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HYPERBOLIC EQUIDISTRIBUTION PROBLEMS
ON SIEGEL 3-FOLDS AND HILBERT
MODULAR VARIETIES

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Abstract
We generalize to Hilbert modular varieties of arbitrary dimension the work of W. Duke [16] on the equidistribution of Heegner points and of primitive positively oriented closed geodesics in the Poincaré upper half-plane, subject to certain subconvexity results. We also prove vanishing results for limits of cuspidal Weyl sums associated with analogous problems for the Siegel upper half-space of degree 2. In particular, these Weyl sums are associated with families of Humbert surfaces in Siegel 3-folds and of modular curves in these Humbert surfaces.

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1. Introduction
In this paper, we use the Maass correspondence for special orthogonal groups \( \text{SO}(p, q) \) with integers \( p, q \geq 1, p + q = m \), together with "accidental" isomorphisms between these groups and certain modular groups in the case \( m = 3, 4, 5 \), to derive explicit formulae expressing cuspidal Weyl sums, associated to hyperbolic
equidistribution problems in Siegel 3-folds and Hilbert modular varieties, in terms of Fourier coefficients for Maass and Hilbert-Maass forms of half-integral weight. The case \( m = 3, (p, q) = (2, 1) \) with base field \( \mathbb{Q} \) was studied in [16]. Convexity and sub-convexity results for these Fourier coefficients, combined with an analogous treatment of the eigenfunctions for the continuous part of the spectrum of the Laplace-Beltrami operator, imply the equidistribution properties stated in this section.

Let \( Q \) be a nondegenerate integral quadratic form whose signature over \( \mathbb{R} \) is \( (p, q), p + q = m, pq \neq 0 \). Let \( \lambda \in \mathbb{R}, \lambda \neq 0 \), and let

\[
W_{\lambda} = \{ x \in \mathbb{R}^m : Q(x) = \lambda \}.
\]

The group \( G = \Omega(Q) \) of \((m \times m)\)-matrices leaving \( Q \) invariant is isomorphic to \( O(p, q) \) and acts transitively on \( W_\lambda \). The stabilizer of any \( x \in W_\lambda \) is isomorphic to \( O(p - 1, q) \) if \( \lambda > 0 \) and to \( O(p, q - 1) \) if \( \lambda < 0 \). A choice of Haar measure on \( G(\mathbb{R}) \) determines an invariant volume form on the majorant space \( \mathcal{H}_{\Omega} \) (see §2). Let \( \Delta_{\Omega} \) be the Laplace-Beltrami operator on \( \mathcal{H}_{\Omega} \), and let \( \Gamma \) be an arithmetic subgroup of \( G \) given by a congruence subgroup of a unit group of \( \mathcal{O} \). In [35], Maass constructed a \( \theta \)-lift on the space of \( \Gamma \)-invariant \( L^2 \)-integrable functions on \( \mathcal{H}_{\Omega} \). This \( \theta \)-lift converges on the \( \Delta_{\Omega} \)-eigenfunctions for the discrete spectrum and has as an image a corresponding Maass cusp form of half-integral weight (see Prop. 2.1).

As mentioned briefly in [16, p. 84], the classical Maass correspondence in the cases of \( m = 4, m = 5 \) leads to the study of other modular groups not treated in that paper. These modular groups arise from considering families of polarized abelian varieties of complex dimension 2. Recall that the complex points \( V(\mathbb{C}) \) of the Siegel 3-fold can be realized as the quotient of the Siegel upper half-space \( \mathcal{H}_2 \) of degree 2 by the integer points \( \text{Sp}(4, \mathbb{Z}) \) of the symplectic group in real dimension 4,

\[
V(\mathbb{C}) \simeq \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}_2.
\]

The underlying variety \( V \) has the structure of a quasi-projective variety defined over \( \mathbb{Q} \). Here

\[
\mathcal{H}_2 = \{ z \in M_2(\mathbb{C}) : z = z^t, \text{Im}(z) > 0 \},
\]

and

\[
\text{Sp}(4, \mathbb{R}) = \{ g \in \text{GL}_4(\mathbb{R}) : g^t J g = J \},
\]

where

\[
J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}.
\]

The group \( \text{Sp}(4, \mathbb{R}) \) acts on \( \mathcal{H}_2 \) by

\[
z' \mapsto \gamma z = (Az + B)(Cz + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]
The projective symplectic group
\[ \text{PSp}(4, \mathbb{R}) = \text{Sp}(4, \mathbb{R})/\{ \pm \text{Id}_2 \} \]
is the full group of analytic automorphisms of the complex domain \( \mathcal{H}_2 \). The matrix \( J \) defines a complex structure and a symplectic form \( E \) on \( \mathbb{R}^4 \).

To every point \( z \in \mathcal{H}_2 \) we can associate the complex torus
\[ A = \mathbb{C}^2 / \mathbb{Z}^2 + z \cdot \mathbb{Z}^2, \]
where \( L = \mathbb{Z}^2 + z \cdot \mathbb{Z}^2 \), and the Riemann form \( E \) determines an \( \mathbb{R} \)-linear nondegenerate alternating form on \( \mathbb{C}^2 \times \mathbb{C}^2 \) taking integer values on \( L \times L \), which gives a principal polarization of \( A \). The complex torus \( A \) has the structure of an abelian surface. In fact, the points of the complex variety \( V(\mathbb{C}) \) correspond bijectively with the complex isomorphism classes of principally polarized abelian surfaces.

For an abelian variety \( A \), we let \( \text{End}(A) \) be the endomorphism ring and put \( \text{End}_{n}(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \). If \( A \) is simple, then \( \text{End}_{n}(A) \) is a division algebra over \( \mathbb{Q} \) with a positive involution induced by the polarization of \( A \).

Albert [1], [2], [3], [4] classified the division algebras over \( \mathbb{Q} \) with positive involution. For the case of abelian surfaces \( \dim(A) = 2 \), this gives the following (see, e.g., [61, Prop. 1.2]).

**Proposition 1.1**
If \( A \) is a simple abelian surface, then \( \text{End}(A) \) is one of the following:

1. The ring \( \mathbb{Z} \).
2. An order in a real quadratic field \( F \).
3. An order in an indefinite quaternion division algebra \( \mathcal{D} \) over \( \mathbb{Q} \).
4. An order in a quartic complex multiplication (CM) field \( K \) (totally imaginary quadratic extension of a real quadratic field).

We restrict ourselves to families of simple abelian surfaces, the nonsimple case being essentially covered by [16], as then the abelian surface is isogenous to a product of elliptic curves. In Proposition 1.1, case (1) is the generic case, which also does not interest us here. Case (2) leads to considering families of Hilbert modular surfaces, case (3) to families of modular curves, and case (4) to families of CM points. Case (4) appears as a special case of Theorem 1.2.

The case \( m = 5 \), \( (p, q) = (3, 2) \), together with a natural isomorphism between \( \text{Sp}(4, \mathbb{R}) \) and \( \text{SO}_0(3, 2) \), enables us to apply the results of §2 to \( \mathcal{H}_2 \) acted on by the lattice \( \text{Sp}(4, \mathbb{Z}) \). Let \( \mathcal{R}_1 \) be a fundamental domain for the action of \( \text{Sp}(4, \mathbb{Z}) \) on \( \mathcal{H}_2 \), and normalize the invariant volume form \( \omega_1 \) on \( \mathcal{H}_2 \) so that \( \mathcal{R}_1 \) has volume \( \omega_1(\mathcal{R}_1) = 1 \).

We describe in §4 subdomains \( S_d \), \( d > 0 \), and \( S_d \), \( d < 0 \), arising from \( \lambda = d \) square-free with \( d \equiv 1 \) mod 4. The complex surfaces \( S_d \) are related to case (2) in that they are
the Humbert surfaces and parameterize abelian surfaces whose endomorphism ring contains an order in the real quadratic field $F = \mathbb{Q}(\sqrt{d})$. On the other hand, $\mathcal{E}_d$ has the structure of a real variety of real dimension $3$.

The case $m = 4$, $(p, q) = (2, 2)$, together with a natural isomorphism between $\text{SL}(2, F) \otimes_{\mathbb{Q}} \mathbb{R}$, where $F = \mathbb{Q}(\sqrt{d})$, and $\text{SO}(2, 2)$, leads to considering $\mathbb{H}^2$ acted on by the lattice $\text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$, where $\mathcal{O}$ is the ring of integers of $F$. Let $\mathcal{A}_2$ be a fundamental domain for the action of $\text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$ on $\mathbb{H}^2$, and normalize the invariant volume form $\omega_2$ on $\mathbb{H}^2$ so that $\omega_2(\mathcal{A}_2) = 1$. We describe in §4 subdomains $\mathcal{A}_{d,n}$, $n > 0$, arising from $\lambda = n$ square-free with $n = N(\alpha)N(\mathcal{O}^\vee)$ for some $\alpha \in F$. (The case where $n < 0$ leads to nothing new as $\text{SO}(2, 1) \simeq \text{SO}(1, 2)$.) The complex curves $\mathcal{A}_{d,n}$ are related to case (3) in that they parameterize abelian surfaces whose endomorphism ring contains an order in the indefinite quaternion algebra $\mathcal{O}_{d,n}$ over $\mathbb{Q}$ with parameters $(d, -n/\delta d)$ for a certain $\delta \in F$.

In §5 we derive in Proposition 5.1 vanishing results for limits of the cuspidal Weyl sums on the side of the orthogonal groups in the above situations, which then apply to the modular side by the discussion of §4. Although we apply our results to families of principally polarized abelian varieties, the same arguments go through without this polarization assumption. In this paper, we do not explore in these same situations the analytically more involved question of how to modify the classical Maass correspondence for the eigenfunctions of the continuous spectrum of $\Delta_0$. Alternatively, one may derive directly in the case $m = 4, 5$ upper bounds for Weyl sums for eigenfunctions of the continuous spectrum, that is, the analogues of the results for $m = 3$ of our §7. We hope to return to this in a later paper. In general, the cuspidal case we treat here is arithmetically more interesting. Together, such results give the following.

**Equidistribution in genus 2**

(i) The family $\{S_d\}_{d > 0}$ of Humbert surfaces and the family $\{\mathcal{E}_d\}_{d < 0}$ of real 3-folds, where $d$ is square-free and $d \equiv 1 \text{ mod } 4$, are equidistributed in $\text{Sp}(4, \mathbb{Z}) \backslash \mathbb{H}_2$. Namely, if $\Omega_1$ is a convex region with smooth boundary in $\mathbb{H}_2$, the ratio

$$\lim_{d \to \infty} \frac{\text{Vol}(S_d \cap \Omega_1)}{\text{Vol}(S_d)} = o_1(1),$$

$$\lim_{-d \to \infty} \frac{\text{Vol}(\mathcal{E}_d \cap \Omega_1)}{\text{Vol}(\mathcal{E}_d)} = o_1(1).$$

(ii) Let $d$ be a positive square-free integer, and let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. The family $\{\mathcal{X}_{d,n}\}_{n > 0}$ of modular curves, where $n$ is congruent mod $d$ to the norm of an ideal in the same class as the inverse different $\mathcal{O}^\vee$ of $\mathbb{Q}(\sqrt{d})$, is equidistributed in $\text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathbb{H}_2$. Namely, if $\Omega_2$ is a convex region with smooth boundary in $\mathbb{H}_2$.

**Hyperbolic equidistribution region with small $r$**

The case $m = 3$, $\text{SL}(2, F) \otimes_{\mathbb{Q}} \mathbb{R}$, where $F = \mathbb{Q}(\sqrt{d})$, leads to considering the ideal $\mathcal{O}$ in $F$. The case of $m = 3$ requires a minor modification of the proof of the theorem, and hence, this leads to our results. Rather, subconvexity allows us to study directly the case $m = 3$.

Let us recall the definition of a complex torus when $\mathcal{O}$ is a rank 1 ring of integers of a field $A$ has real multiplication. A complex torus always has the form $\mathbb{C}/\Lambda$ throughout that $\mathcal{O}$ is a ring of integers in $A$. Let $\mathcal{O} \subseteq \mathcal{O}^\vee$. Let $\mathcal{A}$ be a fractional ideal of $\mathcal{O}$.

The group

$$\text{SL}(\mathcal{O} \oplus \mathcal{A})$$

acts by fractional linear transformation into $\mathbb{R}^8$, and induces a hyperbolic space

is called a Hilbert modular surface. In particular, as a consequence of Theorem 3.3, we recover the embedding

$$\text{SL}(\mathcal{O} \oplus \mathcal{A}) \backslash \mathbb{H}_2$$

for some $z \in \mathbb{H}_2$. The case $m = 2$, $\text{SL}(2, F)$, $g = 2$, we recover the embedding

which is described in detail in §4. When they are referred to as modular surfaces, they are referred to as modular surfaces.
region with smooth boundary in $\mathcal{R}_2$, we have
\[
\lim_{n \to \infty} \frac{\text{Vol}(\mathcal{R}_{d,n} \cap \Omega_2)}{\text{Vol}(\mathcal{R}_{d,n})} = \omega_2(\Omega_2).
\] (1.5)

The case $m = 3$, $(p, q) = (2, 1)$, together with a natural isomorphism between $\text{SL}(2, F) \otimes \mathbb{Q} \cong \mathbb{R}$, where $F$ is a totally real field of degree $g$ over $\mathbb{Q}$, and $\text{SL}(2, \mathbb{R})^g$ leads to considering $\mathcal{H}^g$ acted on by the lattice $\text{SL}(\mathcal{O} \oplus \mathcal{A})$, where $\mathcal{A}$ is a fractional ideal in $F$. The case $g = 1$ was treated in [16]. However, for the case $g > 1$, we need to adapt the classical Maass correspondence to the Hilbert modular situation (see §3), and we need new subconvexity results (see §8 and the footnote on page 93); hence, this leads to our assumption of the generalized Riemann hypothesis (GRH) (or, rather, subconvexity) in Theorem 1.2. In order to treat the continuous spectrum, in §7 we study directly the corresponding Eisenstein Weyl sums.

Let us recall the associated families of abelian varieties. Let $A$ be a $g$-dimensional complex torus where now $g \geq 1$. Let $F$ be a totally real field with $[F : \mathbb{Q}] = g$. Then $A$ has real multiplication (RM) if $\text{End}(A)$ contains an order $\mathcal{O}$ in $F$. Such a complex torus always has the structure of an abelian variety (see [61, p. 207]). We assume throughout that $\mathcal{O}$ is the ring of integers of $F$, and we denote the inverse different by $\mathcal{O}^\vee$. Let $\mathcal{A}$ be a fractional ideal of $F$.

