ASYMPTOTICS OF COMBINATORIAL SUMS AND
THE CENTRAL LIMIT THEOREM

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Abstract. The aim of this paper is to generalize the results of Regev [Adv. in Math., 41 (1981), pp. 115–136] and to simplify the proofs by deducing them from the Central Limit Theorem of probability theory. The generalized results also yield a method for calculating certain multi-integrals, some of which seem highly nontrivial.

Key words. sums, asymptotics, multi-integrals, $S_n$ characters

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1. Introduction. Let $k > 0$ be an integer and $\beta > 0$ a real number. For integers $n \to \infty$, the asymptotic behavior of the sum

$$S_k^{(\beta)}(n) = \sum_{\lambda \in \Lambda_k(n)} d_\lambda^{(\beta)},$$

where $\Lambda_k(n) = \{\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{Z}_k^k | \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_k = n\}$ and $d_\lambda$ is the number of standard Young tableaux of shape $\lambda$, was studied in [12].

This was essentially done as follows. We began with a family $\{\Lambda_k(n, \rho)\}_{\rho \in \mathbb{R}}$ (see § 5) of subsets of $\Lambda_k(n)$, chosen so that for each fixed $\rho$, and each $\lambda \in \Lambda_k(n, \rho)$, the $d_\lambda$, and hence the $d_\lambda^{(\beta)}$, could be evaluated asymptotically as $n \to \infty$. The sums of $d_\lambda^{(\beta)}$ over $\lambda \in \Lambda_k(n, \rho)$, converging to $S_k^{(\beta)}(n)$ as $\rho \to \infty$, were then approximated by certain integrals from which we obtained the asymptotic values of the $S_k^{(\beta)}(n)$.

Examples of such sums, having their origin in combinatorics and in algebras that satisfy polynomial identities, are given in § 5 (see also [2]).

In this paper the set $\Lambda_k(n)$ is replaced by

$$A_k(n) = \{\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}_k^k | \alpha_1 \geq 0, \cdots, \alpha_k \geq 0; \alpha_1 + \cdots + \alpha_k = n\}.$$

We denote

$$\binom{n}{\alpha} = \frac{n!}{\prod_{j=1}^{k} \alpha_j !}$$

and study the asymptotics of sums of the form

$$\sum_{\alpha \in \Lambda_k(n)} f(\alpha) \binom{n}{\alpha}^{(\beta)}$$

for certain functions $f: \bigcup_{n \geq 0} A_k(n) \to \mathbb{R}$.

In § 5 we show that these latter sums generalize the sums $S_k^{(\beta)}(n)$.

In § 2 we state the main result for these generalized sums, the proof of which is given in §§ 3 and 4. In particular, in § 3 we reduce the proof for arbitrary $\beta > 0$ to the case $\beta = 1$, and this case we treat in § 4 using the Central Limit Theorem of probability theory.

2. The main result. For positive integers $k$, $n$, and $\beta > 0$,

$$A_k(n, \rho) = \{\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}_k^k | \alpha_1 \geq 0, \cdots, \alpha_k \geq 0; \alpha_1 + \cdots + \alpha_k = n, \rho \alpha_1 = \cdots = \rho \alpha_k\}$$

and $c_\rho(\alpha) = (c_\rho(\alpha_1), \cdots, c_\rho(\alpha_k))$.

Given a real number $\rho > 0$,

$$C_k(\rho) = \{c_\rho(\alpha) | \alpha \in A_k(n, \rho)\}$$

and

$$A_k(n, \rho) = \bigcup_{c_\rho(\alpha) \in C_k(\rho)} A_k(n, \rho).$$

DEFINITION. A function $h: \mathbb{R} \to \mathbb{R}$ is
(i) There exists a polynomial $p\in A_k(n)$ in $\Lambda_k(n)$

(ii) Given $\rho > 0$,

By this we mean the following: $p(\alpha) / \rho^{\alpha}$, so that if $n \geq N$ then

for all $\alpha \in A_k(n, \rho)$.

Throughout this article, for the notation

$$f(\alpha) = \text{g(\alpha)}$$

means that there exists a permissible $g\in A_k(n)$ such that for $n \geq N_0$ and all $\alpha$ in $\Lambda_k(n)$

$$|f(\alpha) - g(\alpha)| \to 0$$

as $n \to \infty$.
The approach of this article is much simpler than that of [12] and it allows us to deduce a general theorem about the asymptotics of such sums. All asymptotics of related sums known to us are particular cases of that theorem. Our general results also yield a method for calculating certain multi-integrals, some of which seem highly nontrivial. In § 5 we give illustrative examples of such applications.

Recently, Macdonald found some very interesting identities that involve certain multi-integrals [7]. The evaluation of these integrals is done by the Selberg formula [15]. Some of the integrals evaluated in § 5 constitute partial generalizations of these “Macdonald” (or “Mehta”) integrals but it seems that their evaluation cannot be obtained from the Selberg integral.

2. The main result. For positive integers \( k, n \) we write

\[
A_k(n) = \{ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k | \alpha_1 \geq 0, \ldots, \alpha_k \geq 0; \alpha_1 + \cdots + \alpha_k = n \},
\]

\[
A_k = \bigcup_{n \geq 0} A_k(n).
\]

For \( \alpha = (\alpha_1, \ldots, \alpha_k) \) in \( A_k(n) \), let

\[
c_j = c(\alpha) = \frac{1}{\sqrt{n}} \left( \frac{\alpha_j - \frac{n}{k}}{\frac{n}{k}} \right), \quad j = 1, \ldots, k,
\]

and \( c(\alpha) = (c_1, \ldots, c_k) \).

