Multiplicative dependence and bounded height, an example

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Abstract. We show that every algebraic number τ such that τ and $1-\tau$ are multiplicatively dependent has absolute logarithmic height $h(\tau)$ bounded above by $\log 2$, this bound being moreover an isolated value attained only when $\tau = 2, 1/2$. We also find all algebraic $\tau \neq 0, \pm 1$ such that $\tau, \tau+1, \tau-1$ generate a multiplicative group of free rank 1.

1. Statement of results

In a forthcoming paper [BMZ], it will be shown that rational functions on a curve which are multiplicatively independent modulo constants may take multiplicatively dependent values at most in a set of bounded absolute height. In the present note, we treat what may be considered as a particular example of this situation involving the curve $x + y = 1$.

Before stating our results, we define the height we shall use. If K is a number field and $x \in K$, the absolute logarithmic Weil height $h(x)$ of x is defined as follows. Let M_K be the set of places of K. For $v \in M_K$, we normalise the absolute value $|\,.\,|_v$ of K by requiring that, for $x \in K$,

$$|x|_v = \|x\|_v^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}$$

where $\|\,.\,\|_v$ is the unique extension to the completion K_v of the ordinary real or p-adic absolute value in $\mathbb{Q}_v$. For real $a \geq 0$ we set $\log^+ a = \log\max(1, a)$. With these conventions, we define

$$h(x) = \sum_v \log^+ |x|_v$$

where the summation is over all $v \in M_K$. The value of $h(x)$ is independent of the field K containing x used in the above formula. A few basic properties of the height are as follows. For an algebraic number $x \neq 0$,

$$h(x) = h(x^{-1}).$$

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For algebraic numbers $x, y \neq 0$,

$$h(xy) \leq h(x) + h(y)$$

and

$$h(x+y) \leq h(x) + h(y) + \log 2.$$

In this paper we study the set of algebraic numbers τ at which the multiplicatively independent functions $f_1(t) = t$ and $f_2(t) = 1 - t$ take multiplicatively dependent values and find a sharp upper bound for their absolute height. Namely, we prove the following result.

Theorem 1. *Let $\tau \neq 0, 1$ be an algebraic number for which τ and $1 - \tau$ are multiplicatively dependent. Then $h(\tau) \leq \log 2$, with the upper bound being attained exactly when $\tau = 2$ or $\tau = 1/2$.*

We also show the following.

Proposition 1. *Let τ be as in the theorem with $\tau \neq 2, 1/2$. Then there is an absolute constant $0 < c < \log 2$ such that $h(\tau) \leq c$. Therefore $\log 2$ is an isolated value for $h(\tau)$ with τ as in the theorem.*

We have as an immediate consequence of the proposition.

Corollary 1. *Let τ be as in the theorem. Then $h(\tau) + h(1-\tau) \leq 2\log 2$, with the upper bound being attained only when $\tau = 1/2$. If $\tau \neq 1/2$, there is an absolute constant $0 < c' < 2\log 2$ such that $h(\tau) + h(1-\tau) \leq c'$.*

We recall for comparison an unconditional result of Zagier [Z] who showed that for every algebraic number τ with $\tau \neq 0, 1, -\rho, -\bar{\rho}$, with ρ a primitive cubic root of unity, we have

$$h(\tau) + h(1-\tau) \geq (1/2)\log((1+\sqrt{5})/2).$$

The lower bound is in fact achieved when τ is a primitive tenth root of unity.

In Section 2 we give a proof of the theorem, in Section 3 a proof of the proposition and we briefly discuss in Section 4 a higher rank example. We wish to thank D. Masser for suggesting we look at the problem of the theorem and that of Section 4.

2. Proof of the theorem

In what follows, let $\varepsilon_1, \varepsilon_2, \ldots$ and $\eta_1, \eta_2, \ldots$ denote roots of unity. Let τ be an algebraic number such that τ and $1-\tau$ are multiplicatively dependent.

