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ASYMPTOTIC ESTIMATES OF SOME S_n CHARACTERS AND THE IDENTITIES OF THE 2×2 MATRICES

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Abstract

By carefully studying the multiplicity of an S_n character, related to the polynomial identities of the 2×2 matrices, we show $\left(\frac{2\sqrt{2}}{\pi}+\varepsilon\right)\frac{4}{\sqrt{\pi}}\frac{1}{n\sqrt{n}}4^n \lesssim c_n(F_2) \lesssim \frac{4}{\sqrt{\pi}}\frac{1}{n\sqrt{n}}4^n$, where $\varepsilon > 0$ is a certain multi-integral of a positive function, which has not yet been evaluated. The results hint that the upper bound ($\simeq t_n(F_2)$) is probably the asymptotic values of $c_n(F_2)$.

INTRODUCTION

This paper uses the methods introduced in [2], and considerably sharpen the results obtained there, further closing the gap between the lower and the upper bounds for $c_n(F_2)$.

Let $Par(n)$ be the set of partitions of n, and $\Lambda_\ell(n) = \{\lambda \in Par(n) \mid \lambda = (a_1,\dots,a_\ell)\}$ be the partitions with $\le \ell$ parts. Let χ_λ be the S_n-irreducible character corresponding to $\lambda \in Par(n)$, and $\{\chi_n(F_2)\}$ be the co-character sequence of the 2×2 matrices F_2 [2].

Write $\lambda = (\omega_1+\dots+\omega_4, \omega_2+\omega_3+\omega_4, \omega_3+\omega_4, \omega_4) \in \Lambda_4(n)$. In [2] we saw:

$$\chi_n(F_2) = \sum_{\lambda \in \Lambda_4(n)} m_\lambda \chi_\lambda \qquad [2, \text{Introduction}]$$

with the multiplicities m_λ satisfying

$$m_\lambda \ge (\omega_1+1)(\omega_2-1)(\omega_3+1) \quad \text{for } \omega_1,\omega_2,\omega_3 \ge 2 . \qquad [2, \text{Th. } 3.22]$$

In [2,§4] the S_n character, $\psi_\ell(n)$, was introduced, with multiplicities $Y_\ell(\lambda)$:

$$\psi_\ell(n) = \sum_{\lambda \in \Lambda_\ell(n)} Y_\ell(\lambda)\cdot\chi_\lambda .$$

It was shown that

$$Y_4(\omega_1+\dots+\omega_4, \omega_2+\omega_3+\omega_4, \omega_3+\omega_4, \omega_4) \le (\omega_1+1)(\omega_2+1)(\omega_3+1)$$

[2, Th.4.9]

If $\omega_1,\omega_2,\omega_3$ are "large", this implies that

$$Y_4(\lambda) \lesssim m_\lambda ,$$

hence

$$\deg \psi_4(n) \lesssim \deg \chi_n(F_2) = c_n(F_2) . \qquad [2, \text{Th.}4.16]$$

The asymptotic value

$$\deg \psi_4(n) \simeq \frac{\sqrt{2}}{\pi}\cdot\frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n \qquad [2,4.2]$$

thus produced an asymptotic lower bound for $c_n(F_2)$.

The theory of trace identities provided the asymptotic upper bound

$$c_n(F_2) \lesssim t_n(F_2) \simeq \frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n ,$$

where $t_n(F_2)$ are the trace co-dimensions of F_2.

In this paper we study more carefully the multiplicities $Y_4(\lambda)$. Using asymptotic and geometric methods we prove: For most large $\omega_1,\omega_2,\omega_3$, $Y_4(\omega_1+\dots+\omega_4, \omega_2+\dots+\omega_4, \dots, \omega_4)$ is significantly smaller than $\frac{1}{2}\omega_1\omega_2\omega_3$. Thus, for such λ, $2Y_4(\lambda) \lesssim m_\lambda$, and hence

$$\frac{2\sqrt{2}}{\pi}\cdot\frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n \lesssim c_n(F_2) \quad \left(\lesssim \frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n\right) .$$

Most of the asymptotic techniques needed here are developed in [1]. The geometric methods for studying $Y_4(\lambda)$ were introduced in [2,§4].

