TRANSCENDENCE OF VALUES AT ALGEBRAIC POINTS FOR CERTAIN HIGHER ORDER HYPERGEOMETRIC FUNCTIONS (*UPDATED*)

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ABSTRACT. In this paper, we describe the exceptional set of a function studied by Picard. This is the set of algebraic points at which the function takes algebraic values. In particular, we deduce necessary and sufficient conditions for the finiteness of the exceptional set. Some of our results depend on recent conjectures of Pink [21],[22].

1. Introduction

The purpose of this paper is to analyze the exceptional set of a function studied by Picard in the course of his investigation of Appell's hypergeometric function. In particular, we describe the set of algebraic points at which this function assumes algebraic values. Necessary and sufficient conditions for this set to be finite are also derived. Some background for this type of problem is provided in the present section.

One of the recurrent themes in the theory of transcendental numbers is the problem of determining the set of algebraic numbers at which a given transcendental function assumes algebraic values. This set has come to be known as the *exceptional set* of the function. The classical work of Hermite (1873), Lindemann (1882) and Weierstrass (1885) established that the exceptional set associated to the exponential function $\exp(x)$, $x \in \mathbb{C}$, is trivial, that is, consists only of $x = 0$. These results stand among the highlights of 19th century mathematics because they imply that π is a transcendental number, thus proving the impossibility of *squaring the circle*, a problem dating to the ancient Greeks.

Turning to the 20th century, we recall that C.L. Siegel (1929) suggested studying the exceptional set of the classical (Gauss) hypergeometric function of one complex variable $F = F(a, b, c; x)$, where a, b, c are rational

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numbers. An important related step was taken by Th. Schneider (1937), who showed that the exceptional set of a Weierstrass elliptic function with algebraic invariants is void. Subsequently, Schneider (1941) showed that at least one period of an abelian integral of the first or second kind on any curve of positive genus defined over the algebraic numbers is transcendental. More recently, Wüstholz [39] used his analytic subgroup theorem to obtain the definitive result in this direction: each period is transcendental unless it is zero.

Returning to Siegel's question, Wolfart [35] showed that the exceptional set of the classical hypergeometric function corresponds to a subset of complex multiplication (CM or special) points on a moduli space of abelian varieties associated to fixed parameters a, b, c. He also gave sufficient conditions on a, b, c for the exceptional set to be infinite. Next, P.B. Cohen and Wüstholz [9] gave sufficient conditions on a, b, c for the exceptional set to be finite. Their proof, however, assumed the validity of a particular case of the André–Oort conjecture which was subsequently established by Edixhoven and Yafaev in [16].

Research initiated recently [15], [11], [12], [14] extends some of these results to functions of several complex variables. Here, the Appell-Lauricella hypergeometric functions of several complex variables play the role of the Gauss hypergeometric function of one variable. The exceptional set of the Appell-Lauricella function is the collection of all points, each of whose coordinates is an algebraic number, at which the function assumes an algebraic value. Once again, there is a family of abelian varieties associated to an Appell-Lauricella function and a finite to one morphism to its base space (Shimura variety) from the domain of the Appell-Lauricella function.

At this point, an important distinction between functions of one and several complex variables manifests itself for the first time in the theory of transcendence. Namely, we find that the points in the exceptional set of an Appell-Lauricella function corresponding to abelian varieties with complex multiplication in general form a *proper subset* of the exceptional set. The conditions defining this proper subset are quite subtle. This is in sharp contrast to the situation for the Gauss hypergeometric function where, as mentioned earlier, the entire exceptional set corresponds to CM points.

Furthermore, criteria were given in [15] for the exceptional set of an Appell-Lauricella function not to be Zariski dense in its space of regular points. Note that, in the case of several complex variables, the property that replaces finiteness in the one variable case is that the set not be Zariski dense. In particular, we derive in [15] sufficient conditions for this set not to be Zariski dense. Our proof depends on as yet unproved conjectures of Pink, except in some special cases. In [11], [12], [14] sufficient conditions were given to ensure that the proper subset of the exceptional set corresponding to CM points is not Zariski dense. Moreover, a still unproved case of the André–Oort conjecture is utilized in our proofs.

According to Appell and Kampé de Feriet [1], Picard studied the function obtained from Appell's hypergeometric function $F(a,b,b',c;x,y)$ of the complex variables x and y by fixing one of these variables. Setting $x=\lambda$, say, where λ is a fixed number, the resulting function $F_\lambda(y)$ of one complex variable satisfies $F_\lambda(0)=F(a,b,c;\lambda)$, the value of the Gauss hypergeometric function at λ [1]. Assuming that $F_\lambda(0)\neq 0$ and dividing by this number, the resulting function $\Phi_\lambda(a,b,b',c;y)=F_\lambda(y)/F_\lambda(0)$ satisfies $\Phi_\lambda(a,b,b',c;0)=1$. As we follow the spirit of Picard's work on Appell's hypergeometric function, we call this function the Picard function. In the present paper, we use an alternate set of parameters $\mu=\{\mu_i\}_{i=0}^5$, and therefore denote the Picard function by $\Phi_{\mu,\lambda}=\Phi_{\mu,\lambda}(y)$ (see (2.12)). As in [12], [13], our aim is to relate the exceptional set of $\Phi_{\mu,\lambda}$ to isogeny classes of certain abelian varieties and hence to Hecke orbits on a Shimura variety.

Several new features appear in the present paper. First of all, for each rational ball 5-tuple μ, we distinguish two cases depending on λ and refer to them in Theorem 4.2 as the *irreducible* and the *reducible* case, respectively. The irreducible case can be viewed as the generic case. Indeed, for all but finitely many μ, the full André–Oort conjecture for curves predicts that the reducible case arises for only finitely many λ. Next, we use a recent conjecture of Pink [21] to supplement the particular case of the André–Oort conjecture used in [9]. Namely, the subset $\mathcal{T}$ may now be the Hecke orbit of a non-special point. In the irreducible case, this enables us to treat the situation where the fixed isogeny class is of CM type as well as to treat the case where it is not of CM type. In the reducible case, we cannot assume that the Hecke orbit of interest is the orbit of a point, rather it may be the orbit of a Shimura subvariety of positive dimension. This is the origin of the extra conditions defining the exceptional set in [12], [13]. In the present paper, we avoid this complication by appealing to conjectures of Pink [22]. It also turns out that in the reducible case we are sometimes able to apply the unconditional results of [16]. However, in general, Desrousseaux's extra conditions are necessary if we apply the André–Oort conjecture directly.