We can therefore write, for some non-zero algebraic number t, not a root of unity, and certain integers a and b,

$$\tau = \varepsilon_1 t^a, \qquad 1 - \tau = \eta_1 t^b, \tag{1}$$

so that

$$t^a = \varepsilon_2 t^b + \eta_2. \tag{2}$$

It follows that,

$$t^{-a} = \varepsilon_3 t^{-(a-b)} + \eta_3. \tag{3}$$

One may clearly assume in (2) that $a \geq b \geq 0$ and that a and b are coprime.

We first check the cases $b = 0$ and $a = b$. If $b = 0$, then by (2)

$$t^a = \varepsilon_2 + \eta_2. \tag{4}$$

By basic properties of heights (see Section 1), this shows that $h(\tau) \leq \log 2$. If $a = b$, by (3) we can apply the same argument to τ^{-1} and use that $h(\tau) = h(\tau^{-1})$. One checks easily that the upper bound $\log 2$ is only attained here for $\tau = 2$ or $\tau = 1/2$.

We may assume therefore that $a > b > 0$. Then, in particular, both t and t^{-1} are algebraic integers, so that t is a unit. Therefore τ is also a unit. If d denotes the degree of τ and $\sigma_1(\tau), \ldots, \sigma_d(\tau)$ the conjugates of τ, it follows, using $h(\tau) = h(\tau^{-1})$, that τ has absolute logarithmic Weil height given by

$$h(\tau) = \frac{1}{d} \sum_{i=1}^{d} \log^+(|\sigma_i(\tau)|) = \frac{1}{d} \sum_{i=1}^{d} \log^+(|\sigma_i(\tau)^{-1}|), \tag{5}$$

where $|\,.\,|$ is the usual euclidean absolute value on $\mathbb{C}$. By (2), we have

$$|t|^a \leq |t|^b + 1 \tag{6}$$

so that $|t|$ does not exceed the positive real root x_0 of

$$x^a = x^b + 1. \tag{7}$$

Clearly $x_0 > 1$ since $a > b$. We introduce a real parameter $\lambda > 1$ defined by

$$a = \lambda b. \tag{8}$$

Let $\xi = x_0^b$. Then by (7), we have that $\xi = \xi(\lambda)$ is the unique real number greater than 1 satisfying,

$$\xi^\lambda = \xi + 1. \tag{9}$$

The implicit function theorem implies that $\xi = \xi(\lambda)$ is a differentiable function of λ. We have $|t|^b \leq \xi$ so that

$$\log |t| \leq \frac{1}{b} \log \xi$$

and so by (1) and (9)

$$\log|\tau| \le \lambda \log \xi = \log(\xi + 1). \tag{10}$$

We can apply the above discussion also to t^{-1}. Namely by (3),

$$|t^{-1}|^a \le |t^{-1}|^{(a-b)} + 1. \tag{11}$$

Let $\widetilde{\xi}$ be the unique real number greater than 1 satisfying

$$\widetilde{\xi}^{\widetilde{\lambda}} = \widetilde{\xi} + 1, \tag{12}$$

where

$$\widetilde{\lambda} = \frac{\lambda}{\lambda - 1}. \tag{13}$$

We have, as the analogue of (10), that

$$\log|\tau^{-1}| \le \widetilde{\lambda} \log \widetilde{\xi} = \log(\widetilde{\xi} + 1). \tag{14}$$

By a similar discussion to the above, we deduce that both (10) and (14) remain true when we replace τ by any of its conjugates. Let μ with $0 < \mu < 1$ be the rational number such that μd conjugates of τ exceed 1 in absolute value. Then $(1 - \mu)d$ conjugates of τ do not exceed 1 in absolute value. Together with (5), it follows from (10) applied to the conjugates of τ exceeding 1, that

$$h(\tau) \le \mu \log(\xi + 1) \tag{15}$$

and from (14) applied to the conjugates of τ not exceeding 1, that

$$h(\tau) \le (1 - \mu) \log(\widetilde{\xi} + 1). \tag{16}$$

The following lemma completes the proof of the theorem.

