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On Maximal Degrees for Young Diagrams

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INTRODUCTION

In [1] the problem of maximising the degrees associated with Young diagrams in a strip was studied, one of the essential techniques being the application of asymptotic methods. To this end, the first step of [1] in the study of arbitrary partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n with at most k parts (i.e. $\lambda \in \Lambda_k(n)$) was to restrict attention to partitions where all the λ_i's and the $\lambda_i - \lambda_j$'s, $i < j$, are large. This enabled one to asymptotically find the partition λ for which the degree d_λ is maximal.

The present note complements [1] in that it gives a detailed and complete justification of this first step.

In particular, we begin by constructing an explicit and simple algorithm which increases the degree d_λ by transforming an arbitrary $\lambda \in \Lambda_k(n)$ into a final $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda_k(n)$ with all the $\mu_i - \mu_j$, $i < j$, relatively small, in a sense to be made precise.

We then complete our analysis of the first step of [1] by an analytic argument.

THE ALGORITHM

Recall that $\Lambda_k(n) = \{\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k \mid \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0, \lambda_1 + \cdots + \lambda_k = n\}$. Define, for $a \geqslant 0$, $\Lambda_k(a; n) \subseteq \Lambda_k(n)$ by

$$\Lambda_k(a; n) = \{\lambda \in \Lambda_k(n) \mid \lambda_i - \lambda_j \leqslant a\sqrt{n},\ 1 \leqslant i \leqslant j \leqslant k\}.$$

In the following, we shall prove:

(1) PROPOSITION *Let $\lambda \in \Lambda_k(n)$, then there exists $\mu \in \Lambda_k(2\sqrt{k}; n)$ such that $d_\lambda \leqslant d_\mu$.*

(2) ALGORITHM *Let $\lambda \in \Lambda_k(n)$, then*:
(*i*) *if $\lambda \in \Lambda_k(2\sqrt{k}; n)$, we are done*;
(*ii*) *assume now that $\lambda \notin \Lambda_k(2\sqrt{k}; n)$.*

Let l be the smallest integer in $1 \leqslant l \leqslant k-1$ such that $\alpha_l - \alpha_{l+1} \geqslant (2\sqrt{k})\sqrt{n}$ ($\alpha_j = \alpha_j(\lambda) = \lambda_j + k - j, j = 1, \ldots, k$), then we transform λ to λ', where

$$\lambda' = (\lambda'_1, \ldots, \lambda'_k) = (\lambda_1 - 1, \ldots, \lambda_l - 1, \lambda_{l+1} + l, \lambda_{l+2}, \ldots, \lambda_k).$$

Note that, as n is large, $\lambda_l - \lambda_{l+1} \geqslant l + 1$, so that $\lambda' \in \Lambda_k(n)$.

(3) REMARKS (i) Let $\alpha'_j = \lambda'_j + k - j, j = 1, \ldots, k$, then $\alpha' = \alpha(\lambda') = (\alpha_1 - 1, \ldots, \alpha_l - 1, \alpha_{l+1} + l, \alpha_{l+2}, \ldots, \alpha_k)$.

(ii) Note that $\lambda'_1 - \lambda'_k \not\leqslant \lambda_1 - \lambda_k$. Thus, if we apply the algorithm repeatedly to $\lambda \in \Lambda_k(n)$, the algorithm has to stop after $\leqslant \lambda_1 - \lambda_k$ steps. If μ is the biproduct partition then $\mu \in \Lambda_k(2\sqrt{k}; n)$.

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We prove proposition 1 by proving:

(3) LEMMA *Let $\lambda \to \lambda'$ as in step (ii) of the algorithm, then $d_\lambda \leqslant d_{\lambda'}$.*

PROOF By the Young–Frobenius formula,

$$d_\lambda = \frac{n!}{\alpha_1! \cdots \alpha_k!} \prod_{i<j} (\alpha_i - \alpha_j),$$

and similarly for $d_{\lambda'}$.

Thus

$$d_{\lambda'}/d_\lambda = a \cdot b$$

where

$$a = \frac{\alpha_1! \cdots \alpha_{l+1}!}{(\alpha_1 - 1)! \cdots (\alpha_l - 1)!(\alpha_{l+1} + l)!} = \prod_{i=1}^{l} \frac{\alpha_i}{\alpha_{l+1} + i},$$

and

$$b = \prod_{i<j} \frac{\alpha_i' - \alpha_j'}{\alpha_i - \alpha_j}.$$

If $i < j < l + 1$, or $l + 1 < i < j$, then $\alpha_i' - \alpha_j' = \alpha_i - \alpha_j$. Thus,

$$b = \prod_{i<l+1} \left(\frac{\alpha_i - \alpha_{l+1} - (l+1)}{\alpha_i - \alpha_{l+1}} \right) \cdot \prod_{l+1<j} \left(\frac{\alpha_{l+1} - \alpha_j + l}{\alpha_{l+1} - \alpha_j} \right) \cdot \prod_{i<l+1<j} \left(\frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right).$$