The plan of this paper is as follows. In §2, we recall some facts about Appell-Lauricella functions and define the related Picard function. We recall the analytic family of abelian varieties associated to an Appell-Lauricella function in §3. In §4, transcendence techniques are used to describe the exceptional set in terms of certain isogeny classes of abelian varieties. Finally,

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[1] Picard studied Appell's hypergeometric function $F(a,b,b',c;x,y)$. Among other things, he obtained an integral expression for it whose integrand depends on x and y. Setting $x=\lambda$ yields an integral of the type studied by Pochhammer [23] in his investigation of n^{th} order ordinary differential equations whose singular points are all regular and $(n+1)$ in number. It appears that Pochhammer was anticipated by Tissot. The authoritative and comprehensive work by Schlesinger [24] in fact refers to the differential equation under discussion as the "Tissot–Pochhammer differential equation". We refer the reader to Schlesinger's book for a complete account of this fascinating period.

necessary and sufficient conditions for the finiteness of the exceptional set of the Picard function are given in §§5 and 6.

2. The Picard function

In this section, we recall some well-known facts about Appell-Lauricella hypergeometric functions of several complex variables. We also introduce a related function studied by Picard in the course of his investigations of these hypergeometric functions. The main results of the present paper concern the transcendence of values of this Picard function at algebraic arguments.

The Appell–Lauricella hypergeometric functions of $n > 1$ complex variables are generalizations of the classical hypergeometric function of one variable, see [AK], Chapter VII. Appell's name is usually associated to the two variable case, whereas the case of more variables is usually ascribed to Lauricella.

The proper domain for these functions, which are initially defined by a system of partial differential equations, is the weighted configuration space of n distinct points on the projective line. The weights are in general given by $(n+3)$ real numbers. We assume throughout this note that these weights $\mu = \{\mu_i\}_{i=0}^{n+2}$ are rational numbers satisfying the ball $(n+3)$-tuple condition of [10]:

$$0 < \mu_i < 1, \quad i = 0, \ldots, n+2, \qquad \sum_{i=0}^{n+2} \mu_i = 2. \tag{2.1}$$

We refer to such weights as rational ball tuples.

Matching the indices of the variables x_i with those of the weights μ_i, we write the space of regular points for these functions as

$$\mathcal{Q}_n = \{\mathbf{x} = (x_2, \ldots, x_{n+1}) \in \mathbb{P}_1(\mathbb{C})^n : x_i \neq 0, 1, \infty; \quad x_i \neq x_j, i \neq j\}. \tag{2.2}$$

This space has the structure of a quasi-projective variety over $\overline{\mathbb{Q}}$ and can be identified with

$$\mathcal{Q}'_n = \{(x_0, x_1, x_2, \ldots, x_{n+1}, x_{n+2}) \in \mathbb{P}_1(\mathbb{C})^{n+3} : x_k \neq x_l; k \neq l\} / \mathrm{Aut}(\mathbb{P}_1)$$

where $\mathrm{Aut}(\mathbb{P}_1)$ acts diagonally and freely. From this description, one sees that there is a natural action of the symmetric group S_n on $\mathcal{Q}'_n$. Therefore, any result in the sequel that is true for a given $\mu = \{\mu_i\}_{i=0}^{n+2}$ is also true for the rational ball tuple obtained by permuting the μ_i.

The Appell-Lauricella hypergeometric functions are solutions of a system $\mathcal{H}_\mu$ of linear partial differential equations in n variables x_i, $i = 2, \ldots n+1$ with regular singularities along $x_i = 0, 1, \infty$, $x_i = x_j$ $(j \neq i)$: these are also known as the characteristic (hyper)surfaces of $\mathcal{H}_\mu$ (see for example [34]). The system $\mathcal{H}_\mu$ has an $(n+1)$-dimensional solution space. We denote its monodromy group by Δ_μ; it is a subgroup of $\mathrm{PU}(1, n)$. The μ for which Δ_μ is a lattice in $\mathrm{PU}(1, n)$ have been computed (see [10]). When $n = 1$, there are infinitely many such lattices, although only finitely many of them are arithmetic [32]. When $n = 2$, there are 58 such lattices, of which 15 are not arithmetic. For n between 3 and 12, there are 32 such lattices, of which only one is not arithmetic. For $n \geq 13$, the monodromy group is never a lattice (see [10], [17], [19]).

There is a unique solution $F = F_\mu(\mathbf{x})$ of $\mathcal{H}_\mu$ which is holomorphic at the singular point $\mathbf{x} = \mathbf{0}$ and satisfies $F_\mu(\mathbf{0}) = 1$. This generalizes the classical hypergeometric function in one complex variable. For $\mathbf{x} \in \mathcal{Q}_n$, let ω_1 be the differential form

$$\omega_1 = \omega_1(\mu; \mathbf{x}) = u^{-\mu_0}(u-1)^{-\mu_1} \prod_{i=2}^{n+1} (u - x_i)^{-\mu_i} du. \tag{2.3}$$

Even though $\mathbf{x} = \mathbf{0} \notin \mathcal{Q}_n$, we set

$$\omega_1(\mu; \mathbf{0}) = u^{-\mu_0 - \sum_{i=2}^{n+1} \mu_i}(u-1)^{-\mu_1} du. \tag{2.4}$$