Lemma 1. *When $a > b > 0$ in (1), we have $h(\tau) \le S(\lambda)$ where, for $\lambda > 1$,*

$$S(\lambda) = \{\log(\xi + 1)\log(\widetilde{\xi} + 1)\}/\{\log(\xi + 1) + \log(\widetilde{\xi} + 1)\}.$$

The minimum of $S(\lambda)$ is attained when $\lambda = 2$ and $S(2) = \log(\frac{1}{2}(1 + \sqrt{5}))$, whereas $S(\lambda)$ is strictly bounded above by $\log 2$.

Proof. For any positive reals A and B

$$\min(A\mu, B(1 - \mu)) \le \frac{AB}{A + B}.$$

To see this, suppose that $A\mu > \frac{AB}{A+B}$. Then $\mu > \frac{B}{A+B}$ and so $1 - \mu < \frac{A}{A+B}$. A similar argument applies to the case $B(1 - \mu) > \frac{AB}{A+B}$. This shows that the first statement of the lemma follows from (15) and (16). Let

$$R(\lambda) = S(\lambda)^{-1} = \frac{1}{\log(\xi + 1)} + \frac{1}{\log(\widetilde{\xi} + 1)}. \tag{17}$$

As $\lambda \to 1^+$ then $\xi \to \infty$ and as $\lambda \to \infty$ then $\xi \to 1$. Notice that $\lambda \to 1^+$ corresponds to $\widetilde{\lambda} \to \infty$ and *vice versa.* In both these limits $R(\lambda)$ tends to $1/(\log 2)$. We now show that there is a unique solution λ to $\frac{d}{d\lambda}R = 0$, namely $\lambda = 2$. We write

$$R(\lambda) = f(\lambda) + f(\widetilde{\lambda}) \tag{18}$$

where

$$f(\lambda) = \frac{1}{\log(\xi(\lambda) + 1)}. \tag{19}$$

Recall that $\xi = \xi(\lambda)$ is the unique real solution of (9) greater than 1. Using $\widetilde{\lambda} = \lambda/(\lambda - 1)$ and differentiating (18), we see that solving $\frac{d}{d\lambda}R = 0$ amounts to solving

$$\lambda^2 f'(\lambda) = \widetilde{\lambda}^2 f'(\widetilde{\lambda}) \tag{20}$$

where f' is $\frac{d}{d\lambda}f$. Differentiating (9) gives, after some simplifications,

$$\xi' = \frac{-\xi^\lambda \log\xi}{\lambda\xi^{\lambda-1} - 1} \tag{21}$$

where $\xi' = \frac{d}{d\lambda}\xi$. A simple computation using (21) gives

$$g(\lambda) =: \{\lambda^2 f'(\lambda)\}^{-1} = (\lambda\xi^{\lambda-1} - 1)\log\xi. \tag{22}$$

Notice that one solution of (20) is given by $\lambda = \widetilde{\lambda}$ and as we require $\lambda > 1$, this implies $\lambda = 2$. As λ increases, the function $\widetilde{\lambda}$ decreases and *vice versa*, so that if $g(\lambda)$, and therefore $\lambda^2 f'(\lambda)$, is monotonic in $\lambda > 1$ then the only solution of (20) is $\lambda = 2$. To check monotonicity, we differentiate $g(\lambda)$ with respect to λ. After a straightforward computation using (21) we obtain,

$$g'(\lambda) = \frac{\lambda(\log^2\xi)\xi^{\lambda-1}}{\lambda\xi^{\lambda-1} - 1}(\xi^{\lambda-1} - 1)$$

and as $\lambda, \xi > 1$ the right hand side is always positive. Hence, the function $g(\lambda)$ is monotone increasing in the region $\lambda > 1$, so that $\lambda^2 f'(\lambda)$ is monotone decreasing, proving that (20) holds only at $\lambda = 2$. We have shown that there is a unique zero of $\frac{d}{d\lambda}R = 0$ at $\lambda = 2$. Now $R(2) = 1/\log(\frac{1}{2}(1 + \sqrt{5})) > 1/\log 2$, so as $R(2)$ is greater than its limits at 1 and at ∞ we see that $1/(\log 2)$ is a strict lower bound for $R(\lambda)$ when $\lambda > 1$. Therefore $\log 2$ is a strict upper bound for $S(\lambda)$ when $\lambda > 1$. This completes the proof of the lemma. □