Given a real number \( \rho > 0 \) we write

\[
C_k(\rho) = \{(c_1, \ldots, c_k) \in \mathbb{R}^k | |c_j| \leq \rho, j = 1, \ldots, k \}
\]

and

\[
A_k(n, \rho) = \{ \alpha \in A_k(n) | c(\alpha) \in C_k(\rho) \}.
\]

Definition. A function \( h : A_k \to \mathbb{R} \) is defined to be permissible if

(i) There exists a polynomial \( p(x) = p(x_1, \ldots, x_k) \) such that for all \( n \) and all \( \alpha \) in \( A_k(n) \)

\[
|h(\alpha)| \leq |p(c(\alpha))|.
\]

(ii) Given \( \rho > 0 \),

\[
\lim_{\alpha \in A_k(n, \rho)} h(\alpha) = 1.
\]

By this we mean the following: given \( \varepsilon > 0 \), there exists an integer \( N = N(\varepsilon, \rho) \) such that if \( n \geq N \) then

\[
|h(\alpha) - 1| < \varepsilon
\]

for all \( \alpha \in A_k(n, \rho) \).

Throughout this article, for functions \( f : A_k \to \mathbb{R} \), \( g : \mathbb{R}^k \to \mathbb{R} \), and \( \gamma \) a real number the notation

\[
f(\alpha) \approx g(c(\alpha)) \cdot n^\gamma, \quad \alpha \in A_k(n),
\]

means that there exists a permissible function \( h : A_k \to \mathbb{R} \) and real numbers \( \theta > 0, N_0 > 0 \) such that for \( n \geq N_0 \) and all \( \alpha \) in \( A_k(n) \)

\[
|f(\alpha) - h(\alpha)g(c(\alpha)) \cdot n^\gamma| < n^{\gamma - \theta}.
\]
For technical reasons we introduced the above relation \( = \) rather than the conventional \( \cong \); recall \( a_n = b_n \) if \( \lim_{n \to \infty} (a_n/b_n) = 1 \). The results of subsequent sections include the fact that if \( f(\alpha) = g(c(\alpha)) \cdot n^\gamma \), then

\[
\sum_{\alpha \in A_k(n)} f(\alpha) (n/\alpha)^\alpha = \sum_{n=1}^{\infty} g(c(\alpha)) \cdot n^\gamma \cdot \left( \frac{n}{\alpha} \right)^\alpha.
\]

Remark. Let \( p(x_1, \ldots, x_k) \) be a polynomial. By choosing \( \rho \) large enough, the integral

\[
\int_{\mathbb{R}^k} \left| p(x_1, \ldots, x_k) \right| \exp \left( -\sum x_i^2 \right) d^k x
\]

can be made arbitrarily small. This is the only fact about \( p(x) \) which will be used in the sequel (see § 4).

For this reason, one could weaken the condition that \( p(x) \) is a polynomial, by requiring that \( p(x) \) be a function which satisfies the above property.

The main result of the present article is as follows.

Theorem 1. Let \( \gamma \) be a real number and

\[
f : A_k \to \mathbb{R}, \quad g : \mathbb{R}^k \to \mathbb{R}
\]

be functions such that \( g \) is continuous almost everywhere and

\[
f(\alpha) = g(c(\alpha)) \cdot n^\gamma, \quad \alpha \in A_k(n).
\]

Then for \( \beta > 0 \) real

\[
\lim_{n \to \infty} \sum_{\alpha \in A_k(n)} f(\alpha) \left( \frac{n}{\alpha} \right)^\alpha n^{-\gamma + (1/2)(\beta - 1)(k-1)} k^{-\beta n}
\]

\[
= \left( \frac{1}{2\pi} \right)^{k(1/2)(\beta - 1)(k-1)} k^{1/2\beta k} \int_{x_1, \ldots, x_k = 0} \int_{\mathbb{R}^k} g(x) \exp \left( -\frac{1}{2} \beta k (x_1^2 + \cdots + x_k^2) \right) dx_1 \cdots dx_{k-1}
\]

whenever the integral on the right exists.

In § 3 we reduce the proof of Theorem 1 to the case \( \beta = 1 \), which we in turn handle in § 4 using the Central Limit Theorem of probability theory.

We will also encounter the following easy variants of Theorem 1.

Variation 1. In the situation of Theorem 1, let \( D \) be a fixed domain in \( \mathbb{R}^k \) and let

\[
D(n) = \{ \alpha \in A_k(n) \mid c(\alpha) \in D \}.
\]

A simple modification of the proof of Theorem 1 yields

\[
\lim_{n \to \infty} \sum_{\alpha \in A_k(n) \cap D(n)} f(\alpha) \left( \frac{n}{\alpha} \right)^\alpha n^{-\gamma + (1/2)(\beta - 1)(k-1)} k^{-\beta n}
\]

\[
= \left( \frac{1}{2\pi} \right)^{k(1/2)(\beta - 1)(k-1)} k^{1/2\beta k} \int_{x_1, \ldots, x_k = 0} \int_{\mathbb{R}^k} g(x) \exp \left( -\frac{1}{2} \beta k (x_1^2 + \cdots + x_k^2) \right) dx_1 \cdots dx_{k-1}
\]

whenever the integral on the right exists.