For $\mathbf{x} = (x_2, \ldots, x_{n+1}) \in \mathcal{Q}_n$, we then have

$$F = F_\mu(\mathbf{x}) = \int_1^\infty \omega_1(\mu; \mathbf{x}) \Big/ \int_1^\infty \omega_1(\mu; \mathbf{0}), \tag{2.5}$$

the normalizing constant ensuring that $F_\mu(\mathbf{0}) = 1$. With $B(\alpha, \beta)$ the classical Beta function, we may write this as,

$$F = F_\mu(\mathbf{x}) = B(1 - \mu_1, 1 - \mu_{n+2})^{-1} \int_1^\infty \omega_1(\mu; \mathbf{x}). \tag{2.6}$$

In this paper, we will mainly be interested in the cases $n = 1, 2$. For $n = 1$, the space of regular points is given by,

$$\mathcal{Q}_1 = \{x \in \mathbb{C} : x \neq 0, 1\} \tag{2.7}$$

and for $n = 2$, it is given by,

$$\mathcal{Q}_2 = \{(x, y) \in \mathbb{C}^2 : x, y \neq 0, 1; x \neq y\}. \tag{2.8}$$

When $n = 2$, the system $\mathcal{H}_\mu$ has a 3-dimensional solution space, and its monodromy group Δ_μ is a subgroup of $\mathrm{PU}(1,2)$. For $\mu = \{\mu_i\}_{i=0}^4$ a rational ball 5-tuple, let $\mu' = \{\mu'_i\}_{i=0}^3$ be the rational 4-tuple with,

$$\mu'_0 = \mu_0 + \mu_3, \quad \mu'_1 = \mu_1, \quad \mu'_2 = \mu_2, \quad \mu'_3 = \mu_4. \tag{2.9}$$

We assume that μ' is a rational ball 4-tuple, so that, in particular,

$$\mu_0 + \mu_3 < 1. \tag{2.10}$$

In [12], [13], with $n = 2$, it was assumed in addition that $\mu_0 + \mu_2 + \mu_3 < 1$, in order that $\omega_1(\mu; \mathbf{0})$ be a differential of the first kind. In fact, his assumption is not necessary for most of the results of the present paper. The system $\mathcal{H}_{\mu'}$ has a 2-dimensional solution space, and its monodromy group $\Delta_{\mu'}$ is a subgroup of $\mathrm{PU}(1,1)$.

For a fixed $x = \lambda \in \mathcal{Q}_1$, we define the Picard function of a single variable $y \in \mathbb{C} \setminus \{0, 1, \lambda\}$ by,

$$\Phi_{\mu,\lambda} = \Phi_{\mu,\lambda}(y) = \int_1^\infty \omega_1(\mu; \lambda, y) \Big/ \int_1^\infty \omega_1(\mu'; \lambda). \tag{2.11}$$

In view of (2.5), this function is also given by,

$$\Phi_{\mu,\lambda}(y) = F_\mu(\lambda, y)/F_{\mu'}(\lambda). \tag{2.12}$$

For $\lambda \in \mathcal{Q}_1$, we define the exceptional set $\mathcal{E}_{\mu,\lambda}$ of $\Phi_{\mu,\lambda}$ to be

$$\mathcal{E}_{\mu,\lambda} = \{y \in \overline{\mathbb{Q}} \setminus \{0,1,\lambda\} : \Phi_{\mu,\lambda}(y) \in \overline{\mathbb{Q}}^*\}. \tag{2.13}$$

The main focus of this paper is to study criteria for the finiteness of the set $\mathcal{E}_{\mu,\lambda}$.

The function $\Phi_{\mu,\lambda}$ was studied by Picard (see [1], pp71–77). It is the unique solution $\Phi = \Phi(y)$, that extends to a holomorphic function at $y = 0$ and satisfies $\Phi(0) = 1$, of the linear differential equation of order 3 given by:

$$B_0(\lambda; y)\frac{d^3\Phi}{dy^3} + B_1(\lambda; y)\frac{d^2\Phi}{dy^2} + B_2(\lambda; y)\frac{dv}{dy} + B_3(\lambda; y)\Phi = 0. \tag{2.14}$$

Here,

$$\begin{aligned}
B_0(\lambda; y) &= y(y-1)(y-\lambda)\\
B_1(\lambda; y) &= (b-c-1)(y-1)(y-\lambda) + (c-a-b'-2)y(y-\lambda) + (-b-b'-1)y(y-1)\\
B_2(\lambda; y) &= (b'+1)[(a+b+b'+1-c)y + c(y-1) + (a+1-b)(y-\lambda)]\\
B_3(\lambda; y) &= ab'(b'+1),
\end{aligned}$$

and the parameters a, b, b', c are related to the μ_i by:

$$\begin{aligned}
\mu_0 &= c-b-b',\\
\mu_1 &= a+1-c,\\
\mu_2 &= b,\\
\mu_3 &= b',\\
\mu_4 &= 1-a.
\end{aligned} \tag{2.15}$$

The differential equation (2.14) has regular singular points at $y = 0, 1, \lambda, \infty$. For $y \neq 0, 1, \lambda, \infty$, a basis of solutions is given by $\int_g^h \omega(\mu; \lambda, y)$ for $g \neq h$ in $\{0, 1, \lambda, \infty\}$, where $\int_g^h \omega(\mu; x, y)$ is a basis of solutions to $\mathcal{H}_\mu$ at $x = \lambda$.

3. Rational ball tuples and Shimura varieties

The contents of this section appear in previous articles but are reproduced here for the convenience of the reader. We recall in particular the construction of certain analytic families of abelian varieties associated with hypergeometric functions and we identify the Shimura varieties for these families. Following [26], [28] (where $n = 1$), we prefer to call these abelian varieties *Prym varieties* rather than hypergeometric varieties as in [2]. Although some of our discussion is classical, we refer to [7] ($n = 1$), [8] ($n = 2$) and [2], [27] ($n \geq 3$) for more recent descriptions of this construction.