Our calculations show that $c_n(F_2)$ is significantly larger than that lower bound, although we do not know yet by how much (the computer may know). This hints that the upper bound $\frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n \simeq t_n(F_2)$ is probably the asymptotic value for $c_n(F_2)$. This also suggests that $(\omega_1+1)(\omega_2-1)(\omega_3+1)$ is very close to m_λ, which asymptotically determines the co-characters $\chi_n(F_2)$. The remark about $\chi_n(F_k)$, $k \ge 3$, at the end of [2, Introduction], is still valid here.

§1. THE ASYMPTOTIC EXPRESSION FOR $Y_4(\lambda)$

Keep the notations of [1] and [2]: $a,\delta > 0$:

$$Par(n) \supseteq \Lambda_4(n,a,\delta) = \{(\tfrac{n}{4}+c_1\sqrt{n},\dots,\tfrac{n}{4}+c_4\sqrt{n}) \mid a \ge c_1 \ge \dots \ge c_4 \ge -a, c_j - c_{j+1} \ge \delta\}$$

and $$\bar{S}_4(n) = \sum_{\lambda \in \Lambda_4(n)} Y_4(\lambda) d_\lambda \;.$$

By [2,4.14], $\bar{S}_4(n)$ is approximated (as $a \to \infty$) by:

$$\bar{S}_4(n,a,a^{-4}) = \sum_{\lambda \in \Lambda_4(n,a,a^{-4})} Y_4(\lambda) d_\lambda \;.$$

For $\lambda \in \Lambda_4(n,a,a^{-1})$, we approximate $Y_4(\lambda)$ asymptotically by $Y_4'(\lambda)$, to be defined later. The compactness of $P_4(a) \subseteq \mathbb{R}^3$, [1,§2], and the continuity of the functions involved here, imply that $\bar{S}_4(n)$ is approximated by

$$S_4'(n,a,a^{-4}) = \sum_{\lambda \in \Lambda_4(n,a,a^{-4})} Y_4'(n) d_\lambda$$

(moreover, d_λ can be replaced by its asymptotic value, [1,F.1.1]).

Let $\lambda = (\frac{n}{4} + c_1\sqrt{n}, \ldots, \frac{n}{4} + c_4\sqrt{n}) \in \Lambda_4(n,a,\delta)$, then $c_j - c_{j+1} = \gamma_j \geq \delta$, and $\omega_j = \gamma_j\sqrt{n}$, $j = 1,2,3$. By [2, Th. 4.9],

$$Y_4(\lambda) = \sum_{\substack{i_1+i_2+i_3=m \\ 0 \leq i_j \leq \omega_j}} \{\min(\omega_1+i_2-i_1 , \omega_2+i_3-i_2) + 1\}$$

where $$m = \left[\frac{\omega_1 + 2\omega_2 + 3\omega_3}{4}\right] .$$

We are thus led to consider the integral points $(i) = (i_1,i_2,i_3)$ on the intersection of the plane $H_m : x+y+z = m$ with the box

$$B_\omega = \{0 \leq x \leq \omega_1 , 0 \leq y \leq \omega_2 , 0 \leq z \leq \omega_3\} . \qquad [2,§4]$$

For each such point we have the corresponding summand:

$$\min(\omega_1 + i_2 - i_1 , \omega_2 + i_3 - i_2) + 1 \;.$$

Consider the line L on H_m determined by

$$\omega_1 + y - x = \omega_2 + z - y \;.$$

Together with $x+y+z = m$, this yields

$$y = \frac{1}{3}(\omega_2 - \omega_1 + m) = c(\omega_1,\omega_2,\omega_3) = c(\omega) \;.$$

Note that, when $m = \dfrac{\omega_1 + 2\omega_2 + 3\omega_3}{4}$, then

$$c(\omega) = \frac{1}{4}(2\omega_2 - \omega_1 + \omega_3) \;.$$

L splits $H_m \cap B_\omega$ into H_1 and H_2 , $H_1 \cup H_2 = H_m \cap B_\omega$. Let $S_j = H_j \cap \mathbb{Z}^3$, $j = 1,2$.