Variation 2. In the situation where \( g(x_1, \ldots, x_k) \) is a function of the form \( \prod_{i < j} g_{x_i - x_j} \) we may use Lemma 4.3 of [12] to obtain

\[
\int_{x_1, \ldots, x_k = 0} \int_{\mathbb{R}^k} g(x) \exp \left( -\frac{1}{2} \beta (x_1^2 + \cdots + x_k^2) \right) dx_1 \cdots dx_{k-1}
\]

\[
= \frac{\sqrt{\beta}}{2\pi} \int_{\mathbb{R}^k} g(x) \exp \left( -\frac{1}{2} \beta x^2 \right) dx
\]

whenever the integral on the right exists.

We conclude this section with an example which we encounter in the proof of Theorem 1.

Example 1. For \( \alpha = (\alpha_1, \ldots, \alpha_k) \), let \( \alpha_i = 1 \), \( i = 1, \ldots, k \), as above and suppose \( \alpha_j = 1 \) for some \( 1 \leq j < k \).

We verify that

\[
\left( 1 + \frac{1}{2} \beta k \right) \left( 1 + \frac{1}{2} \beta (k-1) \right) \cdots \left( 1 + \frac{1}{2} \beta (k-j+1) \right) = \left( 1 + \frac{1}{2} \beta \right)^{k-j+1}
\]

are permissible.

Example 2. For \( b \) a real number,

\[
\Gamma(n + b + 1)
\]

Write

\[
\Gamma(n + b + 1)
\]

and

\[
h_{k,n}^b(\alpha) = h_{k,n}^b(\alpha_i)
\]

Clearly the \( h_{k,n}^b, j = 1, \ldots, k \), are permissible.

Example 3. For \( \alpha \in A_k(n) \), let

\[
\alpha_i - \alpha_j + i = [c(\alpha_i) - c(\alpha_j)] \sqrt{n};
\]

\[
\left[ c(\alpha_i) - c(\alpha_j) \right] \sqrt{n} = [j - i] < n^{1/2}
\]

Moreover, let \( r = 1/(k-j+1) \).

Let \( M(x_i - x_j) \), let \( M(x_i) \), and check that \( M(\alpha_i - \alpha_j + i) \).

degree \( d \). Clearly, \( M(\alpha_i - \alpha_j + i) \).

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By choosing $p$ large enough, the

$$-(\sum x_i^2)^{(k)} x$$

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in $A_k(n)$.

$-(k-1)\cdot k^{-\beta n} \cdot 
\int_{x_k=0}^{x_k=0} \int_{x_k=0}^{x_k=0} g(x) dx_k \cdots dx_k$,

the case $\beta = 1$, which we in turn probability theory,

ants of Theorem 1. $D$ be a fixed domain in $\mathbb{R}^k$ and let $\alpha \in D$.

$\Gamma(n + b - 1) = \sqrt{2\pi} \cdot e^{-n^{b+1/2}} \cdot n^{b-1/2}$.

Write

$$\Gamma(n + b + 1) = \sqrt{2\pi} \cdot e^{-n^{b+1/2}} \cdot n^{b-1/2}$$

and

$$h_n^{k+1}(\alpha) = h_n^{k+1}(\alpha_j), \quad \alpha = (\alpha_1, \ldots, \alpha_j) \in A_k, \quad j = 1, \ldots, k.$$

Clearly the $h_n^{k+1}, j = 1, \ldots, k$, are permissible functions.

**Example 3.** For $\alpha \in A_k(n)$ and $c(\alpha)$ as before, $\alpha - \alpha_j = [c(\alpha) - c(\alpha_j)] \sqrt{n}$, and $\alpha - \alpha_j = j - i \equiv [c(\alpha) - c(\alpha_j)] \sqrt{n}$; here $h(\alpha) = 1$, and $|[\alpha - \alpha_j]|/\sqrt{n} = |j - i|/n^{(3/2)} - \theta$ for any $0 < \theta < 1$ and $n$ large enough.

Moreover, let $r = \sqrt{k/(k-1)}$ and choose an ordering $\{x_i - x_j, 1 \leq i < j \leq k\} = \{y_1, \ldots, y_r\}$. Let $M(x_i - x_j) = M(y_1, \ldots, y_r)$ be any monomial in the $x_i - x_j$'s of degree $d$. Clearly, $M(\alpha - \alpha_j) = [M(c(\alpha) - c(\alpha_j)] n^{d/2}$, and it is easy to check that $M(\alpha - \alpha_j) = [M(c(\alpha) - c(\alpha_j)] n^{d/2}$.
For example, \( D_k(x_1, \ldots, x_k) = \prod_{i<j} (x_i - x_j) \) is of degree \( r \), hence \( D_k(\alpha_1 + k - 1, \alpha_2 + k - 2, \ldots, \alpha_k) = D_k(c(\alpha)) \cdot n^r = D_k(\alpha_1, \ldots, \alpha_k) \) (see § 5).