Let $\mu = \{\mu_i\}_{i=0}^{n+2}$ be a rational ball $(n+3)$-tuple and let N be the least common denominator of the μ_i. Let $K = \mathbb{Q}(\zeta)$, where $\zeta = \exp(2\pi i/N)$. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$, let σ_s be the Galois embedding of K which maps $\zeta \mapsto \zeta^s$.

By (2.1), for $\mathbf{x} \in \mathcal{Q}_n$, $n \geq 1$, the differential $\omega_1(\mu;\mathbf{x})$ is of the first kind on a projective non-singular curve $\mathcal{C}_{\mu,N} = \mathcal{C}_{\mu,N}(\mathbf{x})$ with singular affine model

$$w^N = u^{N\mu_0}(u-1)^{N\mu_1}\prod_{i=2}^{n+1}(u-x_i)^{N\mu_i}.$$

For $\mathbf{x} \in \overline{\mathbb{Q}}^n$, this curve is defined over a number field.

To every $\mathbf{x} \in \mathcal{Q}_n$, we can associate the Prym variety $T_{\mu,N} = T_{\mu,N}(\mathbf{x})$ of $\mathcal{C}_{\mu,N}(\mathbf{x})$. For proper divisors f of N there is a natural surjection from $\mathrm{Jac}(\mathcal{C}_{\mu,N})$ to $\mathrm{Jac}(\mathcal{C}_{\mu,f})$. Let $T_{\mu,N} = T_{\mu,N}(\mathbf{x})$ be the connected component of the origin in the intersection of all the kernels of these endomorphisms. The automorphism $\chi : (u,w) \mapsto (u, \zeta^{-1}w)$ of the affine model of $\mathcal{C}_{\mu,N}$ induces an action of ζ on $T_{\mu,N}$ which realizes the field K in the endomorphism algebra $\mathrm{End}_0(T_{\mu,N}) = \mathrm{End}(T_{\mu,N}) \otimes \mathbb{Q}$ and an automorphism χ^* of the space of differentials of the first kind $H^0(T_{\mu,N},\Omega)$ on $T_{\mu,N}$. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$, let V_s be the subspace of the elements of $H^0(T_{\mu,N},\Omega)$ which are eigenspaces for the action of K by χ^* with eigenvalue ζ^s, that is, on which K acts by $\sigma_s(K)$. A modern reference for the explicit basis of the subspaces V_s, $s \in (\mathbb{Z}/N\mathbb{Z})^*$ is [2]. The dimension of V_s has been computed by several authors (see, for example, [5]) and is given by

$$r_s = r_s^{(\mu)} = -1 + \sum_{i=0}^{n+2}\langle s\mu_i\rangle. \tag{3.1}$$

For all $s \in (\mathbb{Z}/N\mathbb{Z})^*$, we have $r_s + r_{-s} = (n+1)$. The space $H^0(T_{\mu,N},\Omega)$ is the direct sum of the V_s; therefore the dimension of $T_{\mu,N}$ is $(n+1)\varphi(N)/2$. We say that $T_{\mu,N}$ has (generalized) complex multiplication with (representation) type

$$\Psi = \Psi_\mu = \sum_{s\in(\mathbb{Z}/N\mathbb{Z})^*} r_s\sigma_s. \tag{3.2}$$

Let M be a set of representatives in $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm1\}$ of those s with $r_s r_{-s} \neq 0$. Let m be the cardinality of M and denote its elements by r_{s_j}, $j = 1,\ldots,m$. For $s \in (\mathbb{Z}/N\mathbb{Z})^*$ and $\mathbf{x} \in \mathcal{Q}_n$, we let

$$\omega^{(s)} = \omega^{(s)}(\mu;\mathbf{x}) = u^{-\langle s\mu_0\rangle}(u-1)^{-\langle s\mu_1\rangle}\prod_{i=2}^{n+1}(u-x_i)^{-\langle s\mu_i\rangle}du. \tag{3.3}$$

When $r_{s_j} = 1$, the differential form $\omega^{(s_j)}$ is of the first kind and generates V_{s_j}. We can suppose that $s_1 = 1$, and therefore that $\omega^{(s_1)} = \omega_1 = du/w$. Let $\langle s_j\mu\rangle$ be the rational ball $(n+3)$-tuple given by $\{\langle s_j\mu_i\rangle\}_{i=0}^{n+2}$. The Prym variety of $\mathcal{C}_{\mu,N}(\langle s_j\mu\rangle;\mathbf{x})$ is also $T_{\mu,N}(\mathbf{x})$. When $n=1$, we have $r_s + r_{-s} = 2$, and therefore the number $r_s r_{-s}$ equals 1 if it is non-vanishing.

When $n = 2$, we have $r_s + r_{-s} = 3$, and therefore the number $r_s r_{-s}$ equals 2 if it is non-vanishing.

The abelian variety $T_{\mu,N}$ is principally polarized and, as a complex torus, has lattice $\mathcal{L}$ isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z}[\zeta]^{(n+1)}$. The data $(K, \Psi, \mathcal{L})$ determines a Shimura variety S. Writing

$$\Psi = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} r_s \sigma_s, \tag{3.4}$$

we have

$$\dim(S) = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}} r_s r_{-s} \tag{3.5}$$

The Shimura variety S is the quotient of a complex symmetric domain $\mathcal{H}(\Psi)$ by an arithmetic (or modular) group Γ_i. We have,

$$\mathcal{H}(\Psi) = \prod_{s \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}} \mathcal{H}_{r_s, r_{-s}}, \tag{3.6}$$

where $\mathcal{H}_{u,v}$ is a point if $uv = 0$ and, otherwise,

$$\mathcal{H}_{u,v} = \left\{ z \in M_{u,v}(\mathbb{C}) : 1 - z^t\overline{z} \quad \text{positive and hermitian} \right\}. \tag{3.7}$$

When $s = 1$ we have $r_1 = 1$, $r_{-1} = n$ and $\mathcal{H}_{1,n}$ is the complex n-ball $\mathbb{B}_n$. For details, see [29].