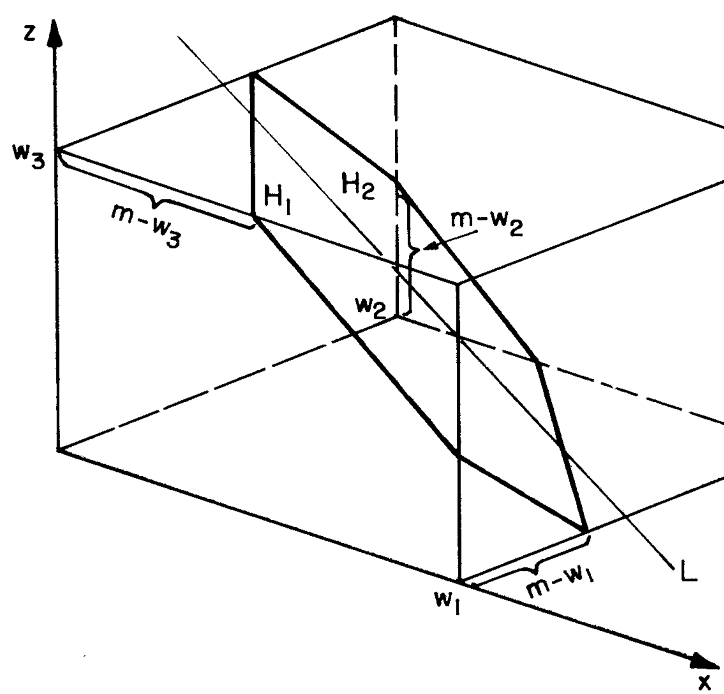

FIG. 1

Clearly, $(i) \in S_1 \Leftrightarrow \min(\omega_1 + i_2 - i_1, \omega_2 + i_3 - i_2) = \omega_1 + i_2 - i_1$, and similarly for $(i) \in S_2$. It follows that

$$Y_4(\lambda) = (\omega_1+1)\#S_1 + (\omega_2+1)\#(S_2 \setminus L) + \sum_{(i)\in S_1} (i_2-i_1) + \sum_{(i)\in S_2 \setminus L} (i_3-i_2). \tag{1.1}$$

We now explain how to replace $Y_4(\lambda)$ by $Y_4'(\lambda) \simeq Y_4(\lambda)$, where the summations are replaced by integrations. This is done as follows:

Let $\mu = \mu(\gamma) = \frac{1}{4}(\gamma_1 + 2\gamma_2 + 3\gamma_3)$, then

$$m = m(\omega) = [\mu\sqrt{n}] .$$

Write $\mu\sqrt{n} = [\mu\sqrt{n}] + \varepsilon_n$, $0 \le \varepsilon_n < 1$, to conclude that

$$\frac{[\mu\sqrt{n}]}{\mu\sqrt{n}} = \frac{\mu\sqrt{n} - \varepsilon_n}{\mu\sqrt{n}} \xrightarrow[n\to\infty]{} 1$$

so $\quad m = [\mu\sqrt{n}] \simeq \mu\sqrt{n}$.

Thus $\quad c = c(\omega) = \frac{1}{3}(\omega_2 - \omega_1 + m) \simeq \frac{1}{3}(\gamma_2 - \gamma_1 + \mu)\sqrt{n}$,

$$\frac{1}{3}(\gamma_2 - \gamma_1 + \mu)\sqrt{n} = \frac{1}{4}(2\gamma_2 - \gamma_1 + \gamma_3)\sqrt{n} = \bar{c}(\gamma)\sqrt{n}$$

where $\quad \bar{c}(\gamma) = \frac{1}{4}(2\gamma_2 - \gamma_1 + \gamma_3)$

$\therefore \quad c = c(\omega) \simeq \bar{c}(\gamma)\sqrt{n}$.

Let $\quad H_\mu = \{x+y+z = \mu\}$

$\quad B_\gamma = \{0 \le x \le \gamma_1, 0 \le y \le \gamma_2, 0 \le z \le \gamma_3\}$,

and let ℓ be the line on H_μ with

$$y = \bar{c}(\gamma) ;$$

ℓ splits $B_\gamma \cap H_\mu$ into the corresponding $h_1 \cup h_2$.

Denote by D_α the dilation by α : $D_\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$.

Finally, let P_1 denote the projection on the x-y plane,

P_2 on the y-z plane.