The group
\[
\text{SL}(\mathcal{O} \oplus \mathcal{A}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, F) : \alpha, \beta \in \mathcal{O}, \beta \in \mathcal{A}^{-1}, \gamma \in \mathcal{A} \right\}
\]
acts by fractional linear transformations on $\mathcal{H}^g$, via the $g$ Galois embeddings of $F$ into $\mathbb{R}$, and induces an action of $\text{PSL}(\mathcal{O} \oplus \mathcal{A}) = \text{SL}(\mathcal{O} \oplus \mathcal{A})/\{\pm 1\}$. The quotient space
\[
\text{PSL}(\mathcal{O} \oplus \mathcal{A}) \backslash \mathcal{H}^g
\]
is called a Hilbert modular variety and corresponds bijectively to the complex isomorphism classes of polarized $g$-dimensional abelian varieties $A$ with RM by $\mathcal{O}$. In particular, as a complex torus we may write
\[
A(\mathbb{C}) = \mathbb{C}^g/\{\mathcal{A} + z \cdot \mathcal{O}\}
\]
for some $z \in \mathcal{H}^g$ with $\mathcal{A} + z \cdot \mathcal{O}$ embedded in $\mathbb{C}^g$ using the Galois embeddings of $F$ into $\mathbb{R}$. The abelian variety $A$ is principally polarized when $\mathcal{A} = \mathcal{O}^\vee$. When $g = 2$, we recover an example of case (2) in Proposition 1.1. There is a natural modular embedding
\[
\text{PSL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathcal{H}^2 \to \text{PSp}(4, \mathbb{Z}) \backslash \mathcal{H}_2
\]
which is described in detail for arbitrary $g$ in [61, Chapter IX], so that the Hilbert modular surfaces can be viewed as subsurfaces of the Siegel 3-fold. In this context, they are referred to as Humbert surfaces.
We consider in §§6–8 the situation arising from \( \lambda = \Delta \in \mathcal{O}, \Delta \neq 0 \). When \( \Delta \) is totally negative, this leads to the set \( \Lambda_\Delta \) of Heegner points (coming from \( \text{SO}(2)^g \)), and when \( \Delta \) is totally positive, this leads to families \( \mathcal{G}_\Delta \) of real g-dimensional varieties (coming from \( \text{SO}(1, 1)^g \)), which for \( g = 1 \) are primitive closed geodesics (see §4). The Heegner points correspond to abelian varieties of dimension \( g \) with complex multiplication by an order in \( F(\sqrt{\Delta}) \), and so the case \( g = 2 \) is related to case (4) in Proposition 1.1. Let \( \mathcal{B}_\mathcal{G} \) be a fundamental domain for the action of \( \text{SL}(2, \mathcal{O}) \) on \( \mathcal{H}^g \), and normalize the invariant volume form \( \mu_\mathcal{G} \) on \( \mathcal{H}^g \) so that \( \mu_\mathcal{G}(\mathcal{B}_\mathcal{G}) = 1 \).

The main application in the case \( m = 3 \) is as follows.

**Theorem 1.2**

Let \( F \) be a totally real number field of degree \( g \geq 1 \) over \( \mathbb{Q} \) with ring of integers \( \mathcal{O} \). Assume that \( F \) has class number 1. Let \( \Delta \in \mathcal{O}, \Delta \neq 0 \), be a generator of the relative discriminant of \( F(\sqrt{\Delta})/F \). Under GRH (or, rather, subconvexity), the families \( \{\Lambda_\Delta\}_{\Delta \leq 0} \) of Heegner points and \( \{\mathcal{G}_\Delta\}_{\Delta \geq 0} \) of real \( g \)-dimensional subvarieties are equidistributed in \( \text{SL}(2, \mathcal{O}) \backslash \mathcal{H}^g \). Namely, if \( \Omega_\mathcal{G} \) is a region with smooth boundary in \( \mathcal{B}_\mathcal{G} \), we have

\[
\lim_{N(\Delta) \to \infty, \Delta \leq 0} \frac{\text{Card}(\Lambda_\Delta \cap \Omega_\mathcal{G})}{\text{Card}(\Lambda_\Delta)} = \mu_\mathcal{G}(\Omega_\mathcal{G}),
\]

\[
\lim_{N(\Delta) \to \infty, \Delta \geq 0} \frac{\sum_{\mathcal{G} \in \mathcal{G}_\Delta} \text{Vol}(\mathcal{G} \cap \Omega_\mathcal{G})}{\sum_{\mathcal{G} \in \mathcal{G}_\Delta} \text{Vol}(\mathcal{G})} = \mu_\mathcal{G}(\Omega_\mathcal{G}). \tag{1.6}
\]

We have made a number of simplifying assumptions that are not essential. In order to reduce the technicalities, we have assumed that \( \mathcal{A} = \mathcal{O} \), that \( F \) has class number 1, and that \( \Delta \) generates a relative discriminant. The technicalities arising from arbitrary class number and arbitrary \( \mathcal{A} \) can be simplified by working in the adelic language. In [10], [11] it was indicated how the fundamental discriminant assumption, appearing also in [16], can be removed for the case \( g = 1 \), and those same ideas may be applicable here (see also [20]).

This paper is organized as follows. In §2 we recall the classical Maass correspondence of [35] and derive formulæ in Proposition 2.2 for Fourier coefficients of Maass forms of half-integral weight in terms of Weyl sums. In §3 we prove new results that generalize, in Propositions 3.2 and 3.3, the Maass correspondence and the Fourier coefficient formulæ to the case \( m = 3 \), \( (p, q) = (2, 1) \) with base field \( F \) a totally real field of degree \( g \geq 1 \) and arbitrary class number. This extends results of [16], [28], and [35], which treat the case \( g = 1 \).

In §4 we use the “accidental” isomorphisms to relate the results of §§2 and 3 to Shimura varieties. The case \( m = 4, 5 \) leads to studying families of Humbert surfaces in Siegel 3-folds and of modular curves in these Humbert surfaces. The case \( m = 3 \) leads to studying families of curves of dimension \( g \), which for \( g = 1 \) are of complex dimension 1.

In §5 we derive phrasing formulae of Hilbert-Mumford type for the modular case. In §6 we derive phrasing formulae of Hilbert-Mumford type for the modular case, in terms of congruence subgroups.

The results of [16] were obtained by constructing a simple example from that of [16], completely generalizing to congruence subgroups in [67], assuming as before the Gross-Prasad conjecture. GRH, so our Theorem 1.2 is a simple consequence of our methods. A more detailed statement of theorems of interest.

For compactness, we refer the reader to the results of [35], [16], [28], and [35].

2. The classical Maass correspondence

As in [16, §4], we define the Shimura varieties of interest, particular, of the Laplace operator.

*Note added in proof: We would like to emphasize that the statements above lead to an understanding of the convergence, using ergodic theory, of the distribution of Heegner points.
leads to studying families of Heegner points and of certain subdomains of real dimension \( g \), which for \( g = 1 \) are primitive closed geodesics, in Hilbert modular varieties of complex dimension \( g \).

In §5 we show how the subconvexity results of [16] (in fact, convexity results would suffice here) can be used to give vanishing of limits of cuspidal Weyl sums.

In §6 we derive in Lemma 6.3 upper bounds for cuspidal Weyl sums in the Hilbert modular case. In §7 we prove in Propositions 7.1 and 7.2 new results that extend classical formulae of Hecke [26] expressing Eisenstein Weyl sums, in the Hilbert modular case, in terms of central values of certain \( L \)-series. These results are of independent interest.

The results of §§6 and 7 combined with subconvexity results for Fourier coefficients of Hilbert-Maass modular forms are then used to prove Theorem 1.2. The corresponding subconvexity results for the holomorphic case have been shown in [14]. We would need (in the notation of §3, where \( \Delta \) is an integer of \( F \) assumed to be square-free or a relative discriminant in the case of class number 1) the Fourier coefficients \( \rho(\Delta, f) \) for \( f \) a cusp form with \( L^2 \)-norm 1 or an Eisenstein series, with eigenvalue \( \lambda \) and half-integral weight \( k \), to have an upper bound in the \( \Delta \)-aspect as good as \( \rho(\Delta, f) \ll_{k, \epsilon} c(\lambda) N_{\mathbb{Q}/\mathbb{Q}}(\Delta)^{-1/4-\delta+\epsilon} \) for a fixed \( \delta > 0 \) and a positive explicit constant \( c(\lambda) \). Partial progress toward subconvexity results in the Maass case have been made by Gergely Harcos [24], but the complete adaptation of the method of [14] to the Maass case remains elusive.* Such results would follow, however, from GRH, so our Theorem 1.2 remains conditional. Although we do not pursue this here, from our methods we can estimate rates of convergence in the above equidistribution statements.

For compact maximal flats of \( \text{SL}_m(\mathbb{Z}) \backslash \text{SL}_m(\mathbb{R})/\text{SO}(n) \), an equidistribution result has been obtained in [39], and this represents a different type of equidistribution result from that of [16], even in the case \( g = 1, n = 2 \). An equidistribution result for Heegner points in Hilbert modular varieties using other methods has been announced by Zhang [67], assuming as-yet-unproven subconvexity results for Hilbert-Maass Fourier coefficients. In [38], subconvexity results are obtained for Rankin-Selberg \( L \)-functions which prove an equidistribution property for incomplete orbits of Heegner points over definite Shimura curves (see also [7], [25]).

2. The classical Maass correspondence

As in [16, §4], we exploit a construction of Maass forms as integrals of certain automorphic eigenfunctions for the ring of invariant differential operators, and, in particular, of the Laplace-Beltrami operator, against Siegel theta functions. We recall in

*Note added in proof: A. Venkatesh, in preprint [62], claims the required subconvexity results, without rates of convergence, using ergodic methods.
outline this construction in order to fix notation, referring the reader to [16], [28], and [35] for details.

Let $Q$ be a symmetric $(m \times m)$-matrix with half-integer off-diagonal elements and integer diagonal elements. Let $(p, q)$, with integers $p, q \geq 0$ satisfying $p + q = m$, be the signature of $Q$. The majorant space $\mathcal{H}_Q$ of $Q$ is defined as

$$\mathcal{H}_Q = \{ H \in M_m(\mathbb{R}) : H = ^t H, H > 0, H Q^{-1} H = Q \},$$

and it is of real dimension $pq$. It is the symmetric space attached to the group $G = \Omega(Q)$ of all real $(m \times m)$-matrices $g$ such that

$$Q(g) = ^t g Q g = Q,$$

where $A[B] = ^t B A B$ for any matrices $A, B$ for which this product makes sense. Indeed, the group $G$ acts transitively on $\mathcal{H}_Q$ by

$$H \mapsto H[g], \quad H \in \mathcal{H}_Q, \ g \in G.$$

The isotropy group in $G$ of any $H \in \mathcal{H}_Q$ is a maximal compact subgroup $K$. Analogous statements hold for the connected component of the identity of $G$. An invariant metric on $\mathcal{H}_Q$ is given by

$$ds^2 = \text{Trace}(H^{-1} dH H^{-1} dH).$$

Let $\Gamma$ be any group of finite index in the unit group

$$\Gamma_Q = \text{SL}(m, \mathbb{Z}) \cap G,$$

and let $\overline{\Gamma}$ be the quotient of $\Gamma$ by $\left\{ \pm I_d \right\} \cap \Gamma$. Then $\overline{\Gamma}$ acts discontinuously on $\mathcal{H}_Q$ and is of finite covolume if $Q$ is not a binary zero form, which we assume from now on. Let $\Delta_Q$ be the Laplace-Beltrami operator on $\mathcal{H}_Q$, and let $d\nu$ be the invariant volume measure induced by $ds^2$. Let $\varphi = \varphi(H)$ be an eigenfunction of $\Delta_Q$ on $\overline{\Gamma} \setminus \mathcal{H}_Q$, with eigenvalue $\lambda^2$ defined by

$$\Delta_Q \varphi + \lambda^2 \varphi = 0.$$

For $\varphi_1, \varphi_2$ functions on $\overline{\Gamma} \setminus \mathcal{H}_Q$, define their inner product by

$$\left\langle \varphi_1, \varphi_2 \right\rangle = \frac{1}{\text{Vol}(\overline{\Gamma} \setminus \mathcal{H}_Q)} \int_{\overline{\Gamma} \setminus \mathcal{H}_Q} \varphi_1 \varphi_2 \, d\nu.$$

For $z = u + iv \in \mathcal{H}$, the complex upper half-plane, and $H \in \mathcal{H}_Q$, let $R = u Q + iv H$.

Following Siegel [58], we define

$$\theta(z) = \theta(z, H) = \sum_{h \in \mathbb{Z}^m} \exp(2\pi i R[h]).$$

PROPOSITION 2.1

Let $\varphi$ be a nonconstant function on $\mathcal{H}_Q$ which is absolutely convergent for $k = p - (m/2)$ and has an $L^2$ norm on $\mathcal{H}_Q$ and it has eigenvalues $\lambda^2$.

In particular, $f(z)$ is given by

$$f(z) = \frac{1}{\sqrt{2}} \sum_{\varphi \in \mathcal{H}_Q} \left\langle \varphi, \varphi \right\rangle \theta(z, \varphi),$$

where

$$\theta(z, \varphi) := \frac{1}{\sqrt{2}} \sum_{h \in \mathbb{Z}^m} \exp(2\pi i R[h]).$$

As a function $\theta(z, \varphi)$ is left $\Gamma$-invariant and $L^2$-finite, as in Proposition 2.1, we have

$$f(z) = \sum_{\varphi \in \mathcal{H}_Q} \left\langle \varphi, \varphi \right\rangle \theta(z, \varphi).$$

On the other hand,
From its definition, it follows that for each fixed $z \in \mathcal{H}$, the function $\theta(z, \cdot)$ on $\mathcal{H}_Q$ is left $\bar{\Gamma}$-invariant. Let the discriminant $D$, the level $N$ for $Q$, and the definition of a Maass form of weight $k$, discriminant $D$ for level $N$ be the same as in [16, §2. 4]. We have the following result, which is [16, Th. 4], except that we use $\bar{\theta}(z)$ instead of $\theta(z)$ (which has the effect of exchanging $p$ and $q$).

**PROPOSITION 2.1**

Let $\varphi$ be a nonconstant eigenfunction of $\Delta_Q$ on $\bar{\Gamma}\backslash \mathcal{H}_Q$ with eigenvalue $\lambda'$ and $\langle \varphi, \varphi \rangle$ finite. Suppose that

$$f(z) = v^{m/4} \langle \varphi, \bar{\theta}(z) \rangle$$

is absolutely convergent for each $z \in \mathcal{H}$. Then $f(z)$ is a Maass cusp form of weight $k = p - (m/2)$ and discriminant $D$ for the congruence subgroup $\Gamma_0(N)$ of $\text{SL}(2, \mathbb{Z})$, and it has eigenvalue $\lambda = (1/4)(\lambda' + m - m^2/4)$.

In particular, $f(z)$ satisfies

$$(\Delta_k + \lambda) f = 0,$$

where

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$

together with a transformation rule for $\Gamma_0(N)$ with automorphy factor depending on $k$ and $D$, and a growth condition at the cusps.