To every $\mathbf{x} \in \mathcal{Q}_n$, we can associate the point (or module) $P_{\mathbf{x}}$ in $S(\mathbb{C})$ which represents the isomorphism class of $T_N(\mathbf{x})$. Let Z be the Zariski closure of $\{P_{\mathbf{x}} : \mathbf{x} \in \mathcal{Q}_n\}$; it is an irreducible closed subvariety of S with dimension $\dim(Z) = n$.

4. Transcendence arguments

Some of the results of this section are implicit in Lemme 2.2, Lemme 2.3, and Lemme 2.4 of [13]. One of the new features in the present paper is that the irreducible and reducible cases of our Theorem 4.2 are treated separately.

For two non-zero complex numbers a, b, we write $a \sim b$ if a/b is algebraic. We say then that a and b are proportional over $\overline{\mathbb{Q}}$.

For two abelian varieties A and B, we write $A \widehat{=} B$ when A and B are isogenous. For an abelian variety C defined over $\overline{\mathbb{Q}}$, let $H^0(C, \Omega_{\overline{\mathbb{Q}}})$ be the space of differential forms which are of the first kind on C and which are defined over $\overline{\mathbb{Q}}$. Denote by $\mathcal{P}_C$ the $\overline{\mathbb{Q}}$-vector space generated by the numbers,

$$\left\{ \int_\gamma \omega : \quad \omega \in H^0(C; \Omega_{\overline{\mathbb{Q}}}); \quad \gamma \in H_1(C; \mathbb{Z}) \right\}.$$

If the endomorphism algebra $\mathrm{End}_0(C)$ of C contains a number field F, then we say that C is F-stable. If C is F-stable and contains no proper non-trivial F-stable abelian subvariety, we say that C is F-irreducible. Otherwise, we say that C is F-reducible. If $F = K = \mathbb{Q}(\zeta)$, let $V_{C,s}$ be the eigenspace of $H^0(C,\Omega)$ on which K acts via $\sigma_s(K)$, where the Galois embedding σ_s maps ζ to ζ^s. In particular σ_1 is the identity embedding.

We begin with the following corollary of Theorem 5 in [38] (for more details see [27], Prop. 1, p.6 and [26], Appendix).

Proposition 4.1. *Let A and B be abelian varieties defined over $\overline{\mathbb{Q}}$. Then $\mathcal{P}_A \cap \mathcal{P}_B \neq \{0\}$ if and only if there exist non-trivial simple abelian subvarieties A' of A, and B' of B, such that $A' \widehat{=} B'$.*

We apply this result to $A = T_{\mu,N}(\lambda, y)$ and $B = T_{\mu',N}(\lambda)$ for $\lambda \in \overline{\mathbb{Q}}$, $\lambda \neq 0, 1$, $F_{\mu'}(\lambda) \neq 0$, and $y \in \mathcal{E}_{\mu,\lambda}$ (for notations, see §§2 and 3). In that case, by (2.11) we have the relation

$$\int_1^\infty \omega_1(\mu; \lambda, y) \sim \int_1^\infty \omega_1(\mu'; \lambda), \tag{4.1}$$

between non-zero numbers, so that $\mathcal{P}_{T_{\mu,N}(\lambda,y)} \cap \mathcal{P}_{T_{\mu',N}(\lambda)}$ is non-trivial and contains a non-zero period of $\omega_1(\mu'; \lambda)$. Therefore, by Proposition 4.1, it follows that $T_{\mu,N}(\lambda, y)$ and $T_{\mu',N}(\lambda)$ share a non-trivial simple factor E up to isogeny. Let E^s, $s \geq 1$, be the smallest power of E that is K-stable. Applying Ex. 3 in §1 of [4] to this situation, we deduce that there are K-stable abelian varieties C and D, which may be trivial, and a positive multiple u of s such that $T_{\mu',N}(\lambda) \widehat{=} E^u \times C$ and $T_{\mu,N}(\lambda, y) \widehat{=} E^u \times D$. Moreover, if $T_{\mu',N}(\lambda)$ is K-irreducible, then the abelian variety C is trivial and $u = s$. Since $\dim(T_{\mu,N}(\lambda, y)) = (3/2)\dim(T_{\mu',N}(\lambda))$, we must have $\dim(D) = \varphi(N)/2$. Therefore D is of CM type and $T_{\mu,N}(\lambda, y) \widehat{=} T_{\mu',N}(\lambda) \times D$. If $T_{\mu',N}(\lambda)$ is K-reducible then, as its dimension is $\varphi(N)$, it must split up to isogeny into a product of two abelian varieties of CM type of dimension $\varphi(N)/2$. Therefore E^s and C are of CM type and there is a simple abelian variety F, which may equal E, such that $C \widehat{=} F^v$. It follows that $T_{\mu',N}(\lambda) \widehat{=} E^s \times C$, and $T_{\mu,N}(\lambda, y) \widehat{=} E^s \times D$ where $\dim(D) = \varphi(N)$.

We summarize the above discussion in the following theorem:

Theorem 4.2. *Suppose that $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$, $F_{\mu'}(\lambda) \neq 0$, and $y \in \mathcal{E}_{\mu,\lambda}$. We have two possibilities:*
1) <u>Irreducible case</u>: assume that the abelian variety $T_{\mu',N}(\lambda)$ is K-irreducible. Then, there is a fixed abelian variety A_3 of CM type such that $V_{A_3,1} = \{0\}$, and such that for all $y \in \mathcal{E}_{\mu,\lambda}$,

$$T_{\mu,N}(\lambda, y) \widehat{=} T_{\mu',N}(\lambda) \times A_3. \tag{4.2}$$

2) <u>Reducible case</u>: assume that the abelian variety $T_{\mu',N}(\lambda)$ is K-reducible, and is therefore of CM type. Then, there are fixed abelian varieties A_1 and A_2 of CM type such that $V_{A_2,1} = \{0\}$, and such that for all $y \in \mathcal{E}_{\mu,\lambda}$,

$$T_{\mu',N}(\lambda) \widehat{=} A_1 \times A_2, \quad T_{\mu,N}(\lambda, y) \widehat{=} A_1 \times D, \tag{4.3}$$

where D is a K-stable abelian variety of dimension $\varphi(N)$ with $V_{D,1} = \{0\}$.