Projections and dilations commute!

We have:

(1) $B_\omega = D_{\sqrt{n}}(B_\gamma)$.

(2) $H_m \simeq D_{\sqrt{n}}(H_\mu)$, in the sense that the two planes are parallel, with distance ≤ 1 between them.

Similarly,

(3) $H_m \cap B_\omega \simeq D_{\sqrt{n}}(H_\mu \cap B_\gamma)$.

(4) $L \simeq D_{\sqrt{n}}(\ell)$ and

(5) $H_j \simeq D_{\sqrt{n}}(h_j)$, $j = 1,2$ (in the above sense).

It follows that

(6) $P_1(H_1) \simeq P_1(D_{\sqrt{n}}(h_1)) = D_{\sqrt{n}}(P_1(h_1))$ in the sense that the two are polygons (in the x-y plane) with parallel boundaries and the distance between two corresponding edges is less than 1. Similarly, $P_2(H_2) \simeq P_2(D_{\sqrt{n}}(h_2)) = D_{\sqrt{n}}(P_2(h_2))$.

To simplify $Y_4(\lambda)$, we need the following trivial:

<u>Lemma 1</u>: Let $\Omega \subset \mathbb{R}^2$ be a "nice" domain (a polygon, for example) and let $f(x,y)$ be continuous and homogeneous of degree ν on Ω : $f(\alpha x, \alpha y) = \alpha^\nu f(x,y)$. Then

$$\iint_{D_{\sqrt{n}}(\Omega)} f(x,y)\,dx\,dy = n\sqrt{n}^{\nu} \iint_{\Omega} f(u,v)\,du\,dv \simeq \sum_{(i_1,i_2)\in D_{\sqrt{n}}(\Omega)\cap\mathbb{Z}^2} f(i_1,i_2) .$$

Proof. Substitute $x = \sqrt{n}\,u$, $y = \sqrt{n}\,v$ to get = . Approximate $\iint_\Omega f(u,v)dudv$ by the Riemann sum determined by the lattice $\frac{1}{\sqrt{n}}\mathbb{Z}^2$ to get $\simeq$.

Lemma 2: Let Ω be a polygon, and A_n a sequence of polygons in $\mathbb{R}^2$ such that $A_n \simeq D_{\sqrt{n}}(\Omega)$ (in the sense defined previously). Let $f(x,y)$ be any continuous function, then

$$\iint_{D_{\sqrt{n}}(\Omega)} f(x,y)dxdy \simeq \iint_{A_n} f(x,y)dxdy .$$

Proof. By Lemma 1 (=), it is enough to show that

$$\iint_{D_{1/\sqrt{n}}(A_n)} f(u,v)dudv \xrightarrow[n\to\infty]{} \iint_\Omega f(u,v)dudv .$$

This is obvious, since by assumptions, $D_{1/\sqrt{n}}(A_n) \xrightarrow[n\to\infty]{} \Omega$, and f is continuous.

Corollary 3: Let Ω , A_n be polygons as in Lemma 2 , $f(x,y)$ as in Lemma 1, then

$$\sum_{(i_1,i_2)\in A_n\cap\mathbb{Z}^2} f(i_1,i_2) \simeq \iint_{D_{\sqrt{n}}(\Omega)} f(x,y)dxdy .$$

Special Cases 4: First, let $A_n = P_1(H_1) \simeq D_{\sqrt{n}}(P_1(h_1)) = P_1(H_1')$ with $f(x,y) = y-x$ $(\nu = 1)$, then (in 1.1)

$$\sum_{(i_1,i_2,i_3)\in S_1} (i_2-i_1) = \sum_{(i_1,i_2)\in P_1(S_1)} (i_2-i_1)$$

$$= \sum_{(i_1,i_2)\in P_1(H_1)\cap\mathbb{Z}^2} (i_2-i_1)$$

$$\simeq \iint_{P_1(H_1)} (y-x)dxdy \simeq \iint_{P_1(H_1')} (y-x)dxdy .$$

Similarly, $\sum_{(i)\in S_2} (i_3-i_2) \simeq \iint_{P_2(H_2)} (z-y)dydz \simeq \iint_{P_2(H_2')} (z-y)dydz$.