We fix $H_0 \in \mathcal{H}_Q$. As $G$ acts transitively on $\mathcal{H}_Q$, we can write $H \in \mathcal{H}_Q$ as $H = H_0[g^{-1}], g \in G$, and then

$$\theta(z, g) := \theta(z, H_0[g^{-1}]) = \sum_{h \in \mathbb{Z}^m} \exp\left(2\pi i u Q[h] - 2\pi v H_0[g^{-1}(h)]\right).$$

As a function $\theta(z, \cdot)$ on $G$, it is left $\bar{\Gamma}$-invariant and right $K$-invariant. Let $\varphi(g)$ be the left $\Gamma$-invariant and right $K$-invariant function on $G$ induced by the eigenfunction $\varphi$ as in Proposition 2.1. Then, for an appropriate choice of Haar measure $dg$ on $G(\mathbb{R})$, we have

$$f(z) = v^{m/4} \sum_{h \in \mathbb{Z}^m} \exp(2\pi i u Q[h]) \int_{\Gamma \backslash G} \varphi(g) \exp\left(-2\pi H_0[\sqrt{v}g^{-1}(h)]\right) \, dg.$$

On the other hand, we can write

$$\lambda = s(1-s) = \frac{1}{4} + \kappa^2, \quad s = \frac{1}{2} + i\kappa, \quad \text{Re}(s) \geq \frac{1}{2}.$$
We know that $f(z) = u + iv, u, v \in \mathbb{R}$, has a Fourier expansion of the form
\begin{equation}
    f(z) = \rho(0)u^{1/2+i\kappa} + \rho'(0)v^{1/2-i\kappa} + \sum_{d \in \mathbb{Z}, d \neq 0} \rho(d)W_{(k/2)\text{sgn}(d),i\kappa}(4\pi |d|v) \exp(2\pi i du),
\end{equation}
where $W_{a,b}(\cdot)$ is the classical Whittaker function (see [36]; in fact, for $f$ as in Proposition 2.1, we have $\rho(0) = \rho'(0) = 0$). Therefore, for $d \neq 0$,
\begin{equation}
    M_d(v) := \rho(d)W_{(k/2)\text{sgn}(d),i\kappa}(4\pi |d|v) = v^{m/4} \sum_{\ell \in \mathbb{Z}^m, Q[\ell] = d} \int_{\Gamma \backslash G} \varphi(g) \exp\left(-2\pi H_0 g^{-1}(\sqrt{v}h)\right) dg.
\end{equation}
For every integer $d \neq 0$, it follows from general results of [58] that the number of orbits, under the action of $\Gamma$, of the solutions of $Q[h] = d, h \in \mathbb{Z}^m$, is finite. Let the cardinality of this orbit be $H(d)$; it is a generalized class number. Let $\{h^{(1)}, \ldots, h^{(H(d))}\}$ be a set of representatives in $\mathbb{Z}^m$ of these orbits, and let $\Gamma_j$ be the stabilizer of $h^{(j)}$ in $\Gamma$. Then
\begin{align}
    v^{-m/4}M_d(v) &= \int_{\Gamma \backslash G} \sum_{Q[\ell] = d} \exp\left(-2\pi H_0 g^{-1}(\sqrt{v}h)\right) \varphi(g) dg \\
    &= \sum_{j=1}^{H(d)} \int_{\Gamma_j \backslash G} \exp\left(-2\pi H_0 g^{-1}(\sqrt{v}h^{(j)})\right) \varphi(g) dg.
\end{align}
We let
\begin{equation}
    I_j = I_j(v) = \int_{\Gamma_j \backslash G} \exp\left(-2\pi H_0 g^{-1}(\sqrt{v}h^{(j)})\right) \varphi(g) dg.
\end{equation}
We can compare directly with the discussion of [35, §5]. In terms of our notation, the notation of that paper becomes $S = Q, u = \varphi, v = -(m-2)/2 + 2i\kappa, x = 2u, y = 2v, \alpha = p/2 - m/4 + 1/2 + i\kappa, \beta = q/2 - m/4 + 1/2 + i\kappa$, and $t = d$. It is shown there that
\begin{equation}
    \exp(2\pi dv) \sum_{j=1}^{H(d)} I_j(v)
\end{equation}
satisfies a second-order differential equation (see [35, (86)]), and by looking at the behavior as $v \to \infty$, one sees that it is a multiple of a standard solution of that equation related to the Whittaker function, which fits with (2.1) and (2.2). Indeed, we have (see [35, (91)])
\begin{equation}
    M_d(v) = v^{m/4} \sum_{j=1}^{H(d)} I_j(v) = A(2\pi |d|)^{-m/4}W_{(k/2)\text{sgn}(d),i\kappa}(4\pi |d|v)
\end{equation}
for some $A \neq 0$ independent of $d$.

We now describe how $I_j$ is computed. The function $I_j$ is an eigenfunction of $Q(E)$, and is an eigenfunction of the operator with the symbol $Q$. Since $G$ acts transitively on $\Gamma \backslash G$, we have
\begin{equation}
    I_j = \int_{\Gamma \backslash G} \varphi(g) dg = \int_{\Gamma \backslash G} \varphi(g) dg.
\end{equation}
We then have
\begin{equation}
    I_j = \int_{\Gamma \backslash G} \varphi(g) dg
\end{equation}

where
\begin{equation}
    \varphi(g) = \varphi(l_jg) \Rightarrow \varphi(l_jg)
\end{equation}

is the stablizer of $h^{(j)}$. The proper choice determines $\varphi(l_jg)$.

We have
\begin{equation}
    I_j = \int_{\Gamma \backslash G} \varphi(l_jg) dg.
\end{equation}
Let $\psi(g) = \varphi(l_jg)$, the operator with the symbol $Q$. Then
\begin{equation}
    \psi(g) = \varphi(l_jg)
\end{equation}

so that $J_j(a)$ is understood to be a multiple of a standard solution of that equation related to the Whittaker function, which fits with (2.1) and (2.2). Indeed, we have (see [35, (91)])
\begin{equation}
    M_d(v) = v^{m/4} \sum_{j=1}^{H(d)} I_j(v) = A(2\pi |d|)^{-m/4}W_{(k/2)\text{sgn}(d),i\kappa}(4\pi |d|v)
\end{equation}
and the identity of $G, \Gamma \backslash G$.
for some $A \neq 0$ independent of $v$.

We now describe this factor $A$. From now on, $c_1, c_2, \ldots$ are positive constants depending only on $Q$ and the sign of $d$; these constants can, in fact, be explicitly computed. The function $\varphi = \varphi(g) \circ G^+ \circ \text{SO}(p, q)$ is $K$-invariant on the right and is an eigenfunction of the appropriately normalized Casimir operator on $G$. Fix a solution $E$ of $Q[E] = \text{sgn}(d)$. We can find $l_j \in G$ such that

$$I_j^{-1}(h^{(j)}) = \sqrt{|d|} E$$

since $G$ acts transitively on the set

$$\{ x \in \mathbb{R}^m : Q[x] = d \neq 0 \}.$$

We then have

$$I_j = \int_{\Gamma' \setminus G} \exp \left( -2\pi H_0 |g^{-1}(\sqrt{|d|} E))| \right) \varphi(l_j g) \, dg,$$

where

$$\Gamma' = \Gamma_j^{-1} \Gamma_j$$

is the stabilizer of $E$ in $G(\mathbb{R})$. Let $d\gamma$ be a fixed Haar measure on $\Gamma' \setminus G(\mathbb{R})$. Such a choice determines a Haar measure $da$ on $\Gamma' \setminus G(\mathbb{R})$ such that

$$dg = d\gamma \, da.$$

We have

$$I_j = \int_{\Gamma' \setminus G(\mathbb{R})} \exp \left( -2\pi d\gamma H_0 |a^{-1}(E))| \right) \int_{\Gamma' \setminus \Gamma(\mathbb{R})} \varphi(l_j \gamma a) \, d\gamma \, da. \quad (2.4)$$

Let $\psi(g) = \varphi(l_j g)$; then $\psi(g)$ is also an eigenfunction of the normalized Casimir operator with the same eigenvalue as $\varphi(g)$. Let

$$J_j(a) = \int_{\Gamma' \setminus \Gamma(\mathbb{R})} \psi(\gamma a) \, d\gamma;$$

then

$$J_j(\gamma a k) = J_j(a), \quad \gamma \in \Gamma' \setminus G(\mathbb{R}), \quad k \in K,$$

so that $J_j(a)$ is uniquely determined by its value on $\Gamma' \setminus G(\mathbb{R}) / K$. In [35] this integral is rewritten in terms of the variable $w = H_0 |a^{-1}(E)|$ and is shown to be a multiple of a standard function in $w$ by using (2.4) and comparing with (2.3). Alternatively, one may use the above discussion together with a uniqueness argument, as is done in [28, (3.7), (3.23)] for the case $(p, q) = (2, 1)$. This enables us to write, for $e$ the identity of $G$,

$$J_j(a) = J_j(e) V_2(a),$$
where \( V_2(a) \) is determined by the condition \( V_2(e) = 1 \). We then have

\[
I_j = J_j(e) \int_{\Gamma_j(\mathbb{R}) \setminus \Gamma(\mathbb{R})} \exp \left( -2\pi d_m H_0[a^{-1}(E)] \right) V_2(a) \, da.
\]

As in [35, (103)], we can compare this directly with (2.3) to deduce

\[
\rho(d) = c_1 |d|^{-m/4} \left\{ \sum_{j=1}^{H(d)} \int_{\Gamma \setminus \Gamma(\mathbb{R})} \varphi(l_j y) \, dy \right\}.
\]

We have shown the following. The notation \( dy \) is used again, now to denote the induced Haar measure on \( \Gamma_j(\mathbb{R}) \).

**Proposition 2.2**

We have

\[
\rho(d) = c_1 |d|^{-m/4} \left\{ \sum_{j=1}^{H(d)} \int_{\Gamma_j \setminus \Gamma(\mathbb{R})} \varphi(y) \, dy \right\}. \tag{2.5}
\]

We can also check this against the formula given in [35, pp. 288–289]. Namely,

\[
\rho(d) = (2\pi)^{-m/4} |d|^{m/4-1} a_d(Q, \varphi),
\]

where

\[
a_d(Q, \varphi) = c_2 |d|^{-m/2+1} \sum_{j=1}^{H(d)} \int_{\Gamma_j \setminus \Gamma(\mathbb{R})} \varphi(l_j y) \, dy
\]

can be interpreted as Siegel's mass (see [58]) of the representation of \( d \) by \( Q \), weighted against \( \varphi \).

3. The Maass correspondence for the Hilbert modular case

In this section we generalize the classical Maass correspondence in the case \((p, q) = (2, 1)\) to the Hilbert modular case. Let \( F \) be a totally real number field of degree \( g \) over \( \mathbb{Q} \). As in §1, we let \( \mathcal{O} \) be the ring of integers of \( F \), and we let \( \mathcal{A} \) be a fractional ideal of \( F \). We define \( \Gamma_{\mathcal{A}} = \text{SL}(\mathcal{O} \oplus \mathcal{A}) \) to be the group of matrices of determinant 1 in the maximal order in \( M_2(F) \) given by

\[
\begin{pmatrix}
\mathcal{O} & \mathcal{A}^{-1} \\
\mathcal{A} & \mathcal{O}
\end{pmatrix}.
\]

For \( z = (z_1, \ldots, z_g) \in \mathbb{C}^g, z_j = u_j + \sqrt{-1}v_j, u_j, v_j \in \mathbb{R}, \) and \( \alpha \in F \), let

\[
\alpha \cdot z = a_1z_1 + \cdots + a_gz_g
\]

with \( a_j, j = 1, \ldots, g \).

Let \( S \in M_2(\mathbb{Z}) \) have

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 \\
\alpha_3 & \alpha_4
\end{pmatrix}
\]

which has signature \((n, 1)\) in the space \( \mathcal{H} \). For \( z \in \mathcal{H} \), the \( n \) coordinates \( H_j \in \mathbb{R} \), \( j = 1, \ldots, n \), of \( z \) in the \( \mathbb{R}^n \) coordinate equal to \( \mathbb{C} \)

\[
\mathcal{L} = \{ z \in \mathcal{H} \mid H_j = 0, j = 1, \ldots, n \}.
\]

Then there are matrices \( \Omega \), \( \Omega_0 \), and \( \Omega_0^{-1} \) such that \( \Omega \Omega_0^{-1} \Omega_0 \) is isomorphic to \( \text{SL}(2, \mathbb{Z}) \). We may write \( \theta(z, H) = \theta(\Omega z, \Omega_0 H) \). For \( \theta, \psi \), \( g_j, j = 1, \ldots, g \), \( g_j \in \mathcal{L} \), \( i(h_1, h_2, h_3) \in \mathcal{L} \),

\[
\theta(\Omega z, \Omega_0 H) = \theta(z, H).
\]

Then \( Q[h] = -4 \).

Letting
with \( a_j, j = 1, \ldots, g \), the Galois conjugates of \( a \), and let

\[
N(\nu) = \prod_{j=1}^{g} \nu_j.
\]

Let \( S \in M_2(\mathbb{Z})^g \) have all its coordinates equal to the matrix

\[
Q = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix},
\]

which has signature (2, 1). The majorant space \( H_Q \) is isomorphic to the upper half-plane \( \mathcal{H} \). For \( z \in \mathcal{H}^g \) with coordinates \( z_j = u_j + iv_j, v_j > 0 \), and \( H \in \mathcal{H}_Q \) with coordinates \( H_j \in \mathcal{H}_Q \), let \( R \) have coordinates \( R_j = u_jQ + iv_jH_j, j = 1, \ldots, g \). Let \( \mathcal{L} \) be the lattice \( \mathcal{A}^{-1} \oplus \mathcal{O} \oplus \mathcal{A} \) in \( F^3 \). We define the theta function

\[
\theta(z, H) := N(\nu)^{3/4} \sum_{h \in \mathcal{L}} \exp(2\pi i (h \cdot R \cdot h)),
\]

where

\[i h \cdot R \cdot h = \sum_{j=1}^{g} i h_j R_j h_j\]

for \( h_j, j = 1, \ldots, g \), the Galois conjugates of \( h \in \mathcal{L} \). Let \( H_0 \in H_Q^g \) have each coordinate equal to the majorant of \( Q \) given by

\[
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

Then there are matrices \( C_j \) in \( \Omega(Q) \) such that \( H_j = H_0[C_j], j = 1, \ldots, g \). As \( \Omega(Q)_0 \) is isomorphic to \( \text{SO}_0(2, 1) \), which is in turn isomorphic to \( \text{SL}(2, \mathbb{R}) \), we may write \( \theta(z, H) \) as a function \( \theta(z, g) \) with \( g \in G = \Omega(Q)^g \), with coordinates \( g_j, j = 1, \ldots, g \), also viewed as elements both of \( \text{SL}(2, \mathbb{R}) \) and \( \text{SO}(2, 1) \). To \( h = (h_1, h_2, h_3) \in \mathcal{L} \) we may associate the matrix

\[
h = \begin{pmatrix} h_1 & h_2/2 \\ h_2/2 & h_3 \end{pmatrix}.
\]

Then \( Q[h] = -4 \det(h) \), and the induced action of \( \text{SL}(2, \mathbb{R}) \) on \( h \) becomes

\[
g : h \mapsto g(h) := ghg^t, \quad g \in \text{SL}(2, \mathbb{R}).
\]

Letting

\[s(x_1, x_2, x_3) = \exp \left( -2\pi \left( 2x_1^2 + x_2^2 + 2x_3^2 \right) \right),\]
from (3.1), we have
\[ \theta(z, g) = N(v)^{3/4} \sum_{h \in \mathcal{H}} \exp \left( 2 \pi i (h_2 z - 4h_1 h_3) \cdot u \right) \mathcal{N}(\sqrt{v} g^{-1}(h)) \] 
(3.3)
where
\[ \mathcal{N}(\sqrt{v} g^{-1}(h)) = \prod_{j=1}^{g} \mathcal{N}(\sqrt{v_j} g_j^{-1}(h_j)). \]

By [51, §7], there is a congruence subgroup \( \Gamma_1 \), and there is a multiplier \( J \) such that for \( \gamma_1 \in \Gamma_1 \),
\[ \theta(\gamma_1 z, g) = J(\gamma_1, z) \theta(z, g), \]
and for \( \gamma \in \Gamma_\infty, k \in K_\infty = K^\mathbb{C} \), where \( K \) is the maximal compact of \( \text{SL}(2, \mathbb{R}) \), we have
\[ \theta(z, \gamma g k) = \theta(z, g). \]

We may adapt the discussion of [28, §2] to our situation. For \( j = 1, \ldots, g \), let \( \Delta_{1/2}^{(j)} \) be the Laplacian in the variable \( z_j = u_j + iv_j \) given by
\[ \Delta_{1/2}^{(j)} = v_j^2 \left( \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2} \right) - \frac{\partial}{\partial u_j}. \]
(3.4)
We have the action of a suitably normalized Casimir operator for each component (see [22]),
\[ D_{\mathcal{H}}^{(j)} \theta(z, g) = 4 \Delta_{1/2}^{(j)} \theta(z, g) + \frac{3}{4} \theta(z, g), \quad j = 1, \ldots, g. \]

