1. Let $A = \begin{bmatrix} 1 & x \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 8 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 \\ 0 & 7 \end{bmatrix}$, and $D = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Compute each of the following.

(a) $B + 3D$

The dimensions of $B$ and $D$ are both $2 \times 3$. Therefore, the operation $B + 3D$ is valid.

$$
B + 3D = \begin{bmatrix} 2 & -3 & 0 \\ 8 & 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 5 & -9 \\ 15 & 3 \end{bmatrix} = \begin{bmatrix} 17 & -27 \\ 33 & 10 \end{bmatrix}
$$

(b) $2C + B$

We cannot add $2C$ and $B$ because they are not the same size.

(c) $4D - 3C^T$

The dimensions of $D$ and $C^T$ are both $2 \times 3$. Therefore, the operation $4D - 3C^T$ is valid.

$$
4D - 3C^T = 4 \begin{bmatrix} 5 & 12 \\ 0 & 7 \end{bmatrix} - 3 \begin{bmatrix} 4 & 0 \\ 6 & -10 \end{bmatrix} = \begin{bmatrix} 10 & 48 \\ 0 & 21 \end{bmatrix} = \begin{bmatrix} 2 \times 3 - 2 \times 3 \\ \end{bmatrix}
$$

(d) $4a_{12} - 2c_{12} + 7d_{13}$

$$
= 4(3) - 2(10) + 7(3a) = 12 + 20 + 21a = 32 + 21a
$$
(e) $DB$ not possible because the # of columns of $D$ does not equal the # of rows of $B$.

(f) $B^TDA$ not possible because the # of columns of $D$ is not equal to the # of rows of $A$.

(g) $CD^T$ not possible

- Size of $C$: $3 \times 2$
- Size of $D^T$: $3 \times 2$
- Size of $D$: $2 \times 3$
(h) $BB^T$

\[ B = \begin{bmatrix} 2 & -3 \\ x & 1 - k \\ 0 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & x \\ -3 & 1 - k \\ 0 & 1 \end{bmatrix} \]

$BB^T$ is $2 \times 2$

\[ C = BB^T = \begin{bmatrix} 2 & 8 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \]

$c_{11} = 4 + 9 + 0 = 13$

\[ c_{12} = \text{row } 1 \times \text{col } 2 = 2 \cdot \begin{bmatrix} 2 & 8 \end{bmatrix} = 16 - 3k + 0 = 16 - 3k \]

\[ c_{21} = 16 - 3k + 0 \]

\[ c_{22} = 64 + k^2 + 36 = k^2 + 100 \]

(i) $A^2 = A \cdot A$

\[ A = \begin{bmatrix} 4 & x \\ 3 & -7 \end{bmatrix} \]

\[ A \cdot A = \begin{bmatrix} 16+3x & -3x \\ 9 - 3x \cdot 14 \end{bmatrix} = \begin{bmatrix} 12 & -2 \end{bmatrix} \]
2) Using matrix algebra, solve for the matrix D:

\[ D = AD + B \]

\[ \frac{3}{2} \frac{2}{3} \times \frac{8}{5} \]

\[ D - AD = B \]

\[ \begin{bmatrix} I_n - A \end{bmatrix} D = (I_n - A)^{-1} B \]

\[ I_n D = (I_n - A)^{-1} B \]

\[ D = (I_n - A)^{-1} B \]
3) Solve for $x$ and $y$:

$$3 \begin{bmatrix} 2 & x \\ 5y & -1 \end{bmatrix} - \begin{bmatrix} -6 & 1 \\ 3y & -5 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ -2x & 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 3x-3y \\ 15y-1 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ -2x & 2 \end{bmatrix}$$

$3x - 3y = -7 \quad \rightarrow \quad 3x - 3y = -7$

$15y - 1 = -2x \quad \rightarrow \quad 2x + 15y = 1$

$$\begin{bmatrix} 3 & -3 & -7 \\ 2 & 15 & 1 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 10 & -2 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$x = -2 \quad y = \frac{1}{3}$$
4) The times (in minutes) required for assembling, testing, and packaging large and small capacity food processors are shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Assembling</th>
<th>Testing</th>
<th>Packaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large</td>
<td>45</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Small</td>
<td>30</td>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

(a) Define a matrix \( \mathbf{T} \) that summarizes the data above.

\[
\mathbf{T} = \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix}
\]

(b) Let \( \mathbf{M} = [100 \ 200] \) represent the number of large and small food processors ordered, respectively. Find \( \mathbf{MT} \) and explain the meaning of its entries.

\[
\mathbf{MT} = \begin{bmatrix} 100 & 200 \end{bmatrix} \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 6500 & 3500 & 2000 \end{bmatrix}
\]

The total minutes required for assembling, testing and packaging large and small food processors is 6500 minutes for large processors and 3500 minutes for small processors.
(c) If assembling costs $3 per minute, testing costs $1 per minute, and packaging costs $2 per minute, find a matrix \( C \) that, when multiplied with \( T \), gives the total cost for making each size of food processor.

\[
C = \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix}
\]

To make a large processor:

\[
45 \times 3 + 15 \times 1 + 10 \times 2
\]

To make a small processor:

\[
30 \times 3 + 10 \times 1 + 5 \times 2
\]
5) If \( \mathbf{A} = \begin{bmatrix} -3 & 7 \\ 8 & 10 \end{bmatrix} \), find \( \mathbf{A}^{-1} \).

\[
\mathbf{A}^{-1} = \begin{bmatrix} \frac{-5}{43} & \frac{7}{86} \\ \frac{4}{43} & \frac{3}{86} \end{bmatrix}
\]

6) If \( \mathbf{B} = \begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix} \), find \( \mathbf{B}^{-1} \).