Next, choose $f(x,y) = 1$ $(\nu = 0)$ to conclude

$$\#S_1 = \sum_{(i)\in S_1} 1 \simeq \iint_{P_1(H_1')} dxdy = \text{Area}(P_1(H_1')) ,$$

and similarly,

$$\#(S_2 \setminus L) \simeq \#S_2 \simeq \text{Area}(P_2(H_2')) .$$

Thus
$$Y_4(\lambda) \simeq Y_4'(\lambda) = \omega_1\,\text{Area}(P_1(H_1')) + \omega_2\,\text{Area}(P_2(H_2')) + \iint_{P_1(H_1')} (y-x)dxdy + \iint_{P_2(H_2')} (z-y)dydz . \quad (1.2)$$

Recall: $P_j(H_j') = D_{\sqrt{n}}(P_j(h_j))$, $j = 1,2$,i.e. m can be replaced by $m' = \frac{1}{4}(\omega_1 + 2\omega_2 + 3\omega_3)$ and c by $c' = \frac{1}{4}(2\omega_2 - \omega_1 + \omega_3)$.

§2. THE MAIN RESULT FOR $Y_4(\lambda)$

The main result of this paper is

Theorem 5: Let $\lambda \in \Lambda_4(n,a,\delta)$ $(a,\delta > 0)$, $\lambda = (\omega_1+\ldots+\omega_4 , \omega_2+\omega_3+\omega_4 , \omega_3+\omega_4 , \omega_4)$, then for large n ,

$$Y_4(\lambda) \simeq \begin{cases} \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{8}\min(\omega_1,\omega_3)(\omega_1-\omega_3)^2 & \text{if } \omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right| \\ \frac{1}{2}\omega_2 \min(\omega_1,\omega_3)(\omega_2+\min(\omega_1,\omega_3)) \le \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2\min(\omega_1,\omega_3) & \text{if } \omega_2 \le \left|\frac{\omega_1-\omega_3}{2}\right| . \end{cases}$$

Note. It follows that for such λ's, $Y_4(\lambda) < \frac{1}{2}\omega_1\omega_2\omega_3$ (large n), hence $2Y_4(\lambda) < \omega_1\omega_2\omega_3$. By [2, Th.3.22], $\omega_1\omega_2\omega_3 \lesssim m_\lambda$ so for large n ,

$$\frac{2\sqrt{2}}{\pi}\cdot\frac{4}{\sqrt{\pi}}\cdot\frac{1}{n\sqrt{n}}\cdot 4^n \simeq 2\deg\psi_4(n) < c_n(F_2) .$$

Moreover, this inequality neglects at least the sums

$$2\left(\sum_{\omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right|} \frac{1}{8}\min(\omega_1,\omega_3)(\omega_1-\omega_3)^2 d_\lambda + \sum_{\omega_2 \leq \left|\frac{\omega_1-\omega_3}{2}\right|} \frac{1}{2}\omega_2^2 \min(\omega_1,\omega_3) d_\lambda\right)$$

which would increase the lower bound significantly. Their actual asymptotic computations involve complicated multi-integrals, yet to be done. This hints that the upper bound ($\simeq t_n(F_2)$) is, probably, the asymptotic value of $c_n(F_2)$.

The proof of Theorem 5 is based on formula (1.2), and is done by geometric methods. The proof itself falls into subdivisions corresponding to the various inequalities among $\omega_1, \omega_2, \omega_3$. By (1.2) we may assume

$$m = \frac{\omega_1 + 2\omega_2 + 3\omega_3}{4}$$

and $$c = \frac{1}{4}(2\omega_2 - \omega_1 + \omega_3) .$$

Therefore $$\left.\begin{array}{l} 0 < c \Leftrightarrow 2\omega_2 > (\omega_1-\omega_3) \\ \omega_2 > c \Leftrightarrow 2\omega_2 > (\omega_3-\omega_1) \end{array}\right\} \Leftrightarrow \omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right| .$$

Thus, the case $\omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right|$ corresponds to $0 < c < \omega_2$,

and $\omega_2 \leq \left|\frac{\omega_1-\omega_3}{2}\right|$ to $c \leq 0$ or $\omega_2 \leq c$.