A Maass-Hilbert form \( \varphi \) may be viewed as a function on \( G \) which is \( K_\infty \)-invariant on the right and which is an eigenfunction of the Casimir operators \( D_{\mathcal{H}}^{(j)} \), satisfying for \( r_j \in \mathbb{R}, j = 1, \ldots, g \),
\[ D_{\mathcal{H}}^{(j)} \varphi(g) = \left( -\frac{1}{4} - (2r_j)^2 \right) \varphi(g), \quad r_j \in \mathbb{R}. \]

Let
\[ U = L^2_{\text{cusp}}(\Gamma_\infty \backslash \mathcal{H}^\mathbb{C}) = \left\{ \varphi : \mathcal{H}^\mathbb{C} \to \mathbb{C} : \varphi(\gamma z) = \varphi(z), \ \gamma \in \Gamma_\infty, \right\} \]
\[ \int_{\Gamma_\infty \backslash \mathcal{H}^\mathbb{C}} |\varphi|^2 \mathcal{N}(v)^{-2} N(du) N(dv) < \infty, \]
\[ \int_0^1 \cdots \int_0^1 \varphi(x, y) N(dx) = 0, \text{ a.e. } y. \]
(3.6)
We can make \( U \) into a Hilbert space using the natural inner product. This space is invariant under the action of the unique self-adjoint extensions of the \( g \) Laplacians
\[ \Delta_{0}^{(j)} = v_j^2 \left( \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2} \right), \quad j = 1, \ldots, g, \]
(3.7)
which provide a basis of the algebra of invariant differential operators on \( \mathcal{H}^g \). The Maass-Hilbert forms are (simultaneous) eigenfunctions for all \( g \) Laplacians \( \Delta^{(j)}_{\lambda} \). Let \( 1/2 \in \mathbb{Q}^g \) be the vector with all its coordinates equal to \( 1/2 \), and let \( \lambda = (\lambda_j)_{j=1}^g \in \mathbb{C}^g \). We may write \( \lambda_j = -s_j(1-s_j), \) \( \text{Re}(s_j) \geq 1/2 \). A Hilbert-Maass form \( f \) for \( \Gamma_1 \) of weight \( 1/2 \) and eigenvalue \( \lambda \) is a function \( f : \mathcal{H} \rightarrow \mathbb{C} \) satisfying

\[
\begin{align*}
& \gamma z = J(\gamma, z)f(z), \quad \gamma \in \Gamma_1, \\
& \Delta^{(j)}_{1/2} f = \lambda_j f, \quad j = 1, \ldots, g,
\end{align*}
\]

with polynomial growth at the cusps. Such a function of \( z = u + iv \in \mathcal{H} \) has a Fourier series development in terms of the classical Whittaker functions of the form

\[
f(u + iv) = \rho_0(v, f) + \sum_{\xi \in \mathcal{O}(f, \Gamma_1), \xi \neq 0} \rho(\xi, f)N(W(1/4, s-1/2(4\pi|\xi|v)), (3.8)
\]

where \( \mathcal{O}(f, \Gamma_1) \) is a certain ideal in \( F \) and

\[
N(W(1/4, s-1/2(4\pi|\xi|v))) = \prod_{j=1}^g W(1/4, s-j, -1/2(4\pi|\xi_j|v_j))
\]

with \( z_1, \ldots, z_g \) the Galois conjugates of \( \xi \in F \). The form \( f \) is cuspidal if it vanishes at the cusps of \( \Gamma_1 \). Let

\[
V = L^2_{\text{cusp}}(\Gamma_1 \backslash \mathcal{H}^g) = \{ f : \mathcal{H}^g \rightarrow \mathbb{C} : f(\gamma z) = f(z), \gamma \in \Gamma_1, \\
f \text{ cuspidal and square integrable} \}. (3.9)
\]

Then, by arguments exactly similar to those of [28, Prop. 2.3], we may derive the analogue of Proposition 2.1.

**Proposition 3.1**

If \( \phi \in U \), viewed as a function on \( G \), is an eigenfunction of the \( D^{(j)}_{\lambda} \) with eigenvalues \(-1/4 + (2r_j)^2\), \( j = 1, \ldots, n \), then

\[
f(z) = \int_{\Gamma_1 \backslash G} \phi(g) \theta(z, g) \, dg
\]

is an element of \( V \) and is an eigenfunction of \( \Delta^{(j)}_{1/2} \) with eigenvalues \(-1/4 + r_j^2\), \( j = 1, \ldots, g \).

Let \( \Delta \in \mathcal{O} \) be totally negative, and let

\[
M_\Delta(v) = \int_0^1 \cdots \int_0^1 \left( \int_{\Gamma_1 \backslash G} \phi(g) \theta(u + iv, g) \, dg \right) N(\exp(-2\pi i (\Delta u))) (3.10)
\]
Then
\[ M_\Delta(v) = N(v)^{3/4} \int_{\Gamma \backslash G} \sum_{h_2^2 - 4h_1h_3 = \Delta} N_s(\sqrt{v} g^{-1}(h)) \varphi(g) \, dg. \]  
(3.11)

Let \( h(\Delta) \) be the number of \( \Gamma \cdot \mathcal{L} \)-orbits of vectors \( h \in \mathcal{L} \) such that \( h_2^2 - 4h_1h_3 = \Delta \), let \( h(i) \) be a representative of the \( i \)th orbit, and let \( \Gamma_i \) be the stabilizer of \( h(i) \). Then we may write
\[ M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} \int_{\Gamma_i \backslash G} N_s(\sqrt{v} g^{-1}(h(i))) \varphi(g) \, dg. \]  
(3.12)

When \( g = 1 \), this corresponds to the situation considered in [28, (3.2)]. We consider two cases: \( \Delta \ll 0 \), totally negative, and \( \Delta \gg 0 \), totally positive.

Case (i). Let \( \Delta \) be totally negative. Let \( h^{(1)}, \ldots, h^{(h(\Delta))} \) be as above. Consider the integral
\[ I_i = \int_{\Gamma_i \backslash G} N_s(\sqrt{v} g^{-1}(h(i))) \varphi(g) \, dg. \]

Then (3.12) becomes
\[ M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_i|} I_i. \]  
(3.13)

The group \( \text{SL}(2, \mathbb{R}) \) acts transitively on the \( g \) hyperboloids of \( \mathbb{R}^3 \) with \( ^t x Q x = \Delta_j \). Therefore, we can find \( \tilde{g}(i) = (\tilde{g}_j(i))_{j=1}^g \in G \) such that
\[ (\tilde{g}_j(i))^{-1}(h_j(i)) = \frac{\sqrt{|\Delta_j|}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]

Let
\[ E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3_g, \]

and let
\[ \psi(g) = \varphi(\tilde{g}(i)g). \]

We have
\[ I_i = \int_{\Gamma_i \backslash G} N_s(\sqrt{v|\Delta|} g^{-1} E) \psi(g). \]

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Using as in [28, after (3.4)], we may write an integral over \( a = \sqrt{v|\Delta|} \)
\[ I_i = \int_{1}^{\infty} \cdots \int_{1}^{\infty} \psi(g) \, dg. \]

Now \( \psi(g) \) is an eigenfunction of \( \Delta \) for the hyperbolic Laplacian, which is the same as in [28], we may use the eigenfunction \( \omega_j(g) \) with
\[ \omega_j(g) = \begin{cases} 1, & \text{if } g \text{ is in the } j\text{th orbit,} \\ 0, & \text{otherwise}. \end{cases} \]

where
\[ Y_{\lambda,j}(r) = \omega_j(g). \]

In conclusion,
\[ M_\Delta \sim N(v)^{3/4} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_i|} I_i. \]

From [28], we have
\[ N(\sqrt{v|\Delta|} \Delta_j \Sigma_i) \sim v \]

and therefore as \( v \to \infty \)
\[ N(\sqrt{v|\Delta|} \Sigma_i) \sim v. \]

On the other hand,

Then \( \rho(\Delta) \) is the \( \Delta \)

We have the as
\[ N(W_{-1/4, \Delta}) \sim 4 \pi^{3/2}. \]

From equations (3.13)
Using as in [28, after (3.6)] the Cartan decomposition of SL(2, ℝ), we may write, as an integral over \( a = (a_j)_{j=1}^g \in \mathbb{R}^g \) with \( \delta(a_j) = (a_j^2 - a_j^{-2})/2 \),

\[
I_i = \int_1^\infty \cdots \int_1^\infty N \left( s \left( \sqrt{\frac{|u| \Lambda_j}{4}} \begin{pmatrix} a_1^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_2^{-2} \end{pmatrix} E \right) \right) 
\times \left( \int_{K_\infty} \int_{K_\infty} \psi(k_1 g k_2) \, dk_1 \, dk_2 \right) N(\delta(a)) N \left( \frac{da}{a} \right). \tag{3.14}\]

Now \( \psi(g) \) is an eigenfunction of the \( D_{g_j}^{(j)} \) with the same eigenvalues \( \lambda_j \) as \( \phi \). As in [28], we may use uniqueness arguments to show that there is a standard spherical function \( \omega_j(g_j) \) with eigenvalue \( \lambda_j \) such that \( \omega_j(e) = 1 \) and

\[
I_i = \phi(g(i)) N \left( Y_j \left( \sqrt{\frac{|u| \Lambda_j}{4}} \right) \right), \tag{3.15}\]

where

\[
Y_{\lambda_j}(t) = \int_1^\infty \left( t \begin{pmatrix} a_j^{-2} \\ 0 \\ a_j^{-2} \end{pmatrix} \right) \omega_{\lambda_j} \left( \begin{pmatrix} a_j & 0 & 0 \\ 0 & a_j^{-1} & 0 \\ 0 & 0 & a_j \end{pmatrix} \right) \delta(a_j) \frac{da_j}{a_j}. \tag{3.16}\]

In conclusion,

\[
M_\Delta(u) = N(u)^{3/4} \sum_{i=1}^{h(\Lambda)} \frac{1}{|G|} \phi(g(i)) N \left( Y_j \left( \sqrt{\frac{|u| \Lambda_j}{4}} \right) \right). \tag{3.17}\]

From [28], we have the asymptotic formula

\[
Y_j(t) \sim \frac{\exp(-8\pi t^2)}{32\pi t^2}, \quad t \to \infty,
\]

and therefore as \( \nu_j \to \infty, j = 1, \ldots, g \),

\[
N \left( Y_j \left( \sqrt{\frac{|u| \Lambda_j}{4}} \right) \right) \sim \exp \left( -2\pi \sum_{j=1}^g \nu_j |\Delta_j| \left( \prod_{j=1}^g 8\pi \nu_j |\Delta_j| \right)^{-1} \right). \tag{3.18}\]

On the other hand, we write

\[
M_\Delta(u) = \rho(\Delta) N \left( W_{-1/4,ir}(4\pi |\Lambda| u) \right). \tag{3.19}\]

Then \( \rho(\Delta) \) is the \( \Delta \)th Fourier coefficient of the function \( f(z) \) of Proposition 3.1.

We have the asymptotic formula as \( \nu_j \to \infty, j = 1, \ldots, g \),

\[
N \left( W_{-1/4,ir}(4\pi |\Lambda| u) \right) \sim \exp \left( -2\pi \sum_{j=1}^g \nu_j |\Delta_j| \left( \prod_{j=1}^g 4\pi \nu_j |\Delta_j| \right)^{-1/4} \right). \tag{3.20}\]

From equations (3.18), (3.19), and (3.20), we deduce the following result.
PROPOSITION 3.2
For \( \Delta \ll 0 \), the \( \Delta \)th Fourier coefficient of the function \( f(z) \) of Proposition 3.1 is given by

\[
\rho(\Delta) = 2^{-g}(4\pi)^{-3g/4}|N(\Delta)|^{-3/4} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_i|} \varphi(\tilde{r}^{(i)}).
\]  

(3.21)

Case (ii). Let \( \Delta \) be totally positive. Let \( h^{(1)}, \ldots, h^{(h(\Delta))} \) be as above. Consider the integral

\[
I_i = \int_{\Gamma_i \setminus G} N\left(s(\sqrt{v}g^{-1}(h^{(i)}))\right) \varphi(g) \, dg.
\]

Then (3.12) becomes

\[
M_\Delta(v) = N(v)^{3/4} \sum_{i=1}^{h(\Delta)} I_i,
\]  

(3.22)

The group \( SL(2, \mathbb{R}) \) acts transitively on the \( g \) hyperboloids of \( x \in \mathbb{R}^3 \) with \( \textup{tr} Qx = \Delta_j \). Therefore, we can find an \( \ell^{(i)} = (\ell_j^{(i)})_{j=1}^g \in G \) such that

\[
(\ell_j^{(i)})^{-1}(h_j^{(i)}) = \sqrt{\Delta_j} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Let

\[
E' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^g \in (\mathbb{R}^3)^g,
\]

and let

\[
\varphi(g) = \varphi(\ell^{(i)} g).
\]

We have

\[
I_i = \int_{\Gamma_i \setminus G} N\left(s(\sqrt{v}g^{-1}(E'))\right) \varphi(g) \, dg,
\]

where

\[
\Gamma_i = \ell^{(i)} \Gamma_i \ell^{(i)}.
\]

The group \( \Gamma'(\mathbb{R}) = \Gamma'_i(\mathbb{R}) \) is the \( g \)th power of the stabilizer of \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \), and it can be written as

\[
\Gamma'(\mathbb{R}) = \prod_{j=1}^g \left\{ \pm \begin{pmatrix} p_j^{1/2} & 0 \\ 0 & p_j^{-1/2} \end{pmatrix} : 0 < p_j < \infty \right\}.
\]
The group $\Gamma'\langle\rangle$ is a discrete free abelian subgroup of $\Gamma'\langle\rangle$ of rank $g$ over $\mathbb{Z}$ (see, e.g., [18, Chap. 1, Sec. 5]). We can decompose each component of $g = (g_j)_{j=1}^g \in G$ as

$$g_j = \begin{pmatrix} 1 & \bar{\xi}_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_j^{1/2} & 0 \\ 0 & p_j^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} k_j$$

$$= \begin{pmatrix} p_j^{1/2} & 0 \\ 0 & p_j^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & \bar{\xi}_j/p_j \\ 0 & 1 \end{pmatrix} k_j, \quad k_j \in K, \quad 0 < p_j < \infty, \quad -\infty < \bar{\xi}_j < \infty.$$ 

We have

$$g_j^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\bar{\xi}_j/p_j \\ 1 \\ 0 \end{pmatrix}.$$ 

Let $t_j = \bar{\xi}_j/p_j$; then $N(dt) := \prod_{j=1}^g dt_j$ is a Haar measure on $G$ and $N\left(\frac{dp_j}{p_j}\right) := \prod_{j=1}^g \frac{dp_j}{p_j}$ is a Haar measure on $G$. We may assume that $dg = N(dt)N\left(\frac{dp_j}{p_j}\right)$.

With this notation, we may write

$$I_i = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( -2\pi \sum_{j=1}^g \nu_j \Delta_j (2t_j^2 + 1) \right)$$

$$\times \int_{\Gamma'\langle\rangle} \varphi \left( \begin{pmatrix} p_j^{1/2} & 0 \\ 0 & p_j^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & t_j \\ 0 & 1 \end{pmatrix} \right) N\left(\frac{dp_j}{p_j}\right) N(dt). \quad (3.23)$$

Let

$$J_i(e) = \int_{\Gamma'\langle\rangle} \varphi \left( \begin{pmatrix} p_j^{1/2} & 0 \\ 0 & p_j^{-1/2} \end{pmatrix} \right) N\left(\frac{dp_j}{p_j}\right) = \int_{\Gamma'\langle\rangle} \varphi(\gamma) d\gamma,$$

where $d\gamma$ is the invariant measure on $\Gamma_1(\mathbb{R})$ induced by $N\left(\frac{dp_j}{p_j}\right)$.