As we will observe in §5, the CM type of A_3 can be given explicitly in terms of the rational ball 5-tuple μ.

We have the following partial converse of Theorem 4.2:

Proposition 4.3. *For $(\lambda, y) \in \mathcal{Q}_2 \cap \overline{\mathbb{Q}}^2$, suppose that $\int_1^\infty \omega(\mu', \lambda) \neq 0$ and $\int_1^\infty \omega(\mu; \lambda, y) \neq 0$.*
1) If we have an isogeny of the form

$$T_{\mu,N}(\lambda, y) \widehat{=} T_{\mu',N}(\lambda) \times A_3, \tag{4.4}$$

with $T_{\mu',N}(\lambda)$ of CM type, then

$$\int_1^\infty \omega_1(\mu; \lambda, y) \sim \int_1^\infty \omega_1(\mu'; \lambda). \tag{4.5}$$

2) Let A_1, A_2 and D be K-stable abelian varieties. Suppose that A_1, A_2 are of CM type and that $V_{A_2,1} = \{0\}$. Moreover, suppose that D has dimension $\varphi(N)$ and that $V_{D,1} = \{0\}$. If we have an isogeny of the form

$$T_{\mu',N}(\lambda) \widehat{=} A_1 \times A_2, \quad T_{\mu,N}(\lambda, y) \widehat{=} A_1 \times D, \tag{4.6}$$

then

$$\int_1^\infty \omega_1(\mu; \lambda, y) \sim \int_1^\infty \omega_1(\mu'; \lambda). \tag{4.7}$$

Proof: We begin by proving part 1). As $r_1^{(\mu)} = r_1^{(\mu')} = 1$, the isogeny in (4.4) induces a K-equivariant map on differential 1-forms which sends $\omega_1(\mu'; \lambda)$ to $\alpha \cdot \omega_1(\mu; \lambda, y)$ where $\alpha \in \overline{\mathbb{Q}}^*$. As $T_{\mu',N}(\lambda)$ is of CM type, by Proposition 5 in [27], the non-zero periods of $\omega_1(\mu'; \lambda)$ are all proportional over $\overline{\mathbb{Q}}^*$. By (4.4), the numbers $\int_\gamma \omega_1(\mu; \lambda, y)$ and $\int_\gamma \omega_1(\mu'; \lambda)$ are therefore proportional over $\overline{\mathbb{Q}}$, where γ is a Pochhammer cycle between 1 and ∞. Replacing the Pochhammer cycle by the line integral from 1 to ∞ multiplies the period by an element of $\overline{\mathbb{Q}}^*$, and so (4.5) follows. The proof of part 2) is analogous to the proof of part 1). ∎

Let $\lambda \in \overline{\mathbb{Q}}$ be a fixed number satisfying the conditions of Theorem 4.2. In the irreducible case, the points of the exceptional set $\mathcal{E}_{\mu,\lambda}$ of the Picard function correspond to abelian varieties $T_{\mu,N}(\lambda, y)$ in a fixed isogeny class represented by $T_{\mu',N}(\lambda) \times A_3$. In §5, this will allow us to apply a conjecture of Pink in [21]. In the reducible case, the isogeny class of the abelian variety D may change as $y \in \mathcal{E}_{\mu,\lambda}$ varies. There is a subset of $\mathcal{E}_{\mu,\lambda}$, defined in Théorème 2 of [13] by introducing a second condition on another Picard function, for which this isogeny class is fixed and of CM type. One may then apply the André–Oort conjecture. In some cases, this conjecture suffices without the second condition. In other cases, we may apply another conjecture of Pink in [22]. We will return to this in §6.

5. The irreducible case

We assume throughout this section that $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$, $F_{\mu'}(\lambda) \neq 0$, and that the abelian variety $T_{\mu',N}(\lambda)$ is K-irreducible. By Theorem 4.2, the exceptional $\mathcal{E}_{\mu,\lambda}$ is then a subset of the set,

$$\overline{\mathcal{E}}_{\mu,\lambda} = \{y \in \overline{\mathbb{Q}} : \quad (\lambda, y) \in \mathcal{Q}_2 \quad \text{and} \quad T_{\mu,N}(\lambda, y) \widehat{=} T_{\mu',N}(\lambda) \times A_3\}. \tag{5.1}$$

By Proposition 4.3, when $T_{\mu',N}(\lambda)$ is of CM type, we see that the complement of $\mathcal{E}_{\mu,\lambda}$ in $\overline{\mathcal{E}}_{\mu,\lambda}$ corresponds to the possible zeros of $\Phi_{\mu,\lambda}$ in $\overline{\mathbb{Q}}$. In the notation of §3, let Z_2 be the complex surface in S given by the Zariski closure of the set $\{P_{(x,y)} : (x, y) \in \mathcal{Q}_2\}$. For $\lambda \neq 0, 1$, let C_λ be the complex curve in S given by the Zariski closure of the set $\{P_{(\lambda,y)} : y \neq \lambda, y \in \mathcal{Q}_1\}$.

Consider Shimura morphisms $\varphi_1 : T \to S$, $\varphi_2 : T \to T'$ and a point $t' \in T'$. An irreducible component of $\varphi_1(\varphi_2^{-1}(t'))$, or of its image under a Hecke operator, is called a weakly special subvariety of S. For a detailed discussion of weakly special subvarieties see [21].