3. Reduction to the case \( \beta = 1 \). In this section we show that the proof of Theorem 1 may be reduced to the case \( \beta = 1 \). By Carlson’s theorem (see for example [3, p. 153, § 9.2]) it suffices to prove Theorem 1 for \( \beta \) a positive integer. To pass from this case to the case \( \beta = 1 \) we show, with notation as in § 2, the following proposition.

**Proposition.** Let \( \beta \) be a positive integer. For \( \gamma \) a real number and

\[
 f: A_k \to \mathbb{R}, \quad g: \mathbb{R}^k \to \mathbb{R},
\]

functions such that \( g \) is continuous almost everywhere and

\[
 f(\alpha) = g(c(\alpha)) \cdot n^\gamma, \quad \alpha \in A_k(n)
\]

we have

\[
 \sum_{\alpha \in A_k(n)} f(\alpha) \left( \begin{array}{c} n \\ \alpha \end{array} \right)^\beta \delta^\beta(k) n^{-\gamma(1/2(k-1)(\beta-1))} = \delta^\beta(k) \sum_{\alpha \in A_k(\beta n)} f^*(\alpha) n^{-\gamma} k^{-\beta n}
\]

where

\[
 \delta^\beta(k) = \left( \frac{1}{\sqrt{2\pi}} \right)^{(k-1)/2} \beta^{(k-1)/2} k^{(1/2)(\beta-1)}
\]

and

\[
 f^* : A_k \to \mathbb{R} \quad \text{satisfies}
\]

\[
 f^*(\alpha) = g\left( \frac{1}{\sqrt{\beta}} c(\alpha) \right) \cdot n^\gamma, \quad \alpha \in A_k(\beta n).
\]

We divide the proof of the proposition into two lemmas.

For \( \beta \) a positive integer and \( \alpha = (\alpha_1, \ldots, \alpha_k) \) in \( A_k(n) \) we denote by \( \beta \alpha \) the element \( (\beta \alpha_1, \ldots, \beta \alpha_k) \) of \( A_k(\beta n) \).

**Lemma 1.** Let \( \beta \) be a positive integer. For \( \alpha = (\alpha_1, \ldots, \alpha_k) \) in \( A_k(n) \) we have as \( n \to \infty \),

\[
 \left( \frac{n!}{\alpha_1! \cdots \alpha_k!} \right)^\beta = h^\beta_0(\alpha) \delta^\beta_0(k) n^{-(1/2(k-1)(\beta-1))} \left( \frac{(\beta n)!}{(\beta \alpha_1)! \cdots (\beta \alpha_k)!} \right)
\]

where

\[
 \delta^\beta_0(k) = \left( \frac{1}{\sqrt{2\pi}} \right)^{(k-1)/2} \beta^{(k-1)/2} k^{(1/2)(\beta-1)}
\]

and

\[
 h^\beta_0 : A_k \to \mathbb{R}
\]

is a permissible function.

**Proof.** In particular, for \( \alpha \in A_k(n) \), since \( 0! = 1 = 1 \), we may assume that \( \alpha_j \geq 1 \).

Applying Stirling’s formula as in Example 2 of § 2, respectively, to \( n!/(\alpha_1! \cdots \alpha_k!) \) and \( (\beta n)!/(\beta \alpha_1)! \cdots (\beta \alpha_k)! \) we deduce that as \( n \to \infty \)

\[
 \left( \frac{n!}{\alpha_1! \cdots \alpha_k!} \right) = h^\beta_0(\alpha) \delta^\beta_0(k) n^{(\beta-1)/2} \left( \prod_{j=1}^k \alpha_j \right)^{-(\beta-1)/2} \left( \frac{(\beta n)!}{(\beta \alpha_1)! \cdots (\beta \alpha_k)!} \right)
\]

where

\[
 h^\beta_0(\alpha) \in A_k \to \mathbb{R}
\]

and \( h^\beta_0(\alpha) \) is a permissible function.

To conclude the proof of the proposition we introduce random variables

\[
 P(X_1 = e_j) = \frac{1}{k}
\]

for \( \alpha \in A_k(n) \) we have

\[
 P(X_1 = e_j) = \frac{1}{k}
\]
is of degree \( r \); hence
\[ n^{r/2} = D_k(\alpha_1, \cdots, \alpha_k) \] (see § 5).

we show that the proof of Theorem 5.1 (see for example [3, p. 153, 4]) is permissible. To pass from this case to the following proposition.

\[ \gamma \text{ a real number and } \rightarrow \mathbb{R}. \]

\[ \text{x in } A_k(n) \]

Theorem

\[ (k) \sum_{a} f^*(a) \left( \frac{b n}{a} \right)^{-\gamma} k^{-\beta n} \]

\[ -1^{(1/2)} k(1/2)(\beta - 1) \]

\[ a \text{ in } A_k(\beta n). \]

\[ \text{a lemmas.} \]

1) in \( A_k(n) \) we denote by \( \beta \alpha \) the \( a_1, \cdots, a_k \) in \( A_k \) we have as \( n \to \infty \),

\[ (\beta n)! / (\beta a_1)_! \cdots (\beta a_k)_! \]

\[ 1^{(1/2)} k(1/2)(\beta - 1) \]

\[ \gamma = 1, \text{ we may assume that } \alpha_j \geq 1. \]

\[ \text{2, respectively, to } (n! / \alpha_1! \cdots \alpha_k!) \]

\[ \gamma - (1/2) / (\beta n)! / (\beta a_1)_! \cdots (\beta a_k)_! \]

where

\[ \delta_2^{(\beta)}(k) = \left( \frac{1}{\sqrt{2 \pi}} \right)^{(k-1)/2} \beta^{(k-1)/2} \]

and \( h_2^{(\beta)} \) is a permissible function.