Arguing as in [28, case (ii)] (where $g = 1$), we can again use the Casimir operators to see that there is, for each $j = 1, \ldots, g$, a unique even function $V_{\bar{\xi}_j}(t_j)$ of $t_j \in \mathbb{R}$ determined by the condition $V_{\bar{\xi}_j}(0) = 1$ and such that

$$I_i = J_i(e) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left( -2\pi \sum_{j=1}^g \nu_j \Delta_j (2t_j^2 + 1) \right)$$

$$\times V_{\bar{\xi}_1}(t_1) \cdots V_{\bar{\xi}_g}(t_g) dt_1 \cdots dt_g. \quad (3.24)$$
From the asymptotics in [28], we have, as \( v_j \to \infty, j = 1, \ldots, g, \)

\[
l_j \sim J_l(e)2^{-g} \prod_{i=1}^{g} \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) (v_j \Delta_j)^{-1/2}.
\]

Hence,

\[
M_{\Delta}(v) \sim N(v)^{1/4} N(\Delta)^{-1/2} 2^{-g} \prod_{i=1}^{g} \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) \sum_{i=1}^{g} J_i(e). \tag{3.25}
\]

We also have the asymptotic formula for \( j = 1, \ldots, g, \)

\[
W_{1/4,ir_j}(4\pi \Delta_j v_j) \sim \exp(-2\pi v_j \Delta_j)(4\pi \Delta_j v_j)^{1/4}, \quad v_j \to \infty.
\]

On the other hand, we write

\[
M_{\Delta}(v) \sim \rho(\Delta) N(W_{1/4,ir_j}(4\pi \Delta_j v_j)). \tag{3.26}
\]

Then \( \rho(\Delta) \) is the \( \Delta \)-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1.

We have the asymptotic formula for \( v_j \to \infty \) and \( j = 1, \ldots, g, \)

\[
M_{\Delta}(v) \sim (4\pi)^{g/4} \rho(\Delta) \exp \left( -2\pi \sum_{j=1}^{g} v_j \Delta_j \right) N(\Delta)^{1/4} N(v)^{1/4}. \tag{3.27}
\]

From equations (3.25), (3.26), and (3.27), we deduce the following result.

**PROPOSITION 3.3**

*For \( \Delta \gg 0 \), the \( \Delta \)-th Fourier coefficient of the function \( f(z) \) of Proposition 3.1 is given by*

\[
\rho(\Delta) = 2^{-g}(4\pi)^{-g/4} N(\Delta)^{-3/4} \sum_{i=1}^{\#(\Delta)} \int_{\Gamma_i \backslash \Gamma_0(\mathbb{R})} \varphi(y) \, dy. \tag{3.28}
\]

4. Families of symmetric domains

We describe the families of symmetric domains to which we apply the Maass correspondence of §§2 and 3. These correspond, in particular, to subvarieties of the Siegel modular variety of genus 2 and to certain Heegner points in arbitrary genus, coming from Hilbert modular varieties.

We exploit a natural isomorphism between \( \text{SO}_0(3, 2) \) and \( \text{Sp}(4, \mathbb{R}) \), following [57, §X]. Let \( Q \) be the quadratic form on \( \mathbb{R}^5 \) of signature \((3, 2)\) given by

\[
Q(x) = Q(x_1, \ldots, x_5) = x_2^2 - 4x_3x_1 - 4x_4x_5, \quad x = (x_1, \ldots, x_5) \in \mathbb{R}^5. \tag{4.1}
\]

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Then \( \mathcal{M}_2 := \mathcal{M}_2 \mathbb{Z} \) is isomorphic to \( \text{Sp}(4, \mathbb{Z}) \) and

\[
Z(\mathcal{M}_2) = \mathcal{M}_2 \mathbb{Z}.
\]

We recover the description of the lower Siegel half-space.

We may also induct through the generating matrices of the group, and this defines \( V \mathcal{M}_2 \), which will be seen via the action of the Maass forms.

Fix \( \lambda \in \mathbb{R}, \lambda \neq 0 \) and \( Q(\lambda) = \lambda \), and the action \( \lambda \cdot \mathcal{M}_2 \to \mathcal{M}_2 \) and to \( \text{SO}(1, 3) \) if

\[
Q(\lambda) < 0, \quad \text{for} \quad \lambda \neq 0.
\]

By checking at \( z \in \mathcal{M}_2 \), \( Q(x) > 0 \) and \( \mathcal{M}_2 \) is complex isomorphic to the upper complex half-space.

Assume from now that \( \mathcal{M}_2 \in \text{Sp}(4, \mathbb{Z}) \setminus \mathcal{M}_2 \).
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Then \( \mathcal{H}_2 \) is isomorphic to the space of vectors \( Z = (z_1, \ldots, z_5) \in \mathbb{C}^5 \) with \( z_5 = 1 \) and
\[
Z^tQZ = 0, \quad \overline{Z}^tQZ < 0, \quad \text{Im}(z_1) > 0.
\]

We recover the description of \$1\$ by setting
\[
z = \begin{pmatrix}
    z_1 \\
    z_2 \\
    z_2 \\
    z_3
\end{pmatrix},
\]
the lower Siegel half-space of degree 2 corresponding to the condition \( \text{Im}(z_1) < 0 \).

We may also introduce \( Q \) as the quinary quadratic form on the space \( V \) of alternating matrices of the form \( M(x), x \in \mathbb{R}^5 \), where
\[
M(x) = \begin{pmatrix}
    0 & -2x_4 & x_2 & -2x_1 & 2x_4 \\
    2x_4 & 0 & 2x_3 & -x_2 & -x_2 \\
    -x_2 & -2x_3 & 0 & 2x_5 & 2x_1 \\
    2x_1 & x_2 & -2x_5 & 0 & 0
\end{pmatrix}.
\]

With \( J \) the standard symplectic (4 \times 4)-matrix as in \$1\$, we have
\[
M(x)J M(x) = Q(x) \cdot J, \quad x \in \mathbb{R}^5,
\]
and this defines \( (V, Q) \). The isomorphism between \( \text{SO}_0(3, 2) \) and \( \text{Sp}(4, \mathbb{R}) \) can then be seen via the action of \( g \in \text{Sp}(4, \mathbb{R}) \) on \( M(x) \in V \) preserving \( Q \) and given by
\[
g : M(x) \rightarrow gM(x) g^{-1}.
\]

Fix \( \lambda \in \mathbb{R}, \lambda \neq 0 \). The group \( \text{SO}(Q) \) acts transitively on the solutions \( x \in \mathbb{R}^5 \) of \( Q(x) = \lambda \), and the isotropy group of any such \( x \) is isomorphic to \( \text{SO}(2, 2) \) if \( \lambda > 0 \) and to \( \text{SO}(1, 3) \) if \( \lambda < 0 \). For \( Q(x) > 0 \), let
\[
\mathcal{R}_\lambda = \left\{ z \in \mathcal{H}_2 : (z^t 1_2) M(x) (z 1_2) = 0 \right\}.
\]

For \( Q(x) < 0 \), let
\[
\mathcal{R}_\lambda^- = \left\{ z \in \mathcal{H}_2 : \overline{(z^t 1_2)} M(x) (z 1_2) = 0 \right\}.
\]

By checking at \( z = \sqrt{-1} I_2 \in \mathcal{H}_2 \) and using transitivity, one sees that the domains \( \mathcal{R}_\lambda \), \( Q(x) > 0 \) are real isomorphic to the symmetric space for \( \text{SO}_0(2, 2) \) and complex isomorphic to \( \mathcal{H}^2_1 \). On the other hand, the domains \( \mathcal{R}_\lambda^- \), \( Q(x) < 0 \) are real isomorphic to the symmetric space for \( \text{SO}_0(3, 1) \). For \( d \in \mathbb{Z}_+ \), let
\[
W_d = \{ h \in \mathbb{Z}^5 : Q(h) = d \}.
\]

Assume from now on that \( d \) is square-free. When \( d > 0 \), let \( S_d \) be the complex surface in \( \text{Sp}(4, \mathbb{Z}) \setminus \mathcal{H}_2 \) given by the union of the images of the \( \mathcal{R}_h, h \in W_d \) (with \( h \) primitive
as \( d \) is square-free). The surface \( S_d \) is nontrivial if and only if \( d \equiv 1 \mod 4 \). Then \( S_d \) is called the Humbert surface of invariant \( d \). For a general reference on Humbert surfaces, see [61, Chap. IX]. The components of the surface \( S_d \) are images of Hilbert modular surfaces induced by the identification of the \( \mathcal{O} \)-module \( \mathcal{O} \oplus \mathcal{O}^\vee \) (with the standard alternating form derived from the trace of \( F = \mathbb{Q}(\sqrt{d}) \) over \( \mathbb{Q} \)) with the \( \mathbb{Z} \)-lattice \( \mathbb{Z}^4 \) (with the standard symplectic form). This amounts to viewing Hilbert modular surfaces as subvarieties of Siegel 3-folds. In particular, by [61, Chap. IX, Prop. (2.3)], the abelian surface

\[
A(\mathbb{C}) = \mathbb{C}^4 / \mathbb{Z}^4 + z \cdot \mathbb{Z}^4
\]

has endomorphism ring \( \text{End}(A) \) containing \( \mathcal{O} \) if and only if \( z \mod \text{Sp}(4, \mathbb{Z}) \) is in \( S_d \).

When \( d < 0 \), let \( \mathcal{O}_d \) be the real 3-dimensional variety in \( \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{M}_2 \) given by the union of the images of the \( \mathcal{H}_h, h \in W_d \) (with \( h \) primitive as \( d \) is square-free). We have \( \mathcal{O}_d \) nontrivial if and only if \( -d \equiv 3 \mod 4 \).

We now turn to studying the Hilbert modular surfaces \( X_d \) with

\[
X_d(\mathbb{C}) \simeq \text{PSL}(\mathcal{O} \oplus \mathcal{O}^\vee) \backslash \mathcal{M}_2
\]

where \( \mathcal{O} \) is the ring of integers of \( F = \mathbb{Q}(\sqrt{d}) \), \( d > 0 \), square-free. There is an isomorphism between \( \text{SO}_0(2, 2) \) and \( \text{SL}(2, F) \otimes \mathbb{Q} \mathbb{R} \). Fix an ideal \( \mathcal{A} \) in the same genus as \( \mathcal{O}^\vee \), and let \( \delta = N(\mathcal{A}) \) be the norm of \( \mathcal{A} \). Let \( \sigma \) be the nontrivial Galois automorphism of \( F \) determined by \( \sigma : \sqrt{d} \mapsto -\sqrt{d} \). As in [32] and in [61, Chap. V] (but with some minor differences in conventions), we let

\[
Y_d = \left\{ M \in M_2(F) : M = \begin{pmatrix} a \sqrt{d} & a \\ -a^\sigma & b \sqrt{d} \end{pmatrix}, a \in F, a, b \in \mathbb{Q} \right\}.
\]

As a \( \mathbb{Q} \)-vector space, \( Y_d \) is isomorphic to \( \mathbb{Q}^4 \). Define \( Q_d : Y_d \to \mathbb{Q} \) to be the quadratic form given by

\[
Q_d[M] = \det M = abd + aa^\sigma.
\]

Then \( Q_d \) has signature \((2, 2)\) and we may embed \( \text{SL}(2, F) \) into \( \text{SO}(Q) \) by the action

\[
g : M \mapsto g^\sigma M g^{-1}, \quad g \in \text{SL}(2, F).
\]

This induces a representation

\[
\text{SL}(2, F) \otimes \mathbb{Q} \mathbb{R} \simeq \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \to \text{SO}(Q) \simeq \text{SO}(2, 2)
\]

and a corresponding isomorphism between \( \mathcal{M}_2 \) and the majorant space \( \mathcal{H}_d = \mathcal{H}_{Q_d} \) of \( Q_d \). In \( Y_d \), we can define the lattice of integral elements given by

\[
Y_d(\mathbb{Z}) = \left\{ M = \begin{pmatrix} a \sqrt{d} & a \\ -a^\sigma & b \sqrt{d}/\delta \end{pmatrix} : a \in \mathcal{A}^{-1}, a, b \in \mathbb{Z} \right\}.
\]
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For \( \lambda \in \mathbb{R} \), \( \lambda \neq 0 \), the group \( \text{SO}(Q_d) \) acts transitively on the \( M \in Y_d \) with \( Q_d[M] = \lambda \), and the isotropy group of any such \( M \) is isomorphic to \( \text{SO}(2, 1) \). Therefore, one may assume that \( \lambda > 0 \).

For \( M \in Y_d \) with \( Q_d[M] > 0 \), let

\[
\mathcal{H}_M = \left\{ (z_1, z_2) \in \mathbb{H}^2 : \begin{pmatrix} z_1 & 1 \\ 1 & 1 \end{pmatrix} M \begin{pmatrix} z_1^* \\ 1 \end{pmatrix} = 0 \right\}.
\]

By checking at \( z = (\sqrt{-1}, \sqrt{-1}) \) and using transitivity, we see that \( \mathcal{H}_M \) is isomorphic to the symmetric space for \( \text{SO}_0(2, 1) \sim \text{SL}(2, \mathbb{R}) \). Namely, it is the graph of a fractional linear transformation and is therefore a copy of \( \mathcal{H} \) embedded into \( \mathbb{H}^2 \).

Let \( n \) be a positive square-free integer. For \( M \in Y_d(\mathbb{Z}) \) with \( Q_d[M] = n \), let

\[
\Gamma_M = \left\{ g \in \text{SL}(\mathcal{O} \oplus \mathcal{O}^*) : g^* M g = M \right\}.
\]

Let \( \mathcal{A}_M \) be the image of the curve \( \Gamma_M \backslash \mathcal{H}_M \) in \( \text{SL}(\mathcal{O} \oplus \mathcal{O}^*) \backslash \mathbb{H}^2 \). Finally, \( \mathcal{A}_{d,n} \) is defined as the curve given by the union of all the \( \mathcal{A}_M, M \in Y_d(\mathbb{Z}) \). \( Q_d[M] = n \). It is called a modular curve and is nontrivial if and only if for some \( \alpha \in F \),

\[
n \equiv N(\alpha)N(\mathcal{A}) \mod d.
\]

Moreover, from [61, p. 102], all irreducible components of \( \mathcal{A}_{d,n} \) have the same volume. The curve \( \mathcal{A}_{d,n} \) corresponds to abelian surfaces whose endomorphism ring contains an order in a quaternion algebra. Namely, let \( \mathcal{O}_{d,n} \) be the quaternion algebra over \( \mathbb{Q} \) with parameters \( (d, -n/d) \): it has basis elements \( 1, i, j, k \), where

\[
i^2 = d, \quad j^2 = -\frac{n}{d}, \quad k = ij = -ji.
\]

For \( M \in Y_d(\mathbb{Z}) \) with \( Q_d[M] = n \), the following algebra is isomorphic to \( \mathcal{O}_{d,n} \) (see [61, Chap. V, Prop. 1.5]):

\[
\mathcal{O}_M = \left\{ g \in \text{Mat}_2(F) : g^* M g = \text{det}(g)M \right\},
\]

and it contains the order of discriminant \( n^2 \) given by

\[
\mathcal{O}_M = \mathcal{O}_M \cap \left( \begin{pmatrix} \mathcal{O} & \mathcal{A}^{-1} \\ \mathcal{A} & \mathcal{O} \end{pmatrix} \right).
\]

For an abelian surface \( A \), we have \( \text{End}(A) \) containing \( \mathcal{O}_M \) if and only if

\[
A(C) \simeq \mathbb{C}^2 / \mathcal{O}^* + z \cdot \mathcal{O}
\]

with \( z = (z_1, z_2) \in \mathcal{H}_M \).
5. Cuspidal Weyl sums

As in §3, let \( Q \) be a prime number, \( r = q \), and \( p + q = m \), and let the function \( f(z) \) depend only on \( Q \) and \( \Gamma \). Let \( H(d) \) be the set \( 1, \ldots, H(d) \).