The following is a special case of Conjecture 1.6 in [21]:

Conjecture 5.1. *Let S be a Shimura variety over $\mathbb{C}$ and $\Lambda \subset S$ the Hecke orbit of a point $P_0 \in S$. Let $C \subset S$ be a closed algebraic curve such that the set $C \cap \Lambda$ has infinite cardinality. Then C is a weakly special curve in S.*

As remarked in [21], it is already known that if C is a weakly special subvariety of S then $C \cap \Lambda$ is infinite if it is non-empty, so it is the other direction of the conjecture that is unknown in general. Points on a Shimura variety in the same Hecke orbit correspond to isogenous abelian varieties. Applying part 1) of Theorem 4.2, Proposition 4.3 and Conjecture 5.1, with P_0 the module in S corresponding to $T_{\mu',N}(\lambda) \times A_3$, we deduce the following:

Theorem 5.2. *Let $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$ with $F_{\mu'}(\lambda) \neq 0$. Suppose that the abelian variety $T_{\mu',N}(\lambda)$ is K-irreducible. Let $\mathcal{E}_{\mu,\lambda}$ be the exceptional set of the Picard function. If we assume the validity Conjecture 5.1, then $\mathcal{E}_{\mu,\lambda}$ is finite if C_λ is a not a weakly special curve in S. If $T_{\mu',N}(\lambda)$ is of CM type and C_λ is a weakly special curve in S, then $\mathcal{E}_{\mu,\lambda}$ is of infinite cardinality.*

In the case where P_0 is a special point and, therefore, corresponds to an abelian variety with complex multiplication, Conjecture 5.1 is a known particular case of the André–Oort conjecture. This uses the fact, proved in [21], that a weakly special curve containing a CM point is a special curve. This particular case was proved in [16]. We therefore have the following unconditional result:

Theorem 5.3. *Let $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$ with $F_{\mu'}(\lambda) \neq 0$. Suppose that the abelian variety $T_{\mu',N}(\lambda)$ is K-irreducible and of CM type. Then C_λ is a special curve in S if and only if $\mathcal{E}_{\mu,\lambda}$ is of infinite cardinality.*

If $T_{\mu',N}(\lambda)$ is K-irreducible and of CM type, its endomorphism algebra is then a field which is a totally imaginary quadratic extension of K. The full strength of the André–Oort conjecture for curves (proved subject to the

Generalized Riemann Hypothesis in [33]) predicts that there are infinitely many such λ if and only if the monodromy group $\Delta_{\mu'}$ is arithmetic.

We conclude this section by remarking that the CM type of A_3 can be given explicitly in terms of the rational ball 5 tuple μ. For $(\lambda, y) \in \mathcal{Q}_2$, let Ψ_μ denote the generalized CM type of $T_{\mu,N}(\lambda, y)$ and $\Psi_{\mu'}$ the generalized CM type of $T_{\mu',N}(\lambda)$. By (3.1), we have

$$\Psi_\mu = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} r_s^{(\mu)} \sigma_s \tag{5.2}$$

where, for all $s \in (\mathbb{Z}/N\mathbb{Z})^*$, we have

$$r_s^{(\mu)} = -1 + \langle s\mu_0 \rangle + \langle s\mu_3 \rangle + \langle s\mu_1 \rangle + \langle s\mu_2 \rangle + \langle s\mu_4 \rangle, \tag{5.3}$$

and $r_s^{(\mu)} + r_{-s}^{(\mu)} = 3$. Again using (3.1),

$$\Psi_{\mu'} = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} r_s^{(\mu')} \sigma_s, \tag{5.4}$$

where, for all $s \in (\mathbb{Z}/N\mathbb{Z})^*$, we have

$$r_s^{(\mu')} = -1 + \langle s(\mu_0 + \mu_3) \rangle + \langle s\mu_1 \rangle + \langle s\mu_2 \rangle + \langle s\mu_4 \rangle, \tag{5.5}$$

and $r_s^{(\mu')} + r_{-s}^{(\mu')} = 2$. Therefore, the CM type of A_3 is given by

$$\Psi_{03} = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} r_s^{(03)} \sigma_s, \tag{5.6}$$

where, for all $s \in (\mathbb{Z}/N\mathbb{Z})^*$, we have

$$r_s^{(03)} = \langle s\mu_0 \rangle + \langle s\mu_3 \rangle - \langle s(\mu_0 + \mu_3) \rangle, \tag{5.7}$$

and $r_s^{(03)} + r_{-s}^{(03)} = 1$. Here, we used the relation $\Psi_\mu = \Psi_{\mu'} + \Psi_{03}$.

We denote by $S^0(03)$ the set of those points $(x, y) \in \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ which lie on the line $y = 0$ but which do not lie on any other characteristic line of $\mathcal{H}_\mu$. As $\mu_0 + \mu_3 < 1$, the points of $S^0(03)$ are in the semi-stable compactification $\mathcal{Q}_2^{sst}$ of $\mathcal{Q}_2$ constructed in [10]. In the same way as described in §§4 and 5 of [8] for the case where Δ_μ is a lattice, we can associate to the points of $S^0(03)$ the points in S which are the modules corresponding to an abelian variety $T_{\mu',N}(\lambda) \times A_{03}$, with A_{03} isogenous to A_3.

6. The reducible case

We assume throughout this section that $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$, $F_{\mu'}(\lambda) \neq 0$, and that the abelian variety $T_{\mu',N}(\lambda)$ is K-reducible and, therefore, of CM type. The full strength of the André–Oort conjecture for curves, proved subject to the Generalized Riemann Hypothesis in [33], predicts that there are only finitely many such λ if and only if $\Delta_{\mu'}$ is not arithmetic. By Theorem 4.2, there are fixed abelian varieties A_1 and A_2 of CM type such that, for all $y \in \mathcal{E}_{\mu,\lambda}$,

$$T_{\mu',N}(\lambda) \widehat{=} A_1 \times A_2, \quad T_{\mu,N}(\lambda, y) \widehat{=} A_1 \times D, \tag{6.1}$$

where D is a K-stable abelian variety of dimension $\varphi(N)$. There is no reason to assume *a priori* that D is fixed as y varies, even though this sometimes happens as we shall see below. As in §5, we shall use the fact that points on a Shimura variety in the same Hecke orbit correspond to isogenous abelian varieties.