3. Proof of the proposition

Let τ be as in the statement of the proposition and adopt the same notations as in the previous section. We write, as in Section 2, for some non-zero algebraic number t, not a root of unity, and certain integers a and b,

$$\tau = \varepsilon_4 t^a, \qquad 1-\tau = \eta_4 t^b.$$

We can assume $a \geq b \geq 0$. The cases $b = 0$ and $a = b$ being easy to settle, we assume that $a > b > 0$ with a and b coprime. Then

$$\varepsilon_4 t^a = -\eta_4 t^b + 1.$$

By replacing t by t^{-1} if necessary (see (3), section 2) and using $h(\tau) = h(\tau^{-1})$, we may further assume that $a \geq 2b > 0$. Let ε_5 satisfy $\varepsilon_5^b = -\eta_4$. Then, setting $u = \varepsilon_5 t$ and $\varepsilon_6 = \varepsilon_4 \varepsilon_5^{-a}$, we have

$$\varepsilon_6 u^a = u^b + 1. \tag{23}$$

It suffices now to estimate $h(u^a) = h(t^a) = h(\tau)$. Let $\lambda = a/b \geq 2$. We have, from the lemma of Section 2,

$$h(u^a) \leq S(\lambda).$$

By (23)

$$(a-b)h(u) \leq \log 2$$

and so

$$h(u^b) \leq \frac{\log 2}{\lambda - 1}. \tag{24}$$

We now appeal to the following consequence of a result of Bilu.

Lemma 2 (Bilu's Lemma, see [B], Theorem 1.1). *Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of non-zero algebraic numbers such that $deg(\alpha_k) \to \infty$ and $h(\alpha_k) \to 0$ as $k \to \infty$. Let $f : \mathbb{C}^* \to \mathbb{R}$ be a bounded continuous function. Then*

$$\lim_{k\to\infty} \frac{1}{[L_k : \mathbb{Q}]} \sum_{\sigma} f(\sigma(\alpha_k)) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})d\theta.$$

Here L_k is any number field containing α_k and the summation is over all distinct embeddings of L_k into $\mathbb{C}$.

Let L be a normal extension of $\mathbb{Q}$ containing ε_6 and u. Let $G = \mathrm{Gal}(L/\mathbb{Q})$ and $D = |G| = [L : \mathbb{Q}]$. The average

$$A(u^b) = \frac{1}{D} \sum_{\sigma\in G} \log^+ |1 + \sigma(u)^b| \tag{25}$$

does not depend on the above choice of L. From (23), since $\lambda \geq 2$, for any conjugate $\sigma(u)$ of u over $\mathbb{Q}$ we have either $|\sigma(u)^{2b}| \leq 1$ or

$$|\sigma(u)^{2b}| \leq |\sigma(u)^a| \leq |\sigma(u)^b| + 1$$

so that in any case $|\sigma(u^b)| \leq 2$. Therefore $\log^+ |1 + \sigma(u)^b| \leq \log 3$. By (24), the height of u^b satisfying (23) tends to zero as λ tends to infinity. Moreover, since we assumed that u is not a root of unity, the degree of u^b satisfying (23) tends to infinity with λ, by (24) and Northcott's Theorem. We therefore have, applying Bilu's Lemma to the u^b and to $f(z) := \min\{\log^+ |1 + z|, \log 3\}$, that

$$\lim_{\lambda \to \infty} A(u^b) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |1 + e^{i\theta}| d\theta = c_1 < \log 2. \tag{26}$$

Let $c_2 = (c_1 + \log 2)/2$. Then there exists a λ_0, independent of τ, such that for $\lambda > \lambda_0$,

$$A(u^b) \leq c_2 < \log 2. \tag{27}$$

On the other hand, since u is an algebraic integer, we have by (23),

$$h(u^a) = \frac{1}{D} \sum_{\sigma \in G} \log^+ |\sigma(u)^a| = A(u^b).$$

Hence, for all $\lambda \geq 2$,

$$h(u^a) \leq \max(S(\lambda_0), c_2) < \log 2,$$

since $S(\lambda_0) \geq S(\lambda)$ for $\lambda_0 \geq \lambda \geq 2$, by the arguments of Section 2. This completes the proof of the proposition.