Clearly, for all $l + 2 \leqslant j$,

$$\frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \geqslant \frac{\alpha_l - \alpha_j - 1}{\alpha_l - \alpha_j}$$

so that

$$\prod_{i=1}^{l} \left(\frac{\alpha_i - \alpha_j - 1}{\alpha_i - \alpha_j} \right) \geqslant \left(\frac{\alpha_l - \alpha_j - 1}{\alpha_l - \alpha_j} \right)^{l}.$$

By a trivial induction on t:

$$\left(\frac{\alpha_l - \alpha_j - 1}{\alpha_l - \alpha_j} \right)^{t} \geqslant \frac{\alpha_l - \alpha_j - t}{\alpha_l - \alpha_j}$$

and hence

$$\left(\frac{\alpha_l - \alpha_j - 1}{\alpha_l - \alpha_j} \right)^{l} \geqslant \frac{\alpha_l - \alpha_j - l}{\alpha_l - \alpha_j}.$$

From the above calculations, it follows that

$$d_{\lambda'}/d_\lambda \geqslant A \, . \, B$$

where

$$A = \prod_{i=1}^{l} A_i, \qquad A_i = \frac{\alpha_i}{\alpha_{l+1} + i} \cdot \frac{(\alpha_i - \alpha_{l+1} - (l+1))}{(\alpha_i - \alpha_{l+1})};$$

$$B = \prod_{j=l+2}^{k} B_j, \qquad B_j = \frac{\alpha_{l+1} - \alpha_j + l}{\alpha_{l+1} - \alpha_j} \cdot \frac{\alpha_l - \alpha_j - l}{\alpha_l - \alpha_j}.$$

We now show that $A_i, B_j \geqslant$
$A_i \geqslant 1$: this is equivalent to s

$(\alpha_i - \alpha_{l+1}$

Since $i, l + 1 < k$ and $\alpha_i \leqslant$
But $x^2 = (\alpha_i - \alpha_{l+1})^2 \geqslant 4kn$, a
Thus (*) holds and $A_i \geqslant 1$ for
$B_j \geqslant 1$: by assumption, n is la
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Therefore, we consider now

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$\Sigma_{i=1}^{k-1} i(c_i - c_{i+1}) + kc_k$ which
$-(k - 1)\sqrt{k}$, and hence

$\mu_1 \geqslant$

(6) COROLLARY. *Let $\mu \in \Lambda_k$*
large. Thus Stirling's formula c
We are now ready to prove:

(7) THEOREM. *Let n be larg*

$\mu_i =$

and denote $g_k(n) = (1/\sqrt{2\pi})^{k-1}$

$\|c\|^2 =$

then

d_μ

$= g_k(n) \cdot e^{-\frac{k}{2}\|c\|^2} \cdot$

In particular, it follows that
the function

$f(c_1, .$

obtains its maximum.

We now show that A_i, $B_j \geqslant 1$ for all i, j

$A_i \geqslant 1$: this is equivalent to showing that

$$(\alpha_i - \alpha_{l+1})^2 - i(\alpha_i - \alpha_{l+1}) - (l+1)\alpha_i \geqslant 0. \qquad (*)$$

Since i, $l + 1 < k$ and $\alpha_i \leqslant n$, (*) holds if $x^2 - kx - kn \geqslant 0$, where $x = \alpha_i - \alpha_{l+1}$. But $x^2 = (\alpha_i - \alpha_{l+1})^2 \geqslant 4kn$, and hence $x^2 - kx - kn \geqslant 3kn - kx > 0$, since $x \leqslant n$. Thus (*) holds and $A_i \geqslant 1$ for $i = 1, \ldots, l$.

$B_j \geqslant 1$: by assumption, n is large, so that $2\sqrt{kn} \geqslant l + 1$, and thus $\alpha_l - \alpha_{l+1} \geqslant l$. Using this, a direct calculation gives $B_j \geqslant 1$ for $j = l + 2, \ldots, k$.

This completes the proof of Lemma 4. □

THE MAIN RESULT

The main result of the present note is the theorem stated below, which is proven using the algorithm of the preceding section, supplemented by an analytic argument.