To conclude the proof of the lemma, we apply Example 1 and the remark preceding it in § 2

\[ \left( \prod_{j=1}^{k} \alpha_j \right)^{-(\beta - 1)/2} \approx \left( \frac{n}{k} \right)^{-(k/2)(\beta - 1)} \]

We now note that to each \( a = (a_1, \cdots, a_k) \) in \( A_k(\beta n) \) we may associate a unique \( \alpha = \alpha(a) = (\alpha_1, \cdots, \alpha_k) \) in \( A_k(n) \) satisfying

\[ \beta a_j = \alpha_j < \beta \alpha_j + \beta, \quad j = 1, \cdots, k - 1, \]

\[ \alpha_k = n - (\alpha_1 + \cdots + \alpha_{k-1}) \]

Each \( \alpha \) in \( A_k(n) \) is associated in this way to exactly \( \beta^{k-1} \) elements of \( A_k(\beta n) \).

We can easily check that as \( n \to \infty \)

\[ c_j(a) = \sqrt{\beta}, j = 1, \cdots, k, \]

in the notation of § 2.

With the correspondence \( \alpha = \alpha(a) \) above we show the following lemma.

**Lemma 2.** For all \( a \) in \( A_k(\beta n) \)

\[ \left( \frac{\beta n}{\beta \alpha} \right) = h_2^{(\beta)}(a) \left( \beta n \right) \]

where \( \alpha = \alpha(a) \) and \( h_2^{(\beta)} \) is a permissible function.

**Proof.** The lemma follows on applying Stirling's formula as in Example 2 of § 2 to both sides of the above equation, and from Example 1 of § 2 and the remark preceding it. The main point is that the distance between \( a \) and \( \beta \alpha \) is uniformly bounded for all \( a \) in \( A_k(\beta n) \).

The proposition now follows after applying Lemmas 1 and 2, being careful to note the remarks preceding Lemma 2.

If Theorem 1 holds for \( \beta = 1 \) we may apply it to the right-hand sum of the proposition to deduce the result of Theorem 1 for \( \beta \) a positive integer, and hence, by the remarks at the beginning of the section, for all real \( \beta > 0 \).

4. **Proof of the theorem in the case** \( \beta = 1 \). By the results of § 3 it suffices to consider the case \( \beta = 1 \). This case is a straightforward application of the Central Limit Theorem of probability and its proof is due to Gideon Schechman.

Let \( e_i, i = 1, \cdots, k \), be the vector in \( \mathbb{R}^k \) with \( i \)th coordinate 1 and zeros elsewhere.

We introduce random variables \( X_1, \cdots, X_n \) that take values in the set \( \{ e_1, \cdots, e_k \} \) with probability

\[ P(X_i = e_j) = \frac{1}{k}, \quad i = 1, \cdots, n, \quad j = 1, \cdots, k. \]

For \( \alpha \) in \( A_k(n) \) we have

\[ P(X_1 + \cdots + X_n = \alpha) = \binom{n}{\alpha} k^{-n} \]
5. Applications. The applications which follow from our main theorem, §1.

**Theorem 2.** Assume \( f : \Lambda_k \to \mathbb{R} \) is continuous almost everywhere. Then

\[
\sum_{\lambda \in \Lambda_k} f(\lambda) \cdot d_\lambda = \left( \frac{1}{2\pi} \right)^{(1/2)B(k)}
\]

\[
\cdot k^{\beta n} \cdot \int_{x_1 + \cdots + x_k = 0} \cdots \int_{x_1, \cdots, x_k = 0} g(x) \cdot D_k(x) \cdot \exp \sum_{i=1}^{k} x_i
\]

**Proof.** It is well known that

\[
d_\lambda = \left( \frac{n}{\lambda_m+j} \right)^{k-1} \cdot \frac{1}{\lambda_m+j} \left( \frac{\lambda_m-j}{\lambda_m+j} \right)
\]

As in Examples 1 and 2 of §2 we have

\[
\frac{1}{\lambda_m+j} = \frac{k}{n} \quad \text{and} \quad \lambda_m+j \geq \lambda_m+j+1
\]

so

\[
\prod_{m=1}^{n} \prod_{j=1}^{k} \left( \frac{\lambda_m-j}{\lambda_m+j} \right)
\]

The proof is now a straightforward

**Remark.** Theorem 2 clearly is obtained for \( \lambda \in \Lambda_k(n, \rho) = \Lambda_k(n) \).

We do have a rigorous algorithm for \( \lambda \) can then be found as in [1].

For further applications, the following

**Corollary.** Let \( f : \Lambda_k \to \mathbb{R} \) be continuous almost everywhere. Let

\[
\sum_{\lambda \in \Lambda_k} f(\lambda) \cdot d_\lambda \]

Then \( p = \gamma = \frac{1}{k}(k-1) \) and

\[
\int_{x_1 + \cdots + x_k = 0} \cdots \int_{x_1, \cdots, x_k = 0} g(x) \cdot D_k(x) \cdot \exp \sum_{i=1}^{k} x_i
\]

**Proof.** The proof follows by

\[
\sum_{\lambda \in \Lambda_k(n)} f(\lambda) \cdot d_\lambda
\]

To apply this corollary, one uses such that for a fixed \( k \) and for all \( n \).