A suitable constant c as in the statement of the proposition may be found using an effective version of Bilu's Lemma. For τ as in the statement of the theorem, a limit point of $\max(h(\tau), h(1 - \tau))$ is

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |1 + e^{i\theta}| d\theta$$

as can be seen by putting $\tau = 1 - e^{2\pi i/p}$ where p is a prime.

4. A higher rank example

There are numerous questions related to those of the present paper that may be asked. As an example consider the following.

Problem 1. *Find all algebraic $\tau \neq 0, \pm 1$ such that $\tau, \tau + 1, \tau - 1$ satisfy two independent multiplicative relations, that is generate a multiplicative group of free rank* 1.

We are able to show:

Answer 1. *If τ is as in Problem* 1, *then τ is either a* 12*-th root of unity not equal* ± 1, *that is a primitive* 12*-th,* 6*-th,* 4*-th or* 3*-rd root of unity, or τ is one of the following:*

(i) $\pm(\frac{1}{2}(1 \pm \sqrt{5}))^{\pm 1}$,
(ii) $\pm s^{\pm 1}$ *where* s *satisfies* $s^3 + s^2 - 1 = 0$,
(iii) *a solution of* $x^2 - \xi x - 1 = 0$ *for any* 6*-th root of unity* $\xi \neq \pm 1$.

In all of the above cases the degree of τ *is at most* 4.

A τ such as in the statement of the problem satisfies $h(\tau) \leq \log 2$, by the result of the theorem in Section 1 of this paper. Once we show that τ has degree at most 4, there is only a finite list of possibilities for τ. We did the direct computation of the possible τ's by hand. Although we only give an outline here, the remaining details are straightforward, if a little tedious, to complete.

Sketch of proof. By the multiplicative dependence assumption, there is a non-zero algebraic number t, not a root of unity, and integers a, b, c such that,

$$\tau = \varepsilon_7 t^a, \qquad \tau + 1 = \varepsilon_8 t^b, \qquad \tau - 1 = \varepsilon_9 t^c,$$

with $\varepsilon_7, \varepsilon_8, \varepsilon_9$ roots of unity. We can suppose that b, c are not both 0, and that a, b, c are not all equal 1. Moreover, we can suppose that the non-zero a, b, c are coprime. Suppose $a \neq 0$. Let τ' be any conjugate of τ over $\mathbb{Q}$, and put

$$\tau' = re^{i\theta}, \qquad r > 0, \quad \theta \in [-\pi, \pi). \qquad \square$$

We have, letting $\alpha = 2b/a$ and $\beta = 2c/a$,

$$|\tau' + 1|^2 = |\tau'|^\alpha,$$
$$|\tau' - 1|^2 = |\tau'|^\beta.$$

Therefore,

$$r^2 + 1 + 2r\cos\theta = r^\alpha,$$
$$r^2 + 1 - 2r\cos\theta = r^\beta.$$

so that

$$2(r^2 + 1) = r^\alpha + r^\beta.$$

A straightforward calculus argument shows that there are at most two positive roots r of this equation for given α and β. Also, for r given, it is clear that $\cos\theta$ is uniquely determined so that $re^{i\theta}$ has at most two possibilities. There are therefore at most 4 possibilities for τ', proving that τ is of degree at most 4. Moreover, τ is necessarily a unit and so the product of the absolute values of its conjugates over $\mathbb{Q}$ equals 1. This further restricts the possibilities. When $a = 0$, so that τ is a root of unity, we may argue in a similar way with $\tau + 1$ replacing τ if $b \neq 0$ and with $\tau - 1$ replacing τ if $c \neq 0$. We again show that τ must have degree at most 4.

It would be interesting, in the light of [BMZ], to find explicit upper bounds for the heights of points on curves where multiplicatively independent rational functions

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[2] D. Zagier, Algebraic
[3] Y. Bilu, Limit distr
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become multiplicatively dependent for more complicated examples than the one considered here. At the same time, the example $x + y = 1$ motivating this paper occurs in many important contexts.

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