrect in [**Mo2**], where it is listed as (2;18;3,3,3,13,10), for which the sum of the μ_i does not equal 2, and it is also falsely asserted there that the associated Γ_μ is arithmetic.

Case (I): (9;6;1,1,1,1,1,1,1,1,1,1,1,1), (8;6;1,1,1,1,1,1,1,1,1,1,2), (7;6;1,1,1,1,1,1,1,1,1,3),(7;6;1,1,1,1,1,1,1,1,2,2), (6;6;1,1,1,1,1,1,1,1,4), (6;6;1,1,1,1,1,1,1,2,3), (6;6;1,1,1,1,1,1,2,2,2), (5;4;1,1,1,1,1,1,1,1), (5;6;1,1,1,1,1,1,1,5), (5;6;1,1,1,1,1,1,2,4), (5;6;1,1,1,1,1,2,2,3) (5;6;1,1,1,1,1,1,3,3), (5;6;1,1,1,1,2,2,2,2), (4;4;1,1,1,1,1,1,2) (4;6;1,1,1,1,2,2,4), (4;6;1,1,1,1,2,3,3), (4;6;1,1,1,2,2,2,3) (4;6;1,1,2,2,2,2,2), (4;10;2,3,3,3,3,3,3), (4;12;2,2,2,2,2,7,7) (3;12;1,3,5,5,5,5), (3;3;1,1,1,1,1,1), (3;4;1,1,1,1,1,3), (3;4;1,1,1,1,2,2) (3;6;1,1,1,3,3,3), (3;6;1,1,2,2,2,4), (3;6;1,1,2,2,3,3), (3;6;1,2,2,2,2,3), (3;8;1,3,3,3,3,3), (3;10;2,3,3,3,3,6), (3;10;3,3,3,3,3,5), (3;12;3,3,3,3,5,7)(NA), (3;12;3,3,3,5,5,5), (2;21;4,8,10,10,10)(NA), (2;24;5,10,11,11,11)(NA), (2;30,7,13,13,13,14)(NA), (2;3,1,1,1,1,2), (2;4;1,1,2,2,2), (2;5;2,2,2,2,2) (2;6;1,2,3,3,3), (2;3;2,2,2,3,3), (2;8;3,3,3,3,4), (2;9;2,4,4,4,4), (2;12;3,5,5,5,6) (2;12;4,4,4,5,7)(NA), (2;12;4,4,5,5,6)(NA), (2;12;4,5,5,5,5), (2;14;5,5,5,5,8) (2;15;4,5,5,5,8)(NA); (2;18;5,7,7,7,10), (2;18;7,7,7,7,8)(NA), (2;20;6,6,9,9,10)(NA) (2;24;7,9,9,9,14)(NA), (2;42;13,15,15,15,26)(NA).

Case (II): (4;6;1,1,1,1,1,2,5), (4;6;1,1,1,1,1,3,4), (3;6;1,1,1,1,3,5) (3;6;1,1,1,1,4,4), (3;6;1,1,1,2,2,5), (3;6;1,1,1,2,3,4), (3;12;2,2,2,2,7,9) (3;12;2,2,2,4,7,7), (2;12;1,3,5,5,10), (2,10;1,1,4,7,7), (2;12;1,2,7,7,7) (2;14;3,3,4,9,9), (2;15;2,4,8,8,8), (2;4;1,1,1,2,3), (2;4;1,1,1,4,5), (2;6;1,1,2,3,5), (2;5;1,1,2,4,4), (2;6;1,1,3,3,4), (2;6;1,2,2,2,5), (2;6;1,2,2,3,4), (2;8;1,3,3,3,6), (2;8;2,2,2,5,5), (2;10;1,4,4,4,7), (2;10,2,3,3,3,9) (2;10;2,3,3,6,6), (2;10;3,3,3,3,8), (2;10,3,3,3,5,6), (2;12;1,5,5,5,8), (2;12;2,2,2,7,11) (2;12;2,2,2,9,9), (2;12;2,2,4,7,9), (2;12;2,2,6,7,7), (2;12;2,4,4,7,7), (2;12;3,3,3,5,10) (2;12;3,3,5,5,8), (2;14;2,5,5,5,11), (2;18;1,8,8,8,11), (2;18;2,7,7,10,10), (2;20;6,6,6,9,13), (2;30;5,5,5,19,26), (2;30;9,9,9,11,22).

Case (III): (2;18;4,5,5,11,11), (2;12;3,3,3,7,8), (2;12;3,3,5,6,7), (2;18;3,3,3,12,15), (2;18;2,7,7,7,13), (2;20;5,5,5,11,14), (2;24;4,4,4,17,19), (2;30,5,5,5,22,23), (2;42;7,7,7,29,34).

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Department of Mathematics, Mailstop 3368, Texas A&M, College Station, TX 77842-3368, USA

E-mail address: ptretkoff@math.tamu.edu