If \( f \) and \( d_n = \deg \chi_n \) satisfy the above

so that Theorem 1 now follows immediately in the case \( \beta = 1. \)
5. Applications. The applications we give here require the following theorem, which follows from our main theorem and is almost equivalent to it ($\Lambda_k = \bigcup_{n \geq 0} \Lambda_k(n)$; § 1).

**Theorem 2.** Assume $f : \Lambda_k \to \mathbb{R}$ satisfies $f(\lambda) \approx g(c(\lambda)) \cdot n^\gamma$ and $g(x_1, \ldots, x_k)$ is continuous almost everywhere. Then

$$\sum_{\lambda \in \Lambda_k} f(\lambda) \cdot D_k^\beta(\lambda) \cdot \exp\left(-\frac{k\beta}{2} \sum_{j=1}^k x_j^2\right) d^{(k-1)}x.$$ 

**Proof.** It is well known that

$$d_\lambda = \left(\begin{array}{c} n \\ \lambda \end{array}\right) \cdot \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(\frac{\lambda_i - \lambda_i + j}{\lambda_i + j}\right).$$

As in Examples 1 and 2 of § 2 we have

$$\frac{1}{\lambda_i + j} \approx \frac{k}{n} \quad \text{and} \quad \lambda_i - \lambda_i + j \approx (c_{m}(\lambda) - c_{m+j}(\lambda)) \sqrt{n},$$

so

$$\prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(\frac{\lambda_i - \lambda_i + j}{\lambda_i + j}\right) \approx D_k^\beta(c(\lambda)) \cdot n^{-(\beta/4)k(k-1)} \cdot k^{(\beta/2)k(k-1)}.$$ 

The proof is now a straightforward use of Theorem 1.

**Remark.** Theorem 2 clearly indicates that the maximum of the $d_\lambda$, $\lambda \in \Lambda_k(n)$, is obtained for $\lambda \in \Lambda_k(n, \rho) = \Lambda_k(n) \cap A_k(n, \rho)$, for some $\rho > 0$ (§ 2).

We do have a rigorous algorithmic proof of this fact (see [16]). The corresponding $\lambda$ can then be found as in [1].

For further applications, the following corollary is very useful.

**Corollary.** Let $f : \Lambda_k \to \mathbb{R}$ and assume $f(\lambda) = g(c(\lambda)) \cdot n^\gamma$, where $g(x_1, \ldots, x_k)$ is continuous almost everywhere. Let $d_n = \sum_{\lambda \in \Lambda_k(n)} f(\lambda) d_\lambda$, $n = 1, 2, \ldots$, and assume

$$d_n = c_n n^p \cdot k^n \quad (c \text{ a constant}).$$

Then $p = \gamma - \frac{1}{2} k(k-1)$ and

$$\int_{X_1 + \cdots + X_k = 0} \int_{X_1 + \cdots + X_k = 0} \ldots \int_{X_1 + \cdots + X_k = 0} g(x) \cdot D_k^\beta(x) \cdot \exp\left(-\frac{k\beta}{2} \sum_{j=1}^k x_j^2\right) d^{(k-1)}x = \left(\frac{1}{k}\right)^{k/2} \cdot (2\pi)^{k(k-1)/2} \cdot c.$$ 

**Proof.** The proof follows by equating the asymptotics of $d_n$ with that of

$$\sum_{\lambda \in \Lambda_k(n)} f(\lambda) d_\lambda,$$ 

which is given by the above theorem. □

To apply this corollary, one usually constructs a series $\{\chi_n\}$, $\chi_n$ an $S_n$ character, such that for a fixed $k$ and for all $n$, $\chi_n = \sum_{\lambda \in \Lambda_k(n)} f(\lambda) \chi_\lambda$.

If $f$ and $d_n = \deg \chi_n$ satisfy the above assumptions, then $\gamma$ and the integral are determined.
Example 4. The sums $S_k^{(0)}(n) = \sum_{\lambda \in \Lambda_k(n)} d_\lambda^k$. Here $f(\lambda) = 1$ (constant), $r = 0$, $g(x) = 1$ and by Theorem 2 we have

$$S_k^{(0)}(n) = \left(\frac{1}{2\pi}\right)^{(1/2)(k-1)} \cdot k^{(1/2)k} \cdot n^{-(\frac{1}{2}k)(k-1)-(1/2)(k-1)(k-1)} \cdot k^m \cdot \int_{x_1, \ldots, x_k \geq 0} \left[ D_k(x) \exp \left( -\frac{k}{2} \sum_{j=1}^k x_j^2 \right) \right]^k d^{k-1}x.$$

This agrees with (F.2.10) of [12].

Example 5. Evaluate

$$\int_{x_1, \ldots, x_k \geq 0} (D_k(x))^2 e^{-\frac{k(\lambda^2)}{2}} d^{k-1}x = J_2(k) \quad (\|x\| = \sum_{j=1}^k x_j^2).$$

As a special case of the corollary we now determine $J_2(k)$. (This is a special case of the Mehta integrals.)