Therefore, we consider now $\mu \in \Lambda_k(2\sqrt{k}; n)$. As usual, we write $\mu = (\mu_1, \ldots, \mu_k)$,

$$\mu_j = \frac{n}{k} + c_j \cdot \sqrt{n}, \qquad j = 1, \ldots, k,$$

and hence $c_i - c_j \leqslant 2\sqrt{k}$ for all $1 \leqslant i, j \leqslant k$. Since $\Sigma_{i=1}^{k} c_i = 0$ and $\Sigma_{i=1}^{k} c_i = \Sigma_{i=1}^{k-1} i(c_i - c_{i+1}) + kc_k$ which is at most $\Sigma_{i=1}^{k-1} i2\sqrt{k} + kc_k$, it follows that $c_k \geqslant -(k-1)\sqrt{k}$, and hence

$$\mu_1 \geqslant \cdots \geqslant \frac{n}{k} - (k-1)\sqrt{kn} \simeq \frac{n}{k}.$$

(6) COROLLARY. *Let $\mu \in \Lambda_k(2\sqrt{k}; n)$ and assume that n is large; then $\mu_1, \ldots, \mu_k$ are all large. Thus Stirling's formula can be applied to $n!$ and to each $(\mu_i + k - i)!$*

We are now ready to prove:

(7) THEOREM. *Let n be large, $\mu \in \Lambda_k(2\sqrt{k}; n)$, $\alpha_i = \mu_i + k - i$,*

$$\mu_i = \frac{n}{k} + c_i\sqrt{n}, \qquad i = 1, \ldots, k$$

and denote $g_k(n) = (1/\sqrt{2\pi})^{k-1} \cdot k^{n+\frac{1}{2}k^2} \cdot (1/n)^{\frac{1}{2}(k^2-1)}$:

$$\|c\|^2 = c_1^2 + \cdots + c_k^2, \qquad \theta = \tfrac{1}{2}k(k-1)$$

then

$$d_\mu \simeq g_k(n) \cdot \mathrm{e}^{-\frac{k}{2}\|c\|^2} \cdot \prod_{i<j} (\alpha_i - \alpha_j)$$

$$= g_k(n) \cdot \mathrm{e}^{-\frac{k}{2}\|c\|^2} \cdot \prod_{i<j} (c_i - c_j)\sqrt{n}^{\theta} + g_k(n) \cdot \mathrm{e}^{-\frac{k}{2}\|c\|^2} \cdot O(\sqrt{n}^{\theta-1}).$$

In particular, it follows that $\max\{d_\lambda | \lambda \in \Lambda_k(n)\}$ is obtained at $\mu \in \Lambda_k(2\sqrt{k}; n)$, for which the function

$$f(c_1, \ldots, c_k) = \mathrm{e}^{-\frac{k}{2}\|c\|^2} \cdot \prod_{i<j} (c_i - c_j)$$

obtains its maximum.

Proof We have

$$\alpha_i - \alpha_j = (c_i - c_j)\sqrt{n} + j - i,$$

so

$$|(\alpha_i - \alpha_j) - (c_i - c_j)\sqrt{n}| = |j - i| < k.$$

By expanding

$$\prod_{i<j} (\alpha_i - \alpha_j) = \prod_{i<j} ((c_i - c_j)\sqrt{n} + j - i)$$

we obtain

$$\left|\prod_{i<j} (\alpha_i - \alpha_j) - \prod_{i<j} (c_i - c_j) \cdot \sqrt{n}^{\frac{1}{2}k(k-1)}\right| \leqslant p(c_1, \ldots, c_k)\sqrt{n}^{\frac{1}{2}k(k-1)-1} \tag{1}$$

where $p(c_1, \ldots, c_k)$ is a polynomial function of the differences $c_i - c_j$. Since the $c_i - c_j$'s are bounded ($\mu \in \Lambda_k(2\sqrt{k}; n)$), $p(c_1, \ldots, c_k)$ is bounded.

It follows that:

$$\prod_{i<j} (\alpha_i - \alpha_j) = \left(\prod_{i<j} (c_i - c_j)\right) \cdot \sqrt{n}^{\theta} + O(\sqrt{n}^{\theta-1}), \qquad \theta = \tfrac{1}{2}k(k-1). \tag{2}$$

Clearly, all the $\mu_1, \ldots, \mu_k$ are large, and so Stirling's formula is valid. By [2, p. 118, Step 3]

$$n! \Big/ \prod_{i=1}^{k} \alpha_j! \simeq (1/\sqrt{2\pi})^{k-1} \cdot k^{n+\frac{1}{2}k^2} \cdot (1/n)^{\frac{1}{2}(k^2-1)} \cdot \mathrm{e}^{-\frac{1}{2}k(c_1^2 + \cdots + c_k^2)}. \tag{3}$$

The theorem now follows on combining (2) and (3) and the Young–Frobenius formula for d_μ. □

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