\( \mathbf{B} \) has no inverse.

\( \mathbf{B} \) is singular.
7) Solve the following system of equations using matrix inverses.

\[ \begin{align*}
3x + 2y &= z + 2 \\
-3y + 2 &= -2x \\
x &= y + z + 4
\end{align*} \]

Use the matrix equation \( AX = B \).

\[ \begin{align*}
3x + 2y - z &= 2 \\
2x - 3y &= -2 \\
x - y - z &= 4
\end{align*} \] equivalent to \( AX = B \)

\[ A = \begin{bmatrix}
3 & 2 & -1 \\
2 & -3 & 0 \\
1 & -1 & -1
\end{bmatrix} \quad \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} \quad \begin{bmatrix}
2 \\
-2 \\
4
\end{bmatrix}
\]

Matrix \( A \) is the coefficient matrix, \( X \) is the variable column, and \( B \) is the constants column.

\[ A^{-1}AX = A^{-1}B \]

\[ X = A^{-1}B \]

\[ \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = X = \begin{bmatrix}
-1 \\
0 \\
-5
\end{bmatrix} \]

\[ x = -1 \quad y = 0 \quad z = -5 \]
9. Steel, electronics

City demands [500] units of steel
[800] units of electronics.

How many units of steel and electronics products should be produced by the village to meet its own needs and those of the nearby city?

Let $X_1 =$ the number of units of steel that should be produced.

Let $X_2 =$ the number of units of electronics.

Input-output matrix $A$: 

\[
\begin{pmatrix}
\text{output (steel)} \\
\text{input (electronics)} \\
\text{input (fuel, etc.)}
\end{pmatrix} = A
\]

Find the demand matrix $D$: 

\[
D = \begin{bmatrix}
500 \\
800
\end{bmatrix}
\]

Let $X = [x_1, x_2]$ 

\[
X = (I_2 - A)^{-1}D
\]

\[
X = \begin{bmatrix}
0.9162 \\
0.2881
\end{bmatrix} = [x_1, x_2]
\]

$x_1 = 501.9162$ units of steel should be produced.

$x_2 = 288.2881$ units of electronics should be produced.
(a) Input-output matrix $A$ is

$$
A = \begin{bmatrix}
0.31 & 0.11 & 0.05 \\
0.05 & 0.13 & 0.07 \\
0.02 & 0.12 & 0.08
\end{bmatrix}
$$

(b) Explain the meaning of the entries in row 1 of this matrix.

$a_{11} = 0.31$: the 0.31 units of crude used to produce one unit of crude product.

$a_{12} = 0.11$: the 0.11 units of crude used to produce one unit of refining product.

$a_{13} = 0.05$: the 0.05 units of crude used to produce one unit of chemical product.

(c) How many units of refining product are constructed in the production of 7500 units of crude product?

$1.0085 \times 7500 = 7500.5$ units of refining product.

(d) How many units of chemical product are required to produce 500 units of each sector in this economy?

$0.11 \times 500 + 0.12 \times 500 + 0.08 \times 500$

$= 55 + 72 + 40$

$= 167$ units of chemical product.
(e) If a neighboring city demands 5500 units of crude, 6750 units of refining, and 1,250 units of chemical products, how much should this economy produce to satisfy internal consumption and meet the demand of the city?

Let \( x_1 \) = the number of crude product that should be produced.

Let \( x_2 \) = the number of refining product.

Let \( x_3 \) = the number of chemical product.

Let \( X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) and \( D = \begin{bmatrix} 5500 \\ 6750 \\ 1250 \end{bmatrix} \).

\( X = (I - A)^{-1} D \)

\( X = \begin{bmatrix} 11400.0671 \\ 9050.9000 \\ 9105.3112 \end{bmatrix} \)

The economy should produce 11,400.0671 units of crude products, 9,050.9000 units of refining product, and 9,105.3112 units of chemical products.

(f) Referring to (e), how many units of each product are consumed internally in meeting the city's demand?

Total production - demand = internal consumption.

\( \begin{bmatrix} 11400.0671 \\ 9050.9000 \\ 9105.3112 \end{bmatrix} - \begin{bmatrix} 5500 \\ 6750 \\ 1250 \end{bmatrix} = \begin{bmatrix} 8400.0671 \\ 2300.9000 \\ 7856.3112 \end{bmatrix} \)

8,400.0671 units of crude product, 2,300.9000 units of refining product, and 7,856.3112 units of chemical products are used internally.