The case $\omega_2 \leq \left|\frac{\omega_1-\omega_3}{2}\right|$

Here there are two subcases, (α) and (β):

(α) $\omega_1 \geq \omega_3$, so $0 < 2\omega_2 \leq \omega_1 - \omega_3$, therefore

$$2\omega_2 + \omega_3 \leq \omega_1 \tag{2.1}$$

and thus $c = \frac{1}{4}(2\omega_2 - \omega_1 + \omega_3) \leq 0$,

hence $P_1(H_1') = \emptyset$

and $$Y_4'(\lambda) = \omega_2 \operatorname{Area}(P_2(H_2')) + \iint_{P_2(H_2')} (z-y)\,dy\,dz .$$

Since $$m - \omega_1 = \frac{1}{4}(2\omega_2 - 3(\omega_1 - \omega_3)) \leq \frac{1}{4}(2\omega_2 - (\omega_1 - \omega_3)) \leq 0$$

and $$m - \omega_3 = \frac{1}{4}(2\omega_2 + \omega_1 - \omega_3) \geq \frac{1}{4}(2\omega_2 + 2\omega_2) = \omega_2 \geq 0 ,$$

therefore $\omega_1 \geq m \geq \omega_2 + \omega_3$.

Thus Figure 1 becomes Figure 2a:

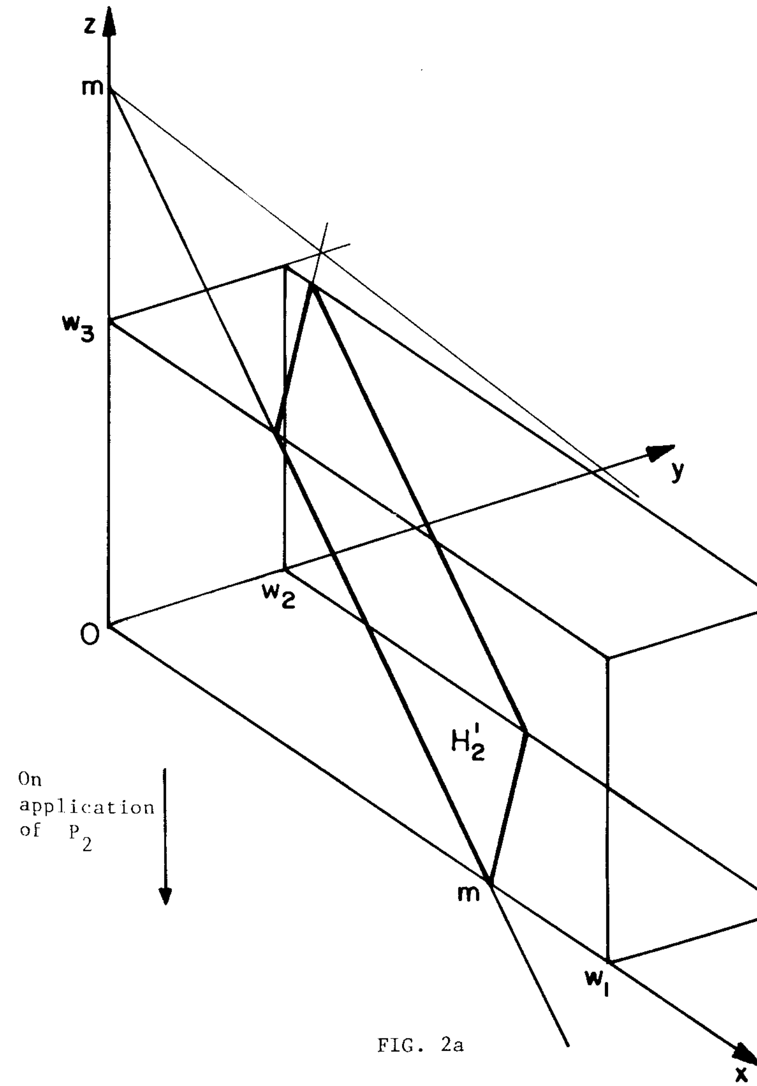

FIG. 2a

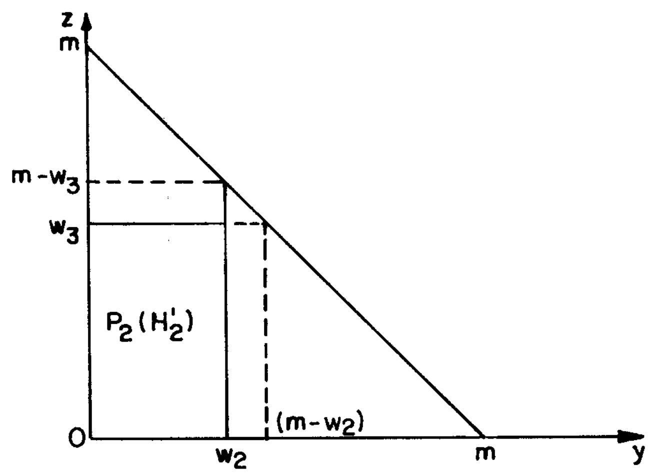

FIG. 2b

Clearly, $\omega_2 \text{ Area}(P_2(H_2')) = \omega_2^2 \omega_3$

and $\iint_{P_2(H_2')} (z-y)dydz = \int_0^{\omega_3}\int_0^{\omega_2} (z-y)dydz =$

$$= \int_0^{\omega_3} (\omega_2 z - \frac{1}{2}\omega_2^2)dz = \frac{1}{2}(\omega_2\omega_3^2 - \omega_3\omega_2^2) ,$$

so that $Y_4'(\lambda) = \omega_2^2\omega_3 + \frac{1}{2}(\omega_2\omega_3^2 - \omega_3\omega_2^2) = \frac{1}{2}(\omega_2\omega_3^2 + \omega_2^2\omega_3) =$

$$= \frac{1}{2}\omega_2\omega_3(\omega_2 + \omega_3) \leq \frac{1}{2}\omega_2\omega_3(\omega_1 - \omega_2) =$$

$$= \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2\omega_3 . \qquad \text{(by (2.1)]}$$

For (α), $\omega_3 = \min(\omega_1,\omega_3)$, therefore

$$Y_4(\lambda) \simeq Y_4'(\lambda) = \frac{1}{2}\omega_2 \min(\omega_1,\omega_3)(\omega_2 + \min(\omega_1,\omega_3))$$

$$\leq \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2 \min(\omega_1,\omega_3) ,$$

as required.

(β) $\omega_3 > \omega_1$, $0 < 2\omega_2 \leq \omega_3 - \omega_1$. Then

$$2\omega_2 + \omega_1 \leq \omega_3 \qquad (2.2)$$

and thus $c \geq \omega_2$ $(c = \frac{1}{4}(2\omega_2 - \omega_1 + \omega_3))$. Hence, for this case,

$P_2(H_2') = \emptyset$, and

$$Y_4'(\lambda) = \omega_1 \text{ Area}(P_1(H_1')) + \iint_{P_1(H_1')} (y-x)dxdy .$$

Here $m - \omega_3 = \frac{1}{4}(2\omega_2 + \omega_1 - \omega_3) \leq 0$

and $m - \omega_1 = \frac{1}{4}(2\omega_2 - 3(\omega_1 - \omega_3)) > \frac{1}{4}(2\omega_2 + 6\omega_2) > 2\omega_2 > 0$ [by(2.2)]

hence $\omega_3 \geq m > \omega_1 + \omega_2$.

Figure 1 now becomes Figure 3a:

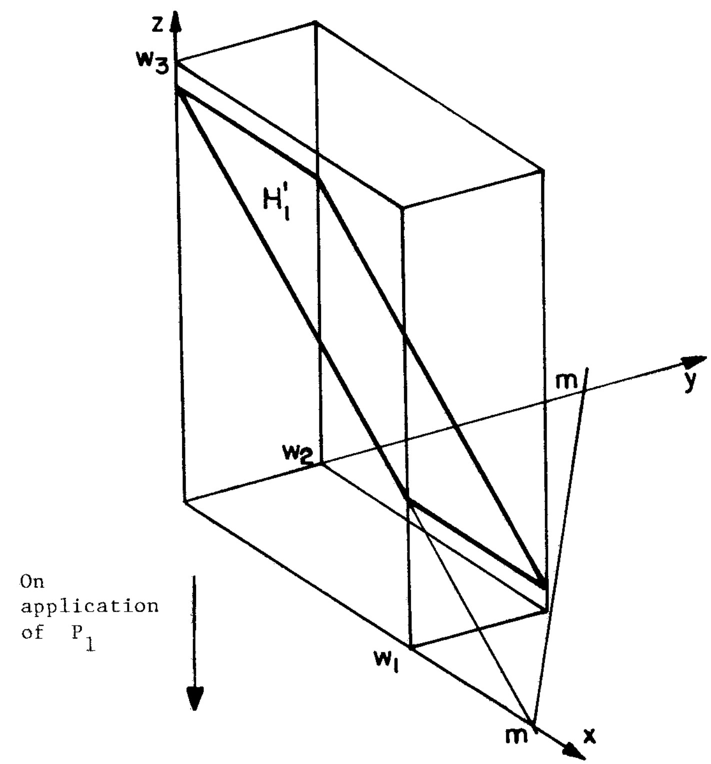

FIG. 3a

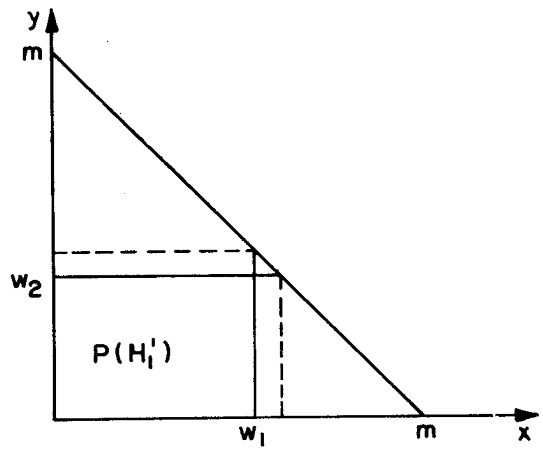

FIG. 3b

Now, $\omega_1 \text{ Area}(P_1(H_1')) = \omega_1^2\omega_2$

and $\iint_{P_1(H_1')} (y-x)\,dxdy = \frac{1}{2}(\omega_1\omega_2^2 - \omega_1^2\omega_2)$

so $Y_4'(\lambda) = \frac{1}{2}\omega_1\omega_2(\omega_2+\omega_1) \leq \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2\omega_1$. [by (2.2)]

For (β), $\omega_1 = \min(\omega_1,\omega_3)$, therefore

$$Y_4(\lambda) \simeq Y_4'(\lambda) = \frac{1}{2}\omega_2 \min(\omega_1,\omega_3)(\omega_2+\min(\omega_1,\omega_3)) \leq \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2 \min(\omega_1,\omega_3)$$

as required.

(α) and (β) complete the case $\omega_2 \leq \left|\frac{\omega_1-\omega_3}{2}\right|$:

$$Y_4(\lambda) \simeq \frac{1}{2}\omega_2[\min(\omega_1,\omega_3)](\omega_2+\min(\omega_1,\omega_3)) \leq \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{2}\omega_2^2\min(\omega_1,\omega_3) \quad (2.3)$$

The case $\omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right|$

Here $0 \lneq c \lneq \omega_2$, hence $P_i(H_i') \neq \emptyset$, $i=1,2$. Also, each $P_i(H_i')$ might be a polygon more complicated than the previous rectangles. For these reasons, the evaluation of $Y_4'(\lambda)$ in this case, although similar in nature to the previous calculations, is longer and much more involved.

It is subdivided into the subcases $\omega_1 \geq \omega_3$ or $\omega_1 < \omega_3$. We did it by drawing $P_i(H_i')$, $i=1,2$, on the same plane, then evaluating the corresponding expression for $Y_4'(\lambda)$. This involved the computation, by Green's theorem, of $\iint_D (x_1-x_2)dx_1dx_2$ over the various possible polygons $D = P_i(H_i')$. We spare the reader the details, which lead to
$Y_4'(\lambda) \simeq \frac{1}{2}\omega_1\omega_2\omega_3 - \frac{1}{8}\min(\omega_1,\omega_3)\cdot(\omega_1-\omega_3)^2$, if $\omega_2 > \left|\frac{\omega_1-\omega_3}{2}\right|$, and thus completes the proof of Theorem 5.

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