Let \( \phi \) be a Maass cusp form of weight \( \frac{1}{2} \) and \( \ell \) be an integer, then we may apply Proposition 2.5.

Therefore, \( f(z) \) is a cusp form of weight \( \frac{1}{2} \) whose derivatives may be defined as Maass cusp forms of weight \( \frac{1}{2} \).

We define

\[
W_{\Delta} = \{ h \in \mathcal{L} : Q(h) = \Delta \}.
\]

From Proposition 2.5,

By Siegel’s mass formula for \( \eta \), the effective lower bound is

Therefore,

By well-known bounds on \( \eta \), integral weight (or \( \ell \) integral, \( \ell \in \mathbb{Z} \)) Maass cusp forms have

This implies the following

PROPOSITION 5.1

For \( m \geq 4 \),

\[ 2a^{(j)} |m_j|^2 + b^{(j)} (\bar{m}_j + \bar{m}_j) + c^{(j)} = 0, \quad j = 1, \ldots, g. \]
5. Cuspidal Weyl sums and equidistribution in genus 2

As in §3, let $Q$ be a quadratic form in $m$ variables of signature $(p, q)$, where $p, q \geq 0$ and $p + q = m$, and let $\Gamma$ be a lattice in $\Omega(Q)$. Let $c_1, c_2, \ldots$ be constants depending only on $Q$ and $\Gamma$. Let $d \in \mathbb{Z}$, $d \neq 0$, be a square-free integer, and let $\Gamma_j$, $j = 1, \ldots, H(d)$ be the stabilizers in $\Gamma$ of a set of representatives mod $\Gamma$ of the set

$$\{x \in \mathbb{Z}^m : Q(x) = d\}.$$

Let $\phi$ be a Maass cusp form of weight zero on $\overline{\Gamma \backslash \mathcal{H}}$ with eigenvalue $\lambda'$. Then we may apply Proposition 2.1 to

$$f(z) = \psi^{m/4}(\phi, \overline{\theta}(z)).$$

Therefore, $f(z)$ is a cuspidal Maass form of discriminant $D$ for $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$, where $D$ and $N$ are determined by $Q$ as in [16, p. 81], and of weight $k = p - m/2$ with eigenvalue $\lambda = 1/4(\lambda' + m - m^2/4)$. Let

$$W_{\text{cusp}}(d, \lambda) = \mu(d)^{-1} \left( \sum_{i=1}^{H(d)} \int_{\Gamma_i \backslash \mathcal{H}} \phi(\gamma) \ d\gamma \right),$$

where we define

$$\mu(d) = \sum_{i=1}^{H(d)} \text{Vol}(\Gamma_i \backslash \mathcal{H}(\mathbb{R})).$$

From Proposition 2.2, we have

$$W_{\text{cusp}}(d, \lambda) = c_3 |d|^{m/4} \mu(d)^{-1} \rho(d).$$

By Siegel's mass formula, $\mu(d)$ is a product of local densities. For $m \geq 4$, we have the effective lower bound

$$\mu(d) \geq c_4 |d|^{m/2-1}.$$

Therefore,

$$W_{\text{cusp}}(d, \lambda) \leq c_5 |d|^{-m/4} \rho(d).$$

By well-known bounds for Fourier coefficients of cusp forms of integral and half-integral weight (or using the stronger [16, Th. 5]), we know that for $m \geq 4$,

$$\lim_{|d| \to \infty} |d|^{-m/4} \rho(d) = 0.$$

This implies the following result.

PROPOSITION 5.1

For $m \geq 4$,

$$\lim_{|d| \to \infty} W_{\text{cusp}}(d, \lambda) = 0.$$
The statement in §1 of equidistribution in genus 2 is a direct corollary of Proposition 5.1 and the discussion of §4 once we treat in a similar way the eigenfunctions of the continuous spectrum of $\Delta_Q$ and show vanishing results for their Weyl sums. We hope to return to this in a later paper.

6. Cuspidal Weyl sums in the Hilbert modular case

In §3 the integral quadratic form $Q$ is in $m = 3$ variables and is of signature $(p, q) = (2, 1)$, and we work over a totally real field $F$ of degree $g$ over $\mathbb{Q}$ and with the lattice $\Gamma_{\mathcal{O}}$ in $G = \text{SL}(2, \mathbb{R})^g$. Recall that $\mathcal{O}$ is a fractional ideal in $F$, and recall that we denote by $\mathcal{O}$ the ring of integers of $F$ and by $\mathcal{L}$ the lattice $\mathcal{O}^{-1} \otimes \mathcal{O} \oplus \mathcal{O}$ in $F^3$.

Let $\Delta \in \mathcal{O}$, $\Delta \neq 0$, and let $\Gamma_i$, $i = 1, \ldots, h(\Delta)$, be the stabilizers in $\Gamma_{\mathcal{O}}$ of a set of representatives mod $\Gamma_{\mathcal{O}}$ of the set

$$\{ h \in \mathcal{L} : Q[h] = \Delta \}.$$ 

Let $\varphi \in U$ be an eigenfunction of $\Lambda(\mathcal{O})$ with eigenvalue $\lambda_j$, $j = 1, \ldots, g$. Then we may apply Proposition 3.1 to

$$f(z) = \int_{\Gamma_{\mathcal{O}} \backslash G} \varphi(g) \theta(z, g) \, dg.$$ 

Suppose $\Delta \ll 0$. With the notation of §3, case (i), let

$$W_{\text{cusp}}(\Delta, \lambda) = h(\Delta)^{-1} \sum_{i=1}^{h(\Delta)} \frac{1}{|\Gamma_i|} \varphi(\xi^{(i)}).$$ 

(6.1)

From the discussion of §4, we have

$$W_{\text{cusp}}(\Delta, \lambda) = h(\Delta)^{-1} \sum_{z \in \Lambda_{\Delta}} \frac{1}{|\Gamma_z|} \varphi(z),$$ 

(6.2)

where $\Gamma_z$ is the stabilizer of $z$ in $\Gamma_{\mathcal{O}}$ and $h(\Delta) = \text{Card}(\Lambda_{\Delta})$. We may take

$$\xi^{(i)} = \left( \begin{array}{cc} 1 & h_2^{(i)} / 2 h_1^{(i)} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} (\sqrt{|\Delta|}/2h_1^{(i)})^{-1} & 0 \\ 0 & (\sqrt{|\Delta|}/2h_1^{(i)})^{-1/2} \end{array} \right).$$

Then, from Proposition 3.2, we have the following proposition.

PROPOSITION 6.1

For $\Delta \in \mathcal{O}$, $\Delta \ll 0$, we have

$$W_{\text{cusp}}(\Delta, \lambda) = 2^g (4\pi)^{3g/4} |N_F/\mathcal{O}(\Delta)|^{3/4} h(\Delta)^{-1} \rho(\Delta).$$
Suppose $\Delta \gg 0$. With notation as in §3, let
\[
W_{\text{cusp}}(\Delta, \lambda) = \mu(\Delta)^{-1} \sum_{i=1}^{h(\Delta)} \int_{\Gamma_i \backslash \Gamma_i(\mathbb{R})} \varphi(\gamma) \, d\gamma,
\] (6.3)
where
\[
\mu(\Delta) = \sum_{i=1}^{h(\Delta)} \int_{\Gamma_i \backslash \Gamma_i(\mathbb{R})} d\gamma.
\]
From the discussion of §4, we have
\[
W_{\text{cusp}}(\Delta, \lambda) = \mu(\Delta)^{-1} \sum_{C \in \mathcal{S}_\Delta} \int_C \varphi(z) \, ds,
\] (6.4)
where
\[
\mu(\Delta) = \sum_{C \in \mathcal{S}_\Delta} \text{Vol}(C).
\]
To explain this last quantity, we consider the Euclidean hyperbolic distance in each copy of $\mathcal{H}$ in $\mathcal{H}_g$ given by
\[
ds_j^2 = y_j^{-2} ((dx_j)^2 + (dy_j)^2), \quad j = 1, \ldots, g.
\]
Then we have the real $g$-form
\[
ds = \prod_{j=1}^{g} ds_j
\]
and
\[
\text{Vol}(C) = \int_C ds.
\]
By (3.28), we have the following proposition.

**Proposition 6.2**
For $\Delta \in \mathcal{O}$, $\Delta \gg 0$, we have
\[
W_{\text{cusp}}(\Delta, \lambda) = 2^g (4\pi)^{g/4} N_{F/\mathbb{Q}}(\Delta)^{3/4} \mu(\Delta)^{-1} \rho(\Delta).
\]
These results enable us to bound the cuspidal Weyl sums from above in terms of the Fourier coefficients of Maass cusp eigenforms of weight $1/2$ and level $4$. From Propositions 6.1 and 6.2, we deduce directly the following.

**Lemma 6.3**
For $\Delta \ll 0$ and as $|N_{F/\mathbb{Q}}(\Delta)| \to \infty$, we have
\[
|W_{\text{cusp}}(\Delta, \lambda)| \ll \frac{|N_{F/\mathbb{Q}}(\Delta)|^{3/4}}{h(\Delta)} \rho(\Delta).
\] (6.5)
For $\Delta \gg 0$ and as $N_{F/\mathbb{Q}}(\Delta) \to \infty$, we have

$$|W_{\text{cusp}}(\Delta, \lambda)| \ll \frac{N_{F/\mathbb{Q}}(\Delta)^{3/4}}{\mu(\Delta)} \rho(\Delta),$$

(6.6)

The implied constants depend only on the field $F$ (and, in fact, can be bounded above explicitly by a function of $g$ only).

7. Eisenstein Weyl sums: The case of the Hilbert modular group for class number 1

We continue with the notation of §§3–5. The methods of Maass, in particular, in Propositions 2.1 and 3.1, do not apply to the eigenfunctions of the continuous spectrum of the Laplacian, which is nontrivial in all cases considered in this paper as the group actions are not cocompact. These eigenfunctions are furnished by the Eisenstein series. We are therefore led to consider averages or Weyl sums as in §5 with cusp forms replaced by Eisenstein series.

We restrict ourselves to the case $m = 3, (p, q) = (2, 1)$ and to the Hilbert modular group $\Gamma = \Gamma_{\mathcal{O}} = \text{PSL}(2, \mathcal{O})$, where $\mathcal{O} = \mathcal{O}_F$ is the ring of integers of a totally real field $F$ of degree $g \geq 1$ with class number 1. We repeatedly use facts about Eisenstein series from [18], [50], [51], [52], and [53]. The generalization to arbitrary class number and to arbitrary $\Gamma_{\mathcal{O}}$ should be straightforward, if somewhat technical, with some necessary material available in [59]. In order to extend to the case $g > 1$ the arguments of [16] on Eisenstein Weyl sums, we generalize in this section some classical arguments due to Hecke [26] and Kronecker [31]. It would be of interest to consider also the cases $m = 4, (p, q) = (2, 2)$ and $m = 5, (p, q) = (3, 2)$ in order, for example, to deduce equidistribution results on the noncuspidal part of the corresponding $L^2$-spaces.

The Eisenstein series are $\Gamma$-automorphic eigenfunctions of the Laplacians $\Lambda_{\mathcal{O}}^{(j)}$, $j = 1, \ldots, g$, of (3.7) corresponding to the continuous part of the spectrum. As we see shortly, they are functions of $(z, s) \in \mathcal{H}^g \times \mathbb{C}$ and $a \in \mathbb{Z}^{g-1}$. Let

$$L^2(\Gamma \setminus \mathcal{H}^g) = \left\{ \varphi : \mathcal{H}^g \to \mathbb{C} : \varphi(\gamma z) = \varphi(z), \gamma \in \Gamma, \right\}
\int_{\Gamma \setminus \mathcal{H}^g} |\varphi|^2 N(u)^{-2} N(du)N(du) < \infty \right\}.$$  

(7.1)

Using general results and more specifically those of [18] and [53], we have a decomposition

$$L^2(\Gamma \setminus \mathcal{H}^g) = L^2_{\text{cusp}}(\Gamma \setminus \mathcal{H}^g) \oplus \mathcal{E} \oplus \mathcal{E},$$

where $\mathcal{E}$ is generated in an appropriate $L^2$-sense by the Eisenstein series evaluated at $s = 1/2 + it, t \in \mathbb{R}$, and $\mathcal{E}$ is generated by the residues of their finitely many poles in $s \in (1/2, 1]$. (The Eisenstein series are not themselves $L^2$-integrable.) For

$$g \geq 1$$
the only such $\mathcal{E}$ is obtained for a given ordering of the stabilizer $\Gamma_\infty$. The series

$$E(z, s) = \sum_{d | \gamma} e_d(z)$$

Here $z = (z_j)_{j=1}^g$ is given by (3.6).

Moreover, if $e_q = (e_q^{(j)})_{j=1}^g$ is the character of the group $\mathcal{O}^*$, the $e_q^{(j)}$ are determined by $e_q^{(j)}(z)$.

$$\mathcal{E}$$

These conditions can be found in [17] for the definition of $E(z, s)$.
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$g \geq 1$, the only such pole occurs at $s = 1$ and has residue given by the volume of the fundamental region of $\Gamma$. The cuspidal part of the $L^2$-decomposition is given by (3.6).