Denote by Ψ_D the generalized CM type of D and by Ψ_i the CM type of A_i, $i = 1, 2$. These types give the representation of K on the holomorphic 1-forms of these abelian varieties. With Ψ_μ as in (5.2) and Ψ_{03} as in (5.6), we have from (6.1) that

$$\Psi_\mu = \Psi_1 + \Psi_2 + \Psi_{03} = \Psi_1 + \Psi_D, \tag{6.2}$$

therefore

$$\Psi_D = \Psi_2 + \Psi_{03} = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} r_s^D \sigma_s, \tag{6.3}$$

where $r_s^D + r_{-s}^D = 2$. If $\Psi_2 = \Psi_{03}$, then we have $\Psi_D = 2\Psi_{03}$, and by the arguments of [29] we can assume that D is always isogenous to A_{03}^2. We may then apply the particular case of the André–Oort conjecture proved in [16]. This is the case in the following example: when the monodromy group Δ_μ of $F_\mu(x, y)$ is a lattice in $\mathrm{PU}(1, 2)$ and $\mu_0 + \mu_2 + \mu_3 < 1$, the point $(x, y) = (0, 0)$ is a stable point (in the sense of [10]). The analytic family of Prym varieties $T_{\mu,N}(x, y)$ can therefore be extended to the stable point $(0, 0)$, as explained in §5 of [8]. Moreover, the corresponding abelian variety $T_{\mu,N}(0, 0)$ is of CM type and isogenous to $A_1 \times A_{03}^2$ as described in Part 1), Théorème 2 of [8]. We see from this same reference that we can take A_1 to have CM type conjugate to:

$$\sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*} \left(\langle s(1 - \mu_1) \rangle + \langle s(1 - \mu_4) \rangle \right) \sigma_s. \tag{6.4}$$

Therefore, Theorem 4.2 and Proposition 4.3 of this paper, together with the particular case of the André–Oort conjecture proved in [16], imply the following:

Theorem 6.1. *Let $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$ with $F_{\mu'}(\lambda) \neq 0$ and suppose in addition that $T_{\mu',N}(\lambda) \widehat{=} A_1 \times A_{03}$. Then the exceptional set $\mathcal{E}_{\mu,\lambda}$ of the Picard function is infinite if and only if C_λ is a special curve in S. Moreover, there are infinitely many $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$ with $T_{\mu',N}(\lambda) \widehat{=} A_1 \times A_{03}$ if and only if the Zariski closure of the image of $S^0(03)$ in S is a special curve.*

In the case where the isogeny class of D is not fixed, we may again apply conjectures of Pink. Let S_D be the Shimura variety associated to the data $(K, \Psi_D, \mathbb{Z}[\zeta])$. From (6.3), the dimension of S_D is given by

$$\dim(S_D) = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}} r_s^D r_{-s}^D. \tag{6.5}$$

Following [22], we define the special closure of an irreducible subvariety Z of S to be the smallest special subvariety containing Z. Moreover, we say that Z is *Hodge generic* if its special closure is an irreducible component of S, so that Z is not contained in any special subvariety of S of positive codimension.

Conjecture 6.2. *(see [22]) Consider a Shimura variety S over $\mathbb{C}$ and an irreducible closed subvariety Z of S. Let S_Z be the special closure of Z. Then the intersection of Z with the union of all special subvarieties of S of dimension strictly less than $\dim(S_Z) - \dim(Z)$ is not Zariski dense in Z.*

In fact, we only need this conjecture in the following weaker form:

Conjecture 6.3. *Consider a Shimura variety S over $\mathbb{C}$ and an irreducible closed subvariety Z. Let S_Z be the special closure of Z. Then the intersection of Z with the Hecke orbit of a special subvariety of S of dimension strictly less than $\dim(S_Z) - \dim(Z)$ is not Zariski dense in Z.*

We deduce the following from Theorem 4.2 and Proposition 4.3 (using (6.1)):

Theorem 6.4. *Let $\lambda \in \mathcal{Q}_1 \cap \overline{\mathbb{Q}}$ with $F_{\mu'}(\lambda) \neq 0$. Let S_λ be the special closure of C_λ. If $\dim(S_\lambda) = 1$, then $\mathcal{E}_{\mu,\lambda}$ is infinite. Now assume Conjecture 6.3. If $\dim(S_\lambda) > \dim(S_D) + 1$, then $\mathcal{E}_{\mu,\lambda}$ is finite.*

For example, if S is the special closure of C_λ we have the following:

Proposition 6.5. *Suppose C_λ is Hodge generic in S. Then $\mathcal{E}_{\mu,\lambda}$ is finite.*

Proof: We must verify the condition

$$\dim(S_\lambda) > \dim(S_D) + 1, \tag{6.6}$$

of Theorem 6.4. As $S_\lambda = S$ by hypothesis, we have

$$\dim(S_\lambda) = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}} r_s^{(\mu)} r_{-s}^{(\mu)} = 2m, \tag{6.7}$$

where m, as in §3, is the number of $s \in (\mathbb{Z}/N\mathbb{Z})^*$ with $r_s^{(\mu)} r_{-s}^{(\mu)} \neq 0$. On the other hand, from (6.1) and (6.3) we see that $r_1^D = 0$, so that

$$\dim(S_D) \leq m - 1. \tag{6.8}$$

The inequality in (6.6) now follows from (6.7) and (6.8). ∎

In [12], [13], additional natural conditions were imposed on $y \in \mathcal{E}_{\mu,\lambda}$ in order to restrict D to a fixed isogeny class of CM type, thereby allowing for the application of [16]. The main idea is to utilize the periods of the differentials of the first kind in the eigensubspace V_{-1} for $T_{\mu,N}(\lambda, y)$. This is in the spirit of the results of Cohen, Shiga, and Wolfart in §5 of [27]. Arguments similar to those of §4 of the present article are then applied to these periods. These arguments imply that A_2 and D in (6.1) share a simple factor up to isogeny. It follows that $D \widehat{=} A_2 \times A_{03}$. We refer to [12], [13] for details.

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