These can be evaluated by the Selberg integral [7].

Let $s_\lambda(\lambda)$ denote the number of the $k$-semistandard tableaux. It is well known that

$$s_\lambda(\lambda) = D_k(\lambda(\lambda))/D_k(k, k-1, \ldots, 1) = [\Gamma(1) \cdots \Gamma(k)]^{-1} \cdot D_k(\lambda(\lambda)).$$

The identity $k^n = \sum_{\lambda \in \Lambda_k(n)} s_\lambda(\lambda)d_\lambda$ can be deduced from either the Knuth-Robinson-Schensted correspondence, or from the $S_n$-character $x_n = \sum_{\lambda \in \Lambda_k(n)} s_\lambda(\lambda)x_\lambda$. It is known that $x_n$ is the character of the natural (permuting coordinates) action of $S_n$ on $V^\otimes m$, for $V = k$, thus $\deg x_n = k^n$. It follows from the corollary that $J_2(k) = \Gamma(1) \cdots \Gamma(k) \cdot \sqrt{2\pi k}^{-1} \cdot (1/k)^{k/2}$. Thus by Variation 2

$$\int_{x} (D_k(x))^2 e^{-\frac{k(\lambda^2)}{2}} d^{k-1}x = \left( \frac{k}{1-k} \right)^{k/2}.$$
Assuming now that \( f \) does satisfy that property, we obtain by Theorem 2 that
\[
\deg \chi_n \approx I_k^2 \cdot \left( \frac{1}{\sqrt{2\pi}} \right)^{k^{3/2}} \cdot \frac{1}{k} \cdot n^{-1/4} k^{1/4(k^2 - 1)} \cdot k^{2(n + 1)},
\]
where
\[
I_k^2 = \left( \prod_{i < j \leq 4} \frac{1}{i_1 \cdots i_4} \int_{x_1, x_2, x_3, x_4} g(x_1, \ldots, x_k) \cdot D_{k^2}(x) \cdot \exp \left( -\frac{k^2}{2} (x_1^2 + \cdots + x_k^2) \right) \right)^{1/2}. \]

On the other hand, since \( \deg \chi_n = \sum_{k \in \Lambda_k(n + 1)} d_n \), hence
\[
\deg \chi_n = \tilde{J}_2(k) \cdot \left( \frac{1}{2\pi} \right)^{k^{3/2}} \cdot k^{k^2} \cdot n^{-(k^2 - 1)/2} \cdot k^{2(n + 1)},
\]
where \( \tilde{J}_2(k) = (1/\sqrt{2\pi})^{k^{3/2}} \cdot J_2(k) \), \( J_2(k) \) as in Example 5. Equating, we deduce the following.

**General case.** Let
\[
\chi_n = \left( \sum_{k \in \Lambda_k(n + 1)} \chi_k \otimes \chi_k \right) \delta_{k^2} = \sum_{\mu \in \Lambda_k,n} f(\mu) \chi_\mu
\]
and assume \( f(\mu) \) satisfies the above property; then
\[
I_k^2 = J_2(k) \cdot \sqrt{2\pi} k^{-2k + 1} \cdot k^{-k^2/2}.
\]

Also, we must have \( \gamma = \gamma(k) = \frac{1}{2}(k^2 - 1)(k^2 - 2) \).

**The case** \( k = 2 \). It follows from [9] and [6] that
\[
f(\mu) \approx (c_1 - c_2)(c_2 - c_3)(c_3 - c_4) \cdot \sqrt{n^2},
\]
where \( c_1 = c(\mu) \). Thus
\[
\int_{x_1, x_2, x_3, x_4} (x_1 - x_2)(x_2 - x_3)(x_3 - x_4) \cdot e^{-2(x_1^2 + \cdots + x_k^2)} \, dx_1, dx_2, dx_3
\]
\[
= \tilde{J}_2(2) \cdot \sqrt{2\pi} \cdot \left( \frac{1}{2} \right)^{10} = \sqrt{2\pi} \cdot \left( \frac{1}{2} \right)^{13}.
\]

**Remarks.** \( I_k^2 \) determines the codimensions of the \( k \times k \) matrices [14], [6] and \( I_k^2 = I_2 \) was therefore calculated in [13, Appendix]. This was a long and complicated calculation by recursive methods, in which a computer was also used; a sketch of it occupies three pages in [13]. Later, William Beckner showed us a very elegant calculation of \( I_2^2 \), which could occupy about two printed pages. The above should be viewed as an "algebraic" (and almost effortless) calculation of \( I_2^2 \). For a higher number \( (\geq 3) \), \( f(\mu) \) and \( g(c(\mu)) \) are unknown, but any conjecture about these can be tested by the above general case.

**Example 7.** Let \( \psi(n) = \sum_{k \in \Lambda_k(n + 1)} Y_k(\lambda) \chi_k \) as in [10].

**The case** \( l = 3 \). By (4.2) of [10] \( \deg \psi(n) = (1/\sqrt{2\pi})^3 \cdot (1/3) \cdot 1/n \cdot 3^n \), while \( Y_3(\lambda) = \min \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\} + 1 = \min \{c(\lambda_1) - c(\lambda_2), c(\lambda_2) - c(\lambda_3)\} \cdot \sqrt{n} \).
Thus, by the corollary,

$$\int \int \min \{x_1-x_2, x_2-x_3\} \cdot D_3(x) \cdot e^{-\beta/3(x_1^2+x_2^2+x_3^2)} dx_1 dx_2$$

$$= \left(\frac{1}{\sqrt{3}}\right)^9 \cdot (\sqrt{2\pi})^2 \cdot (\sqrt{3})^3 \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^2 = \frac{1}{27}.$$  

The case $l = 4$. Here $\deg \psi_4(n) = (1/\sqrt{2p})^2 \cdot 2^4 \cdot (1/\sqrt{n})^3 \cdot 4^4$.