As $F$ is assumed to have class number 1, the group $\Gamma$ has 1 cusp at infinity with stabilizer $\Gamma_\infty$. The Eisenstein series are of the form

$$E(z, s, m) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y^s(\gamma(z)) \lambda_m(y(\gamma(z))), \quad \text{Re}(s) > 1. \quad (7.2)$$

Here $z = (z_j)_{j=1}^g \in \mathbb{C}^g$ with $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}$, and

$$y(\gamma(z)) = \prod_{j=1}^g \text{Im}(\gamma^{(j)}(z_j)),$$

where $\gamma^{(j)}$ is obtained from $\gamma$ by applying the $j$th Galois embedding to its entries for a given ordering of the Galois embeddings of $F$ into $\mathbb{R}$, starting with the identity embedding. Moreover, we have $s \in \mathbb{C}$ and $m = (m_q)_{q=1}^{g-1} \in \mathbb{Z}^{g-1}$ with $\lambda_m$ an exponential sum similar to a grössencharacter. Namely, there are parameters $e_j^{(q)} \in \mathbb{R}$, $q = 1, \ldots, g - 1$, $j = 1, \ldots, g$, determined by the choice of basis of the unit group $\mathcal{O}^* \otimes \mathcal{O}$, such that

$$\lambda_m(z) = \prod_{j=1}^g \prod_{q=1}^{g-1} |z_j|^{2\pi i m_q e^{(q)}_j}.$$

Moreover, if $c_q = (c_q^{(1)}, \ldots, c_q^{(g)})$, $q = 1, \ldots, g - 1$, are totally positive generators of the group $\mathcal{O}^*$ embedded in $(\mathbb{R}^*)^g$ by the Galois embeddings of $F$ into $\mathbb{R}$, then the $e^{(q)}_j$ are determined by the equations

$$\sum_{j=1}^g e^{(q)}_j = 0, \quad q = 1, \ldots, g - 1,$$

and

$$\sum_{j=1}^g e^{(q)}_j \log e^{(j)}_r = \delta_{r,q}, \quad r, q = 1, \ldots, g - 1.$$

These conditions ensure, in particular, that for $u = (u^{(1)}, \ldots, u^{(g)})$ in $\mathcal{O}^*$, we have

$$\prod_{j=1}^g \prod_{q=1}^{g-1} |u^{(j)}|^{2\pi i m_q e^{(q)}_j} = 1. \quad (7.3)$$

The series $E(z, s, m)$ has a meromorphic continuation to all of $s \in \mathbb{C}$. Full details can be found in [18, Chap. II]. We often perform formal manipulations with the series definition of $E(z, s, m)$ without specifying each time the domain of convergence.
We have, using [18, p. 47],

$$E(z, s, m) = \sum_{(c,d) \in \mathcal{O}} g\prod_{j=1}^{g} \frac{y_j^{s_j}}{|c(j)z_j + d(j)|^{2s_j}}, \quad (7.4)$$

where the summation is over $c, d \in \mathcal{O}$ with $(c,d)$ meaning that pairs differing by multiplication by an element of $\mathcal{O}^*$ are identified and $(c,d) = 1$ meaning that $c, d$ generate \mathcal{O}. As usual, $c(j), d(j)$ are the $j$th Galois conjugates of $c, d$. Furthermore,

$$s_j = s + \pi i \sum_{q=1}^{g-1} m_q e^{(q)}_j, \quad j = 1, \ldots, g. \quad (7.5)$$

In that same reference, it is shown that if

$$F(z, s, m) = \sum_{(c,d)} g\prod_{j=1}^{g} \frac{y_j^{s_j}}{|c(j)z_j + d(j)|^{2s_j}}, \quad (7.6)$$

then

$$F(z, s, m) = L(2s, \lambda_{-2m}) E(z, s, m), \quad (7.7)$$

where $L(s, \lambda_m)$ is the Hecke zeta function given by

$$L(s, \lambda_m) = \sum_{(b)} \frac{\lambda_m(b)}{|N_{F/Q}(b)|^s}. \quad (7.8)$$

Here the sum is over the (principal) integral ideals $(b)$ of $F$ and

$$\lambda_m(b) = \prod_{j=1}^{g} \prod_{q=1}^{g-1} |b(j)|^{s_q m_q e^{(q)}_j}. \quad$$

This last expression is well defined, thanks to (7.3).

With the notation of §§4 and 5, the Eisenstein Weyl sums are given, for $\Delta \ll 0$, by

$$W_{Eis}(\Delta, t, m) = \frac{1}{h(\Delta)} \sum_{z_h \in \Lambda_\Delta} E(z_h, \frac{1}{2} + it, m), \quad (7.9)$$

where $h(\Delta)$ is the cardinality of $\Lambda_\Delta$ and, for $\Delta \gg 0$, by

$$W_{Eis}(\Delta, t, m) = \frac{1}{\mu(\Delta)} \sum_{C \in \mathcal{C}_\Delta} \int_C E(z, \frac{1}{2} + it, m) d\mathcal{S}, \quad (7.10)$$

where

$$\mu(\Delta) = \sum_{C \in \mathcal{C}_\Delta} \text{Vol}(C).$$

In order to relate the Eisenstein Weyl sums to Hilbert-Maass Eisenstein series, we make use of results due to Hecke [16]. In particular, we prove that (7.10) extends to Eisenstein series of all levels.

From now on, assume $\Delta$ is odd. Let $\mathcal{Q}_L^n$ denote the discriminant of $L$-extension $L$. From assumption in [16, 6.1, 6.2] and [16, 6.3], we see that any representative of an ideal class $\mathfrak{C}$ of rank 2 in $\mathcal{O}_L$ corresponds to an ideal class $\mathfrak{C}$ of rank 2 in $\mathcal{O}_F$. We may choose these representatives to be prime over $\mathfrak{p}_F$. Let $\mathfrak{C}$ denote the ideal class of $\mathfrak{p}_F$.

Consider first the case $\mathfrak{C}$.

$$L(2s, \lambda_{-2m}) = \frac{1}{\mathcal{Q}_L^n} \sum_{\mathfrak{C}} \mathcal{Q}_L^n q_{\mathfrak{C}}^{(j)}(x, y) \quad \text{in the ideal class} \mathfrak{C},$$

where

$$q_{\mathfrak{C}}^{(j)}(x, y) = \sum_{(c,d) \in \mathfrak{C}}\text{Vol}(C).$$

In the notation of §§4 and 5, we have

$$\text{Vol}(C) = \prod_{j=1}^{g} \prod_{q=1}^{g-1} |b(j)|^{s_q m_q e^{(q)}_j}.$$
In order to relate these Weyl sums to Fourier coefficients of half-integral weight Hilbert-Maass Eisenstein series, we need to generalize for \( g > 1 \) some classical arguments due to Hecke [26] and Kronecker [31] that apply to the special case \( g = 1 \) as in [16]. In particular, the formulae of Propositions 7.1 and 7.2 at the end of this section generalize these classical results.

From now on, we assume that \( \Delta \in \mathcal{O} \) generates the ideal given by the relative discriminant of \( L = F(\sqrt{\Delta}) \) over \( F \). This replaces the fundamental discriminant assumption in [16, Th. 1]. As \( F \) has class number 1, the ideals of \( L \) are free \( \mathcal{O} \)-modules of rank 2. Let \( \rho \to \overline{\rho} \), \( \rho \in L \), denote the nontrivial automorphism of \( L \) over \( F \). We may choose a relative basis \( \{1, \Omega\} \) of \( \mathcal{O}_L \) over \( \mathcal{O} \) such that \( \Delta = (\Omega - \overline{\Omega})^2 \).

Let \( \mathcal{O}_L^* \) denote the units of \( \mathcal{O}_L \), and let \( \chi_{L/F} \) denote the relative field character for \( L \) over \( F \).

Consider first the case \( \Delta \ll 0 \). By (7.7), we have

$$ L(2s, \lambda; -2m) \sum_{z_h \in \Lambda} E(z_h, s, m) = \sum_{z_h \in \Lambda} \sum_{c, d} \prod_{j=1}^g Q_h^{(j)}(c^{(j)}, d^{(j)})^{-sj}, $$

where

$$ Q_h^{(j)}(c^{(j)}, d^{(j)}) = \frac{|c^{(j)} z_j + d^{(j)}|^2}{y_j}, \quad j = 1, \ldots, g. $$

In the notation of §4 (except that we denote \( z_j^+ \) by \( z_j \)), for \( h = (\alpha, \beta, \gamma) \in \mathcal{O}^3 \) we have

$$ Q_h^{(j)}(c^{(j)}, d^{(j)}) = \frac{2}{\sqrt{|\Delta^{(j)}|}} q_h^{(j)}(-d^{(j)}, c^{(j)}), $$

where

$$ q_h^{(j)}(x, y) = a^{(j)} x^2 + b^{(j)} xy + c^{(j)} y^2 = a^{(j)}(x - z_j y)(x - \overline{z}_j y). $$

Therefore,

$$ \prod_{j=1}^g Q_h^{(j)}(c^{(j)}, d^{(j)})^{-sj} = \prod_{j=1}^g \left| \frac{\Delta^{(j)}}{4} \right|^{1/2} \prod_{j=1}^g q_h^{(j)}(-d^{(j)}, c^{(j)})^{-sj}. $$

There is a bijection between the ideal classes of \( \mathcal{O}_L \) and the points \( z_h \in \Lambda_\Delta \). To \( h = (\alpha, \beta, \gamma) \in W_\Delta \) we associate the ideal \( \mathcal{O}_h \) with basis \( (\alpha, (\beta + \sqrt{\Delta})/2) \). The relative norm over \( F \) of \( \mathcal{O}_h \) is generated by \( a \). The relative norms of the integral ideals in the ideal class \( \text{Cl}(\mathcal{O}_h) \) of \( \mathcal{O}_h \) are generated by \( q_h(-d, c) \) for \( c, d \in \mathcal{O} \). Therefore,

$$ \sum_{[c, d]} \prod_{j=1}^g Q_h^{(j)}(c^{(j)}, d^{(j)})^{-sj} = 2^{-8g} |N_{L/F}(\Delta)|^{1/2} \lambda_m^{1/2} \left( \frac{\Delta}{4} \right) \times \sum_{\mathcal{O} \in \text{Cl}(\mathcal{O})} \frac{\lambda_{-m}(|N_{L/F}(\mathcal{O}^s)|)}{N_{L/F}(\mathcal{O}^s)^s}. $$

(7.16)
Combining (7.11) and (7.16), we conclude that

\[ 2^g L(2s, \lambda_{-2m}) \sum_{z_h \in A_\Delta} E(z_h, s, m) = |N_{F/Q}(\Delta)|^{s/2} \lambda_m^{1/2} \left( \frac{\Delta}{4} \right) \times L(s, \lambda_{-m}, L) \]  

(7.17)

where

\[ L(s, \lambda_{-m}, L) = \sum_{\mathfrak{a}} \frac{\lambda_{-m}(N_{L/F}(\mathfrak{a}))}{|N_{L/Q}(\mathfrak{a})|^s} = \sum_{\mathfrak{a} \in \mathfrak{A}} \sum_{\mathfrak{a} \in \mathfrak{A}} \frac{\lambda_{-m}(N_{L/F}(\mathfrak{a}))}{|N_{L/Q}(\mathfrak{a})|^s} \]  

(7.18)

with \( \mathfrak{a} \) ranging over the nonzero integral ideals of \( L \) and \( \mathfrak{A} \) ranging over the ideal classes of \( L \). From (7.17) and (7.18) we deduce

\[ 2^g L(2s, \lambda_{-2m}) \sum_{z_h \in A_\Delta} E(z_h, s, m) = |N_{F/Q}(\Delta)|^{s/2} \lambda_m^{1/2} \left( \frac{\Delta}{4} \right) \times L(s, \lambda_{-m}) L(s, \chi_{L/F} \lambda_{-m}). \]  

(7.19)

where

\[ L(s, \chi_{L/F} \lambda_{-m}) = \sum_{(b)} \frac{\chi_{L/F}(b) \lambda_{-m}(b)}{|N_{F/Q}(b)|^s}, \]  

(7.20)

the sum ranging over the ideals \( (b) \) of \( F \). We deduce finally the following result.

**PROPOSITION 7.1**

For \( \Delta \ll 0 \), we have

\[ W_{Eis}(\Delta, t, m) = 2^{-(1/2 + it)g} \frac{L(1/2 + it, \lambda_{-m})}{L(1 + it, \lambda_{-2m})} \times \frac{|N_{F/Q}(\Delta)|^{1/4 + it} \lambda_m^{1/2} (\Delta/4)^{1/2 + it} \chi_{F(\sqrt{\Delta})/F} \lambda_{-m}}{h(\Delta)}. \]  

(7.21)

Now consider the case \( \Delta \gg 0 \). Combining (7.6) and (7.10), we have

\[ L(2s, \lambda_{-2m}) \sum_{C_h \subseteq \mathfrak{A}_h} \sum_{C_h \subseteq \mathfrak{A}_h} \int_{C_h} E(z, s, m) \, dz = \sum_{C_h \subseteq \mathfrak{A}_h} \sum_{C_h \subseteq \mathfrak{A}_h} \prod_{J=1}^g \left( \frac{y_j}{|c(J)|z + d(J)|z|} \right)^{2s} \, dz. \]  

(7.22)

We adapt some ideas of Hecke [26] for the case \( g = 1 \), as they are explained in [65] and [66], to the general case \( g \geq 1 \). Let \( A \) be a fixed ideal class of \( L = F(\sqrt{\Delta}) \), and let \( B \) be a fixed element of \( A^{-1} \). We have a correspondence \( \mathfrak{a} \mapsto \mathfrak{a} B = (\eta) \) which is a bijection between the set of ideals of \( A \) and the set of principal ideals with \( \eta \in B \).
For a fixed ideal class \( A \), define
\[
L(s, \lambda_m, A) = \sum_{\mathfrak{A} \in \mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathfrak{A}))}{N_{L/Q}(\mathfrak{A})^s}.
\]
(7.23)

Define, as in (7.18),
\[
L(s, \lambda_m, L) = \sum_{\mathfrak{A} \in \mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathfrak{A}))}{N_{L/Q}(\mathfrak{A})^s} = \sum_{A} L(s, \lambda_m, A)
\]
(7.24)

with \( \mathfrak{A} \) ranging over the nonzero integral ideals of \( L \) and \( A \) ranging over the ideal classes of \( L \). For \( B \) a fixed element of \( A^{-1} \), we have
\[
L(s, \lambda_m, A) = N_{L/Q}(\mathfrak{B})^s \lambda_m(N_{L/F}(\mathfrak{B})) \sum_{\mathfrak{A} \in \mathcal{A}} \frac{\lambda_m(N_{L/F}(\mathfrak{A}))}{N_{L/Q}(\mathfrak{A})^s}.
\]
(7.25)

Two numbers \( \eta_1, \eta_2 \in B \) define the same principal ideal if and only if \( \eta_1 = \varepsilon \eta_2 \) for \( \varepsilon \in \mathcal{O}_L^* \). Hence,
\[
L(s, \lambda_m, A) = N_{L/Q}(\mathfrak{B})^s \lambda_m(N_{L/F}(\mathfrak{B})) \sum_{\eta \in \mathcal{O}_L} \frac{\lambda_m(N_{L/F}(\eta))}{|N_{L/Q}(\eta)|^s}
\]
(7.26)

with \( \sum' \) denoting a sum over nonzero elements. We have an exact sequence
\[
1 \to \mathcal{O}_L^{*1} \to \mathcal{O}_L^* \to \mathfrak{g}^*.
\]
(7.27)

where the rightmost arrow is given by the reduced norm from \( L \) to \( F \) and \( \mathcal{O}_L^{*1} \) is the group of units of \( \mathcal{O}_L \) of reduced norm 1. The image \( N_{L/F}(\mathcal{O}_L^*) \) is of finite index in \( \mathfrak{g}^* \).

Moreover, \( \mathcal{O}_L^{*1} \) is a free abelian group of rank \( g \) (as remarked already in \( \S 3 \)). Notice that \( \mathcal{O}_L^{*1} \cap \mathfrak{g}^* = \{ \pm 1 \} \), and notice that the group \( \mathcal{O}_L^{*1} \) is of finite index \( i \) in \( \mathcal{O}_L^* \).

We have, therefore, from (7.26),
\[
L(s, \lambda_m, A) = N_{L/Q}(\mathfrak{B})^s \lambda_m(N_{L/F}(\mathfrak{B}))^{-1} S(s, \lambda_m, \mathfrak{B}),
\]
(7.28)

where
\[
S(s, \lambda_m, \mathfrak{B}) = \sum'_{\eta \in \mathcal{O}_L \mathcal{O}_L^{*1}} \frac{\lambda_m(\eta \mathfrak{B})}{|N_{L/Q}(\eta)|^s}.
\]
(7.29)

Let \( \eta^{(i)}, i = 1, \ldots, g \), be generators of \( \mathcal{O}_L^{*1}/\{\pm 1\} \). Let \( \xi_j \) be the extension to \( L \) of the \( j \)th Galois embedding of \( F \) into \( \mathbb{R} \) chosen so that \( \xi_j(\sqrt{\Delta}) = \sqrt{\Delta^{(j)}} > 0 \), \( j = 1, \ldots, g \). Let \( \eta_j = \xi_j(\eta) \) for \( \eta \in L \). We may suppose that \( \xi_j^{(i)} > 0 \) for \( i, j = 1, \ldots, g \).

We have
\[
S(s, \lambda_m, \mathfrak{B}) = \sum'_{\eta \in \mathcal{O}_L \mathcal{O}_L^{*1}} \frac{\lambda_m(\eta \mathfrak{B})}{|N_{L/Q}(\eta)|^s} \eta_1 \eta_2 \cdots \eta_g \eta_1^{-s}.\]
(7.30)
Hecke observed the following identity (see [65, p. 161]): for $a, b \in \mathbb{R}$, $a, b \neq 0$,

$$\int_{-\infty}^{\infty} \frac{dv}{(a^2e^v + b^2e^{-v})^s} = \frac{c(s)}{|ab|^s},$$  \hspace{1cm} (7.31)

where

$$c(s) = \int_{-\infty}^{\infty} \frac{dv}{(e^v + e^{-v})^s}. \hspace{1cm} (7.32)$$

Let $N(c(s)) = \prod_{j=1}^{g} c(s_j)$. We deduce that

$$N(c(s))S(s, \lambda_{-m}, \mathcal{B}) = \sum_{\eta \in \mathcal{B}/\mathcal{O}} \prod_{j=1}^{g} \int_{-\infty}^{\infty} \frac{dv_j}{(\eta_j e^{v_j} + \eta_j^{-1} e^{-v_j})^s_j}. \hspace{1cm} (7.33)$$

Using the embeddings $\xi_j, j = 1, \ldots, g$, we may embed any element of $L$ into $\mathbb{R}^g$. The transformation $\eta \mapsto e^\eta, \xi \in \mathcal{O}_+^{g}$, with $\xi_j > 0$, $j = 1, \ldots, g$, then corresponds to a vector translation given componentwise by

$$v_j \mapsto v_j + 2\log \xi_j, \hspace{0.5cm} j = 1, \ldots, g.$$  

Let $\mathcal{L}_g$ be the lattice in $\mathbb{R}^g$ generated by the vectors $(2\log \xi_j^i)_{j=1}^{g}, i = 1, \ldots, g$. Then, from (7.33), we deduce that

$$N(c(s))S(s, \lambda_{-m}, \mathcal{B}) = \sum_{\eta \in \mathcal{B}/\mathcal{O}} \int_{\mathcal{L}_g} \frac{N(d\nu)}{\prod_{j=1}^{g} (\eta_j e^{v_j} + \eta_j^{-1} e^{-v_j})^s_j}, \hspace{1cm} (7.34)$$

where $N(d\nu) = \prod_{j=1}^{g} dv_j$.

Now suppose that the ideal $\mathcal{B}$ has basis $[1, w]$ (with $w > \overline{w}$). Then $\eta = cw + d$ for $c, d \in \mathcal{O}, \eta_j = c^{(j)}w_j + d^{(j)}$, and

$$\eta_j^2 e^{v_j} + \eta_j^{-2} e^{-v_j} = (c^{(j)}w_j + d^{(j)}e^{v_j} + (c^{(j)}\overline{w}_j + d^{(j)}e^{-v_j})^2. \hspace{1cm} (7.35)$$

Let $w^+_j = \max(w_j, \overline{w}_j)$, and let $w^-_j = \min(w_j, \overline{w}_j), j = 1, \ldots, g$. Make the change of variables

$$z_j = \frac{w_j^+ + \sqrt{-1} e^{v_j} + w_j^-}{\sqrt{-1} e^{v_j} + 1}, \hspace{0.5cm} j = 1, \ldots, g. \hspace{1cm} (7.36)$$

Then, as $v_j$ ranges from $-\infty$ to $\infty$, the variable $z_j$ runs over the geodesic in $\mathcal{H}$ joining $w^-_j$ to $w^+_j$. A direct calculation shows, with $y_j = \Im(z_j)$, that

$$y_j |c^{(j)}z_j + d^{(j)}|^{-2} = (w_j^+ - w_j^-) \{e^{v_j} (c^{(j)}w_j + d^{(j)}e^{v_j})^2 + e^{-v_j} (c^{(j)}w_j + d^{(j)}e^{-v_j})^2\}^{-1}. \hspace{1cm} (7.37)$$

In the notation of §4, let $h = (\alpha, \beta, \gamma) \in W_{\Delta}$, and let $w^+_j = z^+_j, w^-_j = z^-_j$. Then $\mathcal{B} = \mathcal{B}_h = \mathcal{O} + \mathcal{Z}_a^+$ and $w^+_j - w^-_j = \sqrt{\Delta^{(j)}}/|a^{(j)}|$ if $a \neq 0$. Under the change of variables in (7.36), $\mathcal{O}_h$. From (7.33) and

$$N(c(s))S(s, \lambda_{-m}, \mathcal{B}) \hspace{1cm} (7.34)$$

Now,

$$N_{L/Q}(\mathcal{B})$$

Let $\mathcal{A}_h$ denote the closure of $\mathcal{B}_h$ in $L$. Using (7.39), we deduce

$$L(2s, \lambda_{-2m}) \sum_{\mathcal{A}_h}$$

There is a bijection $\mathcal{E}_h$ corresponding to $\mathcal{O}_h$ and that

$$L(2s, \lambda_{-2m}) \hspace{1cm} \sum_{\mathcal{E}_h}$$

We deduce finally

**PROPOSITION 7.1.**

For $\Delta > 0$, we have

$$W_{Eis}(\Delta, t, m)$$

From Proposition...
variables in (7.36), the quotient $\mathcal{L}_g \backslash \mathbb{R}^g$ becomes the quotient $\Gamma_h \backslash \Gamma_r(\mathbb{R})$ realized as $\mathcal{G}_h$. From (7.33) and (7.37), we deduce

$$N(c(s))S(s, \lambda, -m, \mathcal{B}) = \prod_{j=1}^{g} \left( \frac{\sqrt{A^{(j)}}}{|a^{(j)}|} \right)^{-s_j} \times \sum_{[c,d]} \int_{\mathcal{L}_g} \prod_{j=1}^{g} \gamma_j^{s_j} (c^{(j)} z_j + d^{(j)} |^{-2s_j}) d\mathcal{S}. \hspace{1cm} (7.38)$$

Now,

$$N_{L/Q}(\mathcal{B})^\lambda \lambda_m (N_{L/F}(\mathcal{B})) = \prod_{j=1}^{g} (N_{L/F}(\mathcal{B}))^{s_j} = \prod_{j=1}^{g} |a^{(j)}|^{-s_j} \hspace{1cm} (7.39)$$

Let $A_h$ denote the ideal class of $\mathcal{B}_h$. Combining (7.6), (7.7), (7.28), (7.29), (7.38), and (7.39), we deduce that

$$L(2s, \lambda, -2m) \int_{\mathcal{G}_h} E(z, s, m) d\mathcal{S} = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_m(\Delta)L(s, \lambda, -m, A_h). \hspace{1cm} (7.40)$$

There is a bijection between the ideal classes of $\mathcal{O}_L$ and the representatives of $\mathcal{G}_\Delta$ with $\mathcal{G}_h$ corresponding to $A_h$, $h \in W_\Delta$. We conclude that

$$L(2s, \lambda, -2m) \sum_{\mathcal{G}_h \in \mathcal{G}_\Delta} \int_{\mathcal{G}_h} E(z, s, m) d\mathcal{S} = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_m(\Delta)L(s, \lambda, -m, L) \hspace{1cm} (7.41)$$

and that

$$L(2s, \lambda, -2m) \sum_{\mathcal{G}_h \in \mathcal{G}_\Delta} \int_{\mathcal{G}_h} E(z, s, m) d\mathcal{S} = N(c(s))iN_{F/Q}(\Delta)^{s/2} \lambda_m(\Delta) \times L(s, \lambda, -m)L(s, \chi_{L/F} \lambda, -m). \hspace{1cm} (7.42)$$

We deduce finally the following result.

PROPOSITION 7.2

For $\Delta \gg 0$, we have

$$W_{\text{Eis}}(\Delta, t, m) = N\left(c\left(\frac{1}{2} + it\right)\right) \prod_{l=1}^{\ell} \left( L(1/2 + it, \lambda, -m) L(1 + it, \lambda, -2m) \times \frac{N_{F/Q}(\Delta)^{1/4 + it/2} \lambda_m(\Delta)}{\mu(\Delta)} \right) \prod_{l=1}^{\ell} \left( L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F} \lambda, -m\right) \right). \hspace{1cm} (7.43)$$

From Propositions 7.1 and 7.2, we deduce directly the following.
LEMMA 7.3
For $\Delta \ll 0$ and as $|N_{F/Q}(\Delta)| \to \infty$, we have
\[
|W_{\text{Eis}}(\Delta, t, m)| \ll \frac{|N_{F/Q}(\Delta)|^{1/4}}{h(\Delta)} \frac{1}{|L(1 + it, \chi_{-m})|} \times \left| L\left(\frac{1}{2} + it, \chi_{-m}\right) L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F} \chi_{-m}\right) \right|.
\]
(7.44)

For $\Delta \gg 0$ and as $|N_{F/Q}(\Delta)| \to \infty$, we have
\[
|W_{\text{Eis}}(\Delta, t, m)| \ll \frac{N_{F/Q}(\Delta)^{1/4}}{\mu(\Delta)} \frac{1}{|L(1 + it, \chi_{-2m})|} \times \left| L\left(\frac{1}{2} + it, \chi_{-m}\right) L\left(\frac{1}{2} + it, \chi_{F(\sqrt{\Delta})/F} \chi_{-m}\right) \right|.
\]
(7.45)
The implied constants depend only on the field $F$.

In [16], the Eisenstein Weyl sums for $g = 1$ are shown to be proportional to the Fourier coefficients of Eisenstein series of half-integral weight and level 4 using explicit formulae for these coefficients derived in [23]. In [51], general formulae for Fourier coefficients of Eisenstein series of half-integral weight and level dividing 4 for the group $\text{Sp}(m, F)$ are obtained, where $F$ is a totally real algebraic number field. The case $m = 1$ gives generalizations of the formulae of [23] to the Hilbert modular case. In the notation of [51, Th. 6.1], the product $L(s, \chi_{-m}) L(s, \chi_{F(\sqrt{\Delta})/F} \chi_{-m})$ occurring (at $s = 1/2 + it$) in Propositions 7.1 and 7.2 is proportional to the product of the first Fourier coefficient and the $\Delta$th Fourier coefficient $c(\Delta, s)$ of the Eisenstein series $E' = E'(z, s, 1/2, 0, \chi_{-m}, 4)$ of level 4 and weight $1/2$. The $c(\Delta, s)$ of [51] correspond to $|N_{F/Q}(\Delta)|^{1/2} \rho(\Delta, E')$ with the conventions that we adopt in §3, (3.8).

These results enable us to bound the Eisenstein Weyl sums from above in terms of the Fourier coefficients of Eisenstein series of weight $1/2$ and level 4 or, alternatively, central values of L-functions.

8. Expected subconvexity results and proof of Theorem 1.2
We continue with the assumptions and notation of §§6 and 7. The equidistribution results of Theorem 1.2 would follow, without GRH, from an unconditional proof of
\[
\lim_{|N_{F/Q}(\Delta)| \to \infty} W_{\text{cusp}}(\Delta, \lambda) = 0
\]
and
\[
\lim_{|N_{F/Q}(\Delta)| \to \infty} W_{\text{Eis}}(\Delta, t, m) = 0,
\]
given that the uniformity of the convergence on the left-hand sides is controlled in terms of the other parameters occurring there ($\lambda$, $t$, and $m$). For example, one must provide by the Eulerian identities to the case $\Delta = 1$ (since the eigenvalues of the $*$-operator $f$ a cusp form will have order $k + 1$), and in the case $\Delta > 1$ (since $c(\Delta)|N_{F/Q}(\Delta)|^{-1/2} \rho(\Delta, E')$ in Lemma 7.3, the density of the central value result for $L$-functions of $\text{GL}(2, F)$ in the Maass case has been proved). This method is not sufficient for (8.1) and (8.2) to follow from GRH, so our

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Note. Just after completing the paper, N. Katz and Ullmo [12], independently obtained a result equivalent, even though...
control the denominator \( L(1 + it, \lambda - 2m) \) occurring in Lemma 7.4 as a function of \( m \) and \( t \). This can, in fact, be done, as is discussed in full detail in [12]. The main difficulty is controlling the convergence in the \( \Delta \)-aspect.

As \( \Delta \) is a fundamental (relative) discriminant that is totally definite, \( h(\Delta) \) can be replaced by the class number of \( \mathcal{O}_L, L = F(\sqrt{\Delta}) \) (see, e.g., [15, Chap. 7], [37]). Moreover, for \( \Delta \gg 0 \), we have, by [18, p. 36], \( \mu(\Delta) = h(\Delta)R \), where \( R \) is a regulator associated to \( \mathcal{O}_{L,1}^* \) and given by

\[
R = \det (2 \log \ell_j(t))_{i,j=1}^d.
\]

The results of 

\[ h(\Delta) \gg \varepsilon |N_{F/\mathbb{Q}}(\Delta)|^{1/2 - \varepsilon} \quad \text{as} \quad |N_{F/\mathbb{Q}}(\Delta)| \to \infty
\]

and

\[ h(\Delta)R \gg \varepsilon |N_{F/\mathbb{Q}}(\Delta)|^{1/2 - \varepsilon} \quad \text{as} \quad |N_{F/\mathbb{Q}}(\Delta)| \to \infty,
\]

provided by the Brauer-Siegel theorem (see [8], [9], [56]) together with generalizations to the case \( g > 1 \) of the subconvexity results for \( g = 1 \) in [16, Th. 5], would imply (8.1) and (8.2). The corresponding subconvexity results for the holomorphic case have been shown in [14]. We would need the Fourier coefficients \( \rho(\Delta, f) \) for \( f \) a cusp form with \( L^2 \)-norm 1 or an Eisenstein series, with eigenvalue \( \lambda \) and half-integral weight \( k \), to have an upper bound in the \( \Delta \)-aspect as good as \( \rho(\Delta, f) \ll_k, \varepsilon c(\lambda)|N_{F/\mathbb{Q}}(\Delta)|^{-1/4 + \delta + \varepsilon} \) for a fixed \( \delta > 0 \) and a positive explicit constant \( c(\lambda) \). From Lemma 7.3, the desired result for Eisenstein series would follow from a subconvexity result for \( L \)-functions of \( \text{GL}(1, F) \). Partial progress toward subconvexity results in the Maass case has been made by Gelgely Harcos [24], but the complete adaptation of the \( \text{GL}(2, F) \) methods of [14] to the Maass case remains elusive (softer convexity results sufficient for (8.1), (8.2) have been claimed in [62]). Such results would also follow from GRH, so our Theorem 1.2 remains conditional.

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\section*{Note}

Just after completing the write-up of this paper, we received a preprint of Clozel and Ullmo [12], where similar, and more general, equidistribution results are independently obtained. The methods and language used in their paper are quite different, even though in certain aspects a comparison with this paper is likely implicit.
They use, in particular, methods in ergodic theory due to Ratner [40], formulae of Waldspurger [63], and generalizations of Hecke’s formulae on Eisenstein series due to Wielonsky [64]. Their treatment of results analogous to our Theorem 1.2 also appeals to as-yet-unproven subconvexity results. In another preprint, [13], these authors prove equidistribution results for certain families of Shimura subvarieties of positive dimension.* They use ergodic arguments, which do not give rates of convergence, in contrast to the methods used in this paper.

References


*Note added in proof: There is a sequel to this paper by Ullmo [60]. There are also two new preprints of Zhang [68] and Jiang, Li, and Zhang [27].
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modular forms of genus 2,


[57] ———, Symplectic geometry, Amer. J. Math. 65 (1943), 1 – 86. MR 0008094


http://www.math.u-psud.fr/~ullmo/liste-prepub.html


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