Define

$$g(x_1, \ldots, x_4) = \begin{cases} 
\frac{1}{2} (x_1-x_2)(x_2-x_3)(x_3-x_4) \cdot \frac{1}{8} \left[ \min (x_1-x_2, x_3-x_4) \right] \\
\cdot (x_1-x_2-x_3+x_4)^3 \text{ if } x_2-x_3 > \frac{x_1-x_2-x_3+x_4}{2} \\
\frac{1}{2} (x_2-x_3) \left[ \min (x_1-x_2, x_3-x_4) \right] \\
\cdot (x_2-x_3+\min (x_1-x_2, x_3-x_4)) \text{ if } x_2-x_3 \leq \frac{x_1-x_2-x_3+x_4}{2}.
\end{cases}$$

By Theorem 5 of [5], we have $Y_4(\lambda) = g(c_1, \ldots, c_4) \cdot \sqrt{n}^3$, where $c = c(\lambda)$. Thus,

$$\int \int \int \int g(x_1, \ldots, x_4) D_4(x) \cdot e^{-2(x_1^2+x_2^2+x_3^2)} d^{(3)}x$$

$$= \left(\frac{1}{\sqrt{4}}\right)^{16} \cdot (\sqrt{2\pi})^3 \cdot (\sqrt{2\pi})^3 \cdot 2^2 = \left(\frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^6.$$  

Example 8. Young's rule and the Littlewood–Richardson rule for the outer products of $S_n$-characters provide many "combinatorial" identities; these yield quite interesting integrals. We demonstrate this below.

Let $n = m^2$, $\lambda = (n + \sqrt{n}, n - \sqrt{n}) + (2n)$ and define $\chi_{3n} = \chi_{1} \otimes \chi_{(n)} = \chi_{(n+\sqrt{n}, n-\sqrt{n})} \otimes \chi_{(n)}$ (outer products). By Young's rule,

$$\chi_{3n} = \sum_{\mu \in \lambda(3n)} \chi_{\mu} = \sum_{\mu \in \lambda(3n)} f(\mu) \chi_{\mu}.$$  

Write $\mu_j = (3n/3) + c_j \sqrt{3} n$; then $\mu_1 \leq n + \sqrt{n} \leq \mu_2 \leq n - \sqrt{n} \leq \mu_3$ if and only if $c_1 \leq 1/\sqrt{3} \leq c_2 \leq -1/\sqrt{3} \leq c_3$. Thus

$$f(\mu) = g(c(\mu)) = \begin{cases} 
1, & c_1 \leq 1/\sqrt{3} \leq c_2 \leq -1/\sqrt{3} \leq c_3, \\
0, & \text{otherwise}.
\end{cases}$$

Clearly, $g(x_1, x_2, x_3)$ is continuous almost everywhere. Thus

$$\deg \chi_{3n} = \int \int \int \int D_3(x) e^{-3/2(x_1^2+x_2^2+x_3^2)} d^{(3)}x \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^2 \cdot \sqrt{3}^3 \cdot 4^{3/2} \cdot 3^4.$$  

Now,

$$d_h = \frac{1}{(n^3)}.$$  

Apply Stirling's formula; since

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,$$

we easily find that

$$\deg \chi_{3n} = d_h.$$  

By the corollary,

$$\int \int \int \int D_3(x) e^{-3/2(x_1^2+x_2^2+x_3^2)} d^{(3)}x \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^2 \cdot \sqrt{3}^3 \cdot 4^{3/2} \cdot 3^4.$$  

This example can easily be generalized.

Acknowledgment. We are indebted to the probabilistic aspects of the work of [10].

Now,
\[ \deg \chi_3 = \binom{3n}{2n} \cdot d_n \]
where
\[ d_n = \frac{2n!}{(n + \sqrt{n})! (n - \sqrt{n})!} \frac{2\sqrt{n} + 1}{n + \sqrt{n} + 1} \]

Apply Stirling's formula; since
\[ \lim_{n \to \infty} \left( 1 + \frac{1}{\sqrt{n}} \right)^{n+\sqrt{n}} \cdot \left( 1 - \frac{1}{\sqrt{n}} \right)^{n-\sqrt{n}} = e, \]
we easily find that
\[ \deg \chi_3 \simeq \frac{1}{e} \cdot \frac{\sqrt{3}}{n \sqrt{n}} \cdot 3^{2n}. \]

By the corollary,
\[ \int \int \cdots \int_{x_1 + x_2 + \cdots + x_d = 0} \left( x_1 \cdots x_d \right)^{d/2} e^{x_1^2 + x_2^2 + \cdots + x_d^2} \text{d}^{d/2} x \approx \frac{1}{e} \cdot \frac{\sqrt{3}}{3 \pi}. \]

This example can easily be generalized to higher integrals.

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